

## STOKES' FIRST PROBLEM FOR VISCOELASTIC FLUIDS

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### Summary

The theory given in this paper is based on a generalization of Boltzmann's equation of linear viscoelasticity in which the presence of a Newtonian viscosity is acknowledged. The solution of Stokes' first problem for this kind of fluid, with a viscosity and a relaxation kernel, are derived here for the first time. The formulas given in this paper form a basis for the numerical interpretation of the idea of an effective viscosity and relaxation modulus.

### 1. Introduction

Stokes' first problem is concerned with diffusion of vorticity in a fluid occupying a semi-infinite region above a plate which undergoes a step increase of velocity from rest. This problem is the same as the diffusion of heat when the temperature at the boundary is suddenly increased.

In this paper, we study this problem when the fluids are viscoelastic and satisfy a constitutive equation in which the effects of relaxing elasticity and Newtonian viscosity are simultaneously acknowledged. Our solutions are presented for their intrinsic value and as a contribution to the theory of the wave-speed meter (see Joseph et al. [1]). The wave-speed meter is a rheometrical instrument for measuring the speed of shear waves in liquids. Wave speeds and elastic moduli for many different liquids were measured by Joseph et al. [2].

The constitutive equation used here is a generalization of the one used by Boltzmann, in which the excess stress  $\tau$  is determined by

$$\tau = 2\mu\mathbf{D}[\mathbf{u}(\mathbf{x}, t)] + \int_0^\infty G(s)\mathbf{D}[\mathbf{u}(\mathbf{x}, t-s)] ds, \quad (1.1)$$

where  $\mathbf{u}(\mathbf{x}, t)$  is the velocity,  $\mathbf{D}[\mathbf{u}] = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T)$  is the rate of strain,  $s = t - \tau$  is the lapse time and  $\tau$  the past time,  $G(s)$  is a smooth, positive, decreasing (to zero) function, called a relaxation function and  $\mu$  is a positive constant, called the Newtonian viscosity. The integral expression represents effects of relaxing elasticity. Equation (1.1) is a general linearization of every kind of constitutive model for homogeneous, isotropic fluids which depend on the history of the first spatial gradient of the deformation. It says that such fluids are completely characterized in motions which perturb rest or, more generally, which perturb rigid motion by a viscosity and shear relaxation function. This defines a rheometrical problem which was addressed by the wave-speed meter to which we alluded earlier. A further discussion of the constitutive equation is given by Joseph et al. [1].

When  $\mu = 0$ , the system of equations governing the propagation of vorticity is hyperbolic. Vorticity waves into rest move forward with a constant speed given by  $(G(0)/\rho)^{1/2}$  where  $G(0)$  is the rigidity and  $\rho$  is the density of the liquid [3]. Shear waves are vorticity waves in one dimension.

The solution of Stokes' first problem for elastic fluids  $\mu = 0$  and the special relaxation function  $G(s) = (\eta/\lambda)e^{-s/\lambda}$  can be obtained from the solution of the corresponding problem of electrical transmission lines given in Carslaw and Jaeger [4]. This form of  $G(s)$  which has only one time of relaxation will be called a Maxwell kernel. Solutions of Stokes' first problem with  $\mu \neq 0$  and  $G(s) = (\eta/\lambda)e^{-s/\lambda}$  was given by Morrison [5]. Morrison studied the problem of an impulsively applied and subsequently maintained constant velocity of a modified Voigt material equivalent to a Jeffrey's model. The solution of Stokes' first problem for an elastic fluid  $\mu = 0$  and a general kernel is related to Berry's solution [6] for the propagation of longitudinal stress pulses down a viscoelastic fluid. He obtained formulas for the speed and attenuation of the wave. The formula  $c = (G(0)/\rho)^{1/2}$  appears already in this paper. Chu [7] solved Stokes' problem with  $\mu = 0$  and a general kernel. His was the first definitive general study of this problem. Tanner [8] studied Stokes' first problem for an Oldroyd B fluid. This fluid satisfies a non-linear constitutive equation, but in Stokes' first problem it reduces to the problem studied by Morrison. Tanner says that his form of the solution is not valid when  $\mu = 0$ . It may be hard to compute results from Tanner's solution when  $\mu$  is small. Huilgol [9] gave a solution that is computable for small  $\mu$ . Morrison [5] derived a different form of the solution and he gave some numerical results, but not for small values of  $\mu$ . Step jumps of velocity between two parallel plates in elastic liquids with Maxwell kernels have been studied by Böhme [10], Kazakia and Rivlin [11], Rivlin [12] and Christensen [13]. Stokes' first problem with  $\mu = 0$  and a general relaxation function was studied by Narain and Joseph [14,15]. They also considered the associated problem, considering reflections between

parallel plates. Some of the results of [14,15] were already obtained by Chu [7]. Renardy [16] studied Stokes' first problem for singular kernels  $\mu = 0$ ,  $G(0)$  finite,  $G'(0) = -\infty$ . He found that smooth solutions propagate in this case. Narain and Joseph [14] gave a shock layer analysis for the smoothing effects of viscosity. The part of this analysis which gives the size of the shock layer was corrected by Renardy (see Renardy, Hrusa, Nohel [22]). A complete shock layer analysis is given in [1]. The group of transformations satisfied by Stokes' first problem was given in [1] and was used to develop the theory of the effective viscosity and rigidity (see Section 6).

In the present work, we solve Stokes' first problem for  $\mu \neq 0$  and a general  $G(s)$  by obtaining an explicit computable formula for the inverse Laplace transform. We numerically check the idea of an effective viscosity and relaxation function. In his thesis, Preziosi [17] also studied other geometries that can be interesting for the design of new wave-speed meters.

In one variation of the problem the fluid is confined between parallel walls and the problem is how a steady Couette flow is established from the step change. In another variation, the angular velocity of one of the plates around some perpendicular axis is increased suddenly to a constant value. This spin-up problem models the transition from rest to rigid motion in a semi-infinite region or the development of Couette flow between parallel disks, one of which rotates. In a third variation the fluid is bounded by concentric cylinders. One of the cylinders is suddenly set into sliding motion along its axis; or one of the cylinders is suddenly put into rotation.

## 2. Generation of motion in a fluid in the semi-infinite region above a plane wall by a step change from rest of the velocity of the wall

An incompressible simple fluid fills the semi-infinite region  $y > 0$  above a plate at  $y = 0$ . The plate and fluid are at rest for  $t \leq 0$ . At  $t > 0$  the plate moves parallel to itself with a velocity  $Ue_x$ , where  $U$  is a positive constant. We look for transient solutions  $\mathbf{u} = u(y, t)e_x$ . For such fields there is only one component  $\tau^{12}$  of the excess stress tensor  $\tau$  given by (1.1)

$$\tau^{12} = \mu u_y(y, t) + \int_0^t G(s) u_y(y, t-s) ds,$$

and the equation of motion is  $\rho u_t = \tau_y^{12}$ ; that is,

$$\rho u_t = \mu u_{yy} + \int_0^t G(s) u_{yy}(y, t-s) ds, \quad (2.1)$$

where

$$u(0, t) = UH(t), \quad u(y, 0) = 0,$$

$$u(y, t) \rightarrow 0 \text{ as } y \rightarrow \infty \text{ at fixed } t. \tag{2.2}$$

Equations (2.1) and (2.2) define Stokes' first problem for the generalized Boltzmann constitutive equation (1.1).

The time derivative of (2.1)

$$\rho u_{tt} - G(0)u_{yy} = \mu u_{yyt} + \int_0^t G'(s)u_{yy}(y, t - s) ds \tag{2.3}$$

brings out a wave operator, with wave speed  $c = (G(0)/\rho)^{1/2}$ , on the left. There are no waves, in a strict sense, when  $\mu > 0$ .

The solution of Stokes' first problem for Newtonian fluids,  $\mu \neq 0$ ,  $G(s) = 0$ , is well-known and elementary. The solution of (2.1) and (2.2) for fluids with instantaneous elastic response was given by Chu [7] and by Narain and Joseph [14,15]. There are two papers in which (2.1) and (2.2) are solved for a viscous model with a Maxwell kernel [4,5]. In this case

$$\begin{aligned} \rho u_{tt} - \frac{\eta}{\lambda}u_{yy} &= \mu u_{yyt} - \frac{\eta}{\lambda^2} \int_0^t e^{-s/\lambda}u_{yy}(y, t - s) ds \\ &= \mu u_{yyt} + \frac{\mu}{\lambda}u_{yy} - \frac{\rho}{\lambda}u_t. \end{aligned} \tag{2.4}$$

The solution of Stokes' first problem for viscoelastic liquids in the linearized approximation (small  $U$ ) depends on a dimensionless number which may be identified as a viscosity ratio and a dimensionless relaxation function. Let  $\tilde{\mu} = \mu + \eta$  be the zero shear rate or static viscosity,  $\mu$  is the Newtonian viscosity and  $\eta = \int_0^\infty G(s) ds$  is the elastic viscosity. Then define a dimensionless velocity, time, position and relaxation function

$$[v, \tau, \zeta, g(\tau)] = \left[ \frac{u}{U}, \frac{tG(0)}{\eta}, y \left( \frac{G(0)\rho}{\eta\tilde{\mu}} \right)^{1/2}, \frac{G(t)}{G(0)} \right].$$

When written in these dimensionless variables, (2.1) becomes

$$v_\tau = Jv_{\zeta\zeta} + (1 - J) \int_0^\tau g(s')v_{\zeta\zeta}(\zeta, \tau - s') ds', \tag{2.5}$$

and (2.3) becomes

$$v_{\tau\tau} - (1 - J)v_{\zeta\zeta} = Jv_{\zeta\zeta\tau} + (1 - J) \int_0^\tau g'(\tau)v_{\zeta\zeta}(\zeta, \tau - s') ds' \tag{2.6}$$

where

$$J = \frac{\mu}{\tilde{\mu}} = \frac{\mu}{\mu + \eta} = \frac{\text{Newtonian viscosity}}{\text{static viscosity}}, \tag{2.7}$$

and  $0 \leq J \leq 1$ . When  $J = 0$  the fluid is instantaneously elastic. When  $J = 1$ , the fluid is Newtonian. For the special case of a kernel with one relaxation time,  $H(\tau) = e^{-\tau}$ , (2.5) and (2.6) may be combined to give

$$v_{\tau\tau} + v_\tau - v_{\zeta\zeta} = Jv_{\zeta\zeta\tau}.$$

This was the equation studied by Morrison [5], Tanner [8] and Huilgol [9,18]; it depends on a single parameter, the viscosity ratio. When  $J$  is small, the term on the right-hand side smooths a propagating Heaviside function with a shock layer whose thickness is  $\sqrt{Jx}$ .

### 3. Laplace transform of Stokes' first problem

We now apply the technique used by Narain and Joseph [14,15] for the elastic case ( $\mu = 0$ ) in the general case ( $\mu \neq 0$ ). We shall designate the Laplace transform of  $u(y, t)$  as

$$\bar{u}(y, \omega) = \int_0^{\infty} u(y, t) e^{-\omega t} dt, \quad (3.1)$$

where  $\omega = \xi + i\zeta$  is complex and

$$u(y, t) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \bar{u}(y, \omega) e^{\omega t} d\omega, \quad (3.2)$$

where  $\gamma$  is a positive real number to the right of the singularities of  $\bar{u}(y, \omega)$  in the complex  $\omega$  plane.

The problem satisfied by  $\bar{u}(y, t)$  is

$$\rho\omega\bar{u} = [\mu + \bar{G}(\omega)]\bar{u}'', \quad \bar{u}(0, \omega) = U/\omega,$$

$$\bar{u}(y, \omega) \rightarrow 0 \text{ as } y \rightarrow \infty.$$

We find that

$$\bar{u}(y, \omega) = \frac{U}{\omega} \exp\{-y\eta(\omega)\}, \quad (3.3)$$

$$\eta(\omega) = \left( \frac{\rho\omega}{\mu + \bar{G}(\omega)} \right)^{1/2}, \quad (3.4)$$

where

$$\bar{G}(\omega) = \int_0^{\infty} e^{-\omega t} G(t) dt \quad (3.5)$$

is the Laplace transform of  $G(t)$ . We now have reduced our problem to computing the inverse transform (3.2) when  $\bar{u}(y, \omega)$  is given by (3.3).

*Properties of  $G(s)$ :*

(1)  $G: [0, \infty) \rightarrow \mathbb{R}^+$  is convex and in  $L^1(0, \infty)$ .

(2)  $|G(t) - G(0)| \leq \text{const } t^\alpha$  for any  $t \geq 0$  and some  $\alpha \geq 0$ . (3.6)

*Properties of  $\bar{G}(\omega)$ :*

It is immediate that

$$(1) \bar{G}(\omega)^* = \bar{G}(\omega^*), \quad \bar{G}(\omega) \stackrel{\text{def}}{=} \alpha(\omega) - i\beta(\omega), \quad \bar{G}(\omega)^* \stackrel{\text{def}}{=} \alpha + i\beta; \quad (3.7)$$

and by a standard theorem [19]:

(2)  $\bar{G}(\omega)$  is holomorphic for  $\text{Re } \omega > 0$  and continuous for  $\text{Re } \omega \geq 0$ . (3.8)

Narain and Joseph [14] showed that

(3)  $\lim_{\substack{|\omega| \rightarrow \infty \\ \text{Re } \omega \geq 0}} \omega \bar{G}(\omega) = G(0)$ , (3.9)

$\lim_{\substack{|\omega| \rightarrow \infty \\ \text{Re } \omega > 0}} \arg \bar{G}(\omega) = - \lim_{\substack{|\omega| \rightarrow \infty \\ \text{Re } \omega > 0}} \arg \omega$ , (3.10)

They also stated that:

(4) If  $\arg \omega \in [0, \pi/2]$ , then  $\arg \bar{G}(\omega) \in (-\pi/2, 0]$ ,

If  $\arg \omega \in [-\pi/2, 0]$ , then  $\arg \bar{G}(\omega) \in [0, \pi/2)$ ; (3.11)

when  $G(s)$  is positive monotone decreasing. Fraenkel [20] has shown that (4) may not hold under such a weak hypothesis but it is true provided that  $G(s)$  is convex.

We shall now derive some properties of  $\eta(\omega)$ . From the definition of  $\eta(\omega)$ , we find that

$$\text{Re } \eta(\omega) = \left( \frac{\rho |\omega|}{|\mu + \bar{G}(\omega)|} \right)^{1/2} \cos \frac{1}{2} [\arg \omega - \arg(\mu + \bar{G}(\omega))].$$

Clearly, if  $|\arg \omega| \leq \pi/2$ , then  $|\arg(\mu + \bar{G}(\omega))| \leq |\arg \bar{G}(\omega)| < \pi/2$ . Therefore,  $\frac{1}{2} |\arg \omega - \arg(\mu + \bar{G}(\omega))| < \pi/2$ , and  $\text{Re } \eta(\omega) > 0$ . Since  $|\bar{G}(\omega)|$  goes to 0 when  $|\omega| \rightarrow \infty$ ,  $\lim_{|\omega| \rightarrow \infty} \arg[\mu + \bar{G}(\omega)] = 0$ .

Summarizing our results, we have found that

$\eta(\omega^*) = \eta^*(\omega)$ , (3.12)

and when  $|\arg \omega| \leq \pi/2$

$\text{Re } \eta(\omega) > 0$ , (3.13)

and, if in addition  $|\omega| \rightarrow \infty$ , then

$\text{Re } \eta(\omega) = \left( \frac{\rho |\omega|}{\mu} \right)^{1/2} \cos \frac{\arg \omega}{2}$ , (3.14)

and

$\arg \eta(\omega) = \frac{1}{2} \arg \omega$ . (3.15)

We need these properties to evaluate the contour integral shown in Fig. 1. On lines  $\zeta = \pm R = \text{constant}$ ,  $\omega = \zeta + i\zeta$ ,  $\arg \omega \rightarrow \pm \pi/2$  and  $\arg \eta(\omega) \rightarrow$

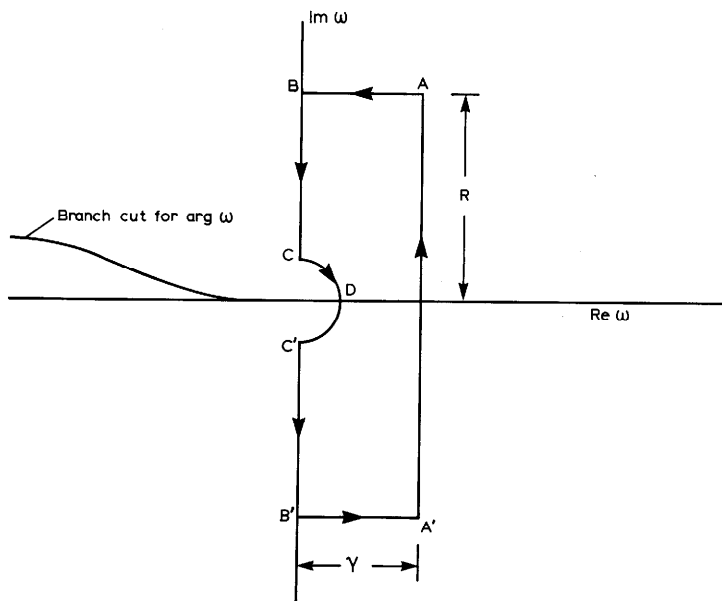


Fig. 1. Contour  $D(\epsilon, R, \gamma)$  for inverting (4.1).

$\pm \pi/4$  as  $R \rightarrow \infty$ . It then follows from (3.14) that  $\operatorname{Re} \eta(\omega) \rightarrow [\rho(\xi^2 + R^2)^{1/2}/2\mu]^{1/2}$ , as  $R \rightarrow \infty$  and, of course,

$$\lim_{\substack{|\omega| \rightarrow \infty \\ \operatorname{Re} \omega \geq 0}} \exp\{-y \operatorname{Re} \eta(\omega)\} = 0 \text{ for any } y > 0. \quad (3.16)$$

#### 4. Inverting the Laplace transform

We turn now to the computation of the inverse integral

$$u(y, t) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{U}{\omega} \exp[\omega t - y\eta(\omega)] d\omega, \quad (4.1)$$

and choose the contour  $D(\epsilon, R, \gamma)$  shown in Fig. 1.

We first show that as  $R \rightarrow \infty$  the contributions to the contour integral along arcs  $A'B'$  and  $AB$  vanish. Along  $AB$   $\omega = \xi + iR$ ,  $R$  is fixed and  $0 \leq \xi \leq \gamma$ ,  $d\omega = d\xi$  and

$$\begin{aligned} \left| \int_{AB} \frac{U}{\omega} \exp[\omega t - y\eta(\omega)] d\omega \right| &= \left| \int_0^\gamma \frac{U}{\xi + iR} \exp[\omega t - y\eta(\omega)] d\xi \right| \\ &\leq \int_0^\gamma \frac{U}{(\xi^2 + R^2)^{1/2}} \exp[\xi t - y \operatorname{Re} \eta(\omega)] d\xi. \end{aligned} \quad (4.2)$$

Using (3.16) we find that (4.2) vanishes when  $R \rightarrow \infty$ . An identical argument shows that the contour integral along  $A'B'$  also vanishes.

In order to evaluate the integral over  $CDC'$  we first note that

$$\lim_{\substack{R \rightarrow \infty \\ \text{Re } \omega > 0}} \omega \left[ \frac{U}{\omega} \exp\{\omega y - y\eta(\omega)\} \right] = U.$$

We may then apply Jordan's lemma

$$\frac{1}{2\pi i} \lim_{\epsilon \rightarrow 0} \int_{CDC'} \frac{U}{\omega} \exp\{\omega t - y\eta(\omega)\} d\omega = \frac{1}{2\pi i} U(-\pi i) = -\frac{U}{2}. \tag{4.3}$$

We are now ready to apply Cauchy's theorem to the closed contour  $D(\epsilon, R, \gamma)$  shown in Fig. 1. We have

$$0 = \int_{\Gamma} \frac{U}{\omega} \exp\{\omega t - y\eta(\omega)\} d\omega.$$

After evaluating this integral using the results implied by (4.2) and (4.3) we get

$$\begin{aligned} & \lim_{R \rightarrow \infty} \frac{1}{2\pi i} \int_{\gamma-iR}^{\gamma+iR} \frac{U}{\omega} f(\omega) d\omega \\ &= \frac{U}{2} + \frac{1}{2\pi i} \lim_{\substack{\epsilon \rightarrow 0 \\ R \rightarrow \infty}} \left[ \int_{\epsilon}^R \frac{U}{\omega} f(\omega) d\omega + \int_{-R}^{-\epsilon} \frac{U}{\omega} f(\omega) d\omega \right], \end{aligned}$$

where  $f(\omega) = \exp[\omega t - y\eta(\omega)]$ . Since  $f(-i\zeta) = f^*(i\zeta)$  we find that

$$\begin{aligned} & \int_{\epsilon}^R \frac{U}{\omega} f(\omega) d\omega + \int_{-R}^{-\epsilon} \frac{U}{\omega} f(\omega) d\omega \\ &= \int_{\epsilon}^R \frac{U}{\zeta} f(i\zeta) d\zeta + \int_{-R}^{-\epsilon} \frac{U}{\zeta} f(i\zeta) d\zeta \\ &= 2i \int_{\epsilon}^R \frac{U}{\zeta} \frac{f(i\zeta) - f(-i\zeta)}{2i} d\zeta = 2i \int_{\epsilon}^R \frac{U}{\zeta} \text{Im } f(i\zeta) d\zeta, \end{aligned}$$

It now follows that

$$\begin{aligned} & \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{U}{\omega} \exp(\omega t - y\eta(\omega)) d\omega \\ &= \frac{U}{2} + \frac{U}{\pi} \int_0^{\infty} \text{Im} \exp\left[ i\zeta t - y(\rho i\zeta / (\mu + \bar{G}(i\zeta)))^{1/2} \right] \frac{d\zeta}{\zeta}. \end{aligned}$$

We may write this result in real variables using

$$\bar{G}(i\zeta) = \alpha(\zeta) - i\beta(\zeta),$$



and

$$\begin{aligned}
 w(\xi) &\stackrel{\text{def}}{=} [\rho\xi/\chi]^{1/2}, \\
 \chi^2(\xi) &\stackrel{\text{def}}{=} [\mu + \alpha(\xi)]^2 + \beta^2(\xi), \\
 \gamma(\xi) &\stackrel{\text{def}}{=} \text{arctg} \frac{\beta(\xi)}{\mu + \alpha(\xi)}.
 \end{aligned}
 \tag{4.4}$$

Thus

$$\begin{aligned}
 \frac{u(y, t)}{U} - \frac{1}{2} &= \frac{1}{\pi} \int_0^\infty \frac{d\xi}{\xi} \exp \left[ -y w(\xi) \cos \left[ \frac{\pi}{4} + \frac{\gamma(\xi)}{2} \right] \right] \\
 &\quad \times \sin \left[ \xi t - y w(\xi) \sin \left[ \frac{\pi}{4} + \frac{\gamma(\xi)}{2} \right] \right] \\
 &= \frac{1}{\pi} \int_0^\infty \frac{d\xi}{\xi} \exp \{ -y w^-(\xi) \} \sin \{ \xi t - y w^+(\xi) \},
 \end{aligned}
 \tag{4.5}$$

where

$$w^\pm(\xi) = \left[ \frac{\xi}{2} \left( \frac{\chi \pm \beta}{\chi^2} \right) \right]^{1/2}.
 \tag{4.6}$$

This solution reduces to the one given by Narain and Joseph [14,15] when  $\mu = 0$ . When there is no elasticity  $\bar{G} = \gamma = 0$ ,  $w(\xi) = \sqrt{\xi/\nu}$ ,  $\nu = \mu/\rho$  and

$$\begin{aligned}
 u(y, t) - \frac{1}{2} &= \frac{1}{\pi} \int_0^\infty \frac{d\xi}{\xi} \exp(-y(\xi/2\nu)^{1/2}) \sin(\xi t - y(\xi/2\nu)^{1/2}) \\
 &= \frac{2}{\pi} \int_0^\infty e^{-z} \sin(az^2 - z) \frac{dz}{z},
 \end{aligned}$$

where  $a = 2\nu t/y^2$ . Using some standard mathematical identities, it can be shown that this function is given by

$$u(y, t) = U \operatorname{erfc}(y/2\sqrt{\nu t}).$$

## 5. Solution of the problem by Fourier transform

The Laplace transform  $\bar{G}(\omega)$  of  $G(s)$  is identical to the complex viscosity  $\eta^*(\Omega)$  when  $\omega = i\Omega$ . The complex viscosity arises in the linearized theory for the response to sinusoidal oscillations of the plate at  $y = 0$ . This shows that our impulse problem may be regarded as an integration over plane waves belonging to different frequencies giving rise to the velocity

$$\begin{aligned}
 u_\omega(y, t) &= c_\omega \exp \left[ i\omega t - y(i\omega\rho/m^*(\omega))^{1/2} \right], \\
 m^*(\omega) &= \mu + \eta^*(\omega).
 \end{aligned}
 \tag{5.1}$$

We want to solve (2.1) and (2.2) by superposing oscillators of the form (5.1) using Fourier transforms

$$\begin{aligned}\tilde{u}(y, \omega) &= \int_{-\infty}^{\infty} u(y, t) e^{-i\omega t} dt \\ &= c_{\omega} \exp\left[-y(i\omega\rho/m^*(\omega))^{1/2}\right], \\ u(y, t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{u}(\omega, y) e^{i\omega t} d\omega.\end{aligned}$$

The first of these integrals is undefined because the data  $u(0, t) = H(t)$  is unsuitable for Fourier transforms. We replace it with data

$$u(0, t) = H(t) - H(t - T)$$

on a compact support. Then

$$\tilde{u}(\omega, 0) = (1 - e^{-i\omega T})/i\omega = c_{\omega},$$

and

$$u(y, t; T) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1 - e^{i\omega T}}{i\omega} \exp\left[i\omega t - y(i\omega\rho/m^*(\omega))^{1/2}\right]$$

After some elementary manipulations we get

$$u(y, t; T) = v(y, t) - v(y, t - T),$$

where  $v(y, t)$  is the function on the right of (4.5). We may use the addition formulas for trigonometric functions to write

$$\begin{aligned}v(y, t - T) &= \frac{1}{\pi} \int_0^{\infty} \frac{d\omega}{\omega} \left[ \exp(-y\omega^-(\omega)) (\sin(\omega t - y\omega^+(\omega)) \cos \omega T \right. \\ &\quad \left. - \cos(\omega t - y\omega^+(\omega)) \sin \omega T \right)].\end{aligned}$$

As  $T \rightarrow \infty$  the first integral tends to zero and, since

$$\frac{1}{\pi} \int_0^{\infty} f(\omega) \frac{\sin \omega\beta}{\omega} d\omega = \frac{f(0)}{2},$$

the second integral is  $-\frac{1}{2}$  and

$$\lim_{T \rightarrow \infty} u(y, t; T) = \frac{1}{2} + v(y, t) = u(y, t).$$

## 6. Instantaneous elasticity and perturbed elasticity

Nearly everyone would agree that liquids will react elastically to forces which change so rapidly that the internal molecular order cannot adjust. The liquid then will not flow. This means  $\mu = 0$  in an exact sense. Viscous effects

effects arise out of relaxing elasticity. This idea goes back to Poisson and was beautifully developed by Maxwell (see Joseph [21] for a recent historical perspective). It can be said that the viscosity in (1.1) represents the effects of fast relaxation associated with small molecules and  $G(s) = G_\mu(s)$  represents more persistent macromolecular relaxations. A theory for this has been given by Joseph et al. [1]. We say that  $\mu$  is an effective viscosity and  $G_\mu(s)$  an effective relaxation function. To understand this we put  $\mu = 0$  and note that the solution of Stokes' first problem corresponding to a given relaxation function may be expressed by the inverse transform

$$u(y, t; G) = \frac{U}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{d\omega}{\omega} \exp\left[\omega t - y(\rho\omega/\bar{G}(\omega))^{1/2}\right]. \quad (6.1)$$

This solution satisfies a certain type of group invariance under radial shifts. The radial shift is given by

$$(y, t) = (\phi\zeta, \phi\tau), \quad 0 < \phi < \infty. \quad (6.2)$$

Joseph et al. [1] showed that

$$u(y, t; G) = u(\zeta, \tau; G_\phi) \quad (6.3)$$

where

$$G_\phi(s) = G(\phi s). \quad (6.4)$$

This says that different observers at different points on a radius in the  $(y, t)$  plane see different scaled relaxation functions. The result follows directly from changing variables.

This result was used by Joseph et al. [1] to show how an effective viscosity can arise when different molecular substructures have different times of relaxation. Suppose there is an additive decomposition

$$G(s) = G_0(s) + G_\mu(s), \quad (6.5)$$

where  $\lambda_0$  is a short mean time of relaxation and

$$\mu = \int_0^\infty G_0(s) ds \quad (6.6)$$

is the viscosity associated with rapidly decaying modes.

To prove the result we use (6.5) and (6.3) to write

$$u(\zeta, \tau; G_\phi) = \frac{U}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{d\Omega}{\Omega} \exp\left[\Omega\tau - \zeta(\rho\Omega/(\bar{G}_0^\phi(\Omega) + \bar{G}_\mu^\phi(\Omega)))^{1/2}\right],$$

where

$$\bar{G}_0^\phi(\Omega) = \int_0^\infty G_0(\phi s) e^{-\Omega s} ds,$$

and

$$\bar{G}_\mu^\phi(\Omega) = \int_0^\infty G_\mu(\phi s) e^{-\Omega s} ds,$$

and  $\tau$  and  $\zeta$  are  $o(1)$ . The function  $G_0(s)$  has a mean relaxation time  $\lambda_0$ . If  $\phi \gg \lambda_0$  then  $G_0(\phi s)$  is crowded at the origin like a  $\delta$  function nearly zero for small values of  $s$  for which  $e^{-\Omega s} \approx 1$ . Then with a small error

$$\bar{G}_0^\phi(\Omega) = \mu/\phi.$$

After rescaling we get

$$u(y, t; G) = \frac{U}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{d\omega}{\omega} \exp\left[\omega t - y\left(\rho\omega/(\mu + \bar{G}_\mu(\omega))\right)^{1/2}\right],$$

when  $(y, t) = \phi(\zeta, \tau)$  are large on the scale defined by  $\lambda_0$ . This leads to an effective viscosity associated with the relaxation of fast modes and an effective rigidity associated with slower modes. The same argument shows that after  $G(s)$  is totally relaxed the solution enters into diffusive limit with viscosity

$$\tilde{\mu} = \mu + \int_0^\infty G_\mu(s) ds.$$

The nature of the small errors which are involved here will be examined in the analysis of the double Maxwell kernel in Section 7. For now it will suffice to note that fast modes need not be all that fast to give rise to viscous effects even in very unsteady processes. It was shown in Joseph et al. [1], using Renardy's analysis of the shock layer perturbation of the propagating Heaviside function, that when  $Jx$  is small

$$u(y, t) = U \exp\left[yG'_\mu(0)/2C_\mu G_\mu(0)\right] \int_{-\infty}^\xi e^{-\psi^2} d\xi + o(J),$$

$$\xi = (t - y/C_\mu)/(JyC_\mu)^{1/2},$$

$$C_\mu = (G_\mu(0)/\rho)^{1/2}$$

The thickness of the shock layer is  $(JyC_\mu)^{1/2} = (Jx)^{1/2}$ .

## 7. Effective viscosity and effective rigidity arising from a relaxation kernel with two relaxation times

We study Stokes' first problem in the dimensionless form (2.5) when

$$g(s) = g_0 e^{-s/\epsilon} + g_1 e^{-s},$$

$$g_0 + g_1 = 1,$$

where  $g_0 \gg g_1$  and  $\epsilon \ll 1$ .

At  $s = 1$  the slow mode is  $e^{-1}$  of its initial value. The Laplace transform of  $g(s)$  is

$$\bar{g}(\omega) = \frac{g_0\epsilon}{1 + \epsilon\omega} + \frac{g_1}{1 + \omega}.$$

The solution of this problem is given by (4.5) and (4.6) with

$$\chi^2 = \alpha^2 + \beta^2, \quad \alpha = \frac{g_0\epsilon}{1 + \epsilon^2\zeta^2} + \frac{g_1}{1 + \zeta^2}, \quad \beta = \frac{g_0\epsilon^2\zeta}{1 + \epsilon^2\zeta^2} + \frac{g_1\zeta}{1 + \zeta^2}.$$

It is easily verified that

$$w^\pm(\zeta) = \left( \frac{\zeta}{2(g_0\epsilon + g_1)} \right)^{1/2}$$

for small  $\zeta$  and that

$$w^-(\zeta) \approx \frac{1}{2} \left( \frac{g_0}{\epsilon} + g_1 \right).$$

and

$$w^+(\zeta) \approx \zeta,$$

for large  $\zeta$ . Moreover,  $w^-(\zeta)$  is a monotone function and  $\sin\{\zeta t - yw^+(\zeta)\}$  is an oscillating function whose frequency increases monotonically without bound as  $\zeta \rightarrow \infty$ .

If  $\epsilon \ll 1$  we may choose a  $\phi < 1$  with a large  $\phi/\epsilon$ . We may replace  $1 + \epsilon^2\zeta^2$  with 1 when  $\zeta < \phi/\epsilon$ . Then

$$\begin{aligned} \frac{u}{U}(y, t) = & \frac{1}{2} + \frac{1}{\pi} \int_0^{\phi/\lambda} \frac{d\zeta}{\zeta} \exp\{-yw^-(\zeta)\} \sin\{\zeta t - yw^+(\zeta)\} \\ & + \frac{1}{\pi} \exp\{-y(g_0/\epsilon + g_1)\} \int_{\phi/\lambda}^{\infty} \frac{d\zeta}{\zeta} \sin\{\zeta(t - y)\}, \end{aligned} \quad (7.1)$$

where

$$\alpha = \mu + \frac{g_1}{1 + \zeta^2}, \quad \beta = \mu\epsilon\zeta + \frac{g_1\zeta}{1 + \zeta^2}$$

and  $\mu = g_0\epsilon = J/(1 - J + J/\epsilon)$  is an effective viscosity. We may neglect the second term on the right-hand side of (7.1) with a small error and we replace  $\phi/\lambda$  with infinity in the first integral, again with a small error.

Therefore we can write the solution as

$$\frac{u}{U}(y, t) = \frac{1}{2} + \frac{1}{\pi} \int_0^{+\infty} \frac{d\zeta}{\zeta} \exp\{-yw^-(\zeta)\} \sin\{\zeta t - yw^+(\zeta)\},$$

where

$$w^{\pm}(\xi) = \left\{ \frac{\xi}{2} \left[ \left( \frac{1 + \xi^2}{\theta^2} \right)^{1/2} \pm \frac{\mu \epsilon \xi (1 + \xi^2) + g_1 \xi}{\theta^2} \right] \right\}^{1/2}, \quad (7.2)$$

and

$$\theta^2 = (\mu + g_1)^2 + \gamma^2 \xi^2, \quad \gamma^2 = \mu^2 (1 + \epsilon^2) + 2\epsilon \mu g_1 + \epsilon^2 \mu^2 \xi^2.$$

## 8. Numerical results

We are going to compute the solution of Stokes' first problem in dimensionless form for a double Maxwell kernel without viscosity and a single Maxwell kernel with viscosity, especially for small viscosities. These two problems are good models of essential processes involved in wave propagation.

The analysis of shear-wave propagation for the double Maxwell kernel shows that even with mild differences between low and fast modes the dynamics is rapidly governed by an effective viscosity and relaxation function.

The analysis of shear-wave propagation for the single Maxwell model with viscosity shows how much smoothing can be expected for different values of  $J$ . We get some sort of wave propagation even with fairly large values of  $J$  (i.e.,  $O(2/10)$ ).

We use these solutions to determine the shape of smoothed wave forms which might be expected in the liquids used in the experiment, reported by Joseph et al. [2].

Finally, we give numerical results for the bell kernel  $G(s) = G_0 e^{-s^2/\lambda^2}$ , as an illustration.

We shall now explain how we did the numerical analysis. We introduce a new dimensionless variable  $x = \xi/(1 - J)^{1/2}$  into (2.5). This gives

$$u_{\tau} = J' u_{xx} + \int_0^{\tau} G(s') u_{xx}(x, \tau - s') ds', \quad (8.1)$$

where  $J' = \mu/\eta$ . The solution of (8.1) can be written as

$$\frac{u}{U}(x, t) = \frac{1}{2} + \frac{1}{\pi} \int_0^{+\infty} f(\xi) d\xi, \quad (8.2)$$

where

$$f(\xi) = \frac{1}{\xi} \exp\{-xw^-(\xi)\} \sin\{\xi t - xw^+(\xi)\},$$

$$w^{\pm}(\xi) = \left[ \frac{\xi}{2} \left( \frac{1}{\theta} + \frac{\beta(\xi)}{\theta^2} \right) \right]^{1/2}, \quad (8.3)$$

$$\theta^2 = [J' + \alpha(\xi)]^2 + \beta^2(\xi).$$

We are going to simplify the integral by computing its head and neglecting its tail.

Near  $\zeta = 0$

$$\alpha(\zeta) = a - J' - \frac{b}{2}\zeta^2 + O(\zeta^4), \quad \beta(\zeta) = c\zeta + O(\zeta^3),$$

where

$$a = J' + \int_0^{+\infty} G(s) \, ds = J' + \eta/\lambda G_0,$$

$$b = \int_0^{+\infty} s^2 G(s) \, ds, \quad c = \int_0^{+\infty} s G(s) \, ds.$$

Therefore the integrand is of the form

$$\begin{aligned} f(\zeta) = & -\frac{x}{\sqrt{2a}} \frac{1}{\sqrt{\zeta}} + \frac{2at + x^2}{2a} - \left( c + \frac{x^2}{3} + 2at \right) \frac{x}{(2a)^{3/2}} \zeta \sqrt{\zeta} \\ & + \left( 2ab + 3c^2 + 2cx^2 + \frac{3}{20}x^4 + 4 \left( c + \frac{x^2}{30} \right) at + 4a^2 t^2 \right) x \zeta^2 \zeta^{1/2} / 2(2a)^3 \end{aligned} \quad (8.4)$$

On the other hand, for large values of  $\zeta$ , the integrand is of the form

$$f(\zeta) = \begin{cases} \frac{\sin \zeta t}{\zeta}, & \text{if } J' = 0, \\ \exp[-x(\zeta/2J')^{1/2}] \frac{\sin \zeta t}{\zeta} & \text{if } J' \neq 0 \text{ } \zeta \gg 1/J'. \end{cases}$$

For the computation we split the integral in three parts and write

$$\frac{v}{U}(x, t) = \frac{1}{2} + \frac{1}{\pi} \left[ \int_0^\epsilon f(\zeta) \, d\zeta + \int_\epsilon^M f(\zeta) \, d\zeta + \int_M^{+\infty} f(\zeta) \, d\zeta \right]. \quad (8.5)$$

The first integral can be evaluated using (8.4) for small  $\epsilon$  with errors of order  $\epsilon^3$  giving

$$\int_0^\epsilon f(\zeta) \, d\zeta = u' + O(\epsilon^3),$$

where

$$\begin{aligned} u' = & -2x\sqrt{\epsilon'} + (t' + x^2)\epsilon' + \left( c + \frac{x^2}{3} + t' \right) \frac{2x}{3} \epsilon' \sqrt{\epsilon'} \\ & + \left[ 2ab + 3c^2 + 2cx^2 + \frac{3}{20}x^4 + 2 \left( c + \frac{x^2}{30} \right) t' + t'^2 \right] \frac{x}{5} \epsilon'^2 \sqrt{\epsilon'}, \end{aligned} \quad (8.6)$$

where

$$\epsilon' = \epsilon/2a, \quad t' = 2at.$$

The third integral can be neglected because

$$\left| \int_M^{+\infty} f(\xi) d\xi \right| \leq \frac{2}{x} (2J'/M)^{1/2} \exp[-x(M/2J')^{1/2}].$$

We first prescribe an allowed error in the computation. Then we choose small  $\epsilon$  and large  $M$  so that the prescription is obeyed. We then compute the second integral using Simpson's method.

To improve the efficiency of Simpson's routine we put  $\xi = z^\gamma, 1 \leq \gamma \leq 4$ . This leads to

$$\frac{u}{U} = \frac{1}{2} + \frac{u'}{\pi} + \frac{\gamma}{\pi} \int_{\epsilon^{1/\gamma}}^{M^{1/\gamma}} \bar{f}'(z^\gamma) \frac{dz}{z}, \tag{8.7}$$

where  $u'$  is given in (8.6) and  $\bar{f}'(\xi) = \xi f(\xi)$ .

This transformation is useful. In fact, if  $N$  operations are needed to integrate the middle term in (8.5), then  $N^{1/\gamma}$  operations will suffice for computing the last term in (8.7). Though we reduce the upper limit of integration to  $M^{1/\gamma}$ , we do not need to increase the mesh size to compute the integral with the desired precision.

The computer time required to compute the complex viscosity is greatly reduced by the preceding transformation because, in any case, it is necessary to compute the other two integrals at each step.

Finally, we remember that Chu [7] has shown that

$$\delta(x) = f(x, x^+/c) = \exp\{xG'(0)/2cG(0)\},$$

when  $\mu = 0$ , where  $c = (G(0)/\rho)^{1/2}$  is the velocity of the shear wave.

*Numerical computation for a Jeffreys' model*

This problem has a viscosity  $\mu$  and  $G(s) = \eta/\lambda e^{-s/\lambda}$ . The dimensionless problem will depend only on the viscosity ratio  $J$  and the dimensionless kernel  $g(s) = e^{-s}$ . In this case

$$\delta(x) = e^{-x/2}, \quad c = 1,$$

$$\alpha(\xi) = 1/(1 + \xi^2), \quad \beta(\xi) = \xi/(1 + \xi^2).$$

When  $J, x$  and  $t$  are small, we may identify a wave, the exponential decay of the wave and the viscous smoothing of the wave. It is easy to verify that the thickness of the shock layer is  $\sqrt{Jx}$ , as predicted by Renardy et al. [22] (see captions of Figs. 2 and 3). These features are harder to identify when  $J$  is large (say,  $J > 0.1$ ). The solution for small  $x$  and  $t$  has the structure of a wave even when the viscosity ratio is not so small. The solution enters a diffusive limit for a large  $x$  and  $t$  for all values of  $0 \leq J \leq 1$  (see Fig. 2 and 3). More examples are given in Preziosi's thesis [17].



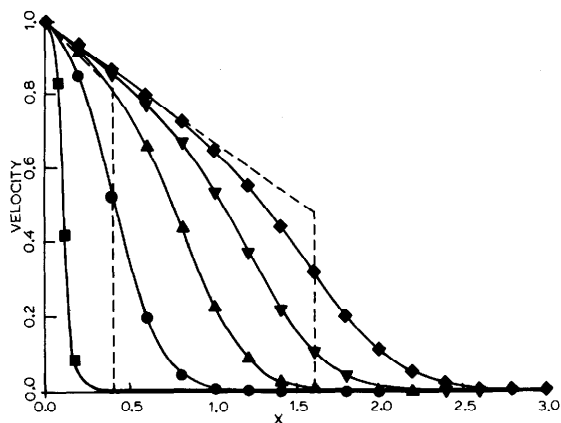


Fig. 2. Flow development for Stokes' first problem as a function of  $x$  for different times  $t$  and when  $\mu/\bar{\mu} = 0.1$ ,  $g(s) = e^{-s}$ . The dashed line through  $x = 0.4$  and  $x = 1.6$  are shocks. The shock strength is the value of  $v$  for  $\mu = 0$  on the line  $x = t$ . The shock layer thickness for  $\mu/\bar{\mu} \neq 0$  is the interval  $\Delta x$  centered on  $x$  in which the change of  $v$  is 95% of its shock value. The thickness is  $\Delta x = 0.710$  for  $x = 0.4$  and  $\Delta x = 0.141$  for  $x = 1.6$ , satisfying the relation  $\Delta x \approx (\mu x/\bar{\mu})^{1/2}$ . ■  $\rightarrow t = 0.0797$ , ●  $\rightarrow t = 0.4$ , ▲  $\rightarrow t = 0.8$ , ▼  $\rightarrow t = 1.2$ , ◆  $\rightarrow t = 1.6$ .

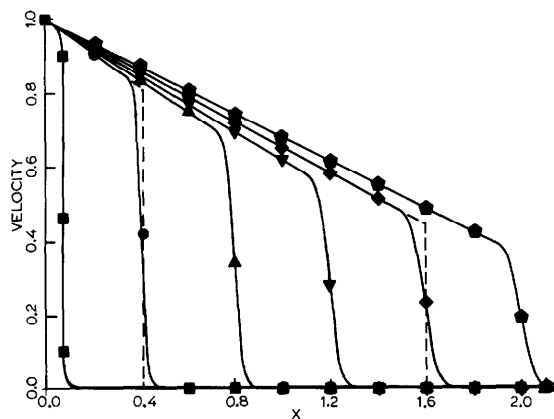


Fig. 3. Flow development for Stokes' first problem as a function of  $x$  for different times  $t$  and when  $\mu/\bar{\mu} = 0.001$ ,  $g(s) = e^{-s}$ . The dashed line through  $x = 0.4$  and  $x = 1.6$  are shocks. The shock strength is the value of  $v$  for  $\mu = 0$  on the line  $x = t$ . The shock layer thickness for  $\mu/\bar{\mu} \neq 0$  is the interval  $\Delta x$  centered on  $x$  in which the change of  $v$  is 95% of its shock value. The thickness is  $\Delta x = 0.0712$  for  $x = 0.4$  and  $\Delta x = 0.142$  for  $x = 1.6$ , satisfying the relation  $\Delta x \approx (\mu x/\bar{\mu})^{1/2}$ . ■  $\rightarrow t = 0.0718$ , ●  $\rightarrow t = 0.4$ , ▲  $\rightarrow t = 0.8$ , ▼  $\rightarrow t = 1.2$ , ◆  $\rightarrow t = 1.6$ , ●  $\rightarrow t = 2.0$ .

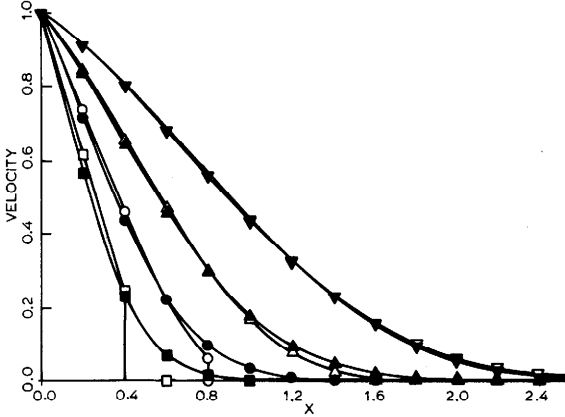


Fig. 4. Comparison of the solution of Stokes' first problem for the double Maxwell kernel (solid symbols)  $g(s) = e^{-s} + 15 e^{-15s}$  and the equivalent Jeffrey's model (open symbols)  $g(s) = e^{-s}$ ,  $\mu/\bar{\mu} = 0.5$ .

#### *Numerical computation for a Maxwell model with two relaxation times*

We argued in Section 7 that the effect of a decayed glassy mode is equivalent to a viscosity. In fact, at times much larger than the fast time of relaxation, the solution for the Maxwell kernel with two modes is identical to the solution with one mode and an effective viscosity. To see any difference we must consider times not so greatly larger than the fast time of relaxation (say less than 10 times). Two examples taken to show the difference are exhibited in Figs. 4 and 5.

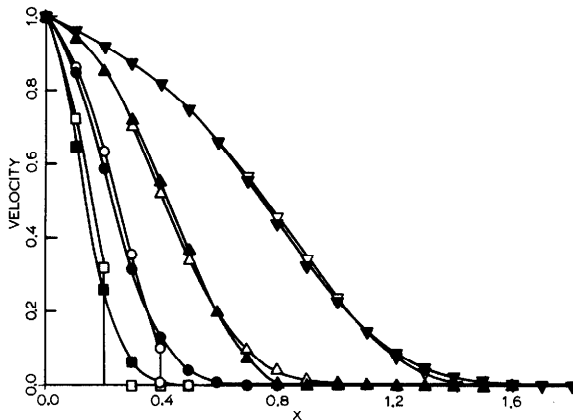


Fig. 5. Comparison of the solution of Stokes' first problem for the double Maxwell kernel (solid symbols)  $g(s) = e^{-s} + 3 e^{-27s}$  and the equivalent Jeffrey's model (open symbols)  $g(s) = e^{-s}$ ,  $\mu/\bar{\mu} = 0.1$ .

*Numerical computation for a bell kernel with viscosity*

When  $G'(0) = 0$  and  $\mu = 0$  there is no attenuation of the amplitude at the front of the travelling wave (see [15], Fig. 1.2). A simple kernel with  $G'(0) = 0$  is the bell kernel  $2\eta e^{-s^2/\lambda^2}/\lambda\sqrt{\pi}$ . The dimensionless form is  $(2/\sqrt{\pi}) e^{-s^2}$  and

$$\alpha(\xi) = e^{-\xi^2/4}, \quad \beta(\xi) = \frac{\xi}{\sqrt{\pi}} \int_0^1 \exp\{(z^2 - 1)\xi^2/4\} dz, \quad c = 1.$$

The thickness of the shock layer is also of order  $\sqrt{Jx}$ .

The smoothing effect of viscosity is perhaps more dramatic because the solution without viscosity is more singular (see Fig. 6). It is perhaps interesting to observe that though the bell kernel fails to satisfy the convexity condition (3.6)<sub>1</sub>, we can compute the solution because the property (3.11) still holds. This apparently shows that the convexity condition is a sufficient but not necessary condition for a solution.

*Comparison with the experimental values of Joseph et al. [2]*

Here, we shall show how to apply these results to interpret the wave forms which may be expected to move the inner cylinder of the wave-speed meter used in the experiments of Joseph et al. [2]. We consider two examples: 1% Polyox in water and olive oil (see Tables 1 and 2).

To do this comparison, we need to assume a value of the viscosity ratio and to choose a distance at which we may view the wave form. The distances

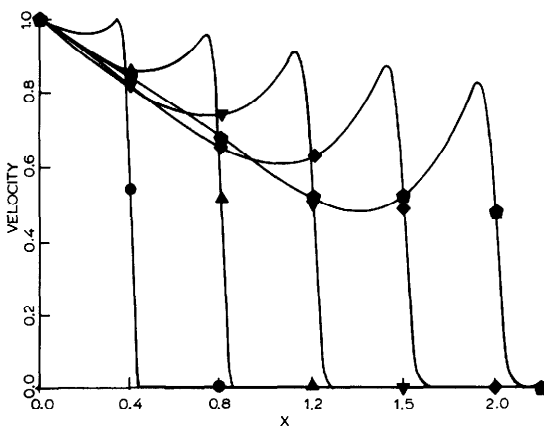


Fig. 6. Flow development for Stokes' first problem as a function of  $x$  for different times  $t$  and when  $\mu/\bar{\mu} = 0.001$ ,  $g(s) = (2/\sqrt{\pi}) e^{-s^2}$ .  $\bullet \rightarrow t = 0.4$ ,  $\blacktriangle \rightarrow t = 0.8$ ,  $\blacktriangledown \rightarrow t = 1.2$ ,  $\blacklozenge \rightarrow t = 1.6$ ,  $\circ \rightarrow t = 2.0$ .

TABLE I

Times of measurement of the wave form for 1% Polyox at 1, 2, 3 mm gaps for wave speeds measured by Joseph et al. [2]  $\bar{\mu} = 61.2$ ,  $c = 14.7$ ,  $G(0) = 215$  in c.g.s. units. The time is given by  $t = 0.239 l / (1 - \mu / \bar{\mu})$  for different values of  $\mu / \bar{\mu}$ . The wave form for times marked with an asterisk is shown in the corresponding Figs. 2 and 3

$l$	$\mu / \bar{\mu}$			
	0.1	0.01	0.001	0.0001
0.1	0.0266	0.0241	0.0239	0.0239
0.2	0.0531	0.0483	0.0478	0.0478
0.3	0.0797*	0.0724	0.0718*	0.0717

TABLE 2

Times of measurement of the wave form for olive oil at 0.25, 0.5, 1, 2, 3 mm gaps for wave speeds measured by Joseph et al. [2].  $\bar{\mu} = 0.6$ ,  $c = 20.5$ ,  $G(0) = 384$  in c.g.s. units. The time is given by  $t = 31.2 l / (1 - \mu / \bar{\mu})$  for different values of  $\mu / \bar{\mu}$ . The wave form for the 0.025 gap is similar to the corresponding graphs in Figs. 2 and 3 for  $t = 0.8$

$l$	$\mu / \bar{\mu}$			
	0.1	0.01	0.001	0.0001
0.025	0.867	0.788	0.781	0.780
0.05	1.734	1.577	1.562	1.561
0.1	3.469	3.153	3.125	3.122
0.2	6.938	6.307	6.250	6.244
0.3	10.406	9.460	9.375	9.367

should correspond to gap sizes used in the experiment with the wave speedmeter. We may determine the time of measurement through the measured values of the wave speed, which in dimensionless terms is given by  $t = G(0)l / c\bar{\mu}(1 - \mu / \bar{\mu})$ .

We do not know  $\mu$ , so the comparison is not definitive. However this procedure may eventually be useful in determining the values of the effective viscosity.

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