

## INSTABILITY OF POISEUILLE FLOW OF TWO IMMISCIBLE LIQUIDS WITH DIFFERENT VISCOSITIES IN A CHANNEL

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**Abstract**—We study the stability of plane Poiseuille flow of two immiscible liquids of different viscosities and equal densities. The problem is like one considered by C. S. Yih who found that flow in two layers of equal thickness was always unstable. We find regions of stability when there are three layers with one of the fluids centrally located. We view our contribution as a study of selection of stable steady flow from a nonunique continuum of Poiseuille flows all of which satisfy the steady Navier–Stokes and which differ from another in the number and thickness of layers of different viscosity. Experiments have shown that there is a tendency for the less viscous fluid to encapsulate the more viscous one. This arrangement of components, with the more viscous fluid in the center of the channel maximizes the mass flux for a fixed pressure gradient. A linear stability analysis of centrally located configuration to long waves is carried out by the analytic methods introduced by Yih [1]. The stability results depend on the viscosity and volume ratio in a fairly complicated way. The flow with the high viscosity fluid centrally located is always stable. Centrally located layers of less viscous fluid, called fingering flows, are always unstable.

### 1. INTRODUCTION

WE CONSIDER the stability of Poiseuille flow in a channel of infinite length in which there are three layers of two fluids, one of which is centrally located. We follow Yih [1] in emphasizing the effect of different viscosities by suppressing the effects of different densities. The density of the two fluids is presumed to be equal and the pressure corresponding to hydrostatic variations in density is removed from the problem. The flow is steady and is driven by a constant pressure gradient.

Our problem however is embedded in a more general one which may be formulated as follows. We look for layered Poiseuille flows, driven by a constant pressure gradient with the volume ratio of the two fluids prescribed. Continuity of stresses and velocity at the interfaces are required in the usual way, and the fluids are assumed to adhere to the stationary walls of the pipe. There are a continuum of steady solutions with fluids of different viscosity in adjacent layers, the number of such layers and their size being indeterminate even within the constraint of prescribed total volumes of each different fluid. Such nonuniqueness appears in the theory of 2-fluid flows for many kinds of flow (Joseph *et al.* [2]). In particular the same kind of nonuniqueness appears in pipe flows in pipes of arbitrary cross section. On the other hand, experiments with the pipe flow indicate that whatever the initial configuration, the low viscosity liquid will eventually encapsulate the high viscosity one. The encapsulation property has been observed for both high and low Reynolds number flows, ranging from oil and water to molten polymers (Everage [3], Southern and Ballman [4], Williams [5]; see also Joseph *et al.* [6] for further references.)

It is necessary to reconcile the existence of a continuum of solutions with the experimentally observed unique configuration. Up to now, explanations have been based on the “viscous dissipation principle” which says that the flow chooses an interface which in some sense minimizes viscous dissipation for a given flow rate, or equivalently, maximizes the volume flux for a given pressure gradient (Everage [3], Southern and Ballman [4], Williams [5], and MacLean [7] and Joseph *et al.* [6]). The last authors also treat the stability of two component flow down a pipe. They find that thin lubricating layers on the pipe wall are stable against long waves and that fingering flows with thin fluid are always unstable. The instability of fingering flows was first established in a study by Hickox [8].

Li [9] studies the stability of plane Couette flow of three layers of liquid with different viscosities against long waves. Tsahalis [10] studies the stability problem of a thin film of

a viscous liquid bounded on one side by another more viscous liquid of semi-infinite extent and on the other side by a fixed wall. Both liquids are in steady motion parallel to their interface and each fluid has a linear velocity profile. Neither of these papers treat Poiseuille flow and they do not pose the question of selection of stable configurations of flow. Fingering and lubricating flows are not defined in these papers.

The paper by Hame and Müller [11] treats the problem of stability of a lubricated Poiseuille flow with a thin lubricating layer. They also find stability.

A definite neutral stability curve is not established in the work reported in the papers of Li [9], Tsalis [10] and Hame and Müller [11].

Joseph *et al.* [2] have shown that the two component flow which maximizes the mass flux for a given pressure drop is also the one in which the more viscous one is centrally located. Here we treat the problem in a more fundamental way, by considering stability and find results like those given by Joseph *et al.* [6] for two component flows in circular pipes. We find that the stability hypothesis embodied in the variational principle is correct because stable lubricating flows (high viscosity fluids centrally located) are always stable. We also find that the configuration with less viscous fluid centrally located is always unstable.

Our stability studies support the criteria for selection which arise from the simple variational studies given in Joseph *et al.* [2, 6]. We note that the mathematical formulation of the general notion that the observed arrangements of components of flow somehow maximize the mass flux is not the most general one possible and in fact is restricted to two restricted classes which include only layered Poiseuille flows, varying in just one dimension. More general formulation of variational ideas of selection should be of interest. We further note that our linear stability results do not guarantee stability against finite disturbances. Stable flows are only weakly stable according to linear theory and they might be unstable to large disturbances.

The variational result which showed that the lubricating flows in three layers maximize the mass flux, suggests stable configurations should be sought among these arrangements. It will be recalled that Yih [1] already showed that the layered Poiseuille flows in two equal layers were unstable against long waves.

In this paper we follow Yih [1] in using Squire's theorem to reduce our stability problem to two dimensions and we use his methods to study stability against long waves. Hooper and Boyd [12] showed that the flow of two fluids with different viscosities is unstable to very short waves when the interfacial tension is zero, but this short wave instability can be stabilized by interfacial tension.

The problem of selection of stable arrangements of two fluids in flow through a channel is only partly settled by this study, and the settled part may be small. Since there are volume and viscosity ratios for which all layered Poiseuille flows of two fluids in three symmetric layers are unstable we are obliged to think about what arrangements are possible. Of the many possibilities we wish to briefly draw attention. In certain circumstances, we might expect to see dynamically maintained emulsions. Such emulsions were reported in experiments of Joseph *et al.* [2]. They are suggested by the homogenized solutions which are required by the variational approach to the problem of selection in pipes of arbitrary cross section (see Joseph *et al.* [6] for a review of the literature and for an example in a pipe of square cross section). The variational problem has so far been posed only on 2-dimensional fields, so that the homogenized solutions cannot represent 3-dimensional emulsions.

A second possibility is the appearance of interfacial waves. Renardy and Joseph [13] have shown that such waves can result as a Hopf bifurcation from the loss of stability of stable arrangements of the two fluids as the volume and viscosity ratio are moved across the border of stability. The bifurcating solution is a travelling wave. The stability of such waves depends in the simplest of cases on whether the bifurcation is subcritical. Traveling interfacial waves do appear in some two fluid experiments between rotating cylinders but the relation between the theory and observations as well as the computation of the direction of bifurcation have not yet been given.

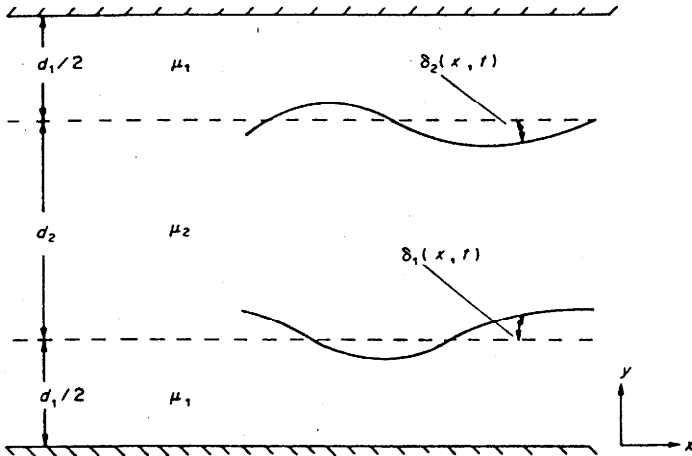


Fig. 1. Poiseuille flow of two fluids in three layers. The two fluids have unequal viscosities:  $\mu_2$  is the viscosity of the fluid in the center,  $\mu_1$  is the viscosity of the fluid on the walls. Lubricating flows are those for which  $\mu_1 < \mu_2$ . Fingering flows are those for which  $\mu_1 > \mu_2$ .

2. MATHEMATICAL FORMULATION

We consider three superimposed layers of fluid with depths  $d_1/2, d_2, d_1/2$  (see Fig. 1). The viscosity of the fluid in the center is  $\mu_2$  and of the fluid on the walls is  $\mu_1$ . The two fluids are assumed to have equal densities. We follow Yih [1] in using Squire's theorem to justify the study of flow and stability in two dimensions with coordinates  $(X, Y)$ . Superscripts with brackets indicate the region of flow:

$$\Omega^{(1)} = \left[ X, Y; -\infty < X < \infty, 0 < Y < \frac{d_1}{2} + \delta_1 \right]$$

$$\Omega^{(2)} = \left[ X, Y; -\infty < X < \infty, \frac{d_1}{2} + \delta_1 < Y < \frac{d_1}{2} + d_2 + \delta_2 \right]$$

$$\Omega^{(3)} = \left[ X, Y; -\infty < X < \infty, \frac{d_1}{2} + d_2 + \delta_2 < Y < d_1 + d_2 \right]$$

The equations of motion are

$$\partial_t \mathbf{U}^{(i)} + \mathbf{U}^{(i)} \cdot \nabla \mathbf{U}^{(i)} = -\nabla P^{(i)} + \mathbf{K} + \mu^{(i)} \nabla^2 \mathbf{U}^{(i)}$$

$$\text{div } \mathbf{U}^{(i)} = 0, \quad i = 1, 2, 3$$

Here  $\mathbf{U}^{(i)} = (U^{(i)}, V^{(i)})$  is the velocity,  $\mathbf{K} = \mathbf{e}_x K$  ( $K > 0$ ) is the applied pressure gradient,  $\mu^{(1)} = \mu^{(3)} = \mu_1, \mu^{(2)} = \mu_2$ . Moreover, at the walls of the channel

$$U_{Y=0}^{(1)} = U_{Y=d_1+d_2}^{(2)} = 0$$

At the interfaces we have continuity of velocity and shear stresses and the jump in the normal stress is balanced by surface tension times the mean curvature and, at each interface,

$$\hat{c}_i \delta + U \partial_x \delta = V$$

where the fields  $\delta_i(X, t), i = 1, 2$  are assumed to be periodic functions of  $X$  of zero mean value. In due course we will write out interface conditions for the disturbance of the basic

Poiseuille flow which is defined below.

The interfaces  $\delta_1 = \delta_2 = 0$  of reference domains  $\Omega_0^{(i)}$ ,  $i = 1, 2, 3$  are flat. The basic flow is a unique symmetric family of steady flows with flat interfaces. This family depends on  $d_1$  and is given by

$$\mathbf{U}^{(i)} = \mathbf{e}_x \tilde{U}^{(i)}(Y), \quad P^{(i)} = 0 \quad (2.1)$$

where

$$\tilde{U}^{(i)} = -\frac{KY^2}{2\mu^{(i)}} + \tilde{c}_1^{(i)}Y + \tilde{c}_2^{(i)}$$

with

$$\tilde{c}_2^{(1)} = \tilde{c}_2^{(3)} = 0$$

$$\tilde{c}_1^{(1)} = \tilde{c}_1^{(3)} = \frac{K}{2\mu_1}(d_2 + d_1)$$

$$\tilde{c}_1^{(2)} = \frac{K}{2\mu_2}(d_1 + d_2)$$

$$\tilde{c}_2^{(2)} = \frac{Kd_1}{4}\left(\frac{d_1}{2} + d_2\right)\left(\frac{1}{\mu_1} - \frac{1}{\mu_2}\right)$$

For nonlinear analysis we extend the functions (2.1) onto  $\Omega^{(i)}$ , call them the extended basic flow, and form the equations for the difference motion.

$$\mathbf{u}^{(i)} = \mathbf{U}^{(i)} - \mathbf{e}_x \tilde{U}^{(i)}(Y), \quad p^{(i)} = P^{(i)}$$

and find that

$$\begin{aligned} \rho(\partial_t \mathbf{u}^{(i)} + \tilde{U}^{(i)} \partial_x \mathbf{u}^{(i)} + \mathbf{e}_x \tilde{U}'^{(i)} v^{(i)} + \mathbf{u}^{(i)} \cdot \nabla \mathbf{u}^{(i)}) &= -\nabla p^{(i)} + \mu^{(i)} \nabla^2 \mathbf{u}^{(i)} \\ \operatorname{div} \mathbf{u}^{(i)} &= 0 \end{aligned} \quad (2.2)$$

in  $\Omega^{(i)}$ . Moreover,

$$\mathbf{u}^{(1)} = \mathbf{u}^{(2)} = 0 \quad (2.3)$$

at  $Y = 0$  and  $d_1 + d_2$  respectively.

The interface conditions are first expressed in terms of unit normal  $\mathbf{n}_i$  and  $\boldsymbol{\tau}_i$  vectors on the interface  $\delta_i$

$$\begin{aligned} \mathbf{n}_i &= \left( -\frac{\delta_i'}{(1 + \delta_i'^2)^{1/2}}, \frac{1}{(1 + \delta_i'^2)^{1/2}} \right) \\ \boldsymbol{\tau}_i &= \left( \frac{1}{(1 + \delta_i'^2)^{1/2}}, \frac{\delta_i'}{(1 + \delta_i'^2)^{1/2}} \right) \end{aligned}$$

We introduce a notation for jumps

$$[[f]] = f_2 - f_1, \quad ((f)) = f_2 - f_3.$$

and write the jump equations for the stress

$$\begin{aligned} [[\tau_1 \cdot \mu D[\mathbf{u}] \cdot \mathbf{n}_1]] &= 0 \\ ((\tau_2 \cdot D[\mathbf{u}] \cdot \mathbf{n}_2)) &= 0 \\ [[\mathbf{n}_1 \cdot \mathbf{T}[\mathbf{u}] \cdot \mathbf{n}_1]] &= \frac{\sigma \delta_1''}{(1 + \delta_1'^2)^{3/2}} \\ ((\mathbf{n}_2 \cdot \mathbf{T}[\mathbf{u}] \cdot \mathbf{n}_2)) &= \frac{\sigma \delta_2''}{(1 + \delta_2'^2)^{3/2}} \end{aligned}$$

where  $\sigma$  is the surface tension, in terms of  $X, Y$  components. Thus with  $\mathbf{u} = (u, v)$  we find that

$$\begin{aligned} (1 - \delta_1'^2)[[\mu(\partial_x v + \partial_y u)]] + 2\delta_1'[[\mu(\partial_y v - \partial_x u)]] &= 0 \\ (1 - \delta_2'^2)[(u(\partial_x v + \partial_y u))] + 2\delta_2'[(\mu(\partial_y v - \partial_x u))] &= 0 \\ \delta_1'^2[[2\mu\partial_x u - p]] + [[2\mu\partial_y v - p]] - \delta_1'[[2\mu(\partial_x v + \partial_y u)]] &= \frac{\sigma \delta_1''}{(1 + \delta_1'^2)^{1/2}} \\ \delta_2'^2((2\mu\partial_x u - p)) + ((2\mu\partial_y v - p)) - \delta_2'((2\mu(\partial_x v + \partial_y u))) &= \frac{\sigma \delta_2''}{(1 + \delta_2'^2)^{1/2}} \end{aligned} \quad (2.4)$$

The following two kinematic conditions have to be added to (2.4):

$$\partial_t \delta_i + (u^{(0)} + \bar{U}^{(0)})\delta_i' = v^{(0)}, \quad (i = 1, 2)$$

Though the total velocity is continuous at the interface, the perturbation velocity is not. Though  $v$  is continuous,

$$[[v]] = ((v)) = 0 \quad (2.5)$$

$u$  is not

$$[[u + \bar{U}]] = ((u + \bar{U})) = 0 \quad (2.6)$$

where

$$\begin{aligned} [[\bar{U}]] &= -\frac{K}{2} \left[ \left[ \frac{1}{\mu} \right] \right] \left( \frac{d_1}{2} + \delta_1 \right)^2 + [[\bar{c}_1]] \left( \frac{d_1}{2} + \delta_1 \right) + [[\bar{c}_2]] \\ &= -\frac{K}{2} \left[ \left[ \frac{1}{\mu} \right] \right] \delta_1 (\delta_1 - d_2) = -[[u]] \end{aligned} \quad (2.7)$$

and, by a similar computation

$$((\bar{U})) = -\frac{K}{2} \left( \frac{1}{\mu} \right) \delta_2 (\delta_2 + d_2) = -((u)) \quad (2.8)$$

Equations (2.2)–(2.8) are the governing nonlinear equations for a disturbance of layered Poiseuille flow. Equations (2.7) and (2.8) can be solved for  $\delta_1$  in terms of  $[[u]]$  and  $\delta_2$  in terms of  $((u))$ . Hence, the unknowns in the problem for the disturbance are  $u^i$  and  $p^i$ . These disturbance equations are in a form convenient for the analysis of bifurcations.

The linearized problem of stability follows easily from the governing nonlinear equation. We treat the problem of linearized stability next.

### 3. LINEARIZED STABILITY EQUATIONS

From now on we work with dimensionless forms of the governing equations. Dimensionless variables

$$[t, x, y, \eta_i, U^{(i)}, u^{(i)}, v^{(i)}, p^{(i)}]$$

are defined by

$$\left[ \frac{2Wt}{d_2}, \frac{2X}{d_2}, \frac{2}{d_2} \left[ Y - \frac{d_1 + d_2}{2} \right], \frac{2\delta_i}{d_2}, \frac{\tilde{U}^{(i)}}{W}, \frac{u^{(i)}}{W}, \frac{v^{(i)}}{W}, \frac{p^{(i)}}{pW^2} \right]$$

where  $W = \tilde{U}((d_1 + d_2)/2)$  is the centerline velocity,

$$U^{(0)} = -h_i = -Kd_2^2/4W\mu^{(i)}$$

$$U^{(1)} = \frac{h_1}{2}(\kappa^2 - y^2), \quad y \in [-\kappa, -1]$$

$$U^{(2)} = \frac{h_2}{2}(1 - m + m\kappa^2 - y^2), \quad y \in [-1, 1]$$

$$U^{(3)} = U^{(1)}, \quad y \in [1, \kappa] \quad (3.1)$$

where  $\kappa = (d_2 + d_1)/d_2$  and  $m = \mu_2/\mu_1$ .

The linearized equations are defined in the reference domain  $[-\kappa, 1]$ ,  $[-1, 1]$ ,  $[1, \kappa]$  and the interface conditions are posed on the flat interfaces at  $y = \pm 1$ . We have

$$\partial_t \mathbf{u}^{(i)} + U^{(i)}(y) \partial_x \mathbf{u}^{(i)} + \mathbf{e}_x U^{(i)}(y) v^{(i)} = -\nabla p^{(i)} + \frac{1}{R^{(i)}} \nabla^2 \mathbf{u}^{(i)}$$

$$\text{div } \mathbf{u}^{(i)} = 0$$

in each of the three layers. Here

$$R^{(i)} = \frac{\rho W d_2}{2\mu^{(i)}}$$

and

$$\mathbf{u}^{(1)} = \mathbf{u}^{(3)} = 0 \quad \text{at } y = \pm \kappa \quad (3.2)$$

The linearized interface conditions, to be evaluated at  $y = -1$ , are

$$\begin{aligned}
 [[R^{-1}(\partial_x v + \partial_y u)]] &= ((R^{-1}(\partial_x v + \partial_y u))) = 0 \\
 [[v]] &= (v) = 0 \\
 [[2R^{-1}\partial_y v - p]] &= T\eta_1'', \quad \left(T = \frac{2\sigma}{\rho W^2 d_2}\right) \\
 ((2R^{-1}\partial_y v - p)) &= T\eta_2'' \\
 [[u]] &= -\eta_1 [[h]] \\
 ((u)) &= +\eta_2 [[h]] \\
 \partial_y \eta_i + U^{(i)} \eta_i' &= v^{(i)} \tag{3.3}
 \end{aligned}$$

where  $h_i$ 's given by (3.1)<sub>1</sub>.

It is clear that  $\eta_1$  and  $\eta_2$  may be eliminated from (3.3). After doing this the linear problem does not involve  $\eta_1$  or  $\eta_2$  any longer.

The spectral problem for the linearized equations may be reduced to a system of ODE's of the Orr-Sommerfeld type by using normal modes. First we introduce a stream function

$$(u^{(i)}, v^{(i)}) = (\partial_y \Psi^{(i)}, -\partial_x \Psi^{(i)})$$

and write

$$\begin{aligned}
 (\Psi^{(i)}, p^{(i)}) &= (\Phi_i, f_i) \exp[i\alpha(x - Ct)] \\
 &= (\Phi_i, f_i) \exp[i\alpha(x - C_r t)] \exp(\alpha C_i t)
 \end{aligned}$$

where  $\alpha$  is real and  $C = C_r + iC_i$ , where  $C_r$  is the wave speed. After eliminating  $f_i^{(i)}$  we get

$$\Phi_i^{(iv)} - 2\alpha^2 \Phi_i'' + \alpha^4 \Phi_i = i\alpha R_i ((U^{(i)} - C)(\Phi_i'' - \alpha^2 \Phi_i) - U''^{(i)} \Phi_i) \tag{3.4}$$

After eliminating  $\eta_1$  and  $\eta_2$  from (3.3), we express (3.2) and (3.3) as

$$\begin{aligned}
 \Phi_1(-\kappa) = \Phi_1'(-\kappa) = \Phi_3(\kappa) = \Phi_3'(\kappa) &= 0 \\
 [[\Phi]] = ((\Phi)) &= 0 \\
 C^* [[\Phi']] &= -\Phi_1(-1) [[h]] \\
 C^* ((\Phi')) &= \Phi_2(1) [[h]] \\
 [[R^{-1}\Phi''']] + \alpha^2 [[R^{-1}\Phi]] &= 0 \\
 ((R^{-1}\Phi''')) + \alpha^2 ((R^{-1}\Phi)) &= 0 \\
 [[R^{-1}\Phi''']] + i\alpha\Theta_1 + \alpha^2\Theta_2 + i\alpha^3\Theta_3 &= 0 \\
 ((R^{-1}\Phi''')) + i\alpha\Lambda_1 + \alpha^2\Lambda_2 + i\alpha^3\Lambda_3 &= 0 \tag{3.5}
 \end{aligned}$$

The eigenvalue appears explicitly in (3.5)<sub>2,3</sub> where

$$\begin{aligned}
 C^* &= C - U^{(1)}(-1) = C - U^{(2)}(1) \\
 R_1 = R_3 &= R, \quad R_2 = R/m \\
 h_2 &= h_1/m
 \end{aligned}$$

The coefficients in (3.5)<sub>7,8</sub> are explicitly given by

$$[\Theta_1, \Theta_2, \Theta_3] = \left[ 0, -3[[R^{-1}\Phi]], \frac{T}{C^*} \Phi_2(-1) \right]$$

$$[\Lambda_1, \Lambda_2, \Lambda_3] = \left[ 0, -3((R^{-1}\Phi)), \frac{T}{C^*} \Phi_2(1) \right]$$

We note that  $\Theta_1$  and  $\Lambda_1$  will not vanish when the two fluids have different densities. The spectral problem was first derived by Yih [1] who applied it to the problem of stability of Poiseuille flow in two layers.

The eigenvalue problem must necessarily decompose into an even and odd problem. We define

$$S_1(y) = \frac{1}{2}(\Phi_1(y) + \Phi_3(-y)), \quad y \in [-\kappa, -1]$$

$$A_1(y) = \frac{1}{2}(\Phi_1(y) - \Phi_3(-y)), \quad y \in [-\kappa, -1]$$

$$S_2(y) = \frac{1}{2}(\Phi_2(y) + \Phi_2(-y)), \quad y \in [-1, 1]$$

$$A_2(y) = \frac{1}{2}(\Phi_2(y) - \Phi_2(-y)), \quad y \in [-1, 1]$$

$$S_3(y) = \frac{1}{2}(\Phi_3(y) + \Phi_1(-y)), \quad y \in [1, \kappa]$$

$$A_3(y) = \frac{1}{2}(\Phi_3(y) - \Phi_1(-y)), \quad y \in [1, \kappa]$$

Obviously

$$\Phi_i = S_i + A_i$$

and

$$S(-y) = S(y), \quad \forall \quad y \in [-\kappa, \kappa]$$

$$A(-y) = -A(y), \quad \forall \quad y \in [-\kappa, \kappa]$$

A simple analysis (omitted for shortness) allows one to see that  $S(y)$  and  $A(y)$  are



determined independently by the uncoupled eigenvalue problems which are stated below. For the even problem we have

$$(S) \left\{ \begin{array}{l} L_y^{(2)} S_2 = 0, \quad y \in (0, 1); \quad L_y^{(3)} S_3 = 0, \quad y \in (1, \kappa) \\ S_3(\kappa) = S_3'(\kappa) = 0 \\ S_2'(0) = S_2''(0) = 0 \\ ((S)) = 0 \\ C^*((S')) = S_2(1)h_1 \left( \frac{1}{m} - 1 \right) \\ ((R^{-1}S'')) + \text{terms in } \alpha^2 = 0 \\ ((R^{-1}S''')) + \text{terms in } \alpha^2, \alpha^3 = 0 \end{array} \right.$$

where

$$L_y^{(i)} = \frac{d^4}{dy^4} - 2\alpha^2 \frac{d^2}{dy^2} + \alpha^4 - i\alpha R_i ((U^{(i)}(y) - C) \left( \frac{d^2}{dy^2} - \alpha^2 \right) + h_i)$$

For the odd eigenvalue problem we have

$$(A) \left\{ \begin{array}{l} L_y^{(2)} A_2 = 0, \quad y \in (0, 1); \quad L_y^{(3)} A_3 = 0, \quad y \in (1, \kappa) \\ A_3(\kappa) = A_3'(\kappa) = 0 \\ A_2(0) = A_2''(0) = 0 \\ ((A)) = 0 \\ C^*((A')) = A_2(1)h_1 \left( \frac{1}{m} - 1 \right) \\ ((R^{-1}A'')) + \text{terms in } \alpha^2 = 0 \\ ((R^{-1}A''')) + \text{terms in } \alpha^2, \alpha^3 = 0 \end{array} \right.$$

The above decomposition of the original problem shows that in order to get the spectrum of problem (3.4)–(3.5) it is both necessary and sufficient to get the spectrum  $\Sigma(S)$  of problem (S) and the spectrum  $\Sigma(A)$  of problem (A).

#### 4. LONG WAVE SOLUTIONS OF THE EIGENVALUE PROBLEM

To solve the eigenvalue problem (S) we follow Yih [1] and use a regular perturbation in which the wave number  $\alpha \ll 1$  is the small parameter. One advantage of this method is that, however large the Reynolds number  $R$ , there always exists a small range of values of  $\alpha$  for which the perturbation procedure is (formally) valid. To this aim put

$$S_i = \hat{S}_i + \check{S}_i \alpha + O(\alpha^2) \\ C_s = \hat{C}_s + \check{C}_s \alpha + O(\alpha^2)$$

At the zeroth order approximation problem  $(\hat{S})$  turns out to be

$$(\hat{S}) \quad \left\{ \begin{array}{l} \hat{S}_2^{(iv)} = 0, \quad y \in (0, 1); \quad \hat{S}_3^{(iv)} = 0, \quad y \in (1, \kappa) \\ \hat{S}_3(\kappa) = \hat{S}'_3(\kappa) = \hat{S}_2(0) = \hat{S}''_2(0) = 0 \\ ((\hat{S})) = 0 \\ \hat{S}_3''(1) = m\hat{S}_2''(1) \\ \hat{S}_3'''(1) = m\hat{S}_2'''(1) \end{array} \right.$$

The condition corresponding to  $(\hat{S})_5$ , i.e.

$$\hat{C}_S^*((\hat{S})) = \hat{S}_2(1)[[h]]$$

will be used subsequently to find  $\hat{C}_S^*$ . Since the solution  $S$  of the general problem  $(S)$  is defined up to a normalizing constant factor, we have fixed this factor by assuming  $S_2(0) = 1$ . The solutions of  $(\hat{S})$  are given by

$$\begin{aligned} \hat{S}_2(y) &= 1 + m^{-1}\Delta^{-1}y^2 \\ \hat{S}_3(y) &= \Delta^{-1}(\kappa - y)^2 \end{aligned}$$

where

$$\Delta = (\kappa - 1)^2 - \frac{1}{m} = \frac{m - n^2}{mn^2}$$

and  $n = d_2/d_1$  is the volume ratio. Thus, there is a curve  $\Gamma$  in the  $(m, n)$ -plane, along which problem  $(\hat{S})$  has no nontrivial solutions. However we shall see that this singularity does not enter into the first order approximation in  $\alpha$ . When  $\Delta \neq 0$ , we can now compute  $\hat{C}_S$ , since

$$\hat{C}_S = \hat{C}_S^* + \frac{h_1}{2}(\kappa^2 - 1)$$

and

$$\hat{S}_2(1) = 1 + m^{-1}\Delta^{-1} = \Delta^{-1}(\kappa - 1)^2 = \Delta^{-1}n^{-2} \quad (4.1)$$

$$((\hat{S})) = 2m^{-1}\Delta^{-1} + 2\Delta^{-1}(\kappa - 1) = 2\Delta^{-1}(m^{-1} + n^{-1}) \quad (4.2)$$

we have

$$\hat{C}_S = \frac{h_1}{2} \left( \frac{(m^{-1} - 1)(\kappa - 1)^2}{(m^{-1} + n^{-1})} + (\kappa^2 - 1) \right) = \frac{h_1}{2} \cdot \frac{(1 + n)(m + 2n)}{n^2(n + m)} \quad (4.3)$$

We are now ready to compute  $\tilde{S}$ , that is the solution of system (S) at the first order approximations in  $\alpha$ . It is easily seen that system ( $\tilde{S}$ ) turns out to be

$$(S) \quad \left\{ \begin{array}{l} \tilde{S}_j^{(iv)} = iR_j((U^{(j)}(y) - \hat{C}_S)\tilde{S}_j'' + h_j\tilde{S}_j), \quad j = 2, 3 \\ \tilde{S}_3(\kappa) = \tilde{S}_3'(\kappa) = \tilde{S}_2(0) = \tilde{S}_2'(0) = 0 \\ ((\tilde{S})) = 0 \\ \tilde{C}_S^*(\tilde{S}') + \hat{C}_S^*(\tilde{S}) = \tilde{S}_2(1)h_1(m^{-1} - 1) \\ \tilde{S}_3(1) = m\tilde{S}_2'(1) \\ \tilde{S}_3'''(1) = m\tilde{S}_2'''(1) \end{array} \right.$$

Let us put

$$iR_j((U^{(j)}(y) - \hat{C}_S)\tilde{S}_j'' + h_j\tilde{S}_j) = ih_1RH_j(y)$$

and let us compute the  $H_j$ 's explicitly. Using (3.1), (4.1), (4.2) and (4.3) we find

$$H_2 = \frac{(n+1)}{mn^2\Delta(n+m)}$$

$$H_3(y) = \frac{(n+1)(2n(n+m)+m)}{n^2\Delta(n+m)} - \frac{2(n+1)}{n\Delta}y$$

If we now set

$$ih_1RH_2 = 4!e_2$$

$$ih_1RH_3(y) = 4!e_3 + 5!f_3y$$

we can express the general solution of ( $\tilde{S}$ ) as follows:

$$\tilde{S}_2(y) = c_2y^2 + e_2y^4$$

$$\tilde{S}_3(y) = a_3 + b_3y + c_3y^2 + d_3y^3 + e_3y^4 + f_3y^5$$

where

$$\tilde{S}_2(0) = 0 \text{ because } S_2(0) = 1$$

and

$$S_2(0) = 1 = \hat{S}_2(0) + \alpha\tilde{S}_2(0) + \text{higher order terms in } \alpha$$

must be satisfied for each  $\alpha$ . Boundary and interface conditions are now applied and after some manipulations we get

$$d_3 = 4me_2 - 4e_3 - 10f_3$$

$$e_3 = \frac{1}{m\Delta} [m(ms+7)e_2 + [m(t-4s)-6]e_3 + [m(v-10s)-20]f_3]$$

where

$$s = -(\kappa-1)^2(2\kappa+1)$$

$$t = -(\kappa-1)^2(3\kappa^2+2\kappa+1)$$

$$v = -(\kappa-1)^2(4\kappa^3+3\kappa^2+2\kappa+1)$$

We do not make the remaining unknowns explicit since we do not need them to compute  $\check{C}_S$ . To this aim first notice that, since  $U$  does not suffer any perturbation,  $\check{C} = \check{C}^*$  and from  $(\check{S})_4$  we get therefore

$$\check{C}_S = h_1 \frac{(1-m)}{m} ((\check{S}))^{-2} [((\check{S}))\check{S}_2(1) - \check{S}_2(1)((\check{S}))]$$

As far as stability is concerned, we need to consider only the imaginary part of  $\check{C}_S$ . On the other hand  $\check{C}_S$  is purely imaginary.

A rather lengthy algebraic computation allows us to write

$$\check{C}_S = i \frac{h_1^2 R}{m} (1-m)((\check{S}))^{-2} Y_S$$

where  $Y_S$  is defined as follows:

$$Y_S = \Delta^{-1} m^{-1} n^{-2} \gamma_S(m, n)$$

and

$$\gamma_S(m, n) = 2n^{-5} \Delta^{-1} (n+1)(n+m)^{-1} \gamma_S^*(m, n)$$

where

$$\gamma_S^*(m, n) = m^2 + 7mn + 16n^2 + 10n^3$$

The curve  $\gamma_S^* = 0$  in the  $(m, n)$ -plane is one branch of the curve of neutral stability; the other one is simply  $m = 1$ . Moreover, bearing in mind the expression for  $((\check{S}))$ , we have

$$\check{C}_S = \frac{ih_1^2 R}{120} \frac{(n+1)}{n^5(m+n)^3} (1-m) \gamma_S^* \quad (4.4)$$

in which, as we said before, the factor  $\Delta$  is no longer involved. In particular we have that, for lubricating flows ( $m > 1$ )

$$\text{sign } \check{C}_S = -\text{sign } \gamma_S^*$$

while, for fingering flows ( $m < 1$ ),

$$\text{sign } \check{C}_S = \text{sign } \gamma_S^*$$

It is worth noting that  $\gamma_S^*(n, m) > 0$  for all  $m, n$ . This indicates that the even mode is always unstable when less viscous fluid is centrally located and stable when more viscous fluid is in the center.

Now we study the eigenvalue problem (A). As before we put

$$A_i = \hat{A}_i + \check{A}_i x + O(x^2)$$

$$C_A = \hat{C}_A + \check{C}_A x + O(x^2)$$

At the zeroth order approximation, system (A) turns out to be

$$(\hat{A}) \left\{ \begin{array}{l} \hat{A}_2^{(iv)} = 0, \quad y \in (0, 1), \quad \hat{A}_3^{(iv)} = 0, \quad y \in (1, \kappa) \\ \hat{A}_3(\kappa) = \hat{A}'_3(\kappa) = \hat{A}_2(0) = \hat{A}'_2(0) = 0 \\ ((\hat{A})) = 0 \\ \hat{A}'_3(1) = m\hat{A}'_2(1) \\ \hat{A}'''_3(1) = m\hat{A}'''_2(1) \end{array} \right.$$

Condition

$$\hat{C}_A^*((\hat{A})) = \hat{A}_2(1)[[h]]$$

will be used later on to find  $\hat{C}_A^*$ .

The normalizing condition for the eigenfunctions A will be

$$A'(0) = 1$$

The solutions of ( $\hat{A}$ ) are given by

$$\begin{aligned} \hat{A}_2(y) &= y + m^{-1}\bar{\Delta}^{-1}y^3 \\ \hat{A}_3(y) &= n^{-3}\bar{\Delta}^{-1}[2(n+1)^3 - 3n(n+1)^2y + n^3y^3] \end{aligned} \tag{4.5}$$

where

$$\bar{\Delta} = m^{-1}n^{-3}[m(3n+2) - n^3] \tag{4.6}$$

therefore there is a curve  $\bar{\Gamma}$  in the (m, n)-plane along which problem ( $\hat{A}$ ) has no nontrivial solutions. However, as in the (S)-case, this singularity does not affect the first order approximation. After some manipulations, the eigenvalue  $C_A$  at the zeroth order is given by

$$\hat{C}_A = \frac{1}{2}h_1 \frac{[2n^4 + (4+3m)n^3 + (2+7m)n^2 + 5mn + m]}{n^2(n^3 + 3mn^2 + 3mn + m)} \tag{4.7}$$

At the first order in  $\alpha$  system (A) is the following

$$(\bar{A}) \left\{ \begin{array}{l} \bar{A}_j^{(iv)} = iR_j[(U^{(0)}(y) - \hat{C}_A)\bar{A}_j'' + h_j\bar{A}_j], \quad j = 2, 3 \\ \bar{A}_3(\kappa) = \bar{A}'_3(\kappa) = \bar{A}_2(0) = \bar{A}'_2(0) = 0 \\ ((\bar{A})) = 0 \\ \bar{C}_A^*((\bar{A}')) + \hat{C}_A^*((\bar{A}')) = \bar{A}_2(1)[[h]] \\ \bar{A}'_3(1) = m\bar{A}'_2(1) \\ \bar{A}'''_3(1) = m\bar{A}'''_2(1) \end{array} \right.$$

Let us put

$$iR_j[(U^{(0)}(y) - \hat{C}_A)\bar{A}_j'' + h_j\bar{A}_j] = ih_1RK_j(y)$$

Using (3.1), (4.5), (4.6) and (4.6) we find

$$K_2(y) = L_1^{-1}(L_2y + L_3y^3)$$

$$K_3(y) = L_4^{-1}(L_5 + L_6y + L_7y^3)$$

where, for simplicity, we have defined

$$L_1 = m^3n^2L_8\bar{\Delta}$$

$$L_2 = 3[(m + 2mn + n^2)L_8 - mn^2L_9] + mn^2L_8\bar{\Delta}$$

$$L_3 = -2n^2L_8$$

$$L_4 = n^3L_8\bar{\Delta}$$

$$L_5 = 2(n + 1)^3L_8$$

$$L_6 = -3n^3L_9$$

$$L_7 = -2n^3L_8$$

$$L_8 = n^2(n^3 + 3mn^2 + 3mn + m)$$

$$L_9 = 2n^4 + (4 + 3m)n^3 + (2 + 7m)n^2 + 5mn + m$$

Let us now set

$$ih_1RK_2(y) = 5!\check{f}_2y + \frac{7!}{3!}\check{h}_2y^3$$

$$ih_1RK_3(y) = 4!\check{e}_3 + 5!\check{f}_3y + \frac{7!}{3!}\check{h}_3y^3$$

so that we can express the general solution of ( $\bar{A}$ ) as follows:

$$\bar{A}_2(y) = \check{b}_2y + \check{d}_2y^3 + \check{f}_2y^5 + \check{h}_2y^7$$

$$\bar{A}_3(y) = \check{a}_3 + \check{b}_3y + \check{c}_3y^2 + \check{d}_3y^3 + \check{e}_3y^4 + \check{f}_3y^5 + \check{h}_3y^7$$

Actually  $\check{b}_2$  is zero since

$$A'_2(0) = 1 = \hat{A}'_2(0) + \bar{A}'_2(0)\alpha + \text{higher order terms in } \alpha$$

must be satisfied for each  $\alpha$ .

The remaining coefficients have to be determined using boundary and interface conditions. After some calculations we get

$$\check{c}_3 = 6\check{e}_3 + 20\check{f}_3 - 20m\check{f}_2 + 84\check{h}_3 - 84m\check{h}_2$$

$$\check{d}_3 = -(ms + 1)^{-1}(x_1\check{e}_3 + x_2\check{f}_3 + x_3\check{h}_3 + x_4\check{f}_2 + x_5\check{h}_2)$$

where  $s$  is the same we used to write  $\check{e}_3$  for system ( $\bar{S}$ ) and

$$x_1 = n^{-4}[4n^4 - m(12n^2 + 8n + 3)]$$

$$x_2 = n^{-5}[10n^5 - m(30n^3 + 20n^2 + 15n + 4)]$$

$$x_3 = n^{-7}[35n^7 - m(105n^5 + 70n^4 + 105n^3 + 84n^2 + 35n + 6)]$$

$$x_4 = mn^{-2}(20m - 9n^2)$$

$$x_5 = mn^{-2}(84m - 34n^2)$$

Explicit knowledge of the other coefficients defining  $\tilde{A}_2$  and  $\tilde{A}_3$  is not needed to compute the eigenvalue. The latter is given by

$$\tilde{C}_A = \tilde{C}_A^* = h_1 \frac{(1-m)}{m} ((\tilde{A}'))^{-2} [((\tilde{A}'))\tilde{A}_2(1) - \tilde{A}_2(1)((\tilde{A}'))]$$

As in the (S)-case,  $\tilde{C}_A$  is purely imaginary. After some manipulations, it turns out that

$$\tilde{C}_A = \frac{iRh_1^2}{840} (1-m)m^{-2}n^3L_8^{-3}\gamma_A^*(n,m) \tag{4.8}$$

where

$$\gamma_A^*(m,n) = \sum_{i=0}^{10} \omega_i(m)n^i$$

Since, for  $m, n > 0$  and  $L_8 > 0$ , we have that, for lubricating flows ( $m > 1$ ),

$$\text{sign } \tilde{C}_A = -\text{sign } \gamma_A^*$$

while, for fingering flows ( $m < 1$ ),

$$\text{sign } \tilde{C}_A = \text{sign } \gamma_A^*$$

It is worth noting that  $\gamma_A^*(n,m) > 0$  for  $m > 1$ . This indicates that the odd mode always gives rise to the stability when the more viscous fluid is centrally located.

One branch of the curve of marginal stability is given, as in the (S)-case, by  $m = 1$ ; the other branch is the curve  $\gamma_A^*(m,n) = 0$ . The coefficients of  $\gamma_A^*$  are explicitly given in the Appendix.

### 5. NUMERICAL RESULTS

The decomposition into even modes (a) and odd modes (b) gives rise to sinuous (a) and varicose (b) interfaces as shown in Fig. 2.

The coefficient of  $\alpha$  in the expansion of the eigenvalues control stability for small  $\alpha$  in both cases. These coefficients are given by (4.4) and (4.8) and are in the form

$$\tilde{C} = iRh_1^2\tilde{C}$$



Fig. 2. Shape of the interface for odd (a) and even (b) disturbances.

where  $\tilde{C}$  is a real function of  $m = \mu_2/\mu_1$  and  $n = d_2/d_1$ . The margin of stability is defined by the equation

$$\tilde{C}(n,m) = 0$$

The values of decay rates for long waves depend on  $\alpha^2Rh_1^2$  but the stability depends only on  $n$  and  $m$ , as in the work of Yih [1]. To have a stable flow we must have stability

against all disturbances, even and odd. Lubricating flows are those with more viscous fluid in the center. Fingering flows have less viscous fluid in the center. We find that the fingering flow is always unstable and lubricating flow is stable against long waves for all possible volume ratios. The algebraic manipulations were done on the CYBER 845 using LISP package REDUCE2.

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#### APPENDIX

$$\omega_0(m) = 5m^4$$

$$\ddot{\omega}_1(m) = 60m^4$$

$$\omega_2(m) = m^3(171m + 106)$$

$$\omega_3(m) = m^2(219m^2 + 351m + 112)$$

$$\omega_4(m) = m^2(252m^2 + 321m + 448)$$

$$\omega_5(m) = m^2(189m^2 + 357m + 470)$$

$$\omega_6(m) = m(441m^2 + 490m - 176)$$

$$\omega_7(m) = 2m(357m - 116)$$

$$\omega_8(m) = 2m(63m + 68)$$

$$\omega_9(m) = 48(m + 1)$$

$$\omega_{10}(m) = 32$$

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