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Nonlinear stability of rotating flow of two fluids

Stabilité non linéaire de l'écoulement de deux fluides en rotation

by

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ABSTRACT. — The stability of Couette flow between concentric cylinders of two immiscible fluids with different viscosities and different densities is studied.

Two approaches are proposed, both based on the energy method.

The first one consists in decomposing the solution at time t into a steady solution defined on the evolution configuration plus a disturbance. An energy estimate is derived, involving an interface term which is not calculable in general. We show that, if this interface term is integrable on R_+, then stability occurs.

We calculate the interface term in the case of rigid motions. We show that under a rather general assumption on the contact angle between the interface and either cylinder, and the end walls, the interface term reduces to a time derivative of some function. Hence the interface, which is solutions, is obtained by minimizing some potential.

The second approach is possible only when an explicit solution is known. When gravity is neglected, a radial solution exists which corresponds to a cylindrical interface. The motion at time t is decomposed into

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this steady solution, which is extended to the time dependent domains, and a perturbation. In the inviscid case, some sufficient stability conditions are deduced from a linearization around the radial solution.

RÉSUMÉ. — Nous étudions la stabilité de l'écoulement de Couette, entre deux cylindres concentriques, de deux fluides newtoniens immiscibles de viscosités et de densités différentes.

Deux approches, toutes deux basées sur la méthode de l'énergie, sont proposées.

La première consiste à décomposer la solution à l'instant t en une solution stationnaire définie sur la configuration à l'instant t, *a priori* inconnue, et une perturbation. Nous en déduisons une estimation d'énergie où intervient un terme d'interface, non calculable en général. Nous montrons que, si le terme d'interface est intégrable sur R_+, alors il y a stabilité.

Nous calculons le terme d'interface dans le cas des mouvements rigides. Nous montrons que, sous une hypothèse assez générale concernant l'angle de contact de l'interface avec l'un ou l'autre cylindre ainsi qu'avec les parois qui limitent les cylindres, le terme d'interface se réduit à une dérivée par rapport au temps d'une certaine fonction. Cela nous permet d'obtenir les interfaces solutions comme minimum d'une certaine fonctionnelle.

La deuxième approche s'applique seulement lorsque l'on connaît explicitement une solution. Lorsque la gravité est négligée, une solution radiale existe et correspond à une interface cylindrique. Nous décomposons la solution en cette solution stationnaire, mais étendue sur la configuration à l'instant t, et une perturbation. Dans le cas non visqueux, nous linéarisons le problème au voisinage de la solution radiale, et nous obtenons des conditions suffisantes de stabilité.

1. Introduction

Flows of two fluids are important and interesting because they are commonplace, they lend themselves to technological application, and they introduce new physical phenomena without counterpart in the flow of one fluid.

Of the many possible applications of the flow of two fluids, the lubrication of one fluid by another would seem to be of special interest. It is possible to lubricate crude oils flowing down pipes by injecting water. The water tends to drift to the wall, lubricating the flow (Charles and Redberger [3]; Hasson and Nir [5]; Hodgson and Charles [7]). This type of lubrication is possible because it is stable (Joseph, Renardy and Renardy [10]). The side-by-side stratified extrusion of polymer melts is used by the synthetic fiber and plastics industry to fabricate important products of commerce, specifically bicomponent fibers and sheets (Lee and White [12]). In these applications it is necessary to control the encapsulation. The encapsulation of more with less viscous fluids appears to be involved in compound jets which are used in ink jet printing (Hertz and Hermanrud [6]).

The lubrication of very viscous liquids passing through dies with less vicous liquids is another application (Macosko, Oscansey and Winter [13]) made viable by the tendency of less viscous liquids to drift into regions of high shear (*see* Joseph, Nguyen and Beavers [9]).

The problem treated here, the stability of two fluids between rotating cylinders, is probably a natural setting for the discussion of the lubricating possibilities of flows of two fluids between rotating cylinders. The linearized problem for the flow of two fluids between rotating cylinders with gravity neglected was studied numerically by Renardy and Joseph [15]. They confined their attention to the case, not generic, in which the interface between the two liquids in the basic flow is of constant radius. They found that it is possible to have such a basic flow with a flat cylindrical interface even when the heavy fluid is inside, provided that the heavy fluid is in a thin layer and has a relatively low viscosity. This is a stable lubricating flow, which runs against intuition, which would suggest instability due to centrifuging.

The problem of two rotating fluids, like other two fluid problems, is characterized by a high degree of nonuniqueness which is associated with the fact that the placements of the two fluids even in steady flow is not determined by the equations and interface conditions for steady flow (see Joseph, Nguyen and Beavers [9]). The mathematical problem of uniqueness of steady flow of two Navier-Stokes liquids is basically wide open. One of the problems, perhaps central, is that many stable configurations of two fluids are possible, ranging from flows with smooth interfaces to various emulsified configurations.

The flow of two fluids is more nonlinear than the flow of one fluid because of the geometric nonlinearities introduced at the interface. For example, the surface tension is very nonlinear and in the problem of two rotating fluids the centripetal acceleration field which enters into the potential is also nonlinear.

Nonlinear and nonunique features introduced by the placement problem in the flow of two fluids has a certain implication for the linear theory of stability of flow of two fluids. The new feature here is that the basic flow whose stability needs to be studied may not be unique even at vanishing Reynolds number. For example, in a static fluid in the absence of gravity, the two fluids will form small spheres whose size and distribution are determined by initial conditions, as in the experiments of Plateau (see, e. g., Joseph, Nguyen and Beavers [11]). In principle, a continuum of static solutions could be linearly unstable. Another example is the stability of a thin film coating a rotating cylinder. This problem was first considered by Yih [16]. He showed that a film of constant radius, in the absence of gravity, would be unstable and tend to develop rings before being destroyed by instabilities. This result, though correct, does not imply that stable films rotating rigidly on a cylinder in air are unstable. In fact, corrugated films rotating rigidly on a cylinder in air are possible and stable. This shows that in the flow of two fluids the choice of the basic flow whose stability is to be studied has an importance without parallel in the flow of a single fluid. The same remark applies to the study of stability of Couette flow of two fluids by Renardy and Joseph [15]. They found some conditions under which flow with perfectly circular uncorrugated interfaces would be stable. Questions about the stability of steady flow with corrugated or otherwise deformed interfaces are left unanswered.

In this paper we shall attempt to address some of these new nonlinear features which are introduced by the placement problem associated with the flow of two rotating fluids. We have formulated two different approaches, both based on energy methods, to this problem. In the first approach we decompose the motion into an abstract stationary motion and a disturbance. The stationary motion has a non-stationary interface, the same interface as the disturbance, but there are no time derivatives in the problem governing this motion. We have a stationary problem in a fixed region at each instant. The solution of each stationary problem exists and is unique when the Reynolds number is small. The stability of this type of stationary problem is finally decided by the decay of the disturbance energy. When this energy decays, we get two things, the conditions for nonlinear stability and the domain, the interface, for the stationary flows which are stable. We establish some results for this new type of decomposition but we are only partially successful in determining explicit conditions for stability.

The second approach to nonlinear stability through the study of energy is at first glance more traditional but it also involves in some sense the definition of an abstract basic motion. In the case treated here we may write down a basic Couette flow when the interface is a cylinder of constant radius and gravity is neglected. To study the stability of this flow we must, however, extend it into a deformed domain. In fact, we really study the stability of this extended motion. This type of extension is evidently required for nonlinear studies of the validity of linearization and is perfectly well suited to the study of bifurcation. However, this type of analysis is very restricted in the sense that it does not address the problem of finding the stable placements of two fluids.

In sections 2 and 3, we describe the physical situation and the equations of motion. Section 4 to 7 are devoted to the study of the first decomposition of motion. This decomposition is introduced in section 4. In section 5, we derive an energy identity, and we identify explicitly the destabilizing term at the interface. In section 6 we derive an energy inequality. Unfortunately, we are not able to estimate the destabilizing term, but we prove that if this term is merely integrable, then global stability holds. In section 7, we consider the problem of rigid motions which was studied by Joseph, Renardy, Renardy and Nguyen [12]. Our contribution here is to identify explicitly energy terms which arise at contact lines on the cylinders and on the end walls. We specify conditions, much more general than assuming that the contact angle is constant, under which the contact lines give rise to a potential. Sections 8 to 10 are devoted to the study of the second decomposition of motion. In section 8 and 9, this decomposition of motion is defined and energy equalities are derived. In section 10, we study the linearized stability of inviscid flows in circles and we find some sufficient conditions for stability.

2. Description of the problem

We consider two immiscible fluids filling the region between two coaxial cylinders, the axis being horizontal. The inner cylinder has radius R_1 and angular velocity Ω_1; the outer cylinder has radius R_2 and angular velocity Ω_2 (cf. *Fig.* 1).

Two cases are studied: either the cylinders are infinitely long, or they are bounded by end walls perpendicular to the axis. In the case of infinite cylinders, we shall restrict our attention to motions which are spatially periodic, a cell of periodicity having length $\Lambda = 2\pi/\alpha$, where α is a wave number.

The interface between the two fluids is assumed to be a sufficiently smooth surface, possibly unconnected. This means, for example, that one fluid may be a collection of disjoint bubbles or drops, as in a water-oil emulsion. The interface may touch either

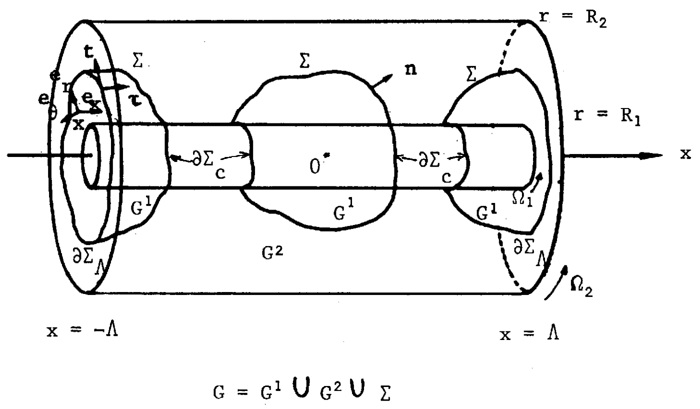


Fig. 1. – Sketch of the apparatus.

Fig. 1. – Schéma de l'appareil.

cylinder, and, in the case of finite cylinders, the interface may touch the end walls. The locus of touching is called a contact line. In section 7, a complete analysis of rigid motion, in which $\Omega_1 = \Omega_2$, is presented and a rather general set of interfaces is considered.

3. Equations of motion

Let us denote G_Λ, or G, the domain occupied by the two fluids: for Λ positive,

$$G_\Lambda = \{\mathbf{x} = (r, \theta, x),\ R_1 < r < R_2,\ 0 \leqq \theta < 2\pi,\ -\Lambda < x < \Lambda\};$$

the x-axis is the axis of the two coaxial cylinders. Let ρ_i, μ_i, $i = 1, 2$, be the density and the viscosity of the fluids. Let $G^i_\Lambda(t)$ or $G^i(t)$, $i = 1, 2$, denote the region in $G_\Lambda = G$ occupied by the fluid i, at time t.

The motion of each fluid is described by the Navier-Stokes equations, that is, for $i = 1, 2$, the velocity $\mathbf{u}_i = u_i \mathbf{e}_r + v_i \mathbf{e}_\theta + w_i \mathbf{e}_x$ and the pressure p_i of the fluid i satisfy,

$$(1) \qquad \begin{cases} \rho_i (\partial_t \mathbf{u}_i + (\mathbf{u}_i . \nabla) \mathbf{u}_i) = -\nabla p_i + \operatorname{div} \mathbf{S}_i + \rho_i \mathbf{g} & \text{in } G^i, \\ \operatorname{div} \mathbf{u}_i = 0 \quad \text{in } G^i, \end{cases}$$

where $\mathbf{S}_i = 2\mu_i \mathbf{D}[\mathbf{u}_i]$, $\mathbf{D}[\mathbf{u}] = (\nabla \mathbf{u} + (\nabla \mathbf{u})^T)/2$; $\mathbf{g}$ denotes the gravity. The right-hand side of the first equation (1) can also be written as

$$-\nabla \varphi_i + \operatorname{div} \mathbf{S}_i,$$

with $\varphi_i = p_i + \rho_i g r \sin \theta$.

As a boundary condition, we will consider a nonslip condition, say

$$(2) \qquad \mathbf{u}_i = \mathbf{u}^* \quad \text{on } \partial G^i \cap \partial G, \qquad i = 1, 2,$$

where $\mathbf{u}^*$ is defined below. First, on the cylinders, we have

$$\mathbf{u}^*(R_i, \theta, x) = R_i \Omega_i \mathbf{e}_\theta, \qquad (\theta, x) \in [0, 2\pi] \times [-\Lambda, \Lambda], \quad i = 1, 2, \tag{3}$$

In the case of finite cylinders, we assume that ∂G_Λ is a rigid body, i. e., that the two cylinders rotate at the same speed $\Omega = \Omega_1 = \Omega_2$; the boundary condition is then:

$$\mathbf{u}^*(r, \theta, \pm\Lambda) = r\Omega \mathbf{e}_\theta, \qquad (r, \theta) \in [R_1, R_2] \times [0, 2\pi]. \tag{4}$$

In the case of infinite cylinders, we assume that the motion is spatially periodic and that G_Λ denotes a cell of periodicity; the periodic boundary condition is then ([1]):

$$\mathbf{u}_i(r, \theta, \Lambda) = \mathbf{u}_i(r, \theta, -\Lambda), \qquad (r, \theta) \in [R_1, R_2] \times [0, 2\pi], \quad i = 1, 2. \tag{5}$$

The interface Σ between regions 1 and 2 is given by

$$F(\mathbf{x}(t), t) = 0, \qquad t \in R_+,$$

and, because it is an identity in t,

$$\frac{\partial F}{\partial t} + \mathbf{u} . \nabla F = 0 \quad \text{on } \Sigma, \tag{6}$$

where we have assumed that the normal component of velocity $d\mathbf{x}/dt$ of the surface Σ and of the particles of fluid on either side of Σ are the same. The velocity $\mathbf{u}$ is continuous across Σ, and the jump across Σ of the traction is equal to the surface tension, that is

$$\left\{ \begin{array}{l} [\mathbf{u}] = 0 \quad \text{on } \Sigma, \\ [-p\mathbf{n} + S . \mathbf{n}] = 2 H T \mathbf{n} \quad \text{on } \Sigma, \end{array} \right. \tag{7}$$

where $[.] = (.)_1 - (.)_2$ denotes the jump across Σ, $\mathbf{n} = \nabla F / |\nabla F|$ denotes the unit normal vector pointing from G^1 to G^2, T denotes the surface tension coefficient, a given positive constant, and 2 H is the mean curvature of Σ.

We complete the boundary value problem (1) through (7) by prescribing the velocity of each fluid and the position Σ^0 of Σ at time $t = 0$:

$$\left\{ \begin{array}{ll} \mathbf{u}_i(0) = \mathbf{u}_i^0 \quad \text{in } G^i, & i = 1, 2, \\ F(0) = F^0 \quad \text{in } G. & \end{array} \right. \tag{8}$$

The second condition (8) specifies in particular the volumes occupied by each fluid.

([1]) More generally, the periodic condition can be written in the following way

$$\left. \frac{\partial^k u_i(r, \theta, x)}{\partial x^k} \right|_{x=\Lambda} = \left. \frac{\partial^k u_i(r, \theta, x)}{\partial x^k} \right|_{x=-\Lambda}, \qquad k = 0, 1, \ldots, l,$$

l depending on the regularity of the solution.

Remark 3.1. — Since the total volume of each incompressible fluid is conserved, we may deduce that

$$(9) \qquad \langle \mathbf{u}.\mathbf{n} \rangle_\Sigma \overset{\text{def}}{=} \int_\Sigma \mathbf{u}.\mathbf{n}\, d\Sigma = 0. \ \square$$

In what follows, we investigate two different approaches for the study of uniqueness and stability of the solutions of problem (1) through (8).

4. Decomposition of the motion (I)

Following the ideas given by Joseph, Renardy, Renardy and Nguyen [11], we propose an unusual notion of stability based on some decomposition of motion into an *unsteady* basic motion plus a disturbance. This notion of stability is weaker than the classical one which is based on a decomposition of motion into an *equilibrium* plus a disturbance. This approach has the advantage that the interface between the two fluids is the same for the basic motion and for the disturbance.

We assume that problem (1) through (8) has a solution $(\mathbf{u}(t), p(t), \Sigma(t))$, $t \in \mathrm{R}$ and we decompose it in the following way:

$$(10) \qquad \mathbf{u} = \mathbf{u}_0 + \hat{\mathbf{u}}, \qquad p = p_0 + \hat{p},$$

where $(\mathbf{u}_0, p_0)$ is a solution of the steady problem:

$$(11) \qquad \begin{cases} \rho_i (\mathbf{u}_{0i} . \nabla) \mathbf{u}_{0i} = -\nabla p_{0i} + \operatorname{div} \mathbf{S}_{0i} + \rho_i \mathbf{g} & \text{in } \mathrm{G}^i, \\ \operatorname{div} \mathbf{u}_{0i} = 0 \quad \text{in } \mathrm{G}^i, \qquad i = 1, 2, \end{cases}$$

$$(12) \qquad \mathbf{u}_{0i} = \mathbf{u}_i^* \quad \text{on } \partial \mathrm{G}^i \cap \partial \mathrm{G}, \qquad i = 1, 2,$$

$$(13) \qquad \mathbf{u}_0 . \mathbf{n} = 0 \quad \text{on } \Sigma,$$

$$(14) \qquad \begin{cases} [\mathbf{u}_0] = 0 \quad \text{on } \Sigma, \\ [\boldsymbol{\tau}_k . \mathbf{S}_0 . \mathbf{n}] = 0, \qquad k = 1, 2, \quad \text{on } \Sigma, \end{cases}$$

where $\mathbf{S}_0 = 2\mu \mathbf{D}[\mathbf{u}_0]$ and $(\boldsymbol{\tau}_1, \boldsymbol{\tau}_2)$ is an orthonormal basis of the tangent plane to Σ. As in (1), we may introduce the notation $\varphi_{0i} = p_{0i} + \rho_i g r \sin\theta$, $i = 1, 2$.

In problem (11) through (14), the surface $\Sigma = \Sigma(t)$ is fixed and $(\mathbf{u}_0, p_0)$ depends on t through $\Sigma(t)$ only. So, to a solution $(\mathbf{u}(t), p(t), \Sigma(t))_{t \in \mathrm{R}_+}$ of problem (1) through (8), we associate a family of unsteady basic motions $(\mathbf{u}_0(t), p_0(t), \Sigma(t))_{t \in \mathrm{R}_+}$, parametrized by t.

Let $\mathscr{S}$ be a set of regular surfaces Σ, containing the set $\{\Sigma(t)\}_{t \in \mathrm{R}_+}$. This definition supposes that we know, or, at least, that we have an idea of, the set of interfaces when t varies (*cf.* section 7, for a particular case).

DEFINITION 4.1. — *Solutions* $(\mathbf{u}_0, p_0)$ *of problem* (11) *through* (14) *with* $\Sigma \in \mathscr{S}$ *are called candidates for Couette motions.*

Candidates $(\mathbf{u}_0, p_0, \Sigma)$ which also satisfy the condition

$$[-p_0+\mathbf{n}.\mathbf{S}_0.\mathbf{n}]=2\,\mathrm{HT} \quad \text{on } \Sigma \tag{15}$$

are the Couette motions whose uniqueness and stability are under investigation.

The disturbance $(\hat{\mathbf{u}}, \hat{p})$ satisfies the following equations:

$$\left\{\begin{array}{l} \rho_i(\partial_t\hat{\mathbf{u}}_i+(\hat{\mathbf{u}}_i+\mathbf{u}_{0i}).\nabla\hat{\mathbf{u}}_i+(\hat{\mathbf{u}}_i.\nabla)\mathbf{u}_{0i})=-\nabla\hat{p}_i+\operatorname{div}\hat{\mathbf{S}}_i, \\ \operatorname{div}\hat{\mathbf{u}}_i=0 \quad \text{in } \mathrm{G}^i, \qquad i=1, 2, \end{array}\right. \tag{16}$$

$$\left\{\begin{array}{l} \hat{\mathbf{u}}_i=0 \quad \text{on } \partial\mathrm{G}^i\cap\partial\mathrm{G}, \qquad i=1, 2, \\ \text{in the case of finite cylinders,} \\ \hat{\mathbf{u}}_i(r=\mathrm{R}_i)=0, \qquad \hat{\mathbf{u}}_i(x=\Lambda)=\hat{\mathbf{u}}_i(x=-\Lambda), \qquad i=1, 2, \\ \text{in the spatially periodic case,} \end{array}\right. \tag{17}$$

$$\hat{\mathbf{u}}.\mathbf{n}=-\partial_t\mathrm{F}/|\nabla\mathrm{F}| \quad \text{on } \Sigma, \tag{18}$$

$$\left\{\begin{array}{l} [\hat{\mathbf{u}}]=0, \\ [-\hat{p}\,\mathbf{n}+\hat{\mathbf{S}}.\mathbf{n}]=2\,\mathrm{HT}\,\mathbf{n}+[p_0\,\mathbf{n}-\mathbf{S}_0.\mathbf{n}], \quad \text{on } \Sigma, \end{array}\right. \tag{19}$$

$$\left\{\begin{array}{l} \hat{\mathbf{u}}_i(0)=\mathbf{u}_i^0-\mathbf{u}_{0i}(0) \quad \text{in } \mathrm{G}^i, \qquad i=1, 2, \\ \mathrm{F}(0)=\mathrm{F}^0 \quad \text{in } \mathrm{G}, \end{array}\right. \tag{20}$$

where $\hat{\mathbf{S}}=2\,\mu\,\mathbf{D}[\hat{\mathbf{u}}]$.

The conservation of volume implies, in particular, that

$$\langle\hat{\mathbf{u}}.\mathbf{n}\rangle_\Sigma=-\langle|\nabla\mathrm{F}|^{-1}\partial_t\mathrm{F}\rangle_\Sigma=0,$$

(*cf.* Remark 3.1).

Our aim is to give some ideas on the uniqueness and stability of solutions to problem (1) through (8). For this, we first define the notion of stability that we are going to consider, and then we derive an energy identity satisfied by the disturbance $\hat{\mathbf{u}}$.

5. Energy identity for decomposition (I)

As in the classical analysis of stability, we shall define some Reynolds number Re such that, if Re is small enough, the candidates for Couette motions (*cf.*, Definition 4.1) are uniquely determined by the data.

Let us first define some functional spaces. For an integrable function f in G, which equals f_1 in G^1, and f_2 in G^2, we set

$$<f>=\int_{\mathrm{G}^1}f_1\,d\mathbf{x}+\int_{\mathrm{G}^2}f_2\,d\mathbf{x}.$$

Let us denote **H** the space of solenoidal vectors with components in $\mathrm{L}^2(\mathrm{G})$, and with $\mathbf{u}_i.\mathbf{n}=\mathbf{0}$ on $\partial\mathrm{G}^i\cap\partial G$ (or satisfying the corresponding boundary conditions in the spatially periodic case). Let us denote **X** the space of vectors **u** with components in the

Sobolev space H^1 (G) and satisfying the boundary conditions (17). Let **V** be the space of solenoidal vectors, which are in **X**.

DEFINITION 5.1. — *For a vector* **v** *in* **X**, *we define*

(i) *the energy*, $\mathscr{E}(\mathbf{v})=(1/2)<\rho|\mathbf{v}|^2>$,

(ii) *the production*, $\mathscr{I}_{\mathbf{u}_0}(\mathbf{v})=<\rho\,\mathbf{v}.\mathbf{D}[\mathbf{u}_0].\mathbf{v}>$, *where* $\mathbf{D}[\mathbf{u}_0]$ *is in* $L^\infty(G)^3$,

(iii) *the dissipation*, $\mathscr{D}(\mathbf{v})=<\nabla\mathbf{v}.\mathbf{S}>=2<\mu\mathbf{D}[\mathbf{v}]^2>$.

We will say that a steady solution to problem (1) through (8) is (asymptotically) *stable* if the vector $\hat{\mathbf{u}}$, defined by (10), goes to **0** in **H** as $t\to\infty$, for all initial data ($\mathbf{u}^0$, F^0) close enough to this steady solution.

LEMMA 5.2. — *Regular solutions* $\hat{\mathbf{u}}$ of problem (16) *through* (20) *satisfy the following energy identity:*

$$\frac{d}{dt}\mathscr{E}+\mathscr{I}_{\mathbf{u}_0}+\mathscr{D}=<\hat{u}.\mathbf{n}\,N>_\Sigma \tag{21}$$

with $N=2HT+\left[p_0-\mathbf{n}.\mathbf{S}_0.\mathbf{n}\right]=2HT+\left[\varphi_0-\mathbf{n}.\mathbf{S}_0.\mathbf{n}\right]-[\rho]\,gr\sin\theta.$

Proof. — Multiplying equation $(16)_1$ by $\hat{\mathbf{u}}_i$ in $[L^2(G^i)]^3$, and adding the two equations obtained for $i=1, 2$, we get the identity:

$$\frac{d}{dt}\mathscr{E}+\mathscr{I}_{\mathbf{u}_0}+\mathscr{D}=\langle\left[\hat{\mathbf{u}}.(-\hat{p}\,\mathbf{n}+\hat{\mathbf{S}}.\mathbf{n})\right]\rangle_\Sigma. \tag{22}$$

The boundary term may be written as

$$\langle\hat{\mathbf{u}}.\left[-\hat{p}\,\mathbf{n}+\hat{\mathbf{S}}.\mathbf{n}\right]\rangle_\Sigma,$$

for $\hat{\mathbf{u}}$ is continuous across Σ, next as

$$\langle\hat{\mathbf{u}}.(2HT\,\mathbf{n}+\left[p_0\mathbf{n}-\mathbf{S}_0.\mathbf{n}\right])\rangle_\Sigma$$

because of condition $(19)_2$. Moreover, because of $(14)_2$, we have

$$\hat{\mathbf{u}}.[\mathbf{S}_0].\mathbf{n}=\hat{\mathbf{u}}.\mathbf{n}\,\mathbf{n}.[\mathbf{S}_0].\mathbf{n}\quad\text{on }\Sigma.$$

Thus, from (22), we deduce the energy identity (21). □

Remark 5.3. — If, for some t, ($\mathbf{u}_0(t)$, $p(t)$, $\Sigma(t)$) is a candidate which satisfies the normal stress conditions (15), then $N(t)=0$ and ($\mathbf{u}_0(t)$, $p(t)$, $\Sigma(t)$) *is a Couette motion.* □

Remark 5.4. — Parts of the boundary term in (21) can be expressed as time derivatives. We find, using (10) and (13), that

$$\langle 2HT\,\hat{\mathbf{u}}.\mathbf{n}\rangle_\Sigma=\langle 2HT\,\mathbf{u}.\mathbf{n}\rangle_\Sigma=T\left(-\frac{d}{dt}|\Sigma|+\int_{\partial\Sigma}\boldsymbol{\tau}.\mathbf{U}\,dl\right) \tag{23}$$

(*cf.*, e.g., Aris [1], p. 230, Dussan [4], Joseph [8], II, p. 243), where $|\Sigma|$ denotes the measure of Σ, $\partial\Sigma$ is the line of contact of Σ on the walls at $x=\pm\Lambda$ and on the cylinders at $r=R_1$ or R_2, $\boldsymbol{\tau}$ is the normal to $\partial\Sigma$ lying on Σ, and $\mathbf{U}$ the velocity of $\partial\Sigma$; so identity (21) may be written as

$$\frac{d}{dt}(\mathscr{E}+T|\Sigma|)+\mathscr{I}_{\mathbf{u}_0}+\mathscr{D}=\langle\hat{\mathbf{u}}.\mathbf{n}\,M\rangle_\Sigma+T\left(\int_{\partial\Sigma}\boldsymbol{\tau}.\mathbf{U}\,dl\right), \tag{24}$$

with $M=[p_0-\mathbf{n}.\mathbf{S}_0.\mathbf{n}]$. Let us notice that the last term in (24) vanishes if the interface is fixed ($\mathbf{U}=0$), if the interface Σ is perpendicular to both walls ($\boldsymbol{\tau}.\mathbf{U}=0$), or if $\partial\Sigma$ has measure 0 (e.g., one liquid is completely confined in bubbles).

The term $\langle\hat{\mathbf{u}}.\mathbf{n}\,[\rho]\,gr\sin\theta\rangle_\Sigma$ in (21), due to the presence of gravity, can also be expressed as the time derivative of some functional (*cf.*, Remark 7.8). □

Remark 5.5. — There are examples where the whole interface term in (21) and in (24) can be expressed as the time derivative of some potential (*cf.*, [11] and section 7). □

Remark 5.6. — As a particular case let us mention the situation where Σ is a graph in cylindrical coordinates, i.e., Σ has an equation, $F(\mathbf{x},t)=r-R(x,\theta,t)=0$. Then, we can write some explicit formulae:

$$\begin{gathered} M=[p_0]-[S^{rr}]+[S^{r\theta}]+\partial_\theta R/R-[S^{rx}]\,\partial_x R,\\ \langle\hat{\mathbf{u}}.\mathbf{n}\,M\rangle_\Sigma=\int_{-\Lambda}^{\Lambda}\int_0^{2\pi} M\,R\,\partial_t R\,d\theta\,dx, \end{gathered} \tag{25}$$

where M has been defined in Remark 5.4. □

The next section is devoted to some applications of the energy identity (21).

6. Some remarks regarding uniqueness and stability of solutions

We now seek some stability criteria framed in terms of the Reynolds number.

If endowed with the norm $(2\langle\mathbf{D}[.]^2\rangle)$, spaces $\mathbf{X}$ and $\mathbf{V}$, introduced in section 5, are Hilbert spaces. This follows from Korn's inequality,

$$\langle|\mathbf{v}|^2\rangle\leqq 2\kappa\langle\mathbf{D}[\mathbf{v}]^2\rangle,\qquad\forall\,\mathbf{v}\in\mathbf{X}, \tag{26}$$

for some positive constant $\kappa=\kappa(G)$. Actually, because of the identity,

$$2\langle\mathbf{D}[\mathbf{v}]^2\rangle=\langle|\nabla\mathbf{v}|^2\rangle+\langle(\operatorname{div}\mathbf{v})^2\rangle,\qquad\forall\,\mathbf{v}\in\mathbf{X},$$

the constant κ for (26) restricted to $\mathbf{V}$ is equal to Poincaré's constant (i.e., the first eigenvalue of the Stokes problem in G, with suitable boundary conditions). So, we have

$$\mathscr{D}(\mathbf{v})\geqq 2\lambda\mathscr{E}(\mathbf{v}),\qquad\forall\,\mathbf{v}\in\mathbf{X}, \tag{27}$$

where $\lambda=\kappa^{-1}\min(\mu_1,\mu_2)/\max(\rho_1,\rho_2)$. This inequality holds for connected configurations, as well as for emulsions.

Let us *assume* that

$$m = \text{Sup}\,\{\,\|\mathbf{D}(\mathbf{u}_0)\|_\infty,\ \mathbf{u}_0 \textit{ solution of problem } (11) \textit{ through } (14) \textit{ with } \Sigma \in \mathcal{S}\}$$

is finite. (The space $\mathcal{S}$ has been introduced in section 4).

Let us define a *Reynolds number*, $\text{Re} = \kappa\, m \max(\rho_1, \rho_2)/\min(\mu_1, \mu_2)$.

We now state a uniqueness result.

PROPOSITION 6.1. — *Let* Σ *be in* $\mathcal{S}$. *If* $\text{Re} < 1$, *the problem* (11) *through* (14) *has at most one solution and this solution is stable.*

Proof. — Let us denote $\hat{\mathbf{u}}$ the difference of a solution to problem (11) through (14) and a solution of the same problem for which a term $\rho_i \partial_t \mathbf{u}_{0i}$ has been added in $(11)_1$. Σ is fixed, so we have $\hat{\mathbf{u}}.\mathbf{n} = 0$, and the energy identity (21) implies

$$\frac{d\mathcal{E}}{dt} + \sigma\mathcal{E} \leqq 0, \tag{28}$$

where because of $\text{Re} < 1$,

$$\sigma = \min_{\mathbf{v}} \frac{\mathcal{D} + \mathcal{I}_{\mathbf{u}_0}}{\mathcal{E}}$$

is bounded below by a positive constant, $2\,m(\text{Re}^{-1} - 1)$. Thus $\mathcal{E}(t) \leqq \mathcal{E}(0)\exp(-\sigma t)$, $\forall t > 0$. This inequality proves the uniqueness and stability of a solution to problem (11) through (14). □

Remark 6.2. — As for the stability of the solutions of problem (1) through (8), we may prove that the disturbances, solutions of problem (16) through (20), have the following property:

(29) *If* $\text{Re} < 1$ *and* $\langle \hat{\mathbf{u}}.\mathbf{n}\,\mathrm{N} \rangle_\Sigma$ *belongs to* $\mathrm{L}^1(\mathrm{R}_+)$,

then $\hat{\mathbf{u}}$ *belongs to* $\mathrm{L}^\infty(\mathrm{R}_+; \mathbf{H})$ *and to* $\mathrm{L}^2(\mathrm{R}_+; \mathbf{V})$.

This means that, if the assumptions in (29) are satisfied, then $\mathbf{u}$ is weakly stable (in the sense that there exists a sequence t_j, $t_j \to +\infty$, such that $\hat{\mathbf{u}}(t_j)$ goes to $\mathbf{0}$ in $\mathbf{V}$).

To prove (29), we first deduce from the energy identity (21) that

$$\frac{d\mathcal{E}}{dt} + 2\,m(\text{Re}^{-1} - 1)\,\mathcal{E} \leqq \langle \hat{\mathbf{u}}.\mathbf{n}\,\mathrm{N} \rangle_\Sigma; \tag{30}$$

integrating this inequality on R_+, we deduce that $\mathcal{E}$ belongs to $\mathrm{L}^\infty(\mathrm{R}_+)$ and to $\mathrm{L}^1(\mathrm{R}_+)$. But we have

$$\int_0^{+\infty} \mathcal{I}_{\mathbf{u}_0}\, dt \geqq -2\,m \int_0^{+\infty} \mathcal{E}\, dt.$$

So, integrating (21) on $(0, +\infty)$, we obtain also

$$\mathscr{E}(+\infty)-2m\int_0^{+\infty}\mathscr{E}\,dt+\int_0^{+\infty}\mathscr{D}\,dt\leqq\mathscr{E}(0)+\int_0^{+\infty}\langle\hat{\mathbf{u}}.\mathbf{n}\,\mathrm{N}\rangle_\Sigma\,dt,$$

which shows that $\mathscr{D}$ belongs to $L^1(R_+)$. □

In the next section, we shall introduce an example where the condition « $\langle\hat{\mathbf{u}}.\mathbf{n}\,\mathrm{N}\rangle_\Sigma \in L^1(R_+)$ » is reduced to the boundedness from below of some potential. Moreover, we shall see that, in some particular cases, this property of boundedness is satisfied.

7. Global stability of rigid rotations of two fluids

We consider here the problem of stability of rigid rotations of two fluids with different densities and different viscosities, when gravity is neglected.

Joseph, Renardy, Renardy, Nguyen [11], hereinafter referred to as JRRN, treated the case in which the fluids fill the region between two infinite cylinders rotating with same velocity $\Omega\ (=\Omega_1=\Omega_2)$, and in which the interface is a graph in cylindrical coordinates, i. e., is represented by an equation

$$r=\mathrm{R}(\theta, x, t), \qquad \text{for} \quad \theta\in(0, 2\pi), \qquad x\in(-\Lambda, \Lambda),$$

where the spatial average of R^2 is a given constant $d^2>0$,

$$\int_{-\Lambda}^{\Lambda}\int_0^{2\pi}\mathrm{R}^2\,d\theta\,dx=d^2.$$

In this section, we extend the results of JRRN to the more realistic case in which the cylinders are bounded by end walls at $x=\pm\Lambda$. It was shown in JRRN that there are cases in which the stable interface touches the end walls, e. g., when $J=d^3[\rho]\Omega^2/T$ is larger than four, the stable interface has a constant radius d. But there are also cases in which the interface touches the inner cylinder, as in Figure 2. We also improve certain mathematical arguments of the previous work [11].

Let us first define what will be called « candidates » in this section.

DEFINITION 7.1. — *Candidates for rigid motions, with gravity neglected, are axisymmetric and are defined for* $R_1<r<R_2$ *by*

$$(\mathbf{u}_0(r),\ p_0(r))=(\Omega r\,\mathbf{e}_\theta,\ \rho\,\Omega^2 r^2/2+C), \tag{31}$$

where C *is some constant depending on the fluid.*

We draw attention to the fact that the definition for candidates in this section is different from the one we considered in section 4: candidates here satisfy $S_0=0$, and $\mathbf{u}_0.\mathbf{n}=\Omega r\,\mathbf{e}_\theta.\mathbf{n}\neq 0$ on Σ. A (steady) rigid motion $(\mathbf{u}_0, p_0, \Sigma)$ must satisfy the interface condition, $[p_0]=2\,\mathrm{HT}$ on Σ.

Let us now describe the set of interfaces that we shall consider in this section (*cf.* Figure 1).

We assume that the interface Σ between the two fluids has a finite number of components Σ^l, $l=1,\ldots,p$, each of them is represented locally by a finite number of equations $r=R^l(\theta, x, t)$, where R^l is a $\mathscr{C}^1$ function such that $R^l(\theta=0)=R^l(\theta=2\pi)$ (the so-called « periodicity in θ » condition); these local charts satisfy also suitable continuity conditions at common points.

We assume also that the boundary $\partial\Sigma$ of the interface has the following properties: either $\partial\Sigma$ has measure zero (case of emulsions or suspensions), or $\partial\Sigma$ has a finite number of components, say $\partial\Sigma=\partial\Sigma_\Lambda\cup\partial\Sigma_c$; $\partial\Sigma_\Lambda$ lies on the end walls at $x=\pm\Lambda$ and has an equation $r=R(\theta, \pm\Lambda, t)$; $\partial\Sigma_c$ lies on the cylinders with radius R_1 or R_2 and is made of a finite number of contact lines, each of them having an equation $x=\chi(R_i, \theta, t)$, $i=1$ or 2.

We may now calculate the boundary term in (21)

$$\langle \hat{\mathbf{u}}.\mathbf{n}([p_0]+2\,\mathrm{HT})\rangle_\Sigma,$$

with $\hat{\mathbf{u}}=\mathbf{u}-\mathbf{u}_0$. Making use of formula (23), we obtain

$$(32)\quad \langle \hat{\mathbf{u}}.\mathbf{n}([p_0]+2\,\mathrm{HT})\rangle_\Sigma=\langle \mathbf{u}.\mathbf{n}[p_0]\rangle_\Sigma - \langle \mathbf{u}_0.\mathbf{n}([p_0]+2\,\mathrm{HT})\rangle_\Sigma+\mathrm{T}\left(-\frac{d}{dt}|\Sigma|+\int_{\partial\Sigma}\tau.\mathbf{U}\,dl\right).$$

(We have observed above that $\mathbf{u}_0.\mathbf{n}$ is non-zero in this section.) A first expression for this boundary term is given in the following Lemma.

Lemma 7.2. — *Let α_Λ (resp. α_c) denote the angle between the interface Σ and the end walls (resp. the cylinders). Then the boundary term in* (21) *may be expressed as*

$$(33)\quad \langle \hat{\mathbf{u}}.\mathbf{n}([p_0]+2\,\mathrm{HT})\rangle_\Sigma = -\frac{d}{dt}\left\{\mathrm{T}|\Sigma|-\frac{\Omega^2}{8}[\rho](\langle R^3|\nabla F|^{-1}\rangle_\Sigma+\int_{\partial\Sigma_c}\varphi\,dl)\right\} -\mathrm{T}\left(\sum_{l=1,\ldots,p}\int_0^{2\pi}\left\{\left(\varepsilon_\Lambda^l\frac{\mathrm{D}R^l}{\mathrm{D}t}R^l\cos\alpha_\Lambda\right)\Bigg|_{x=\pm\Lambda} - \sum_{i=1}^{2}R_i\left(\varepsilon_c^l\frac{\mathrm{D}\chi^l}{\mathrm{D}t}\cos\alpha_c\right)\Bigg|_{r=R_i}\right\}d\theta\right),$$

where φ is some function defined on $\partial\Sigma_c$ and ε_Λ^l, $\varepsilon_c^l=-1$, 0 or 1.

Proof. — For fixing the ideas, we make the calculation of the boundary terms in (32) on a part of Σ which is a graph in cylindrical coordinates, and which touches the inner cylinder and the end wall at $x=\Lambda$. So we assume that Σ has an equation $r=R(\theta, x, t)$, for $x_1(\theta)<x<\Lambda$, $0\leq\theta<2\pi$, with $R(\theta, x_1(\theta), t)=R_1$, $\forall\,\theta\in[0, 2\pi]$; the function R is $\mathscr{C}^1$ and satisfies $R(0)=R(2\pi)$. Here $\partial\Sigma_c$ is the curve $x=\chi(R_1, \theta, t)=x_1(\theta, t)$, where χ is $\mathscr{C}^1$ and satisfies $\chi(0)=\chi(2\pi)$, and $\partial\Sigma_\Lambda$ is the curve $r=R(\theta, \Lambda, t)$.

First of all, we show that the term $\langle \mathbf{u}_0 . \mathbf{n}\, [p_0] \rangle_\Sigma$ is zero. Because of $\langle \mathbf{u}_0 . \mathbf{n} \rangle_\Sigma = 0$, we have

$$\langle \mathbf{u}_0 . \mathbf{n}\, [p_0] \rangle_\Sigma = \frac{1}{2} \Omega^2 [\rho] \langle \mathbf{u}_0 . \mathbf{n}\, R^2 \rangle_\Sigma.$$

Now, the term

$$I_1 = \langle \mathbf{u}_0 . \mathbf{n}\, R^2 \rangle_\Sigma = -\Omega \left(\int_0^{2\pi} \int_{x_1(\theta)}^{\Lambda} R_\theta R^3\, dx\, d\theta \right)$$

is calculated by using Leibnitz's rule in the following way:

$$4 \left(\int_{x_1(\theta)}^{\Lambda} R_\theta R^3\, dx \right) = \frac{\partial}{\partial \theta} \left(\int_{x_1(\theta)}^{\Lambda} R^4\, dx \right) + \frac{\partial x_1}{\partial \theta} R^4 (\theta, x_1(\theta)),$$

where $R(\theta, x_1(\theta)) = R_1$, $\forall \theta$. Integrating this relation for $\theta \in (0, 2\pi)$ and using the periodicity of x_1, we deduce that I_1 is zero, so is $\langle \mathbf{u}_0 . \mathbf{n}\, [p_0] \rangle_\Sigma$.

The calculation of $\langle \mathbf{u} . \mathbf{n}\, [p_0] \rangle_\Sigma$ is similar to the preceding one. It is shown in Appendix 1 that $\langle \mathbf{u} . \mathbf{n}\, [p_0] \rangle_\Sigma$ is the time derivative of some function,

$$\langle \mathbf{u} . \mathbf{n}\, [p_0] \rangle_\Sigma = -\frac{d}{dt} \left\{ -\frac{\Omega^2}{8} [\rho] \left(\langle R^3 |\nabla F|^{-1} \rangle_\Sigma + \int_{\partial \Sigma_c} \varphi\, dl \right) \right\}, \tag{34}$$

with

$$\varphi = \pm R_i^3 (\chi\, \mathbf{t} . \mathbf{e}_\theta) \Big|_{r = R_i}, \qquad i = 1 \text{ or } 2; \tag{35}$$

$\mathbf{t}$ is a unit vector tangent to $\partial \Sigma_c$ (*cf.* Appendix); in the expression of φ, the sign is + (resp. −) on the parts of $\partial \Sigma_c$ being on the right (resp. left) side of a component of fluid 1 (in (35) $i = 1$ in the present case).

The calculation of the terms

$$\langle \mathbf{u}_0 . \mathbf{n}\, 2 H T \rangle_\Sigma \text{ and } T \int_{\partial \Sigma} \boldsymbol{\tau} . \mathbf{U}\, dl$$

are given in Appendix 1, and the following expressions are obtained:

$$\langle \mathbf{u}_0 . \mathbf{n}\, 2 H \rangle_\Sigma = -\Omega \left\{ \int_0^{2\pi} \left(\frac{R R_x R_\theta}{|\nabla F|} \Big|_{x = \Lambda} - \frac{R_1 \chi_r \chi_\theta}{|\nabla F|} \Big|_{r = R_1} \right) d\theta \right\}, \tag{36}$$

$$\int_{\partial \Sigma} \boldsymbol{\tau} . \mathbf{U}\, dl = \int_0^{2\pi} \left(\frac{R R_x R_t}{|\nabla F|} \Big|_{x = \Lambda} - \frac{R_1 \chi_r \chi_t}{|\nabla F|} \Big|_{r = R_1} \right) d\theta, \tag{37}$$

where

$$|\nabla F| = (1 + R_\theta^2 / R^2 + R_x^2)^{1/2} \quad on\ \{x = \Lambda\},$$

$$(1+\chi_\theta^2/R_1^2+\chi_r^2)^{1/2} \quad \text{on } \{r=R_1\}.$$

But the contact angles α_c and α_Λ are given by

$$\text{(38)} \qquad \begin{cases} \cos\alpha_\Lambda = \mathbf{n}\,.\,\mathbf{e}_x = -R_x\,|\nabla F|^{-1}, \\ \cos\alpha_c = \mathbf{n}\,.\,\mathbf{e}_r = \chi_r\,|\nabla F|^{-1}, \end{cases}$$

so that

$$\text{(39)} \qquad -\langle \mathbf{u}_0\,.\,\mathbf{n}\,2H\rangle_\Sigma + \int_{\partial\Sigma} \boldsymbol{\tau}\,.\,\mathbf{U}\,dl$$
$$= -\int_0^{2\pi}\left\{\left(\frac{DR}{Dt}R\cos\alpha_\Lambda\right)\Big|_{x=\Lambda} - \left(\frac{D\chi}{Dt}R_1\cos\alpha_c\right)\Big|_{r=R_1}\right\}d\theta,$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \Omega\frac{\partial}{\partial\theta}$$

is a derivative following rigid motions.

Collecting all these results, we may write the result of the calculation in the form (33), with $\varepsilon_\Lambda^l = 0$ if Σ^l does not touch the end walls, 1 (resp. -1) if Σ^l touches the wall $\{x=\Lambda\}$ (resp. $\{x=-\Lambda\}$), and with $\varepsilon_c^l=0$ if Σ^l does not touch the cylinder $\{r=R_i\}$, 1(resp. -1) if Σ^l touches the cylinder $\{r=R_i\}$ and fluid 1 is on the right side (resp. left side) of fluid 2. □

In order to reduce the last term of (33) to potential form, we have to make an assumption on the contact angles α_c and α_Λ of the interface Σ with the end walls and with the cylinders. The result is the following one:

PROPOSITION 7.3. — *Let us assume that the following assertions hold:*

(40) (*a*) *The contact angle α_Λ of the interface with the end walls depends only on the distance r of the contact line to the axis of the cylinders.*

(41) (*b*) *The contact angle α_c of the interface with the cylinders depends only on the distance $\Lambda - x$ of the contact line to the end wall at $x=\Lambda$.*

Then

$$\text{(42)} \qquad \langle \hat{\mathbf{u}}\,.\,\mathbf{n}([p_0]+2HT)\rangle_\Sigma = -\frac{d}{dt}\mathcal{P},$$

with

$$\text{(43)} \qquad \mathcal{P} = T\left(|\Sigma| - \int_{\partial\Sigma}\psi\,dl\right) - \frac{\Omega^2}{8}[\rho]\left(\langle R^3|\nabla F|^{-1}\rangle_\Sigma + \int_{\partial\Sigma_c}\varphi\,dl\right)$$

for some function ψ defined on $\partial\Sigma$.

Proof. — Assumptions (40) and (41) mean that there exist two functions $\psi_\Lambda = \psi_\Lambda(r)$ and $\psi_c = \psi_c(x)$ such that

$$r \cos \alpha_\Lambda(r) = \psi'_\Lambda(r), \qquad R_1 \cos \alpha_c(x) = \psi'_c(x). \tag{44}$$

The calculation of the last term of (33) is now straightforward. Namely, conditions (44) imply:

$$\begin{cases} \langle \mathbf{u}_0 . \mathbf{n}\, 2\mathrm{HT} \rangle_\Sigma = \mathrm{T}\Omega \displaystyle\int_0^{2\pi} \frac{\partial}{\partial \theta} (\psi_\Lambda(\mathrm{R}) + \psi_c(x))\, d\theta = 0, \\ \displaystyle\int_{\partial\Sigma} \boldsymbol{\tau} . \mathbf{U}\, dl = \frac{d}{dt}\left(\int_{\partial\Sigma} \psi\, dl \right), \end{cases} \tag{45}$$

with

$$\psi = \begin{cases} \pm (\psi_\Lambda(\mathrm{R})\, \mathbf{t} . \mathbf{e}_\theta) \Big|_{x = \pm\Lambda} & \text{on } \partial\Sigma_\Lambda, \\ \pm (\psi_c(x)\, \mathbf{t} . \mathbf{e}_\theta) \Big|_{r = \mathrm{R}_i} & \text{on } \partial\Sigma_c, \qquad i = 1 \text{ or } 2; \end{cases} \tag{46}$$

in the expression of ψ, the sign is $+$ (resp. $-$) on the parts of $\partial\Sigma$ being on the right (resp. left) side of a component of fluid 1.

Hence, we have shown that, under the assumptions (40) and (41) on the contact angles of the interface with the end walls and with the cylinders, the whole boundary term in (21) can be expressed as the time derivative of some function $\mathscr{P}$, defined by (43). □

Remark 7.4. — Assumptions (40) and (41) are equivalent to assuming a functional relation for the contact angle determining it by its position on the contact line. Precisely, such a relation is a differentiable map $\mathscr{F}$ between the set $\mathscr{X}$ of contact lines and the set $\mathscr{Y}$ of "contact angles". For instance, in the case of end walls $\{x = \pm\Lambda\}$, $\mathscr{X}$ is the space of 2π-periodic functions $\mathrm{R} = \mathrm{R}(\theta)$, which are $\mathscr{C}^1$, and with values in $[\mathrm{R}_1, \mathrm{R}_2]$; $\mathscr{Y}$ is the space of 2π-periodic functions which are $\mathscr{C}^0$, and with values in $[-\mathrm{R}_2, \mathrm{R}_2]$; and

$$\mathscr{F} : \begin{cases} \mathscr{X} & \to & \mathscr{Y} \\ \mathrm{R} & \to & \mathrm{R} \cos \alpha_\Lambda \\ \text{"contact line"} & & \text{"contact angle"} \end{cases} \tag{47}$$

Young-Dupré's relation is given by $\mathscr{F}(\mathrm{R}) = c\mathrm{R}$, $\forall \mathrm{R} \in \mathscr{X}$, where c ($= \cos \alpha_\Lambda$) is a fixed constant in $[-1, 1]$. Assumption (40) and, similarly, assumption (41) is equivalent to the more general functional relation given by

$$\mathscr{F}(\mathrm{R})(\theta) = \varphi_\Lambda(\mathrm{R}(\theta)). \quad \forall \theta \in [0, 2\pi],$$

where φ_Λ is a given absolutely continous function from $[\mathrm{R}_1, \mathrm{R}_2]$ into $(-\mathrm{R}_2, \mathrm{R}_2)$. □

Following the lines of analysis laid down in JRRN we may now prove the following result.

THEOREM 7.5. — *Assume that the assertions* (40) and (41) *on the contact angle of the interface of the two fluids with the end walls and with the cylinders hold. Assume that the function* $\mathscr{P}$ *given by* (43), *is bounded from below on the set of interfaces.*

Then the following hold:

(i) *Rigid motions are weakly stable in the sense that the energy of disturbances is integrable on* R_+.

(ii) *Stable Couette flows, with* $\Omega_1=\Omega_2$, *are rigid motions and the interface is obtained by minimizing the function* $\mathscr{P}$ *on* $\mathscr{S}$.

Proof. — (i) Under the assumptions (40) and (41), the energy identity (21) reduces to

$$\frac{d}{dt}(\mathscr{E}+\mathscr{P})=-\mathscr{D}. \tag{48}$$

First, we have (27); i.e., $\mathscr{D}\geqq 2\lambda\mathscr{E}$, for some $\lambda>0$, and, from relation (44), we deduce that

$$\mathscr{E}\in L^1(R_+), \mathscr{D}\in L^1(R_+), \qquad \lim_{t\to\infty}(\mathscr{E}(t)+\mathscr{P}(t))<+\infty, \tag{49}$$

since $\mathscr{P}$ is bounded from below on the set of interfaces. This proves point (i) of the theorem.

(ii) We next show that the only stable configurations (in the sense defined in Section 5) are those which minimize $\mathscr{P}$ on $\mathscr{S}$. By definition, a stable equilibrium $(\mathbf{u}_e, p_e, \Sigma_e)$ satisfies

$$\lim_{t\to\infty}\mathscr{E}(t)=0$$

and, because of (49),

$$\lim_{t\to\infty}\mathscr{P}(t)=\mathscr{P}(\infty)\,(\text{finite}),$$

for all $(\mathbf{u}^0, p^0, \Sigma^0)$ in a neighborhood of $(\mathbf{u}_e, p_e, \Sigma_e)$. Let us choose some $\mathbf{u}^0$ in a neighborhood of $\mathbf{u}_e$ such that

$$\hat{\mathbf{u}}(0)=\mathbf{u}^0-\mathbf{u}_e$$

has a "small" energy $\mathscr{E}(0)$ and a "large" dissipation $\mathscr{D}(0)$. Integrating (48) on $(0,+\infty)$, we obtain

$$\mathscr{P}(\infty)-\mathscr{P}(0)=\mathscr{E}(0)-\int_0^\infty \mathscr{D}(t)\,dt<0,$$

wich proves that $\mathscr{P}(\infty)$ minimizes $\mathscr{P}$ on a neighborhood of Σ_e. (This argument improves the one following (6.3) in JRRN.)

Finally, we note that critical points of $\mathscr{P}$ are steady solutions of problem (1) through (8). Namely, the Euler equations for the minimization problem $\min_{\mathscr{S}}\mathscr{P}$ is

$$-[p_0]=2\,TH,$$

together with the boundary conditions (44). This means that stable solutions (as defined in section 5), are stationary solutions to problem (1) through (8).

This, proves point (ii) of the theorem. □

Remark 7.6. — An analysis of the definition (43) of $\mathscr{P}$ shows that $\mathscr{P}$ is bounded from below, in particular, if the set of interfaces $\mathscr{P}$ is such that $|\Sigma|$, $|\partial \Sigma_c|$, $|\partial \Sigma_\Lambda|$, for $\Sigma \in \mathscr{S}$, is bounded. If stability is studied in a $\mathscr{C}^1$-neighborhood of a steady state, this condition is satisfied. □

Remark 7.7. — The decay of the energy $\mathscr{E}$ implies that the rest state is the terminal form of every disturbance. But the rest state needs not be unique. The problem of uniqueness of stable configurations of steady flows of two fluids even at small Reynolds number (defined in Section 6) is an open problem. □

Remark 7.8. — The term $\langle \mathbf{u}.\mathbf{n}[\rho]\,gr\sin\theta\rangle_\Sigma$ arising in the pressure, and due to the presence of gravity in (21), can also be expressed, for any motion, as the time derivative of some functional. Assuming, as in section 5, that

$$\mathbf{u}_0 \cdot \mathbf{n} = 0,$$

and making the calculation on a part of Σ which is a graph in cylindrical coordinates, and which touches the inner cylinder and the end wall at $x = \Lambda$, we obtain

$$\langle \hat{\mathbf{u}}.\mathbf{n}[\rho]\,gr\sin\theta\rangle_\Sigma = \frac{1}{2}\frac{d}{dt}\left\{[\rho]g\left(\langle \mathrm{R}^2\sin\theta\,|\nabla \mathrm{F}|^{-2}\rangle_\Sigma + \int_{\partial\Sigma_c} \mathrm{R}^{-1}\,\varphi\sin\theta\,d\theta\right)\right\},$$

where φ has been defined in (35). □

Remark 7.9. — The working of the contact line cannot always be represented by a potential; the assumptions (40) and (41) need not hold. They hold in the case of constant contact angle, a fixed contact angle and, in general, whenever implicit relations of the form $f_1(\alpha, \mathrm{R}) = 0$ or $f_2(\alpha, \chi) = 0$. In our derivation we have assumed that these relations could be solved for α; this assumption could be relaxed. □

8. Decomposition of the motion (II)

The following decomposition is a natural one for studies of linearized and nearly linearized bifurcation theories of free surface problems, though it does not seem useful for more global consideration. This decomposition underlies for example the works of [2], [11], [15] and [16].

Let $(\mathbf{u}_e, p_e, \Sigma_e)$ be an equilibrium state, i. e., a stationary solution of problem (1) through (8), and let $(\mathbf{u}, p, \Sigma)$ be a time dependent solution. Another way to study stability is to consider the following decomposition of motion,

$$\mathbf{u} = \tilde{\mathbf{u}}_e + \tilde{\mathbf{u}}, \qquad p = \tilde{p}_e + \tilde{p}, \tag{50}$$

where $(\tilde{\mathbf{u}}_e, \tilde{p}_e)$ is some modification of $(\mathbf{u}_e, p_e)$ and is defined in the time dependent regions $\mathrm{G}^1(t)$ and $\mathrm{G}^2(t)$ in which $\mathbf{u}(t)$ and $p(t)$ are defined.

In this section, and the following ones, we assume that gravity is negligible, and we consider the stability of steady Couette flows in which the two liquids are arranged in two infinite concentric circular rings with a cylindrical interface Σ_e *at* $r=d\,(R_1<d<R_2)$. Problem (1) through (8) admits a radial solution $(\mathbf{u}_e, p_e)$ given by the relations:

$$\text{(51)}\quad \left\{\begin{aligned} &\mathbf{u}_e=V(r)\,\mathbf{e}_\theta, \qquad V(r)=A\,r+\frac{B}{r},\\ &p_e=p_e(r)=\rho\left(\frac{A^2 r^2}{2}+2\,AB\ln r-\frac{1}{2}\frac{B^2}{r^2}+C\right),\end{aligned}\right.$$

where

$$\text{(52)}\quad \left\{\begin{aligned} &A_i=(-d^{-2}[\mu]\Omega_i+[\mu\Omega R^2]R_1^{-2}R_2^{-2})\,q^{-1},\\ &B_i=-\mu_1\mu_2[\Omega]\mu_i^{-1}q^{-1}, \qquad i=1, 2,\\ &q=-d^{-2}[\mu]+[\mu R^2]R_1^{-2}R_2^{-2},\\ &[C]=T\,d^{-1}-[\rho A^2]\,d^2/2-[2\rho AB]\ln d+[\rho B^2]/(2\,d).\end{aligned}\right.$$

The subscripts 1 and 2 denote, respectively, quantities evaluated in the equilibrium regions:

$$G_e^1=\{\mathbf{x}=(r, \theta, x),\ R_1<r<d,\ 0\leqq\theta<2\pi,\ 0\leqq x<2\pi/\alpha\},$$
$$G_e^2=\{\mathbf{x}=(r, \theta, x),\ d<r<R_2,\ 0\leqq\theta<2\pi,\ 0\leqq x<2\pi/\alpha\}.$$

Let $(\tilde{\mathbf{u}}_e, \tilde{p}_e)$ be defined by relations (51), in which the coefficients A_i and B_i are given by relations (52) in $G^i(t)$ instead of G_e^i, for $i=1, 2$. Such a $(\tilde{\mathbf{u}}_e, \tilde{p}_e)$ is sometimes called an extension of $(\mathbf{u}_e, p_e)$ to the time dependent regions $G^1(t)$ and $G^2(t)$. Let us notice that $(\tilde{\mathbf{u}}_e, \tilde{p}_e)$ is not a steady solution of problem (1) through (8) and that $\tilde{\mathbf{u}}_e$ is discontinuous across $\Sigma(t)$.

The equations governing $\tilde{\mathbf{u}}=(u, v, w)$ and $\tilde{p}$, defined by relations (50), are

$$\text{(53)}\quad \left\{\begin{aligned} &\rho_i(\partial_t\tilde{\mathbf{u}}_i+(\tilde{\mathbf{u}}_i+\tilde{\mathbf{u}}_{ei})\,.\,\nabla\tilde{\mathbf{u}}_i+\tilde{\mathbf{u}}_i\,.\,\nabla\tilde{\mathbf{u}}_{ei})=-\nabla\tilde{p}_i+\operatorname{div}\tilde{\mathbf{S}}_i,\\ &\operatorname{div}\tilde{\mathbf{u}}_i=0 \quad \text{in } G^i, \qquad i=1, 2,\end{aligned}\right.$$

$$\text{(54)}\quad \tilde{\mathbf{u}}_i(r=R_i)=0, \qquad \tilde{\mathbf{u}}_i(x=0)=\tilde{\mathbf{u}}_i(x=2\pi/\alpha), \qquad i=1, 2,$$

$$\text{(55)}\quad u=R_t+\frac{v+V}{R}R_\theta+w\,R_x \quad \text{on } \Sigma,$$

$$\text{(56)}\quad \left\{\begin{aligned} &[u]=[w]=0, \qquad [v]=[B](R\,d^{-2}-R^{-1}),\\ &\tau_k\,.\,[\tilde{\mathbf{S}}]\,.\,\mathbf{n}=0, \qquad k=1, 2,\\ &[-\tilde{p}+\mathbf{n}\,.\,\tilde{\mathbf{S}}\,.\,\mathbf{n}]=Q \quad \text{on } \Sigma,\end{aligned}\right.$$

$$\text{(57)}\quad \left\{\begin{aligned} &\tilde{\mathbf{u}}_i(0)=\mathbf{u}_i^0-\tilde{\mathbf{u}}_{ei} \quad \text{in } G^i, \qquad i=1, 2,\\ &R(0)=R^0 \quad \text{in } (0, 2\pi)\times\left(0, \frac{2\pi}{\alpha}\right),\end{aligned}\right.$$

where $\tilde{\mathbf{S}}_i = 2\mu_i \mathbf{D}[\tilde{\mathbf{u}}_i]$, and $\mathrm{Q} = 2\mathrm{HT} + [\tilde{p}_e]$ is a function of R,

$$\mathrm{Q(R)} = \mathrm{T}(2\mathrm{H} + d^{-1}) + [\rho \mathrm{A}^2]\frac{\mathrm{R}^2 - d^2}{2} + 2[\rho \mathrm{AB}]\ln\frac{\mathrm{R}}{d} - \frac{[\rho \mathrm{B}]^2}{2}(\mathrm{R}^{-2} - d^{-2}). \tag{58}$$

Here we are assuming that the deformed interface $\Sigma = \Sigma(t)$ is a graph in cylindrical coordinates, with an equation $r = \mathrm{R}(x, \theta, t)$, for some $\mathscr{C}^1$ function R satisfying the volume constraint condition $((\mathrm{R}^2)) = ((d^2))$, where

$$((.)) = \int_0^{2\pi/\alpha} \int_0^{2\pi} .\, d\theta\, dx.$$

Moreover, the function R is assumed to be $(2\pi/\alpha)$-periodic in x, 2π-periodic in θ.

Let us note that this method of decomposition introduces a discontinuity for the $\mathbf{e}_\theta$ component of the velocity [*cf.* (56)].

In the next section, we derive different energy identities satisfied by the disturbances.

9. Energy identities for decomposition (II)

To form the energy equation, we multiply $(53)_1$ by $\tilde{\mathbf{u}}_i$, integrate over G^i, then add for $i = 1, 2$, and we find

$$\frac{1}{2}\frac{d}{dt}\langle \rho |\tilde{\mathbf{u}}|^2 \rangle + \langle \rho \tilde{\mathbf{u}}.\mathbf{D}[\tilde{\mathbf{u}}_e].\tilde{\mathbf{u}} \rangle + \langle 2\mu \mathbf{D}[\tilde{\mathbf{u}}]^2 \rangle = \langle [-\tilde{p}\tilde{\mathbf{u}}.\mathbf{n} + \tilde{\mathbf{u}}.\tilde{\mathbf{S}}.\mathbf{n}] \rangle_\Sigma \tag{59}$$

Using the interface conditions (56), we may write

$$\begin{aligned}[-\tilde{p}\tilde{\mathbf{u}}.\mathbf{n} + \tilde{\mathbf{u}}.\tilde{\mathbf{S}}.\mathbf{n}] &= [(\mathbf{u} - \tilde{\mathbf{u}}_e)(-\tilde{p}\mathbf{n} + \tilde{\mathbf{S}}.\mathbf{n})] \\ &= \mathbf{u}.[-\tilde{p}\mathbf{n} + \tilde{\mathbf{S}}.\mathbf{n}] - [\tilde{\mathbf{u}}_e.(-\tilde{p}\mathbf{n} + \tilde{\mathbf{S}}.\mathbf{n})] \\ &= \mathbf{u}.\mathbf{n}([\tilde{p}_e] + 2\mathrm{HT}) - [\tilde{\mathbf{u}}_e.(-\tilde{p}\mathbf{n} + \tilde{\mathbf{S}}.\mathbf{n})]\end{aligned}$$

As shown in relation (58), $[\tilde{p}_e]$ is a function of R alone. So there exists some function $\varphi_e = \varphi_e(\mathrm{R})$ such that

$$\langle \mathbf{u}.\mathbf{n}[\tilde{p}_e] \rangle_\Sigma - ((\mathrm{R}\mathrm{R}_t[\tilde{p}_e])) - \frac{d}{dt}((\varphi_e)),$$

namely,

$$\varphi_e(\mathrm{R}) = \frac{[\rho \mathrm{A}^2]}{8}(\mathrm{R}^2 - d^2)^2 + \frac{[\rho \mathrm{AB}]}{2}\mathrm{R}^2 \ln\frac{\mathrm{R}^2}{d^2} - \frac{[\rho \mathrm{B}^2]}{4}\ln\frac{\mathrm{R}^2}{d^2},$$

where we have made use of the volume condition $((\mathrm{R}^2)) = ((d^2))$. So equation (59) may be written as

$$\frac{d}{dt}(\mathscr{E} + \mathrm{T}|\Sigma| - ((\varphi_e))) + \mathscr{I}_{\tilde{\mathbf{u}}_e} + \mathscr{D} = \langle \mathbf{e}_\theta.[\mathrm{V}(\tilde{p}\mathbf{n} - \tilde{\mathbf{S}}.\mathbf{n})] \rangle_\Sigma, \tag{60}$$

with the notations of section 5.

In the particular case of axisymmetric disturbances, equation $(53)_2$ projected on $\mathbf{e}_\theta$ is independent of $\tilde{p}$. So we obtain an energy equation for v, and another one for $\mathbf{q}=(u, w)$:

$$(61)\quad \frac{d}{dt}\left\langle \rho\frac{v^2}{2}\right\rangle+\left\langle \rho\left(\frac{uv^2}{r}+\zeta uv\right)\right\rangle+\left\langle \mu\left(|\nabla_2 \mathbf{v}|^2+\left(\frac{v}{r}\right)^2\right)\right\rangle=-\langle \mathbf{e}_\theta.\tilde{\mathbf{S}}.\mathbf{n}[\mathrm{V}]\rangle_\Sigma.$$

$$(62)\quad \frac{d}{dt}\left(\left\langle \rho\frac{|\mathbf{q}|^2}{2}\right\rangle+\mathrm{T}|\Sigma|-((\varphi_e))\right)-\left\langle \rho\left(\frac{uv^2}{r}+2\Omega uv\right)\right\rangle$$
$$+\left\langle \mu\left[|\nabla_2 \mathbf{q}|^2+2\left(\frac{u}{r}\right)^2+\left(\frac{\partial u}{\partial r}\right)^2+\left(\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial u}{\partial x}+\frac{\partial w}{\partial r}\right)^2\right]\right\rangle=0.$$

A derivation of (61) and (62) is given in Appendix 2.

Energy identities (60) through (62) will be applied in the case of linear disturbances in section 10.

10. Linearized stability of inviscid flows in circles

In this last section, we shall consider only inviscid flows: these flows are classically considered in the literature (*cf.*, e. g., [2], [16]). The case of viscous flows has been studied by Renardy and Joseph [15] for disturbances in normal modes.

When $\mu=0$, we may consider hydrodynamic equilibria $(\mathbf{u}_e, p_e)$ given by

$$(63)\qquad \mathbf{u}_e(\mathrm{r})=\mathbf{e}_\theta \mathrm{V}(r), \qquad p'_e(r)=\rho \mathrm{V}^2(r) r^{-1},$$

with any continuous function V. We again suppose that the steady flow is arranged in annular rings. Because of $\mu=0$, the prescribed conditions on the azimuthal component of velocity and on the shear stress must be relaxed.

Setting $\mathrm{R}=d+\delta$, with small δ's, we linearize the system (53) through (57) around the equilibrium $(\mathbf{u}_e, p_e, \Sigma_e=\{r=d\})$. The linearized interface conditions, evaluated on the unperturbed surface $\Sigma_e=\{r=d\}$, are

$$u(d)=\delta_t+\Omega(d)\,\delta_\theta,$$
$$[u(d)]=0,$$
$$[-\tilde{p}(d)]=[\rho]\mathrm{V}^2(d)\,d^{-1}\delta+\mathrm{T}(d^{-2}\delta+d^{-2}\delta_{\theta\theta}+\delta_{xx});$$

so the linearized form of the energy estimate (60) is

$$(64)\qquad \frac{d}{dt}\left\{\mathscr{E}+\frac{1}{2\mathrm{T}}((d^{-1}\delta_\theta^2+d\delta_x^2))-\frac{1}{2}\left(\frac{\mathrm{T}}{d}+[\rho]\mathrm{V}^2(d)\right)((\delta^2))\right\}+\mathscr{I}_{\tilde{\mathbf{u}}_e}=0.$$

A substantial simplification arises if we assume normal disturbances, i. e., disturbances of the form

$$(\tilde{\mathbf{u}}, \tilde{p})=(\tilde{\mathbf{u}}(r), \tilde{p}(r))\exp(\sigma t+i\alpha x+ik\theta).$$

The flat interface $\{r=d\}$ is also perturbed with the factor $\exp(\sigma t+i\alpha x+ik\theta)$. The linearized equations of motion and boundary conditions become:

$$\text{(65)}\qquad \begin{cases} \rho(\sigma+ik\Omega)w=-i\alpha\tilde{p}, \\ \rho(\sigma+ik\Omega)u-2\rho\Omega v=-\tilde{p}', \\ \rho(\sigma+ik\Omega)v-\rho\zeta u=-ikr^{-1}\tilde{p}, \\ (ru)'+ikv+i\alpha w=0, \quad \text{in } \{R_1<r<d\} \text{ and in } \{d<r<R_2\}, \end{cases}$$

$$\text{(66)}\qquad u(R_1)=u(R_2)=0,$$

$$\text{(67)}\qquad u(d)=(\sigma+ik\Omega(d))\delta,$$

$$\text{(68)}\qquad \begin{cases} [u(d)]=0, \\ [-\tilde{p}(d)]=\{[\rho]V^2(d)d^{-1}+T(d^{-2}(1-k^2)-\alpha^2)\}\delta, \end{cases}$$

where

$$\Omega(r)=V(r)/r, \qquad \zeta(r)=(rV(r))'/r.$$

Here, δ is an unknown constant which may be calculated from $u(d)$ by relation (67).

Axisymmetric disturbances (i. e., $k=0$) satisfy the following energy identity:

$$\text{(69)}\quad \sigma\langle\rho(|u|^2+|w|^2)\rangle+\sigma^{-1}\langle 2\rho\Omega\zeta|u|^2\rangle=\sigma^{-1}\{[\rho]V^2(d)+T(d^{-1}-\alpha^2 d)\}|u(d)|^2,$$

where we have denoted

$$|f|^2=f\bar{f}, \qquad \bar{f}=\text{complex conjugate of } f,$$

and if $f=f_1$ in $\{R_1<r<d\}$ and $f=f_2$ in $\{d<r<R_2\}$.

$$\langle f\rangle=\int_{R_1}^{d} f_1\, r\, dr+\int_{d}^{R_2} f_2\, r\, dr.$$

Solving equation (69) in σ, we obtain

$$\text{(70)}\quad \sigma^2=\{-\langle 2\rho\Omega\zeta|u|^2\rangle+([\rho]V^2(d)+T(d^{-1}-\alpha^2 d))|u(d)|^2\}/(\langle\rho(|u|^2+|w|^2\rangle).$$

The flow is unstable to axisymmetric disturbances if the numerator in (70) is positive, and neutrally stable if it is negative.

In the absence of the interface (single fluid flow) we retrieve the well known result of Rayleigh (*see* e. g., Joseph [8], I, p. 139). When one of the fluid is air, our result reduces to that of Boudourides and David [2]. In the particular case where the basic flow is of the form

$$V_l=A_l r+B_l/r, \qquad A_l,\ B_l \text{ constant}, \quad l=1,\ 2,$$

we have

$$\Omega_l\zeta_l=2A_l V_l;$$

so, a configuration with the heavier fluid inside ($\rho_1 > \rho_2$) may be stable for short waves, i. e., for large α, if there is enough of the fluid (i. e., d large) and if the speeds of rotation of both cylinders are such that

$$A_l V_l > 0, \qquad l = 1, 2.$$

More generally, the term $[\rho] V^2(d)$ is stabilizing if $\rho_1 < \rho_2$, i. e., if the heavier fluid is outside. Surface tension is stabilizing for short waves, i. e., for large α's, and destabilizing for long waves.

Remark 10.1. — The linearized form of the energy estimate (60) in the case of viscous flows is

$$\frac{d}{dt}\left\{ \mathscr{E} + \frac{1}{2} T((d^{-1}\delta_\theta^2 + d\delta_x^2)) - \frac{1}{2}\left(\frac{T}{d} + [\rho] V^2(d)\right)((\delta^2))\right\} + \mathscr{I}_{\tilde{\mathbf{u}}_e} + \mathscr{D} = 2\frac{[B]}{d}((\delta \tilde{S}^\theta(d))). \ \square$$

APPENDIX 1

Proof of Lemma 7.2 (continued). — Here we prove formulae (34) through (37), by making the calculations on a part of $\partial\Sigma$ such that $\partial\Sigma_\Lambda$ lies on the wall at $x = \Lambda$, and $\partial\Sigma_c$ on the cylinder at $r = R_1$. We first recall the notations in the general case.

(i) *Notations.* — The interface $\Sigma = \{F(\mathbf{x}) = 0\}$ is described locally by an equation

$$r = R(\theta, x, t), \qquad x_1(\theta) \leqq x \leqq x_2(\theta), \qquad 0 \leqq \theta < 2\pi,$$

or

$$x = \chi(r, \theta, t), \qquad r_1(\theta) \leqq r \leqq r_2(\theta), \qquad 0 \leqq \theta < 2\pi.$$

We assume that $\partial\Sigma$ has a part on the end walls $\partial\Sigma_\Lambda = \{\mathbf{x} = R(\theta, \pm\Lambda, t)\mathbf{e}_r\}$ and a part on the cylinders

$$\partial\Sigma_c = \{\mathbf{x} = R_i \mathbf{e}_r + \chi(R_i, \theta, t)\mathbf{e}_x\}, \qquad i = 1 \text{ or } 2.$$

Let us denote

$$\mathbf{n} = \nabla F |\nabla F|^{-1}, \tag{71}$$

the unit vector normal to Σ, pointing from fluid 1 to fluid 2, with

$$F(\mathbf{x}) = r - R(\theta, x), \qquad \text{or} \qquad F(\mathbf{x}) = \chi(r, \theta) - x;$$

(72) $\mathbf{t} = d\mathbf{x}/dl$, a unit vector tangent to $\partial\Sigma$;

(73) $\boldsymbol{\tau} = \mathbf{n} \times \mathbf{t}$, a unit vector normal to $\partial\Sigma$ and lying on Σ;

(74) $\mathbf{U} = d\mathbf{x}/dt$, the velocity of the contact line $\partial\Sigma$.

On the end walls at $x=\pm\Lambda$, we have

$$\nabla F=\mathbf{e}_r-\mathbf{e}_\theta R_\theta/R-\mathbf{e}_x R_x,$$

$$\mathbf{t}=(\mathbf{e}_r R_\theta+\mathbf{e}_\theta R)\frac{d\theta}{dl},$$

$$U=\mathbf{e}_r R_t+(\mathbf{e}_r R_\theta+\mathbf{e}_\theta R)\frac{d\theta}{dt},$$

(ii) *Calculation of* $\langle \mathbf{u}.\mathbf{n}[p_0]\rangle_\Sigma$. First of all, because of $\langle \mathbf{u}.\mathbf{n}\rangle_\Sigma=0$, we have

$$\langle \mathbf{u}.\mathbf{n}[p_0]\rangle_\Sigma=\frac{1}{2}\Omega^2[\rho]\langle \mathbf{u}.\mathbf{n}R^2\rangle_\Sigma.$$

Using the kinematic condition (6), we define

$$I_2=\langle \mathbf{u}.\mathbf{n}R^2\rangle_\Sigma=\int_0^{2\pi}\int_{x_1(\theta)}^{\Lambda} R_t R^3\,dx\,d\theta$$

and calculate it by using Leibnitz's rule in the following way:

$$4\int_0^{2\pi}\int_{x_1(\theta)}^{\Lambda} R^3 R_t\,dx\,d\theta=\frac{d}{dt}\left\{\int_0^{2\pi}\int_{x_1(\theta)}^{\Lambda} R^4\,dx\,d\theta\right\}+\int_0^{2\pi}\frac{\partial x_1}{\partial t}R^4|_{x=x_1(\theta)}\,d\theta.$$

$$=\frac{d}{dt}\left\{\int_0^{2\pi}\int_{x_1(\theta)}^{\Lambda} R^4\,dx\,d\theta+R_1^4\left(\int_0^{2\pi}x_1(\theta)\,d\theta\right)\right\}.$$

Hence, I_2 is the time derivative of some functional, so is $\langle u.\mathbf{n}[p_0]\rangle_\Sigma$. Using notations (71) and (72) we find formulae (34) and (35).

(iii) *Calculation of* $\langle \mathbf{u}_0.\mathbf{n}\,2HT\rangle_\Sigma$. With the assumptions on the part of Σ where we make the calculation, we have

$$\langle \mathbf{u}_0.\mathbf{n}\,2HT\rangle_\Sigma=-T\Omega\left(\int_0^{2\pi}\int_{x_1(\theta)}^{\Lambda} 2HRR_\theta\,d\theta\right).$$

We next make use of the formula, derived in JRRN,

$$RR_\theta 2H=\frac{\partial}{\partial\theta}\left(\frac{-R(1+R_x^2)}{|\nabla F|}\right)+\frac{\partial}{\partial x}\left(\frac{RR_xR_\theta}{|\nabla F|}\right).$$

The term coming from the $\partial/\partial x$-derivative is integrated by parts, so that

$$(75)\quad \int_0^{2\pi}\int_{x_1(\theta)}^{\Lambda}\frac{\partial}{\partial x}\left(\frac{RR_xR_\theta}{|\nabla F|}\right)dx\,d\theta=\int_0^{2\pi}\left\{\frac{RR_xR_\theta}{|\nabla F|}\bigg|_{x=\Lambda}-\frac{RR_xR_\theta}{|\nabla F|}\bigg|_{x=x_1(\theta)}\right\}d\theta.$$

The term coming from the $\partial/\partial\theta$-derivative is calculated by using Leibnitz's rule, so that

$$\int_{x_1(\theta)}^{\Lambda}\frac{\partial}{\partial\theta}\left(-\frac{R(1+R_x^2)}{|\nabla F|}\right)dx$$

$$= \frac{\partial}{\partial\theta}\left\{\int_{x_1(\theta)}^{\Lambda}\left(-\frac{R(1+R_x^2)}{|\nabla F|}\right)dx\right\} - \frac{\partial x_1}{\partial\theta}\left(\frac{R(1+R_x^2)}{|\nabla F|}\right)\Bigg|_{x=x_1(\theta)}.$$

From the relation $R(\theta, x_1(\theta)) = R_1$, $\forall\theta$, we deduce $\partial x_1/\partial\theta = -R_\theta/R_x$ so that

$$(76)\qquad \int_0^{2\pi}\int_{x_1(\theta)}^{\Lambda}\frac{\partial}{\partial\theta}\left(\frac{-R(1+R_x^2)}{|\nabla F|}\right)dx\,d\theta = R_1\left(\int_0^{2\pi}\frac{R_\theta(1+R_x^2)}{R_x|\nabla F|}\Bigg|_{x=x_1(\theta)}d\theta\right).$$

From formulae (75) and (76), we deduce that $\langle \mathbf{u}_0.\mathbf{n}\,2HT\rangle_\Sigma$ reduces to integrals on $\partial\Sigma$,

$$(77)\qquad \langle \mathbf{u}_0.\mathbf{n}\,2HT\rangle_\Sigma = -T\Omega\left\{\int_0^{2\pi}\frac{RR_xR_\theta}{|\nabla F|}\Bigg|_{x=\Lambda} + \frac{R_1R_\theta}{R_x|\nabla F|}\Bigg|_{x=x_1(\theta)}d\theta\right\}.$$

Actually, the term on the inner cylinder can be expressed in terms of the function χ, so that we obtain formula (36).

(iv) *Calculation of* $\int_{\partial\Sigma}\tau.\mathbf{U}\,dl$. The function $\tau.\mathbf{U}$ is calculated as follows,

$$\tau.\mathbf{U} = \mathbf{U}.(\mathbf{n}\times\mathbf{t}) = \mathbf{n}.(\mathbf{t}\times\mathbf{U});$$

on the end walls (resp. the cylinder at $r = R_i$), we have

$$\tau.\mathbf{U} = RR_xR_t|\nabla F|^{-1}\frac{d\theta}{dl}\qquad\left(\text{resp. }\tau.\mathbf{U} = R_i\chi_r\chi_t|\nabla F|^{-1}\frac{d\theta}{dl}\right),$$

so that we obtain formula (37).

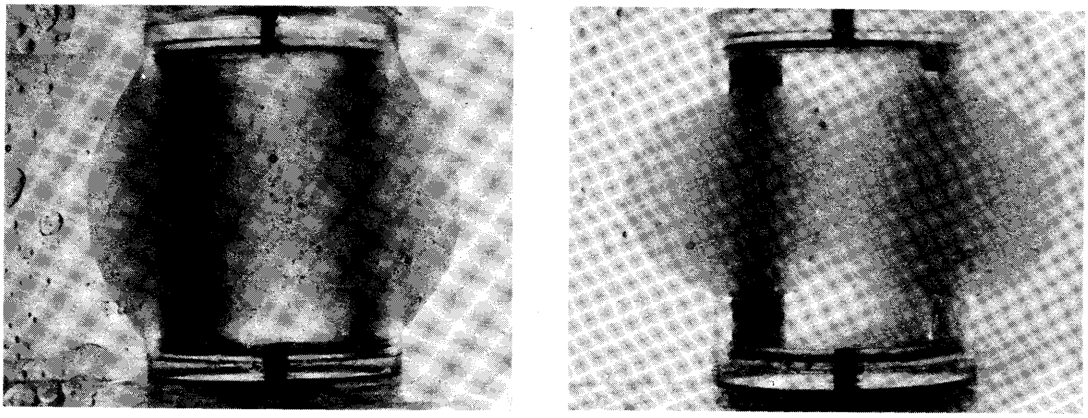
APPENDIX 2

Derivation of energy identities (61) and (62)

In the case of axisymmetric disturbances, the perturbation equations (53) through (57) become

$$(78)_a\qquad \rho\left(\partial_t v + (\mathbf{q}.\nabla_2)v + \frac{uv}{r} + \zeta u\right) = \mu\left(\nabla_2^2 - \frac{1}{r^2}\right)v,$$

$$(78)_b\qquad \left\{\begin{array}{l} \rho\left(\partial_t u + (\mathbf{q}.\nabla_2)u - \dfrac{v^2}{r} - 2\Omega v\right) = -\tilde{p}_r + \mu\left(\nabla_2^2 - \dfrac{1}{r^2}\right)u, \\ \rho(\partial_t w + (\mathbf{q}.\nabla_2)w) = -\tilde{p}_x + \mu\nabla_2^2 w, \\ \nabla_2.\mathbf{q} = 0,\quad \text{in } G^i,\qquad i = 1, 2, \end{array}\right.$$



(*a*) (*b*)

Fig. 2. — Silicone oil (viscosity = 125 Po, density = 0.975 g/cm³) roller on a 7 cm diameter plexiglas rod rotating steadily at .73 rad/sec. The box is filled with water. The effect of gravity on the roller is very small because the difference in densities is very small and the viscosity of the oil is very large. The shape of the interface is determined by minimizing the potential $\mathscr{P}$ defined by (43). It is clear from (*b*) that the contact angle et the contact line varies with θ and hence does not satisfy the Young-Dupré equation. (*a*) Shortly after the start of the experiment. (*b*) Three days later.

Fig. 2. — Un rouleau d'huile de silicone (viscosité = 125 Po, densité = 0,975 g/cm³ sur une barre en plexiglas de 7 cm de diamètre tournant uniformément à la vitesse de .73 rad/sec. La boîte est remplie d'eau. L'influence de la gravité sur le rouleau est très faible car la différence des densités est très faible et la viscosité de l'huile eest très grande. La configuration de l'interface est déterminée par la minimisation du potential $\mathscr{P}$ défini en (43). D'après la photo (b), il est clair que l'angle de contact le long de la ligne de contact dépend de θ et donc ne satisfait pas la loi de Young et Dupré. (a) Peu de temps après le début de l'expérience. (b) Trois jours plus tard.

$$\left\{\begin{array}{l} u=v=w=0 \quad \text{at } r=R_i, \qquad i=1, 2, \\ (u, v, w)(x=0)=(u, v, w)(x=2\pi/\alpha), \end{array}\right. \tag{79}$$

$$u=R_t+wR_x \quad \text{on } \Sigma, \tag{80}$$

$$\left\{\begin{array}{l} [u]=[w]=0, \qquad [v]=[B](R\,d^{-2}-R^{-1}), \\ \mathbf{e}_\theta.[\tilde{\mathbf{S}}].\mathbf{n}=(\mathbf{n}\times\mathbf{e}_\theta)[\tilde{\mathbf{S}}].\mathbf{n}=0, \\ [-\tilde{p}+\mathbf{n}.\tilde{\mathbf{S}}.\mathbf{n}]=Q, \quad \text{on } \Sigma, \end{array}\right. \tag{81}$$

where

$$\mathbf{q}=\mathbf{e}_r u+\mathbf{e}_x w, \qquad \Omega(r)=V(r)/r, \qquad \zeta(r)=(r\,V(r))'/r,$$
$$\nabla_2=\mathbf{e}_r\,\partial/\partial r+\mathbf{e}_x\,\partial/\partial x, \qquad \nabla_2.\mathbf{q}=r^{-1}((ru)_r+(rw)_x).$$

Primes denote differentiation with respect to r.

To form the energy identity (61) for v, we multiply $(78)_a$ by v, integrate over G^1 and G^2, and add these two equations.

On the other hand, identity (60) for axisymmetric disturbances may be written as

$$(82)\quad \frac{d}{dt}\left\{\left\langle \rho\frac{|\tilde{\mathbf{u}}|^2}{2}\right\rangle + T|\Sigma| - ((\varphi_e))\right\} + \left\langle 2\rho uv\frac{B}{r^2}\right\rangle + \langle 2\mu \mathbf{D}[\mathbf{u}]^2\rangle = -\langle \mathbf{e}_\theta . \tilde{\mathbf{S}} . \mathbf{n}[V]\rangle_\Sigma .$$

Now subtracting (61) from (82), we obtain the energy equality (62) satisfied by $\mathbf{q}=(u, w)$.

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