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# HOPF BIFURCATION IN TWO-COMPONENT FLOW*

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**Abstract.** The stability of viscosity-stratifed bicomponent flow has been studied by long wave asymptotics, by short wave asymptotics and numerically. These studies have shown that interfacial instabilities arise from the viscosity difference between the two fluids. If the surface tension between the fluids is nonzero, then Hopf type bifurcatinos leading to traveling interfacial waves are expected. In this paper, we prove a rigorous theorem establishing the existence of bifurcating solutions of this nature.

**Key words.** two-component flow, Hopf bifurcation

**AMS(MOS) subject classifications.** Primary 35B32, 35Q10, 76E05, 76V05

**1. Introduction.** The stability of two-component parallel shear flows has been analyzed by long-wave asymptotics [5], [16], short-wave asymptotics [6], [13] and numerically [11], [13]. These studies show that, if the fluids have different viscosities, then instabilities can arise at all Reynolds numbers.

This raises the question of possible alternative flow patterns which might be stable. Yih [16] has conjectured that wavy interfaces might develop. The analysis of Hooper and Boyd [6] reveals a crucial difference between the cases of zero and nonzero surface tension between the fluids. If the surface tension is zero, then sufficiently short waves are always unstable, i.e., there is an infinite number of unstable modes. This situation is very much unlike the usual problems of bifurcation theory, and we believe it is possible that no smooth interface, steady or unsteady, would be stable in this situation. (In reality, of course, the surface tension is not zero, but there will be instability for very short waves when the surface tension is small and the Reynolds number is large. We think that this instability mechanism may be relevant in the formation of emulsions.)

In the case of nonzero surface tension, however, one can establish a bifurcation theorem. If the bifurcation turns out supercritical, this provides a basis for Yih's conjecture. Whether or not the bifurcation is supercritical will in general have to be decided by a numerical calculation. To our knowledge, such calculations have not yet been done. The computations referred to above concern only the eigenvalues of the linearized problems. For the sake of simplicity, we confine attention to plane Couette flow, but it is clear that similar techniques can be applied to more complicated geometries such as concentric flow in pipes or between rotating cylinders. We consider plane Couette flow of two fluids with equal density, but different viscosities, and an interface parallel to the plates. Periodic boundary conditions are imposed in the streamwise direction. Evidently, this configuration is stable at rest. If there is a flow,

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however, then instabilities can develop [6], [16]. Since, however, surface tension will damp short waves, there can be only a finite number of unstable modes. Generically, as the flow rate is increased, one specific mode will be the first to become unstable. Since the eigenvalues are complex, one expects a bifurcation of the Hopf type [7], leading to traveling interfacial waves.

The Hopf bifurcation thoerem in infinite dimensions [4], [8]–[10], [14] relies on coercive estimates for the linearized equations. For one-component free surface flows such estimates were derived by Beale [2], [3], and we shall, in §3, derive analogous estimates for two-component flow. Our proof differs from Beale's and is slightly simpler. Using these coercive estimates, we can then establish a bifurcation theorem in §4. In §§5–8, we outline an algorithm for the computation of bifurcating solutions.

**2. Formulation of the problem.** We consider two-dimensional flow of two fluids with different viscosities and equal densities between parallel plates; see Fig. 1. The motion in each fluid is described by the Navier–Stokes equations:

$$\left.\begin{aligned} &\rho(\dot{\mathbf{u}}+(\mathbf{u}\cdot\nabla)\mathbf{u})=\eta_1\Delta\mathbf{u}-\nabla p,\\ &\nabla\cdot\mathbf{u}=0,\end{aligned}\right\}\qquad 0<y<h(x), \tag{2.1}$$

$$\left.\begin{aligned} &\rho(\dot{\mathbf{v}}+(\mathbf{v}\cdot\nabla)\mathbf{v})=\eta_2\Delta\mathbf{v}-\nabla q,\\ &\nabla\cdot\mathbf{v}=0,\end{aligned}\right\}\qquad h(x)<y<1. \tag{2.2}$$

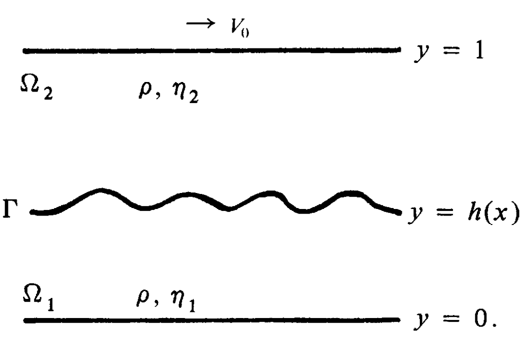


FIG. 1

We have no slip conditions at the walls:

$$\mathbf{u}=0 \qquad \text{at } y=0, \tag{2.3}$$

$$\mathbf{v}=(V_0,0) \qquad \text{at } y=1. \tag{2.4}$$

Across the interface, there must be continuity of velocity,

$$\mathbf{u}=\mathbf{v} \quad \text{at } y=h(x), \tag{2.5}$$

continuity of shear stress

$$\text{(2.6)}\quad \eta_1\left\{(1-h'^2)\left(\frac{\partial u_2}{\partial x}+\frac{\partial u_1}{\partial y}\right)+2h'\left(\frac{\partial u_2}{\partial y}-\frac{\partial u_1}{\partial x}\right)\right\}$$
$$=\eta_2\left\{(1-h'^2)\left(\frac{\partial v_2}{\partial x}+\frac{\partial v_1}{\partial y}\right)+2h'\left(\frac{\partial v_2}{\partial y}-\frac{\partial v_1}{\partial x}\right)\right\}\quad \text{at } y=h(x),$$

and balance of the normal stress difference by surface tension

$$\text{(2.7)}\quad 2\eta_1\frac{\partial u_2}{\partial y}-p-2h'\eta_1\left(\frac{\partial u_2}{\partial x}+\frac{\partial u_1}{\partial y}\right)+h'^2\left(2\eta_1\frac{\partial u_1}{\partial x}-p\right)$$
$$=2\eta_2\frac{\partial v_2}{\partial y}-q-2h'\eta_2\left(\frac{\partial v_2}{\partial x}+\frac{\partial v_1}{\partial y}\right)+h'^2\left(2\eta_2\frac{\partial v_1}{\partial x}-q\right)$$
$$+T\frac{h''}{(1+h'^2)^{1/2}}\quad \text{at } y=h(x).$$

Here $T$ is the surface tension parameter. Finally, we have the kinematic boundary condition

$$\text{(2.8)}\qquad \dot{h}+u_1h'=u_2.$$

We are interested in solutions to (2.1)–(2.8) which have a given period $L$ in the $x$-direction and are periodic in $t$. We denote by $\Omega_1$ the set $\{(x,y)\,|\,0\leqq x|L,\,0\leqq y\leqq h(x)\}$, by $\Omega_2$ the set $\{(x,y)\,|\,0\leqq x\leqq L, h(x)\leqq y\leqq 1\}$ and by $\Gamma$ the interface $\{(x,y)\,|\,0\leqq x\leqq L, y=h(x)\}$. The spaces $H^k(\Omega_1)$, $H^k(\Omega_2)$, $H^k(\Gamma)$ consist of those functions which have $k$ square integrable derivatives and satisfy periodic boundary conditions in the $x$-direction.

**3. The linearized problem.** In this chapter, we obtain coercive estimates on the linear problem, which we shall need later. We put $V_0=0$ and linearize (2.1)–(2.8) at the rest state $\mathbf{u}=\mathbf{v}=0$, with a flat interface $h=h_0$. We include inhomogeneous terms in (2.1), (2.2), (2.6) and (2.7). The goal of this section is to derive estimates for a resolvent operator, i.e. the time dependence is assumed to be exponential, and $\partial/\partial t$ can be replaced by a constant factor $\lambda$. This leads to the problem:

$$\text{(3.1)}\qquad \left.\begin{array}{l}\lambda\rho\mathbf{u}=\eta_1\Delta\mathbf{u}-\nabla p+\mathbf{f}_1,\\ \nabla\cdot\mathbf{u}=0,\end{array}\right\}\quad 0<y<h_0$$

$$\text{(3.2)}\qquad \left.\begin{array}{l}\lambda\rho\mathbf{v}=\eta_2\Delta\mathbf{v}-\nabla q+\mathbf{f}_2\\ \nabla\cdot\mathbf{v}=0,\end{array}\right\}\quad h_0<y<1,$$

$$\text{(3.3)}\qquad \mathbf{u}=0,\quad y=0,$$
$$\text{(3.4)}\qquad \mathbf{v}=0,\quad y=1,$$
$$\text{(3.5)}\qquad \mathbf{u}=\mathbf{v},\quad y=h_0,$$

$$\text{(3.6)}\qquad \eta_1\left(\frac{\partial u_2}{\partial x}+\frac{\partial u_1}{\partial y}\right)-\eta_2\left(\frac{\partial v_2}{\partial x}+\frac{\partial v_1}{\partial y}\right)=f_3,\qquad y=h_0,$$

$$\text{(3.7)}\qquad 2\eta_1\frac{\partial u_2}{\partial y}-p-2\eta_2\frac{\partial v_2}{\partial y}+q-Th''=f_4,\qquad y=h_0,$$

$$\text{(3.8)}\qquad \lambda h=u_2,\ y=h_0.$$

We seek solutions periodic in $x$ with period $L$, and of course the same periodicity is assumed for $\mathbf{f}_1$, $\mathbf{f}_2$, $f_3$ and $f_4$. Our goal is the following estimate:

THEOREM 3.1. *Let* $\gamma>0$. *For* $\operatorname{Re}\lambda \geqq \gamma$, *the following estimate holds for solutions of* (3.1)–(3.8):

$$(3.9)\qquad \begin{aligned}&\|\mathbf{u}\|_{H^2(\Omega_1)}+\|\mathbf{v}\|_{H^2(\Omega_2)}+\|p\|_{H^1(\Omega_1)}+\|q\|_{H^1(\Omega_2)}+|\lambda|\,\|\mathbf{u}\|_{L^2(\Omega_1)}\\&\quad+|\lambda|\,\|\mathbf{v}\|_{L^2(\Omega_2)}+\|h\|_{H^{5/2}(\Gamma)}+|\lambda|\,\|h\|_{H^{3/2}(\Gamma)}\\&\quad\leqq C\left[\|\mathbf{f}_1\|_{L^2(\Omega_1)}+\|\mathbf{f}_2\|_{L^2(\Omega_2)}+\|f_3\|_{H^{1/2}(\Gamma)}+|\lambda|^{1/4}\|f_3\|_{L^2(\Gamma)}+\|f_4\|_{H^{1/2}(\Gamma)}\right].\end{aligned}$$

(*Here $C$ can depend on $\gamma$ but not on $\lambda$.*)

*Proof*. We multiply (3.1) by $\bar{\mathbf{u}}$ (the complex conjugate of $\mathbf{u}$), and (3.2) by $\bar{\mathbf{v}}$, add them and integrate over the domain. Integrating by parts, and using the boundary and interface conditions, we find

$$(3.10)\qquad \begin{aligned}&\lambda\rho\left(\|\mathbf{u}\|_{L^2}^2+\|\mathbf{v}\|_{L^2}^2\right)+\frac{1}{2}\eta_1\langle\mathbf{u},\mathbf{u}\rangle+\frac{1}{2}\eta_2\langle\mathbf{v},\mathbf{v}\rangle+\bar{\lambda}T\|h'\|_{L^2}^2\\&\quad=(\mathbf{f}_1,\bar{\mathbf{u}})_{L^2}+(\mathbf{f}_2,\bar{\mathbf{v}})_{L^2}+\int_0^L f_3\bar{u}_1(y=h_0)\,dx+\int_0^L f_4\bar{u}_2(y=h_0)\,dx.\end{aligned}$$

Here $\langle\mathbf{u},\mathbf{u}\rangle=\int(\nabla\mathbf{u}+(\nabla\mathbf{u})^T):(\nabla\bar{\mathbf{u}}+(\nabla\bar{\mathbf{u}})^T)$.

Since $\mathbf{u}$ is divergence free, $u_2$ has a trace on the interface whose $H^{-1/2}$-norm can be bounded by $\|\mathbf{u}\|_{L^2(\Omega_1)}$ [15], hence the last term on the right side can be bounded by $\|f_4\|_{H^{1/2}(\Gamma)}\|\mathbf{u}\|_{L^2(\Omega_1)}$. The third term is bounded by

$$\|f_3\|_{L^2(\Gamma)}\|\mathbf{u}\|_{L^2(\Gamma)}\leqq\|f_3\|_{L^2(\Gamma)}\|\mathbf{u}\|_{L^2(\Omega_1)}^{1/2}\|\mathbf{u}\|_{H^1(\Omega_1)}^{1/2}.$$

We assume that the right side of (3.9) is bounded by a constant of order one, and we wish to bound the left side. From (3.10), we immediately obtain bounds for $\|\mathbf{u}\|_{H^1}$, $\|\mathbf{v}\|_{H^1}$, $\|h\|_{H^1}$ (we have used Korn's inequality here).

In the following, we make repeated use of the following estimates

$$(3.11)\qquad \begin{aligned}&\|\mathbf{u}\|_{H^2}+\|\mathbf{v}\|_{H^2}+\|p\|_{H^1}+\|q\|_{H^1}\\&\quad\leqq C\left(\|\mathbf{f}_1\|_{L^2}+\|\mathbf{f}_2\|_{L^2}+\|f_3\|_{H^{1/2}}+|\lambda|\,\|h\|_{H^{3/2}}+|\lambda|\,\|\mathbf{u}\|_{L^2}+|\lambda|\,\|\mathbf{v}\|_{L^2}\right),\end{aligned}$$

$$(3.12)\qquad \|h\|_{h^{5/2}}\leqq C\left(\|\mathbf{u}\|_{H^2}+\|\mathbf{v}\|_{H^2}+\|p\|_{H^1}+\|q\|_{H^1}+\|f_4\|_{H^{1/2}}\right),$$

$$(3.13)\qquad \|h\|_{H^{3/2}}\leqq\|h\|_{H^1}^{2/3}\|h\|_{H^{5/2}}^{1/3}.$$

Estimate (3.11) follows from (3.1)–(3.6) and (3.8), which form an elliptic system in the sense of Agmon, Douglis and Nirenberg [1]. This can be seen as follows: We can formally regard (3.1), (3.2) as being posed in the same domain by mapping the strip occupied by fluid 2 onto the strip occupied by fluid 1. We do this in such a way that the interface is mapped onto itself and the solid boundaries are mapped onto each other. This yields a system for the six unknowns $(u_1,u_2,p,v_1,v_2,q)$, which are now defined on the same domain. That the Stokes equations are an elliptic system is well known. The same holds of course for two sets of Stokes equations. It is also well known that Dirichlet boundary conditions satisfy the complementing condition. Showing the complementing nature of the interface condition is a straightforward calculation, which we omit. Equation (3.12) is a trivial consequence of (3.7) and the trace theorem, and (3.13) follows from the convexity property of Sobolev norms.

In the following, we start from the energy equation and then use (3.11)–(3.13) in an iterative fashion to obtain better and better estimates. This will lead to the following preliminary result.

*Lemma* 3.2. *For any* $\varepsilon>0$, *the following quantities can be estimated by the right-hand side of* (3.9) (*with constants allowed to depend on* $\varepsilon$):

$$\|h\|_{H^1}|\lambda|^{1-\varepsilon},\quad \|\mathbf{u}\|_{L^2}|\lambda|^{1-\varepsilon},\quad \|\mathbf{v}\|_{L^2}|\lambda|^{1-\varepsilon},\quad \|\mathbf{u}\|_{H^1}|\lambda|^{1/2-\varepsilon},\quad \|\mathbf{v}\|_{H^1}|\lambda|^{1/2-\varepsilon},$$
$$(\|\mathbf{u}\|_{H^2}+\|p\|_{H^1})|\lambda|^{-1/2-\varepsilon},\quad (\|\mathbf{v}\|_{H^2}+\|q\|_{H^1})|\lambda|^{-1/2-\varepsilon}.$$

*Proof*. We prove by induction that

$$(3.14)_n \qquad \|h\|_{H^1}\leqq C|\lambda|^{2^n/3^n-1},$$
$$(3.15)_n \qquad \|\mathbf{u}\|_{L^2},\ \|\mathbf{v}\|_{L^2}\leqq C|\lambda|^{2^n/3^n-1},$$
$$(3.16)_n \qquad \|\mathbf{u}\|_{H^1},\ \|\mathbf{v}\|_{H^1}\leqq C|\lambda|^{1/2(2^n/3^n-1)},$$
$$(3.17)_n \qquad \|\mathbf{u}\|_{H^2}+\|p\|_{H^1},\ \|\mathbf{v}\|_{H^2}+\|q\|_{H^1}\leqq C|\lambda|^{1/2+2^n/3^n}.$$

Obviously the lemma follows by letting $n\to\infty$. We already have (3.14)–(3.16) for $n=0$. By combining (3.11)–(3.13), we obtain

$$(3.18)\quad \|\mathbf{u}\|_{H^2}+\|p\|_{H^1}+\|\mathbf{v}\|_{H^2}+\|q\|_{H^1}$$
$$\leqq C\Big[\|\mathbf{f}_1\|_{L^2}+\|\mathbf{f}_2\|_{L^2}+\|f_3\|_{H^{1/2}}+|\lambda|\|\mathbf{u}\|_{L^2}+|\lambda|\|\mathbf{v}\|_{L^2}$$
$$+|\lambda|\|h\|_{H^1}^{2/3}\cdot\big(\|\mathbf{u}\|_{H^2}+\|\mathbf{v}\|_{H^2}+\|p\|_{H^1}+\|q\|_{H^1}+\|f_4\|_{H^{1/2}}\big)^{1/3}\Big].$$

With $\beta=\|\mathbf{u}\|_{H^2}+\|p\|_{H^1}+\|\mathbf{v}\|_{H^2}+\|q\|_{H^1}$, we find, using $(3.14)_n$–$(3.16)_n$

$$(3.19)\qquad \beta\leqq C\big(1+|\lambda|^{2^n/3^n}+|\lambda|^{2^{n+1}/3^{n+1}+1/3}(\beta+c')^{1/3}\big).$$

From this, $(3.17)_n$ follows easily.

Next, we wish to show that $(3.14)_{n+1}$–$(3.16)_{n+1}$ follow from $(3.17)_n$. Using (3.8), the trace theorem and the convexity property of Sobolev norms, we find

$$(3.20)\qquad \|h\|_{H^1}\leqq \frac{C}{|\lambda|}\sqrt{\|\mathbf{u}\|_{H^2}}\sqrt{\|\mathbf{u}\|_{H^1}}.$$

Moreover (3.10) implies

$$(3.21)\qquad \|\mathbf{v}\|_{H^1}+\|\mathbf{u}\|_{H^1}\leqq C\big(\|\mathbf{u}\|_{L^2}+\|\mathbf{v}\|_{L^2}+|\lambda|^{-1/4}\|\mathbf{u}\|_{L^2}^{1/2}\|\mathbf{u}\|_{H^1}^{1/2}\big)^{1/2},$$

and

(3.22)
$$\|\mathbf{v}\|_{L^2}+\|\mathbf{u}\|_{L^2}\leqq C\Big[\|h\|_{H^1}+|\lambda|^{-1/2}\big(\|\mathbf{u}\|_{L^2}+\|\mathbf{v}\|_{L^2}+|\lambda|^{-1/4}\|\mathbf{u}\|_{L^2}^{1/2}\|\mathbf{u}\|_{H^1}^{1/2}\big)^{1/2}\Big].$$

With $x=\|\mathbf{u}\|_{H^1}+\|\mathbf{v}\|_{H^1}$, $y=\|\mathbf{u}\|_{L^2}+\|\mathbf{v}\|_{L^2}$ and using $(3.17)_n$, we find that (3.21), (3.22) take the following form

$$(3.23)\qquad x\leqq C\big(y^{1/2}+|\lambda|^{-1/8}x^{1/4}y^{1/4}\big),$$

$$(3.24)\qquad y\leqq C\big(|\lambda|^{2^{n-1}/3^n-3/4}x^{1/2}+|\lambda|^{-1/2}y^{1/2}+|\lambda|^{-5/8}y^{1/4}x^{1/4}\big).$$

From (3.23) it follows that $x \leqq Cy^{1/2}$ or $x \leqq C|\lambda|^{-1/6}y^{1/3}$, depending on whether the first or second term on the right is bigger. In the first case, (3.24) yields

$$(3.25)\qquad y \leqq C\Big(|\lambda|^{2^{n-1}/3^n-3/4}y^{1/4}+|\lambda|^{-1/2}y^{1/2}+|\lambda|^{-5/8}y^{3/8}\Big).$$

From this $(3.15)_{n+1}$ is immediate. In the second case, we get

$$(3.26)\qquad y \leqq C\Big(|\lambda|^{2^{n-1}/3^n-3/4-1/12}y^{1/6}+|\lambda|^{-1/2}y^{1/2}+|\lambda|^{-5/8-1/24}y^{1/3}\Big).$$

From this, $(3.15)_{n+1}$ is also immediate. From $(3.15)_{n+1}$ and (3.21) follows $(3.16)_{n+1}$, and using (3.20) and $(3.17)_n$ we find $(3.14)_{n+1}$. This concludes the proof of the lemma.

We now return to the proof of Theorem 3.1.

To proceed further, we take difference quotients in the $x$-direction. These satisfy the same equations (3.1)–(3.8) with the $f$'s replaced by their difference quotients. From (3.10), we then see that the $H^1$-norms of all $x$-derivatives of $\mathbf{u}$ and $\mathbf{v}$ can be estimated by terms of order 1. The divergence condition now yields

$$\|u_2\|_2,\ \|v_2\|_2 \leqq C.$$

Equation (3.8) and the trace theorem imply that $\|h\|_{H^{3/2}} \leqq C/|\lambda|$, and from (3.11), and Lemma 3.2 we conclude that

$$\|\mathbf{u}\|_{H^2}+\|\mathbf{v}\|_{H^2}+\|p\|_{H^1}+\|q\|_{H^1} \leqq C|\lambda|^{\varepsilon}.$$

Using this in (3.20), we get $\|h\|_{H^1} \leqq C|\lambda|^{-5/4+\varepsilon}$, and by inserting this in (3.21), (3.22), we find $\|\mathbf{u}\|_{L^2}+\|\mathbf{v}\|_{L^2} \leqq C/|\lambda|$. By using (3.11) again, we obtain the theorem.

*Remarks.* We have so far only given estimates for a solution that was assumed to exist and have the regularity implied by the left-hand side of (3.9). Such estimates show that (3.1)–(3.8) for $\operatorname{Re}\lambda \geqq \gamma > 0$ is solvable for a closed set of $f$'s (in the topology indicated by the right side of (3.9)). Solvability for a dense set of $f$'s can be shown in a straightforward manner by separation of variables. (Separation of variables leads to a system of ODE's, and it is easy to show that a Fredholm alternative holds for these ODE's. The absence of eigenfunctions follows from the energy equation.) From this we see that in fact, for any $\lambda$ with $\operatorname{Re}\lambda > 0$, (3.1)–(3.8) has a unique solution. Inequality (3.9) holds uniformly in any closed subset of any right half-plane, if this subset contains no eigenvalues. Moreover, by compactness, the number of eigenvalues is countable, and there can be only finitely many in any bounded set. Equation (3.10) implies that all eigenvalues have negative real parts.

**4. Bifurcation to travelling waves.** It is convenient to use a domain mapping which takes the domain occupied by each fluid to a fixed one. The most straightforward way to construct such a mapping is to stretch or contract vertical lines. We shall in addition transform the velocity fields in such a way that the divergence condition is preserved and (2.8) reduces to the linearized form even in the nonlinear case. In doing this, we essentially follow Beale [2].

Let $y=h_0$ be the flat interface of the rest state, and let $y=h(x,t)$ be the actual interface, which we assume periodic in $x$ with period $L$.

Let $P$ be any linear extension operator that maps functions $h(\zeta)\in H^s(\Gamma)$, into functions $\tilde{h}(\zeta,\eta)$ such that $\tilde{h}(\zeta,h_0)=h(\zeta)$ and $\tilde{h}\in H^{s+1/2}(\Omega)$ ($P$ exist according to the trace theorem). For simplicity, we also assume that $P$ takes $h\equiv h_0$ to $\tilde{h}\equiv h_0$. Let then $f_0(\eta)$ be a $C^\infty$-function of $\eta$ such that $f_0=1$ near $\eta=h_0$ and $f_0=0$ near $\eta=0$ and $\eta=1$. Define $\bar{h}(\zeta,\eta,t)=\tilde{h}(\zeta,\eta,t)\cdot f_0(\eta)+h_0(1-f_0(\eta))$.

We now define

$$\theta(\zeta,\eta,t)=\left(\zeta,\eta\cdot\frac{\bar h(\zeta,\eta,t)}{h_0}\right). \tag{4.1}$$

Evidently, $\theta$ maps the strip $0\leqq\eta\leqq h_0$ to $\Omega_1$, and the strip $h_0\leqq\eta\leqq 1$ to $\Omega_2$. The velocities are transformed as follows:

$$u_i(\theta(\zeta,\eta,t))=\frac{\partial\theta_i}{\partial\zeta_j}\tilde u_j/J, \tag{4.2}$$

where $J$ is the Jacobian of $\theta$. Of course, $\mathbf{v}$ in $\Omega_2$ is transformed in the same way. Explicitly, (4.2) reads

$$\mathbf{u}=\left(\frac{\bar h}{h_0}+\eta\frac{\bar h_\eta}{h_0}\right)^{-1}\left(\tilde u_1,\eta\frac{\bar h_\zeta}{h_0}\tilde u_1+\left(\frac{\bar h}{h_0}+\eta\frac{\bar h_\eta}{h_0}\right)\tilde u_2\right). \tag{4.3}$$

Formula (4.2) is set up in such a way that $\mathbf{u}$, $\mathbf{v}$ are divergence-free in the $x$, $y$-plane, if $\tilde{\mathbf{u}}$, $\tilde{\mathbf{v}}$ are divergence-free in the $\zeta$, $\eta$-plane. Moreover, (2.8) now assumes the simple form

$$\dot h(\zeta,t)=\tilde u_2(\zeta,h_0,t). \tag{4.4}$$

It is also clear that (2.5) does not change its form, i.e. $\mathbf{u}=\mathbf{v}$ at $y=h(x)$ simply becomes $\tilde{\mathbf{u}}=\tilde{\mathbf{v}}$ at $\eta=h_0$. The boundary conditions at the walls are also preserved. When these substitutions are inserted into (2.1)–(2.8), we obtain a new set of equations, which we do not write down explicitly. We shall refer to these new equations as (2.1)*–(2.8)*. If we have a flat interface $h=h_0$, then of course (4.3) reads $\mathbf{u}=\tilde{\mathbf{u}}$, and (2.1)*–(2.8)* have the same form as (2.1)–(2.8).

Plane Couette flow is the following solution of (2.1)–(2.8):

$$\hat h(x)\equiv h_0,\ \hat u_1=\frac{\eta_2V_0}{\eta_1+h_0(\eta_2-\eta_1)}y,\quad \hat u_2=0,\quad \hat v_1=\frac{\eta_1V_0y+V_0h_0(\eta_2-\eta_1)}{\eta_1+h_0(\eta_2-\eta_1)},$$

$$\hat v_2=0,\quad \hat p=0,\quad \hat q=0.$$

We can linearize at this solution, and obtain a set of linearized equations analogous to (3.1)–(3.8). As usual, we call $\lambda$ an eigenvalue, if the homogeneous linearized problem has nontrivial solutions. The estimates in §3 imply that, for $V_0=0$ (rest), there is a countable sequence of eigenvalues, all in a sector of the left half plane bounded away from the imaginary axis. All these eigenvalues have finite multiplicity and Fredholm index zero. Estimates like those in §3 can easily be extended to the linearization at Couette flow with finite $V_0$. If we linearize (2.1)–(2.8) at this flow, then there are a number of terms perturbing (3.1)–(3.8). All these terms are relatively compact except the one resulting from $u_1h'$ in (2.8). This latter term vanishes in a frame moving with the fluid on the interface. Standard perturbation theory [12] can now be used to show that estimates like in §3 hold for $\lambda$ in any closed set that lies in a right half plane and contains no eigenvalues. However, there can, and as [6], [16] show, there will be a finite number of eigenvalues with positive real parts if $V_0$ is large enough. Generically, there will be a critical value $V_{0c}$, such that, for $V_0<V_{0c}$, all eigenvalues have negative real parts, but a pair of simple complex conjugate eigenvalues crosses the imaginary axis transversally as $V_0$ increases past $V_{0c}$. Let us denote these imaginary eigenvalues by $\pm i\omega_0$.

We introduce the following substitutions in (2.1)*–(2.8)*:

$$\tau=\omega t,\quad \mathbf{u}^*=\tilde{\mathbf{u}}-\hat{\mathbf{u}}(V_0),\quad \mathbf{v}^*=\tilde{\mathbf{v}}-\hat{\mathbf{v}}(V_0),\quad h^*=(h-h_0)\frac{\omega}{\omega_0}.$$

We use the following notation for function space: $H^k_{\tilde{\Omega}_1}$ denotes the space of functions defined on $-\infty<\zeta<\infty$, $0\leqq\eta\leqq h_0$, which have period $L$ in $\zeta$ and $H^k$-regularity. Similarly, we define $H^k_{\tilde{\Omega}_2}$. Finally $H^k_{\tilde{\Gamma}}$ is the set of $k$ times differentiable periodic functions depending on $\zeta$ alone.

$H^k(X)$ denotes the spaces of all $2\pi$-periodic functions defined on $-\infty<\tau<\infty$, taking values in $X$, and having $k$ square integrable derivatives.

For the analysis of (2.1)*–(2.8)*, we choose the following space $V$:

$$\begin{aligned} V=\Big\{&(\mathbf{u}^*,\mathbf{v}^*,p,q,h^*)\,|\,\mathbf{u}^*\in H^1\big(H^2_{\tilde{\Omega}_1}\big)\cap H^2\big(L^2_{\tilde{\Omega}_1}\big),\ v^*\in H^1\big(H^2_{\tilde{\Omega}_2}\big)\cap H^2\big(L^2_{\tilde{\Omega}_2}\big),\\ &p\in H^1\big(H^1_{\tilde{\Omega}_1}\big),\ q\in H^1\big(H^1_{\tilde{\Omega}_2}\big),\ h^*\in H^1\big(H^{5/2}_{\tilde{\Gamma}}\big)\cap H^2\big(H^{3/2}_{\tilde{\Gamma}}\big);\\ &\operatorname{div}\mathbf{u}^*=0,\ \operatorname{div}\mathbf{v}^*=0,\\ &\iint_{\tilde{\Omega}_1} p+\iint_{\tilde{\Omega}_2} q=0,\ \mathbf{u}^*=0 \text{ at } \eta=0,\ \mathbf{v}^*=0 \text{ at } \eta=1,\ \mathbf{u}^*=\mathbf{v}^* \text{ at}\\ &\qquad \eta=h_0,\ \omega_0\frac{\partial h^*}{\partial\tau}=\mathbf{u}^* \text{ at } \eta=h_0,\ \int_0^L h^*(\zeta)\,d\zeta=0\Big\}. \end{aligned}$$

Functions in this space have sufficient regularity such that all the nonlinearities in (2.1)*–(2.8)* are defined. We can now prove a Hopf bifurcation result based on the implicit function theorem. This relies essentially on an iterative scheme which at each stage solves the linearized problem with the nonlinear terms as inhomogeneities. It is important that such an iteration takes the space $V$ into itself, i.e. that by inverting the linearized operator we gain at least as much regularity as we lose by evaluating the nonlinear terms. This is guaranteed by the coercive estimate of Theorem 3.1. In this way, we obtain the following.

THEOREM 4.1. *Assume that, at $V_0=V_{0c}$, there is a pair of algebraically simple complex conjugate eigenvalues $\pm i\omega_0$, $\omega_0\neq 0$, and no other eigenvalue is an integral multiple of $i\omega_0$. Moreover, assume that those eigenvalues cross the imaginary axis transversally, i.e. if $\lambda(V_0)$ denotes the branch of eigenvalues for which $\lambda(V_{0c})=i\omega_0$, then $(d/dV_0)\mathrm{Re}\lambda(V_0)|_{V_0=V_{0c}}\neq 0$. Then there is an analytic branch of nontrivial time-periodic solutions $(\mathbf{u}^*(\varepsilon),\mathbf{v}^*(\varepsilon),p(\varepsilon),q(\varepsilon),h^*(\varepsilon))=:Y(\varepsilon)$, such that $Y(\varepsilon)\in V$ is a solution of (2.1)*–(2.8)* for $V_0=V_0(\varepsilon)$ with temporal frequency $\omega=\omega(\varepsilon)$. For $\varepsilon=0$, we have $V_0=V_{0c}$, $\omega=\omega_0$ and $Y=0$. This branch of periodic solutions is unique except for phase shift or changes of parametrization.*

If, at $V_0=V_{0c}$, all eigenvalues other than $\pm i\omega_0$ have negative real parts and we have $\frac{d}{dV_0}\mathrm{Re}\lambda(V_0)|_{V_0=V_{0c}}>0$, then the bifurcating periodic solutions are stable if $V_0(\varepsilon)>V_{0c}$ for small $\varepsilon\neq 0$, and unstable if $V_0(\varepsilon)<V_{0c}$.

*Remark.* It is easy to show higher regularity of the bifurcating solutions by choosing function spaces of higher regularity for the bifurcation analysis.

**5. Reduction of the bifurcation problem to local form.** In the previous two sections, we have provided the analytical tools and the estimates needed to establish a bifurcation theorem. In the following, we now describe an algorithm for calculating approximations to this bifurcating solution.

As before, we study bifurcation from plane Couette flow, and we consider the velocity of the upper plate as the bifurcation parameter. Plane Couette flow is the following solution of (2.1)–(2.8):

$$
\begin{aligned}
&\hat{h}(x)=h_0,\\
&\hat{u}_1=\frac{\eta_2 V_0 y}{\eta_1+h_0(\eta_2-\eta_1)},\\
(5.1)\qquad &\hat{u}_2=0,\qquad \hat{p}=0,\\
&\hat{v}_1=\frac{[\eta_1 y+h_0(\eta_2-\eta_1)]V_0}{\eta_1+h_0(\eta_2-\eta_1)},\\
&v_2=0,\qquad q=0.
\end{aligned}
$$

For the bifurcation problem, it is convenient to introduce new variables representing the perturbation of this solution. We therefore replace $u_1$ and $v_1$ by $u_1+\hat{u}_1$, $v_1+\hat{v}_1$, where $\hat{u}_1$, $\hat{v}_1$ are given by formula (5.1) in the regions $0\leqq y\leqq h(x,t)$ and $h(x,t)\leqq y\leqq 1$, respectively. Moreover, we set

$$(5.2)\qquad h(x,t)=h_0+\delta(x,t),$$

and $\delta(x,t)$ has zero mean value as a function of $x$. With this change of variables, (2.1) and (2.2) take the form

$$(5.3)\quad \rho\left[\dot{\mathbf{u}}+\hat{u}_1\frac{\partial \mathbf{u}}{\partial x}+\mathbf{e}_x u_2\frac{\partial \hat{u}_1}{\partial y}+(\mathbf{u}\cdot\nabla)\mathbf{u}\right]=\eta_1\Delta\mathbf{u}-\nabla p,\quad 0\leqq y\leqq h(x,t),\quad \operatorname{div}\mathbf{u}=0,$$

$$(5.4)\quad \rho\left[\dot{\mathbf{v}}+\hat{v}_1\frac{\partial \mathbf{v}}{\partial x}+\mathbf{e}_x v_2\frac{\partial \hat{v}_1}{\partial y}+(\mathbf{v}\cdot\nabla)\mathbf{v}\right]=\eta_2\Delta\mathbf{v}-\nabla q,\quad h(x,t)\leqq y\leqq 1,\quad \operatorname{div}\mathbf{v}=0.$$

On the walls, we have

$$(5.5)\qquad \mathbf{u}=0 \text{ at } y=0,\qquad \mathbf{v}=0 \text{ at } y=1.$$

In the normal and shear stress conditions, the terms replaced by $\hat{u}_1$, $\hat{v}_1$ are such that they cancel. Moreover, $h'=\delta'$, so $h$ can be replaced by $\delta$. Across the interface $y=h(x,t)$, we thus obtain the following conditions resulting from (2.6) and (2.7)

$$
\begin{aligned}
(5.6)\quad &(1-\delta'^2)\eta_1\left(\frac{\partial u_2}{\partial x}+\frac{\partial u_1}{\partial y}\right)+2\eta_1\delta'\left(\frac{\partial u_2}{\partial y}-\frac{\partial u_1}{\partial x}\right)\\
&=(1-\delta'^2)\eta_2\left(\frac{\partial v_2}{\partial x}+\frac{\partial v_1}{\partial y}\right)+2\eta_2\delta'\left(\frac{\partial v_2}{\partial y}-\frac{\partial v_1}{\partial x}\right),
\end{aligned}
$$

$$
\begin{aligned}
(5.7)\quad &2\eta_1\frac{\partial u_2}{\partial y}-p-2\delta'\eta_1\left(\frac{\partial u_2}{\partial x}+\frac{\partial u_1}{\partial y}\right)+\delta'^2\left(2\eta_1\frac{\partial u_1}{\partial x}-p\right)\\
&=2\eta_2\frac{\partial v_2}{\partial y}-q-2\delta'\eta_2\left(\frac{\partial v_2}{\partial x}+\frac{\partial v_1}{\partial y}\right)+\delta'^2\left(2\eta_2\frac{\partial v_1}{\partial x}-q\right)+T\delta''/(1+\delta'^2)^{1/2}.
\end{aligned}
$$

The condition (2.5) is replaced by

$$u_1 - v_1 = \hat{v}_1(h_0+\delta) - \hat{u}_1(h_0+\delta) = \frac{(\eta_1-\eta_2)\delta V_0}{\eta_1 + h_0(\eta_2-\eta_1)}, \tag{5.8}$$

$$u_2 = v_2. \tag{5.9}$$

Equation (5.8) shows that, for $\eta_1 \neq \eta_2$ and $V_0 \neq 0$, $\delta(x,t)$ can be eliminated and expressed by the jump in the $x$-component of velocity at $y = h_0 + \delta(x,t)$. Finally, equation (2.8) assumes the form

$$\dot{\delta} + \hat{u}_1(h_0)\delta' + u_1\delta' + \frac{\eta_2 V_0 \delta\delta'}{\eta_1 + h_0(\eta_2-\eta_1)} = u_2. \tag{5.10}$$

Our bifurcation problem is now given by (5.1)–(5.10). The null solution corresponds to plane Couette flow. We consider $\delta$ in (5.6), (5.7) and (5.10) as having been eliminated from (5.8), and look for solutions $(\mathbf{u}, \mathbf{v}, p, q)$ which are periodic in $x$.

**6. The spectral problem and its adjoint.** The spectral problem for the stability of the null solution is

$$\rho\left[\lambda\mathbf{u} + \hat{u}_1\frac{\partial \mathbf{u}}{\partial x} + \mathbf{e}_x u_2 \hat{u}_1'\right] + \nabla p - 2\eta_1 \operatorname{div}\underline{\underline{D}}[\mathbf{u}] = 0, \quad 0 \leqq y \leqq h_0, \quad \operatorname{div}\mathbf{u} = 0, \tag{6.1}$$

$$\rho\left[\lambda\mathbf{v} + \hat{v}_1\frac{\partial \mathbf{v}}{\partial x} + \mathbf{e}_x v_2 \hat{v}_1'\right] + \nabla q - 2\eta_2 \operatorname{div}\underline{\underline{D}}[\mathbf{v}] = 0, \quad h_0 \leqq y \leqq 1, \quad \operatorname{div}\mathbf{v} = 0. \tag{6.2}$$

Here $\underline{\underline{D}}[\mathbf{u}]$ is the symmetric part of $\nabla\mathbf{u}$. $\hat{u}_1'$ and $\hat{v}_1'$ denote derivatives with respect to $h$, while $h'$ and $\delta'$ will continue to denote derivatives with respect to $x$. On the walls we have

$$\mathbf{u} = 0 \text{ at } y = 0, \qquad \mathbf{v} = 0 \text{ at } y = 1. \tag{6.3}$$

From (5.8), we have $\delta = k(u_1 - v_1)$, where $k = (\eta_1 + (\eta_2 - \eta_1)h_0)/V_0(\eta_1 - \eta_2)$.

By inserting this into the remaining interface conditions and linearizing, we find the following conditions at $y = h_0$:

$$u_2 - v_2 = 0, \tag{6.4}$$

$$2\eta_1 D_{12}[\mathbf{u}] - 2\eta_2 D_{12}[\mathbf{v}] = 0, \tag{6.5}$$

$$-p + 2\eta_1 D_{22}[\mathbf{u}] + q - 2\eta_2 D_{22}[\mathbf{v}] - Tk(u_1'' - v_1'') = 0, \tag{6.6}$$

$$\left(\lambda + \hat{u}(h_0)\frac{\partial}{\partial x}\right)k(u_1 - v_1) - u_2 = 0. \tag{6.7}$$

We turn next to the computation of the spectral problem, which is adjoint to (6.1)–(6.7). We multiply (6.1) by $\bar{\mathbf{u}}^*$, (6.2) by $\bar{\mathbf{v}}^*$, the complex conjugates of the adjoint velocities, and integrate the resulting expressions over their domain of definition. We assume periodicity in $x$ with period $L$, and integrate by parts using periodicity, solenoidality and (6.3) to derive

(6.8)
$$\int_{\Omega_1}\left\{\rho\lambda\bar{\mathbf{u}}^*-\rho\hat{u}_1\frac{\partial\bar{\mathbf{u}}^*}{\partial x}+\rho\bar{u}_1^*\hat{u}_1'\mathbf{e}_y-2\eta_1\operatorname{div}\underline{\underline{D}}[\bar{\mathbf{u}}^*]\right\}\cdot\mathbf{u}\,dx\,dy$$
$$+\int_{\Omega_2}\left\{\rho\lambda\bar{\mathbf{v}}^*-\rho\hat{v}_1\frac{\partial\bar{\mathbf{v}}^*}{\partial x}+\rho\bar{v}_1^*\hat{v}_1'\mathbf{e}_y-2\eta_2\operatorname{div}\underline{\underline{D}}[\bar{\mathbf{v}}^*]\right\}\cdot\mathbf{v}\,dx\,dy$$
$$-\int_{\Omega_1}p\operatorname{div}\bar{\mathbf{u}}^*-\int_{\Omega_2}q\operatorname{div}\bar{\mathbf{v}}^*$$
$$=-\int_0^L\{(-q+2\eta_2D_{22}[\mathbf{v}])\bar{v}_2^*-(-p+2\eta_1D_{22}[\mathbf{u}])\bar{u}_2^*$$
$$+2\eta_2\bar{v}_1^*D_{12}[\mathbf{v}]-2\eta_2v_1D_{12}[\bar{\mathbf{v}}^*]-2\eta_2v_2D_{22}[\bar{\mathbf{v}}^*]$$
$$+2\eta_1u_1D_{12}[\bar{\mathbf{u}}^*]-2\eta_1\bar{u}_1^*D_{12}[\mathbf{u}]+2\eta_1u_2D_{22}[\mathbf{u}^*]\}\,dx.$$

By considering special forms of **u**, **v**, $p$, $q$, we find that in $0\leqq y\leqq h_0$, we have

(6.9)
$$\rho\lambda\bar{\mathbf{u}}^*-\rho\hat{u}_1\frac{\partial\bar{\mathbf{u}}^*}{\partial x}+\rho\mathbf{e}_y\bar{u}_1^*\hat{u}_1'-2\eta_1\operatorname{div}\underline{\underline{D}}[\bar{\mathbf{u}}^*]=-\nabla\bar{p}^*,\qquad\operatorname{div}\bar{\mathbf{u}}^*=0,$$

whilst in $h_0\leqq y\leqq 1$,

(6.10)
$$\rho\lambda\bar{\mathbf{v}}^*-\rho\hat{v}_1\frac{\partial\bar{\mathbf{v}}^*}{\partial x}+\rho\mathbf{e}_y\bar{v}_1^*\hat{v}_1'-2\eta_2\operatorname{div}\underline{\underline{D}}[\bar{\mathbf{v}}^*]=-\nabla q^*,\qquad\operatorname{div}\bar{\mathbf{v}}^*=0.$$

We insert (6.9) and (6.10) back into (6.8), and compute

$$\int_{\Omega_1}\mathbf{u}\cdot\nabla\bar{p}^*\,dx\,dy+\int_{\Omega_2}\mathbf{v}\cdot\nabla\bar{q}^*\,dx\,dy=\int_0^L(\bar{p}^*u_2-\bar{q}^*v_2)\,dx.$$

This term is added to the right-hand side of (6.8), leading to

(6.11)
$$0=\int_0^L\{(-q+2\eta_2D_{22}[\mathbf{v}])\bar{v}_2^*+(p-2\eta_1D_{22}[\mathbf{u}])\bar{u}_2^*$$
$$+(\bar{q}^*-2\eta_2D_{22}[\bar{\mathbf{v}}^*])v_2+(-\bar{p}^*+2\eta_1D_{22}[\bar{\mathbf{u}}^*])u_2$$
$$+2\eta_2\bar{v}_1^*D_{12}[\mathbf{v}]-2\eta_1\bar{u}_1^*D_{12}[\mathbf{u}]$$
$$+2\eta_1u_1D_{12}[\bar{\mathbf{u}}^*]-2\eta_2v_1D_{12}[\bar{\mathbf{v}}^*]\}\,dx.$$

We use (6.6) to reduce the first line of (6.11), (6.4) for the second line and (6.5) for the third line. Thus we find

(6.12)
$$0=\int_0^L\{(-q+2\eta_2D_{22}[\mathbf{v}])(\bar{v}_2^*-\bar{u}_2^*)-T\delta''\bar{u}_2^*$$
$$+u_2(\bar{q}^*-2\eta_2D_{22}[\bar{\mathbf{v}}^*]-\bar{p}^*+2\eta_1D_{22}[\bar{\mathbf{u}}^*])$$
$$+(\bar{v}_1^*-\bar{u}_1^*)2\eta_1D_{12}[\mathbf{u}]$$
$$+(u_1-v_1)\cdot2\eta_1D_{12}[\bar{\mathbf{u}}^*]+v_1(2\eta_1D_{12}[\bar{\mathbf{u}}^*]-2\eta_2D_{12}[\bar{\mathbf{v}}^*])\}\,dx.$$

We next write

$$\int_0^L \delta'' \bar{u}_2^* \, dx = \int_0^L \delta \bar{u}_2^{*\prime\prime} \, dx,$$

and set $\delta = k(u_1 - v_1)$, $u_2 = (\lambda + \hat{u}(h_0)\partial/\partial x)k(u_1 - v_1)$.

This leads to

$$\begin{aligned} 0 = \int_0^L \Big\{ &(-q + 2\eta_2 D_{22}[\mathbf{v}])(\bar{v}_2^* - \bar{u}_2^*) \\ &+ (u_1 - v_1)\Big\langle -Tk\bar{u}_2^{*\prime\prime} + \Big(\lambda - \hat{u}(h_0)\frac{\partial}{\partial x}\Big)k\big(\bar{q}^* - 2\eta_2 D_{22}[\bar{\mathbf{v}}^*] \\ &\qquad - \bar{p}^* + 2\eta_1 D_{22}[\bar{\mathbf{u}}^*]\big) + 2\eta_1 D_{12}[\bar{\mathbf{u}}^*] \Big\rangle \\ &+ (\bar{v}_1^* - \bar{u}_1^*)\cdot 2\eta_1 D_{12}[\mathbf{u}] + v_1\big(2\eta_1 D_{12}[\bar{\mathbf{u}}^*] - 2\eta_2 D_{12}[\bar{\mathbf{v}}^*]\big)\Big\} \, dx. \end{aligned}$$

This yields the following four conditions on $y = h_0$:

$$2\eta_1 D_{12}[\bar{\mathbf{u}}^*] = 2\eta_2 D_{12}[\bar{\mathbf{v}}^*], \tag{6.13}$$

$$\bar{u}_1^* = \bar{v}_1^*, \tag{6.14}$$

$$\bar{u}_2^* = \bar{v}_2^*, \tag{6.15}$$

$$\begin{aligned} -Tk\bar{u}_2^{*\prime\prime} + \Big(\lambda - \hat{u}(h_0)\frac{\partial}{\partial x}\Big)k\big(\bar{q}^* - 2\eta_2 D_{22}[\bar{\mathbf{v}}^*] \\ - \bar{p}^* + 2\eta_1 D_{22}[\bar{\mathbf{u}}^*]\big) + 2\eta_1 D_{12}[\bar{\mathbf{u}}^*] = 0. \end{aligned} \tag{6.16}$$

Thus the adjoint problem is given by the differential equations (6.9), (6.10), the Dirichlet conditions $\bar{\mathbf{u}}^* = 0$, $\bar{\mathbf{v}}^* = 0$ on the walls and conditions (6.13)–(6.16) on the interface.

It is easy to establish a necessary and sufficient condition for the solvability of the inhomogeneous problem corresponding to (6.1)–(6.7). Suppose that the zeros on the right of the first equation in (6.1) and (6.2) and on the right of (6.4)–(6.7) are replaced by

$$\mathbf{g}_1(x,y), \quad \mathbf{g}_2(x,y), \quad g_3(x), \quad g_4(x), \quad g_5(x), \quad g_6(x), \tag{6.17}$$

respectively. This inhomogeneous problem is solvable if and only if the data (6.17) are orthogonal to the kernel of the adjoint, that is, when (6.17) is such that

$$\begin{aligned} \int_{\Omega_1} \mathbf{g}_1 \cdot \bar{\mathbf{u}}^* \, dx\,dy + \int_{\Omega_2} \mathbf{g}_2 \cdot \bar{\mathbf{v}}^* \, dx\,dy \\ = \int_0^L \big\{ g_3\big(\bar{q}^* - 2\eta_2 D_{22}[\bar{\mathbf{v}}^*]\big) + g_4 \bar{u}_1^* + g_5 \bar{u}_2^* \\ + g_6\big(\bar{q}^* - 2\eta_2 D_{22}[\bar{\mathbf{v}}^*] - \bar{p}^* + 2\eta_1 D_{22}[\bar{\mathbf{u}}^*]\big)\big\} \, dx. \end{aligned} \tag{6.18}$$

We are interested in the neighborhood of a critical point, where a loss of stability occurs. There is a critical plate velocity $V_0 = \hat{V}_0$ such that the real part of $\lambda$ vanishes, and $\lambda = i\omega_0$. We put

$$V_0 = \hat{V}_0(1 + R),$$

so criticality is when $R=0$. We get Hopf bifurcation when the loss of stability is strict

$$\operatorname{Re} \left.\frac{\partial \lambda}{\partial R}\right|_{R=0} \neq 0.$$

Let

$$\zeta_0=\begin{cases} \mathbf{u} & \text{in } \Omega_1(y<h_0), \\ \mathbf{v} & \text{in } \Omega_2(y>h_0) \end{cases}$$

be a right eigenvector satisfying (6.1)–(6.7) at criticality. Then $\bar{\zeta}_0$ is the right eigenvector belonging to $-i\omega_0$. $\bar{\zeta}_0^*$ and $\zeta_0^*$ are the left eigenvectors belonging to $i\omega_0$ and $-i\omega_0$.

**7. Domain perturbations and Hopf bifurcation.** We have already demonstrated that Couette flow can bifurcate into a time-periodic solution in which we have travelling interfacial waves. To compute this solution we would, following Lindstedt, map the solution into a fixed frequency domain ($2\pi$ periodic in $s$)

$$\omega\, dt = ds,$$

and replace

$$(\dot{\mathbf{u}}, \dot{\mathbf{v}}, \dot{\delta}) \quad \text{with } \omega\left(\frac{\partial \mathbf{u}}{\partial s}, \frac{\partial \mathbf{v}}{\partial s}, \frac{\partial \delta}{\partial s}\right) \tag{7.1}$$

in (5.3), (5.4) and (5.10). We then map our problem into a fixed spatial domain, using a one-to-one linear mapping, which takes boundary points into boundary points

$$y=(h(x,t)-h_0)\frac{y_0-1}{h_0-1}+y_0\begin{cases} h\leqq y\leqq 1, \\ h_0\leqq y_0\leqq 1, \end{cases} \tag*{(7.2)$_1$}$$

and

$$y=\left(1+\frac{h(x,t)-h_0}{h_0}\right)y_0\begin{cases} 0\leqq y\leqq h(x,t), \\ 0\leqq y_0\leqq h_0. \end{cases} \tag*{(7.2)$_2$}$$

We then change variables, putting $x=x_0$ and $y=\tilde{y}(x_0,y_0)$, where $\tilde{y}$ is defined by (7.2) in (5.3)–(5.10). The form of these equations changes under the change of variables. However, following ideas introduced by Joseph [18], [19] we find many simplifications. We shall now explain these simplifications.

First we introduce an amplitude parameter which is conveniently set as a projection

$$\varepsilon=[\mathbf{u}, \mathbf{z}^*],$$

where

$$[\mathbf{a},\mathbf{b}] \overset{\text{def}}{=} \frac{1}{2\pi}\int_0^{2\pi} \langle \mathbf{a},\mathbf{b}\rangle\, ds,$$

and $\langle \mathbf{a},\mathbf{b}\rangle$ are integrals over both regions of the type displayed in (6.8). We are working in the frame of Iooss-Joseph [17, §VIII. 3] and

$$\mathbf{z}^*=e^{is}\zeta_0^*, \tag{7.3}$$

where $\zeta_0^*$ is the left eigenvector at criticality which was introduced at the end of the last section.

The bifurcating solution may be computed in a series of powers of $\varepsilon$. Thus

$$\omega(\varepsilon)=\omega_0+\frac{\varepsilon^2}{2}\omega_2+\frac{\varepsilon^4}{4!}\omega_4+\cdots, \tag{7.4}$$

$$V_0(\varepsilon)=\hat{V}_0\left(1+\frac{\varepsilon^2}{2}\hat{V}_2+\frac{\varepsilon^4}{4!}\hat{V}_4+\cdots\right). \tag{7.5}$$

It follows from the classical theory of Hopf bifurcation that $\omega$ and $V_0$ are even functions of $\varepsilon$. We have assumed this in writing (7.4) and (7.5). Moreover,

$$\left.\begin{array}{l}\mathbf{u}(x,y,s,\varepsilon)\\ \mathbf{v}(x,y,s,\varepsilon)\\ p(x,y,s,\varepsilon)\\ q(x,y,s,\varepsilon)\end{array}\right\}=\sum_{l=0}^{\infty}\frac{\varepsilon^{l+1}}{(l+1)!}\quad\begin{array}{l}\mathbf{u}^{[l]}(x_0,y_0,s)\\ \mathbf{v}^{[l]}(x_0,y_0,s)\\ p^{[l]}(x_0,y_0,s)\\ q^{[l]}(x,y,s).\end{array} \tag{7.6}$$

The functions of $x$ and $y$ are defined in deformed domains with a wavy interface $h(x,s,\varepsilon)-h_0=\delta(x,s,\varepsilon)$. The functions of $x_0$ and $y_0$ were defined in the reference domain with a flat interface at $y_0=h$. The perturbation of the interface $\delta(x,s,\varepsilon)$ can be eliminated from (5.8); that is,

$$\delta(x,s,\varepsilon)=k(u_1-v_1)(x,s,\varepsilon)$$

is an identity for all $x$, $s$, $\varepsilon$. The square brackets on the left of (7.6) indicate differentiation following the mapping evaluated at $\varepsilon=0$. For example,

$$\begin{aligned}\mathbf{u}^{[n]}(x_0,y_0,s)&=\left.\frac{d^n}{d\varepsilon^n}\right|_{\varepsilon=0}u(x,y,s,\varepsilon)\\ &=\frac{\partial^n}{\partial\varepsilon^n}u(x_0,y(x_0,y_0,s,\varepsilon),s,\varepsilon).\end{aligned} \tag{7.7}$$

There is a simple differential calculus for domain perturbations. The partial derivatives, holding $x_0,y_0$ fixed, at $\varepsilon=0$ are also important. For these

$$\mathbf{u}^{\langle n\rangle}(x_0,y_0,s,\varepsilon)=\left[\frac{\partial^n}{\partial\varepsilon^n}u(x_0,y_0,s,\varepsilon)\right]_{\varepsilon=0}. \tag{7.8}$$

The two types of derivatives are related by the chain rule

$$\begin{aligned}&\mathbf{u}^{[1]}=\mathbf{u}^{\langle 1\rangle}+y^{\langle 1\rangle}\frac{\partial\mathbf{u}^{\langle 0\rangle}}{\partial y},\\ &\mathbf{u}^{[2]}=\mathbf{u}^{[2]}+2y^{\langle 1\rangle}\frac{\partial\mathbf{u}^{\langle 1\rangle}}{\partial y}+\left(y^{\langle 1\rangle}\right)^2\frac{\partial^2\mathbf{u}^{\langle 0\rangle}}{\partial y^2}+y^{\langle 2\rangle}\frac{\partial\mathbf{u}^{\langle 0\rangle}}{\partial y},\\ &\mathbf{u}^{[n]}(x_0,y_0,s)=\mathbf{u}^{\langle n\rangle}(x_0,y_0,s)+\text{lower order terms},\end{aligned} \tag{7.9}$$

where

$$y^{\langle n\rangle}(x_0,y_0,s)=\begin{cases}\delta^{\langle n\rangle}(x_0,s)\dfrac{y_0-1}{h_0-1}, & h_0\leqq y_0\leqq 1,\\[2mm] \delta^{\langle n\rangle}(x_0,s)\dfrac{y_0}{h_0}, & 0\leqq y_0\leqq h_0.\end{cases} \tag{7.10}$$

On the free surface $y_0=h_0$ of the reference domain, we have

$$y^{\langle n\rangle}(x_0,h_0,s)=\delta^{\langle n\rangle}(x_0,s). \tag{7.11}$$

The equations governing the coefficients in (7.6) are very complicated because the differential operators with derivatives with respect to $x$ and $y$ in the field equations must be reexpressed by derivatives with respect to $x_0$, $y_0$ under the change of variables $x$, $y\to x_0$, $y_0$. Since this mapping is invertible we can continue (7.6) as

$$\left.\begin{matrix}\mathbf{u}(x,y,s,\varepsilon)\\ \mathbf{v}(x,y,s,\varepsilon)\\ \text{etc.}\end{matrix}\right\}=\sum_{l=0}^{\infty}\frac{\varepsilon^{l+1}}{(l+1)!}\quad\begin{matrix}\mathbf{u}^{[l]}(x,y_0(x,y,s,\varepsilon),s)\\ \mathbf{v}^{[l]}(x,y_0(x,y,s,\varepsilon),s)\\ \text{etc.}\end{matrix} \tag{7.12}$$

Fortunately it is never necessary to solve the complicated equations which govern the derivatives $[l]$ on the right of (7.6). In fact we need only to do much simpler calculations for the partial derivatives $\langle l\rangle$. When the partial derivatives $\langle l\rangle$ are known the total derivatives $[l]$ may be computed by the chain rule (7.9). The point of simplicity of partial derivatives is that they do not perturb the operators which are defined on region $\Omega_1$ and $\Omega_2$, below and above the free surfaces, see [18] and [19]. For example,

$$\operatorname{div}\mathbf{u}^{\langle n\rangle}(x_0,y_0,s)=\frac{\partial u_1^{\langle n\rangle}}{\partial x_0}+\frac{\partial u_2^{\langle n\rangle}}{\partial y_0}=0, \tag{7.13}$$

whereas

$$\operatorname{div}\mathbf{u}^{[n]}(x_0,y_0,s)\neq 0.$$

The same type of simplification holds for the perturbation equations which arise from (7.1), (5.3) and (5.4). For example,

$$\begin{aligned}\rho\Bigg[\omega_0\frac{\partial\mathbf{u}^{\langle 2\rangle}}{\partial s}+\hat{u}_1\frac{\partial\mathbf{u}^{\langle 2\rangle}}{\partial x}+\mathbf{e}_x u_2^{\langle 2\rangle}\hat{u}_1'+\hat{u}_1^{\langle 2\rangle}\frac{\partial\mathbf{u}^{\langle 0\rangle}}{\partial x}&\\ +\mathbf{e}_x u_2^{\langle 0\rangle}\hat{u}_1'^{\langle 2\rangle}+\mathbf{u}^{\langle 1\rangle}\cdot\nabla\mathbf{u}^{\langle 1\rangle}+\omega_2\frac{\partial\mathbf{u}^{\langle 0\rangle}}{\partial s}\Bigg]&=\eta_1\Delta\mathbf{u}^{\langle 2\rangle}-\nabla p^{\langle 2\rangle}.\end{aligned} \tag{7.14}$$

The unknowns here are $\mathbf{u}^{\langle 2\rangle}$, $p^{\langle 2\rangle}$, $\omega_2$ and $\hat{V}_2$.

It is not possible to avoid the total derivatives $[l]$ in (5.6)–(5.10) because these are defined on a manifold, the interface, and not in a region. The interface conditions are of the form

$$g(x,y=h(x,s,\varepsilon),s,\varepsilon)=0 \tag{7.15}$$

and the perturbation of $y=h$ with $\varepsilon$ cannot be avoided. In fact the interface conditions are identities on the interface so that tangential derivatives on them vanish (see [19]).

The following is a recipe for perturbations of the domain in bifurcation problems. First, we introduce the frequency $\omega(\varepsilon)$ into (5.3), (5.4) and (5.10) using (7.1). We then insert the series (7.4), (7.5) and the series

$$\left.\begin{matrix}\mathbf{u}(x_0,y_0,s,\varepsilon)\\ \mathbf{v}(x_0,y_0,s,\varepsilon)\\ p(x_0,y_0,s,\varepsilon)\\ q(x_0,y_0,s,\varepsilon)\\ \delta(x_0,s,\varepsilon)\end{matrix}\right\}=\sum_{l=0}\frac{\varepsilon^{l+1}}{(l+1)!}\quad\begin{matrix}\mathbf{u}^{\langle l\rangle}(x_0,y_0,s),\\ \mathbf{v}^{\langle l\rangle}(x_0,y_0,s),\\ p^{\langle l\rangle}(x_0,y_0,s),\\ q^{\langle l\rangle}(x_0,y_0,s),\\ \delta^{\langle l\rangle}(x_0,s)\end{matrix} \tag{7.16}$$

into $\operatorname{div}\mathbf{u}=\operatorname{div}\mathbf{v}=0$, (5.3), (5.4) and (5.5) and identify the perturbation equations. These equations hold in the reference domain. To get the equations which govern the interface conditions (5.6)–(5.10) we insert the series (7.6) and identify. Then we express the derivatives $[l]$ with partial derivatives $\langle l\rangle$, using the chain rule. The perturbation equations arising from interface conditions are thus defined on the flat interface $y=h_0$.

The series on the right of (7.16) may be expressed on the deformed domain by inverting the mapping, as in (7.12). In fact, the series on the right of (7.16) is equal to the series on the right of (7.6), though the partial sums of these two series are not equal (see equations of [18]).

**8. Solvability of the perturbation equations.** We must solve perturbation problems of the following form:

(i) All functions of $s$ are $2\pi$ periodic in $s$.

(ii) All functions of $x=x_0$ are $L$ periodic in $x_0$.

(iii) In the region $0\leqq y_0\leqq h_0$,

$$\frac{\partial u_1^{\langle n\rangle}}{\partial x_0}+\frac{\partial u_2^{\langle n\rangle}}{\partial y_0}=0, \tag{8.1}$$

$$\rho\left[\omega_0\frac{\partial u_1^{\langle n\rangle}}{\partial s}+\hat{u}\frac{\partial \mathbf{u}^{\langle n\rangle}}{\partial x_0}+\mathbf{e}_x u_2^{\langle n\rangle}\hat{u}'\right]-\eta_1\Delta\mathbf{u}^{\langle n\rangle}+\nabla p^{\langle n\rangle}=\boldsymbol{\theta}_1(\omega_n,\hat{V}_n,x_0,s). \tag{8.2}$$

(iv) In the region $h_0\leqq y_0\leqq 1$ we have the same equations with $\mathbf{v}^{\langle n\rangle}(x_0,y_0,s)$, $\hat{v}(y_0)$, $q^{\langle n\rangle}$ and $\boldsymbol{\theta}_2$ replacing $\mathbf{u}^{\langle n\rangle}$, $\hat{u}$, $p^{\langle n\rangle}$ and $\boldsymbol{\theta}_1$.

(v) $u^{\langle n\rangle}=0$ at $y_0=0$, $v^{\langle n\rangle}=0$ at $y_0=1$.

(vi) On the interface $y=h_0+\delta$, we have by (5.8)

$$\delta=k[[u_1]],\ [[u_1]]\overset{\text{def}}{=}u_1-v_1. \tag{8.3}$$

We have eliminated $\delta$ from the interface equations (5.6), (5.7) and (5.10) with (8.3). Of course

$$\delta^{\langle n\rangle}=k\left[\left[u_1^{\langle n\rangle}\right]\right]\quad\text{on } y_0=h_0.$$

(vii) The four interface conditions on $y_0=h_0$ are of the form

$$u_2^{\langle n\rangle}-v_2^{\langle n\rangle}=\theta_3(\hat{V}_n,x_0,s),$$
$$\eta_1 D_{12}[\mathbf{u}^{\langle n\rangle}]-\eta_2 D_{12}[\mathbf{v}^{\langle n\rangle}]=\theta_4(\hat{V}_n,x_0,s),$$
$$-p^{\langle n\rangle}+2\eta_1 D_{22}[\mathbf{u}^{\langle n\rangle}]+q^{\langle n\rangle}-2\eta_2 D_{22}[\mathbf{v}^{\langle n\rangle}]-kT\frac{\partial^2}{\partial x_0^2}\left[\left[u_1^{\langle n\rangle}\right]\right]=\theta_5(\hat{V}_n,x_0,s),$$
$$k\left(\omega_0\frac{\partial}{\partial s}+\hat{u}_1(k_0)\frac{\partial}{\partial x_0}\right)\left[\left[u_1^{\langle n\rangle}\right]\right]-u_2^{\langle n\rangle}=\theta_6(\omega_n,\hat{V}_n,x_0,s).$$

The inhomogeneous terms $\theta$ are linear in the unknown parameters $\omega_n$ and $\hat{V}_n$ and are otherwise known from computations at orders $l<n$.

These inhomogeneous problems can be solved uniquely in the space orthogonal to the null space of the adjoint operator introduced at the begining of §6. This null space is two-dimensional and is spanned by

$$\mathbf{z}^*\quad\text{and}\quad\bar{\mathbf{z}}^*$$

defined by (7.3) and explained in Iooss and Joseph [17, §VIII.3]. There are therefore two solvability conditions to be used in the determination of the parameters $\omega_n$ and $\hat{V}_n$.

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