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CHANGE OF TYPE AND LOSS OF EVOLUTION IN THE FLOW OF VISCOELASTIC FLUIDS

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Summary

In this paper we discuss concepts associated with viscosity, elasticity, hyperbolicity, Hadamard instability and ill posedness of Cauchy problems in the flow of viscoelastic fluids. We frame the analysis in terms of vorticity and develop relations between change of type in steady flow and the ill posedness of the unsteady problem. We also consider the problem of regularizing Hadamard instabilities by the addition of Newtonian contributions to the constitutive equations.

1. Introduction

In this paper we discuss important concepts in the theory of dynamics of viscoelastic fluids. We shall confine our attention to fluids with instantaneous elasticity. Such fluids have zero retardation times (zero Newtonian viscosity) and they support various kinds of hyperbolic dynamics as well as catastrophic loss of stability to short waves. These phenomena are related to the type of a system of partial differential equations.

(1) The unsteady quasilinear problem is called evolutionary if, roughly speaking, the Cauchy problem for it is well posed (this is a notion strictly weaker than hyperbolicity). The loss of evolution is an instability of the Hadamard type in which short waves will sharply increase in amplitude. For

many models, those treated here, the problem of evolution may be conveniently framed in terms of vorticity.

(2) The steady quasilinear system may be analyzed for type. It is neither elliptic nor hyperbolic. On the other hand, the vorticity is either hyperbolic or elliptic, and it may change type, hyperbolic in some regions of flow and elliptic in others, as in transonic flow. We shall show that the full unsteady quasilinear system will undergo a loss of stability in the sense of Hadamard when the steady vorticity equation for inertialess flow is hyperbolic.

We consider a number of examples. Some models are always evolutionary and do not change type in unsteady flow. The vorticity equation for steady flow of such models can and does change type. Other models can become non-evolutionary and therefore undergo Hadamard instability. Some flows of these models are evolutionary, e.g.; shearing flows, while others are not evolutionary. In either case, the steady problem can undergo a change of type. Loss of evolution is impossible in flow perturbing uniform motion. To lose evolution it is necessary that certain stresses should exceed critical values. In this sense the loss of evolution can be identified with the problem of failure of numerical simulations at high Weissenberg numbers.

2. Loss of evolution

The loss of evolution is a concept associated with the well posedness of the Cauchy problem. Let us consider a quasilinear system of the form

$$\boldsymbol{A}\frac{\partial \boldsymbol{u}}{\partial t} + \sum_{j=1}^{n} \boldsymbol{B}_j \frac{\partial \boldsymbol{u}}{\partial x_j} + \boldsymbol{b} = 0, \tag{2.1}$$

where $\boldsymbol{A}$, $\boldsymbol{B},\ldots,$ $\boldsymbol{B}_n$ are $m \times m$ matrix valued functions and $\boldsymbol{b}$ is an m vector depending on $\boldsymbol{u}$, $\boldsymbol{x}$, t. The system (2.1) is evolutionary in some domain D of $\mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}$ in the t direction if for every fixed $\boldsymbol{u}$, $\boldsymbol{x}$, t in D and any unit n-vector $\boldsymbol{\nu}$, the eigenvalue problem

$$\left[-\lambda \boldsymbol{A} + \sum_{j=1}^{n} \nu_j \boldsymbol{B}_j\right] \boldsymbol{v} = 0 \tag{2.2}$$

has only real eigenvalues. The system (2.1) is hyperbolic in the t direction if (2.2) has m real eigenvalues, not necessarily distinct, and a set of m linearly independent eigenvectors.

To justify these definitions, let us consider the simple case where $\boldsymbol{b} = 0$ and the matrices $\boldsymbol{A}$, $\boldsymbol{B},\ldots,$ $\boldsymbol{B}_m$ are independent of $\boldsymbol{u}$, $\boldsymbol{x}$, t. Let $\boldsymbol{\nu}$ be a unit vector in $\mathbb{R}^n$. If (2.1) is evolutionary, then any plane wave solution of (2.1) propagating in the $\boldsymbol{\nu}$-direction $\boldsymbol{v}(\boldsymbol{x}, t) = \boldsymbol{v}\exp(\mathrm{i}k(\boldsymbol{\nu}\cdot\boldsymbol{x} - \lambda t))$, k real, has necessarily λ real. This prevents the so called Hadamard instability, i.e., the fact that at any time t the amplitude of $\boldsymbol{u}$ could become arbitrarily large, even if u is bounded (but highly oscillatory) at time $t = 0$. In this context, hyperbolicity means that in every direction in space, m independent plane waves can propagate.

The quasilinear systems for the velocity, stresses and pressure of fluids with instanteneous elasticity are not hyperbolic in the usual sense. For these systems it is proper to think of the loss of evolution. On the other hand, the unsteady equation for the vorticity is hyperbolic when it is evolutionary and is evolutionary when hyperbolic.

The use of plane waves to study the well posedness of the Cauchy problem is justified, in general, for waves so short that $\boldsymbol{A}$, $\boldsymbol{B}_j$ and $\boldsymbol{b}$ have constant components in a small period cell defined by the wave.

Hadamard instabilities are much stronger than those studied in bifurcation theory. One cannot expect a secondary flow whose dynamics would be governed by the evolution of one or several modes. Rather, if the initial value problem becomes ill-posed, there are flow fields which would not occur even as transient states, since "random" disturbances containing all modes would blow up instantly. The importance of loss of evolution in the flow of viscoelastic fluids was first recognized by Rutkevitch [1,2,31]. Many other results were given by Joseph et al. [3]. Independent results, following in part out of discussions leading to [3] were given by Ahrens et al. [4], and Renardy [5,6]. Dupret and Marchal [7,8] have also given some new results on the problem of loss of evolution in three dimensional problems. It has been suggested that models of viscoelastic fluids which lose evolution are no good and should be discarded; this idea is wrong, the loss of evolution can actually warn us about certain physical instabilities like melt fracture. On the other hand, it is certain that the loss of evolution will produce a disaster in numerical simulations.

Some systems of equations never lose evolution. This is true of the dynamical system generated by Newtonian fluids and by upper and lower convected Maxwell models. These systems are not always closer to physics than systems which can lose evolution.

The loss of evolution is associated with the initial value problem. It can occur for some problems and not for others. This is the case for incompressible inviscid fluids governed by Euler's equations in two space dimensions. It is known that when the initial data are smooth this equation has a unique smooth solution defined for all times. However, certain well-known problems associated with Euler's equations can become non-evolutionary. H. Aref [9] has commented on this:

> Historically, this issue seems first to have arisen in the context of the Kelvin–Helmholtz problem. When the "common" vortex sheet rolls up, it apparently becomes singular after a finite time. One can gauge this feature already from the vantage point of linearized stability analysis: In the

> Kelvin–Helmholtz problem the amplification rate of a wave of wavelength λ is inversely proportional to λ. Thus, short waves amplify faster than longer waves at all wavelengths, and so, except for some delay in actually exciting the short waves, the outcome is almost inevitably headed for a singularity of some kind. The ill-posedness of the problem consists in the loss of a certain degree of analyticity after a finite time. (This is sometimes called ill-posedness in the sense of Hadamard.)
>
> This feature in itself is of interest, at least mathematically. One may ask, for example, what kind of "weak solution(s)" to the problem are available after the singularity time? The physical significance of such solutions (should they arise in some reasonable systematic way) is not at all clear at present....
>
> From a more pragmatic point of view one must regard the emergence of a singularity as physically unacceptable, a feature that shows an inadequacy in the description of the problem. Clearly, the basic equations must be augmented in some way that will remove the singularity. The notion that perturbations of arbitrarily short wavelength grow arbitrarily fast cannot be aphysically meaningful statement within the framework of a hydrodynamic theory. And, sure enough, any physically motivated mechanism that provides a cut-off at small length scales will also lend a regularizing aspect to the solution. Sometimes, indeed, the problem can (apparently) be completely regularized so that smooth solutions exist for all time given smooth initial data. Regularizing mechanisms of this kind include diffusion, interfacial tension and the introduction of a small but finite thickness of the interfacial transition region itself.

The famous fingering instability of Saffman and Taylor [10] is an instability to short waves of the Kelvin–Helmholtz type. When the less viscous fluid in the saturated porous materials fingers into the more viscous liquid, the short waves are the most unstable; the system based on Darcy's law is not evolutionary. We do not advocate throwing out Darcy's law because of this.

It is known that some popular hydrodynamic models of flowing composites, used in the study of fluidized beds and for other applications arising in mixture theories, lead to ill-posed initial value problems [11,12].

An interesting discussion of the loss of evolution in the equation of magnetohydrodynamics with the Hall effect taken into account is given by Brushlinskii and Morozov [13]. The equations of magnetohydrodynamics for a non-dissipative plasma plane flow will not be evolutionary if the Hall effect is taken into account. The loss of evolution seems to be associated with certain physical instabilities.

On the other hand, Kulikovskii and Regirer [14] have shown that electrodynamic equations which change type in steady flow can lose evolution and go unstable. Such solutions therefore cannot be realized. In the words of Kulikovskii and Regirer

> Owing to the rapid increase of perturbations, nonevolutionary equations cannot describe correctly changes of any physical quantity in time. Nonevolutionary solutions of the nonlinear equations in many cases can be regarded as an oversimplification in the derivation of these equations by discarding terms which are small for evolutionary solutions, but they can be essential for the perturbations which display a rapid increase. As the short wave disturbances increase most rapidly, then these could be the terms containing space derivatives of higher order or mixed derivatives with respect to space of time.

A similar point of view was adopted by Rutkevitch [2] in his discussion of loss of evolution in viscoelastic fluids. He says

> In order to describe the development of small perturbations in the region where evolutionarity of the initial conditions is not possible, the effect of supplementary physical parameters should be taken

into account. In a real system, these parameters can be extremely small, but they play a definite role in establishing a finite upper limit for the rate of buildup of perturbations.

One such parameter, used now extensively in numerical simulations of non-Newtonian flows, is to add a Newtonian viscosity. In Section 8 we shall address this question from the point of view of rheometrical science.

The paper by Ahrens et al. [4] reports a study of the stability of viscometric flow using the type of short memory introduced by Akbay et al. [15]. The instability found by Akbay et al. can be identified as a loss of evolution leading to the catastrophic short wave instability of Hadamard type whenever

$$\frac{[(N_1(\kappa)/\kappa)']^2\kappa^3}{\tau(\kappa)\tau'(\kappa)} > 16, \tag{2.3}$$

where κ is the shear rate, $\tau(\kappa)$ is the shear stress and $N_1(\kappa)$ is the first normal stress. Catastrophic instabilities to short waves of this type may be characteristic for some of the types of instability called "melt fracture". Ahrens et al. [4] addressed the question of justification for the short memory assumption and find that it cannot be justified for some of the more popular rheological models. The left-hand side of (2.3) reduces to the square of the recoverable shear (N_1^2/τ^2) when the variation of N_1/κ^2 and τ/κ is small. W. Gleissle [16] found that flow instabilities (melt fracture) commenced in 14 very different type polymer melts and solutions when the recoverable shear varied 4.36–5.24 with a mean 4.63. This seems to be in rather astonishing agreement with the criterion (2.3}.

Hadamard instabilities may be endemic in the theory of flow of viscoelastic fluids with instantaneous elasticity.

3. Loss of evolution and change of type for models with hyperbolic vorticity

We would consider all the models which give rise to a vorticity equation which can be hyperbolic in the steady case. These were identified by Joseph, Renardy and Saut [3] as JRS models. They include all constitutive models whose principal parts are of Oldroyd type.

$$\lambda\frac{\mathscr{D}\tau}{\mathscr{D}t} = 2\eta \boldsymbol{D}[\boldsymbol{u}] + \boldsymbol{l}, \tag{3.1}$$

where $\boldsymbol{l}$ is of lower order (see JRS) and where λ is the relaxation time, τ is the extra stress, λ is the elastic viscosity, $\boldsymbol{D}[\boldsymbol{u}]$ is the symmetric part of the velocity gradient, and

$$\frac{\mathscr{D}\tau}{\mathscr{D}t} = \frac{\partial\tau}{\partial t} + (\boldsymbol{u}\nabla)\tau + \tau\Omega - \Omega\tau - a(\boldsymbol{D}\tau + \tau\boldsymbol{D}), \tag{3.2}$$

where $\boldsymbol{\Omega} = \frac{1}{2}(\nabla \boldsymbol{u} - \nabla \boldsymbol{u}^{\mathrm{T}})$ is the skew symmetric part of $\nabla \boldsymbol{u}$ of $a \in [-1, 1]$.

The lower order terms $\boldsymbol{l}$ in (3.1) may depend on $\boldsymbol{u}$ and $\boldsymbol{\tau}$, but not on their derivatives. The upper convected, corotational and lower convected Maxwell models arise when $\boldsymbol{l} = -\boldsymbol{\tau}$ and $a = [1, 0, -1]$. Different models can be obtained by different theories of the lower order terms $\boldsymbol{l}$. One version of the model by Phan-Thien and Tanner [17] may be expressed by (3.1) with $\boldsymbol{l} = -(1 + c \operatorname{tr} \boldsymbol{\tau})\boldsymbol{\tau}$ where c is a constant. The Johnson–Segalman model [18] with an exponential kernel is a special case of Phan-Thien and Tanner with $c = 0$ and in fact is one of the Maxwell type of Oldroyd models. A simple Giesekus model [19] is given by (3.1) with $a = 1$, $\boldsymbol{l} = -(\boldsymbol{\tau} + (\alpha/\mu)\boldsymbol{\tau}^2)$ where $0 \leqslant \alpha \leqslant 1$ is a constant related to the relative mobility tensor.

We now turn to the analysis of evolution for systems governed by (3.1). The method used is a kind of linearized stability analysis for short waves of the type already given in Section 2.

We shall study the problem of evolution of the ten field variables $[\boldsymbol{u}, \boldsymbol{\tau}, p]$ satisfying

$$\begin{aligned} &\lambda \frac{\mathscr{D}\boldsymbol{\tau}}{\mathscr{D}t} = \eta(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^{\mathrm{T}}) + \boldsymbol{l}, \\ &\rho\left[\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u}\right] + \nabla p - \operatorname{div} \boldsymbol{\tau} = 0, \\ &\operatorname{div} \boldsymbol{u} = 0, \end{aligned} \tag{3.3}$$

where $\mathscr{D}\boldsymbol{\tau}/\mathscr{D}t$ is given by (3.2). We decompose the motion

$$[\boldsymbol{u}, \boldsymbol{\tau}, p] = [\hat{\boldsymbol{u}}, \hat{\boldsymbol{\tau}}, \hat{p}] + [\boldsymbol{u}', \boldsymbol{\tau}', p'],$$

where the roof functions satisfy (3.3) and the prime functions are small. We take the liberty of calling the roof flow basic; but in fact it is an arbitrary solution of (3.3). After linearizing, we find that

$$\begin{aligned} &\lambda\left[\frac{\partial \boldsymbol{\tau}'}{\partial t} + (\hat{\boldsymbol{u}} \cdot \nabla)\boldsymbol{\tau}' + \hat{\boldsymbol{\tau}}\boldsymbol{\Omega}' - \boldsymbol{\Omega}'\hat{\boldsymbol{\tau}} - a(\boldsymbol{D}'\hat{\boldsymbol{\tau}} + \hat{\boldsymbol{\tau}}\boldsymbol{D}')\right] - 2\eta \boldsymbol{D}' + \boldsymbol{l}', \\ &\rho\left[\frac{\partial \boldsymbol{u}'}{\partial t} + (\hat{\boldsymbol{u}} \cdot \nabla)\boldsymbol{u}'\right] + \nabla p' - \operatorname{div} \boldsymbol{\tau}' - \rho(\boldsymbol{u}' \cdot \nabla)\hat{\boldsymbol{u}}, \\ &\operatorname{div} \boldsymbol{u}' = 0, \end{aligned} \tag{3.4}$$

where l' does not involve derivatives of $\boldsymbol{u}'$, $\boldsymbol{\tau}'$.

We next fix our attention at a point $\boldsymbol{x}$, of the field and define $\chi = \boldsymbol{x} - \boldsymbol{x}_0$. Then we imagine that the basic flow and the derivatives of it in (3.3) are constant and equal to their value at $\boldsymbol{x}_0$. We may then represent the cartesian components of the disturbance as

$$[\boldsymbol{u}', \boldsymbol{\tau}', p'] = [\omega_i, \sigma_{ij}, q] \exp \mathrm{i}(k_l \chi_l - \omega t),$$

where the ten amplitudes $[\omega_i, \sigma_{ij}, q]$ depend on the basic flow at $\boldsymbol{x}_0$. The amplitudes are governed by

$$c\sigma_{ij} - \tfrac{1}{2}\hat{\tau}_{il}\big[(1-a)\omega_l n_j - (1+a)\omega_j n_l\big] - \tfrac{1}{2}\big[\omega_i n_l(1+a) - \omega_l n_i(1-a)\big]\hat{\tau}_{lj}$$
$$+\mu(n_j\omega_i + n_i\omega_j) = \mathrm{O}(1/k),$$
$$-\rho c\omega_i = -qn_i + \sigma_{ij}n_j, \tag{3.5}$$
$$\omega_i n_i = 0,$$

where $\boldsymbol{n} = \boldsymbol{k}/k$, $k = (k_1^2 + k_2^2 + k_3^2)^{1/2}$ and

$$c = \omega/k - \hat{\mathbf{u}} \cdot \mathbf{n}. \tag{3.6}$$

These equations were first derived by Rutkevitch [2] for the three values $a = [1, 0, -1]$. When $k = \infty$, the problem (3.5) can be regarded as an eigenvalue problem for ω/k or c (see (2.2)).

We can find the ten amplitude $[\omega_i, \sigma_{ij}, q]$ if and only if the determinant Δ of the coefficients vanishes, where

$$\Delta = \big[-\rho c^2 + \mu - \tfrac{1}{2}\tau_{22}(1-a) + \tfrac{1}{2}\tau_{11}(1+a)\big]$$
$$\times\big[-\rho c^2 + \mu + \tfrac{1}{2}\tau_{11}(1+a) - \tfrac{1}{2}\tau_{33}(1-a)\big]. \tag{3.7}$$

The derivation of (3.7) follows along lines set down by Rutkevitch; special coordinates are selected such that $n_1 = 1$, $n_2 = n_3 = 0$, $\hat{\tau}_{23} = 0$ and $\hat{\tau}_{22} > \hat{\tau}_{33}$.

The nontrivial values c^2 are then given by

$$c_+^2 = \frac{1}{\rho}\big[\mu + \tfrac{1}{2}\hat{\tau}_{11}(1+a) - \tfrac{1}{2}\hat{\tau}_{33}(1-a)\big],$$
$$c_-^2 = \frac{1}{\rho}\big[\mu + \tfrac{1}{2}\hat{\tau}_{11}(1+a) - \tfrac{1}{2}\hat{\tau}_{22}(1-a)\big]. \tag{3.8}$$

Departing slightly now from Rutkevitch, using the result proved in Section 4, we call c the velocity of propagation of wave fronts of short waves of vorticity.

In any case, one has $c^2 = f$, where f is real valued. If $f > 0$, we get propagation. If $f < 0$, then

$$c = \pm \mathrm{i}\sqrt{f} = \pm \mathrm{i}\,\mathrm{Im}\left[\frac{\omega}{k}\right]. \tag{3.9}$$

Equation (3.9) shows that if $f < 0$, then there exist short waves of rapidly growing amplitude, the flow undergoes a Hadamard instability.

If we now suppose that at $\boldsymbol{x}_0$ the system is in principal coordinates of $\hat{\boldsymbol{\tau}}$, the eigenvalues of $\hat{\boldsymbol{\tau}}$ satisfying:

$$\hat{\tau}_1 \geqslant \hat{\tau}_2 \geqslant \hat{\tau}_3, \tag{3.10}$$

then $f > 0$, (and the system is of evolution type, stable to short waves) if and only if

$$\mu + \tfrac{1}{2}a(\hat{\tau}_1 + \hat{\tau}_3) - \left[\tfrac{1}{2}(\hat{\tau}_1 - \hat{\tau}_3)\right] > 0. \tag{3.11}$$

Among the Maxwell models ($\boldsymbol{l} = -\boldsymbol{\tau}$) only the upper ($a = 1$) and lower ($a = -1$) convected models are always evolutionary. This follows from the integral form of these two models. The integrals are expressed by positive definite tensors restricting the range of $\boldsymbol{\tau}$ to evolutionary regions (see [3]). Dupret and Marchal [7] have shown that if the criterion for evolution is satisfied initially it will not fail subsequently. We will show in Section 7 that Maxwell fluids $\boldsymbol{l} = -\boldsymbol{\tau}$ and with $a \neq \pm 1$ can lose evolution in certain flows. The models Phan-Thien and Tanner [14], Johnson and Segalman [18], Leonov [20] and Giesekus [19] may also lose evolution in certain flows.

4. Evolution of the vorticity

The vorticity equation for (3.3) in 3-D flows may be written as (see (6.4) in Joseph [30]):

$$\rho\left[\frac{\partial^2 \zeta_k}{\partial t^2} + 2(\boldsymbol{u}\cdot\nabla)\frac{\partial \zeta_k}{\partial t} + u_e u_j \frac{\partial^2 \zeta_k}{\partial x_e \partial x_j}\right] + \tfrac{1}{2}(a-1)e_{kej}\tau_{jq}\frac{\partial[\operatorname{curl}\zeta]_q}{\partial x_e}$$
$$-\tfrac{1}{2}(a+1)\tau_{mp}\frac{\partial^2 \zeta_k}{\partial x_m \partial x_p} - \frac{\eta}{\lambda}\nabla^2\zeta_k + l_k = 0. \tag{4.1}$$

where e_{kej} is the alternating tensor and l_k are all terms of order lower than two derivatives of the vorticity $\zeta = \operatorname{curl} \boldsymbol{u}$.

The analysis for stability to short waves which was given in Section 3 may be applied to (4.1). We find exactly the same formula $\Delta = 0$ given by (3.7).

It follows that *the quasilinear system (3.3) is of evolution type if and only if the vorticity equation (4.1) is of evolution type.*

5. First-order quasilinear systems for plane flow

In plane flow, we have six equations in six unknowns.

$$\sigma_t + u\sigma_x + v\sigma_y + \tau(v_x - u_y) - a\left[2\sigma u_x + \tau(u_y + v_x)\right] - 2\mu u_x = l_1,$$
$$\tau_t + u\tau_x + v\tau_y + \tfrac{1}{2}(\sigma - \gamma)(u_y - v_x) - \tfrac{1}{2}a(\sigma + \gamma)(u_y + v_x) - \mu(u_y + v_x) = l_2,$$
$$\gamma_t + u\gamma_x + v\gamma_y + \tau(u_y - v_x) - a\left[2\gamma v_y + \tau(u_y + v_x)\right] - 2\mu v_y = l_3,$$
$$\rho(u_t + uu_x + vu_y) + p_x - \sigma_x - \tau_y = 0,$$
$$\rho(v_t + uv_x + vv_y) + p_y - \tau_x - \gamma_y = 0,$$
$$u_x + v_y = 0 \tag{5.1}$$

where

$$\boldsymbol{\tau} = \begin{bmatrix} \sigma & \tau \\ \tau & \gamma \end{bmatrix},$$

$\boldsymbol{u} = (u, v)$, where l_1, l_2, l_3 depend on $\boldsymbol{\tau}$ and possibly on $\boldsymbol{u}$, but not on their derivatives.

The analysis of evolution follows exactly along the lines laid out in Section 3 and Section 6. Since there are only two normal stresses, we replace (3.10) with $\hat{\tau}_1 \geqslant \hat{\tau}_2$ and (3.11) becomes

$$\mu + \tfrac{1}{2}a(\hat{\tau}_1 + \hat{\tau}_2) - \tfrac{1}{2}(\hat{\tau}_1 - \hat{\tau}_2) > 0. \tag{5.2}$$

There is only one speed of propagation (3.8) in the plane. The condition (5.2) for evolution is necessary and sufficient for the stability of solutions of (5.1) to short waves. The same condition, but written relative to general coordinates:

$$\begin{aligned} &\tau^2 - \left[\mu - \tfrac{1}{2}\gamma(1-a) + \tfrac{1}{2}\sigma(1+a)\right]\left[\mu - \tfrac{1}{2}\sigma(1-a) + \tfrac{1}{2}\gamma(1+a)\right] < 0, \\ &\tfrac{1}{2}\gamma(1-a) - \tfrac{1}{2}\sigma(1+a) - \mu < 0, \end{aligned} \tag{5.3}$$

was derived by Joseph et al. [3] for the vorticity equation associated with (5.1) (see also Section 6):

$$\begin{aligned} \lambda\Bigg[&\rho\frac{\partial^2\zeta}{\partial t^2} + 2\rho(\boldsymbol{u}\cdot\nabla)\frac{\partial\zeta}{\partial t} + \left[\rho u^2 - \mu - \tfrac{1}{2}\sigma(1+a) + \tfrac{1}{2}\gamma(1-a)\right]\frac{\partial^2\zeta}{\partial x^2} \\ &+ 2(\rho uv - \tau)\frac{\partial^2\zeta}{\partial x\partial y} + \left[\rho v^2 - \mu + \tfrac{1}{2}\sigma(1-a) - \tfrac{1}{2}\gamma(a+1)\right]\frac{\partial^2\zeta}{\partial y^2}\Bigg] \\ &= \bar{l}_1 \text{ (of lower order)}. \end{aligned} \tag{5.4}$$

Condition (5.2) (or (5.3)) implies that the unsteady equation for vorticity is hyperbolic.

The loss of evolution of system (5.1) should not be confused with the possible change of type of the steady problem. We shall examine the connection with these two phenomena now. The difference between the steady and unsteady problem is most easily explained in terms of the vorticity. The analysis of steady problems for the quasilinear system (5.1) may be written as

$$\boldsymbol{H}\boldsymbol{q}_x + \boldsymbol{J}\boldsymbol{q}_y + \boldsymbol{l} = 0.$$

where $\boldsymbol{q}$ is a column vector with components $[u, v, \sigma, \gamma, \tau, p]$ and $\boldsymbol{H}$, $\boldsymbol{J}$, $\boldsymbol{l}$ depend on $\boldsymbol{q}$ but not on its derivatives. To analyze the type of this system we look for characteristics $\theta(x, y) = \text{const.}$, $\theta_x \mathrm{d}x + \theta_y \mathrm{d}y = 0$.

The analysis is straightforward. The characteristics are given by $\mathrm{d}y/\mathrm{d}x = \alpha$, where α is a solution of

$$\det[-\alpha\boldsymbol{H} + \boldsymbol{J}] = 0. \tag{5.5}$$

This leads us to (11.2) of [3]:

$$(1+\alpha^2)(-\alpha u+v)^2\left\{\rho(-\alpha u+v)^2+\tfrac{1}{2}(\gamma-\sigma)(\alpha^2-1)\right.$$
$$\left.+2\tau\alpha-(\alpha^2+1)(\mu+\tfrac{1}{2}a(\gamma+\sigma))\right\}=0. \qquad (5.6)$$

There are imaginary roots $\alpha=\pm 1$, double real roots along streamlines $\alpha=v/u$ and two roots for the last factor:

$$\alpha=B/A\pm(B^2-AC)^{1/2}/A, \qquad (5.7)$$

where

$$A=\mu-\rho u^2+\tfrac{1}{2}\sigma(1+a)-\tfrac{1}{2}\gamma(1-a),$$
$$B=\tau-\rho uv, \qquad (5.8)$$
$$C=\mu-\rho v^2+\tfrac{1}{2}\sigma(a-1)+\tfrac{1}{2}\gamma(a+1).$$

Wherever

$$B^2-AC=-\mu^2+\rho[\mu+a\sigma+a\gamma](u^2+v^2)+\tfrac{1}{2}\rho(\gamma-\sigma)(u^2-v^2)$$
$$+\tau^2+\tfrac{1}{4}\sigma^2(1-a^2)+\tfrac{1}{4}\gamma^2(1-a^2)-\mu a(\sigma+\gamma)-2\rho\tau uv>0.$$

we have two more real characteristics.

Our system (5.5) is therefore of a mixed type: it has imaginary characteristics and therefore is not hyperbolic; it has real characteristics and therefore is not elliptic. This is not an unusual situation in fluid mechanics (the steady Euler equations of incompressible inviscid fluids are of mixed type) but gives rise to mathematical difficulties: the study of such systems is not well developed (see [22]). What is new is the fact that the characteristics associated with the last factor in (5.6) can be real or complex in the flow. Joseph et al. [3] showed that the roots (5.7) are in fact associated with the steady vorticity equation. This equation is either hyperbolic ($B^2-AC>0$) or elliptic ($B^2-AC<0$). The roots can be elliptic in one region of flow and hyperbolic in other regions, as in the case of transonic flow. The other characteristics have a simple interpretation: the imaginary roots $\alpha=\pm 1$ are associated with the equation $\Delta\psi=-\zeta$, where ζ is the vorticity and ψ the stream function.

Many well-known methods used in hyperbolic problems can be used in our problem. For example, in the region in which $B^2-AC>0$ we may find ordinary differential equations along the two characteristic curves associated with (5.7). (For hyperbolic systems, we "reduce the system to the characteristic form"). Let $\boldsymbol{Y}_\alpha$ be a left eigenvector associated to one of the two real eigenvalues (5.7) in the "hyperbolic" case $B^2-AC>0$. Then

$$\boldsymbol{Y}_\alpha(-\alpha\boldsymbol{H}+\boldsymbol{J})=0 \qquad (5.9)$$

and

$$Y_\alpha(Hq_x + Jq_y) + Y_\alpha l = m(q_\alpha + \alpha q_y) + Y_\alpha l = 0 \tag{5.10}$$

where $m = Y_\alpha H$. Let us write the characteristic associated to α in the form $y = f(x)$. Then

$$m(x)\frac{\mathrm{d}q}{\mathrm{d}x}(x, f(x)) + Y_\alpha l(x) = 0 \tag{5.10}$$

is an ordinary differential equation on the characteristic. The left eigenvector Y_α can be easily computed.

One finds that

$$H = \begin{bmatrix} -2(a\sigma+\mu) & \tau(1-a) & u & 0 & 0 & 0 \\ 0 & (1-a)\frac{1}{2}\gamma - (1+a)\frac{1}{2}\sigma - \mu & 0 & 0 & u & 0 \\ 0 & -(1+a)\tau & 0 & u & 0 & 0 \\ \rho u & 0 & -1 & 0 & 0 & 1 \\ 0 & \rho u & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$J = \begin{bmatrix} -\tau(1+a) & 0 & v & 0 & 0 & 0 \\ \frac{1}{2}\sigma(1-a) - \frac{1}{2}\gamma(1+a) - \mu & 0 & 0 & 0 & v & 0 \\ \tau(1-a) & -2(\mu+a\gamma) & 0 & v & 0 & 1 \\ \rho v & 0 & 0 & 0 & -1 & 0 \\ 0 & \rho v & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$-\alpha H + J =$

$$\begin{bmatrix} 2\alpha(a\sigma+\mu) - \tau(1+a) & -\tau\alpha(1-a) & -\alpha u + v & 0 & 0 & 0 \\ \frac{1}{2}\sigma(1-a) - \frac{1}{2}\gamma(1+a) - \mu & -\alpha\left[\frac{1}{2}\gamma(1-a) - \frac{1}{2}\sigma(a+1) - \mu\right] & 0 & 0 & -\alpha u + v & 0 \\ \tau(1-a) & \alpha\tau(1+a) - 2(\mu+a\gamma) & 0 & -\alpha u + v & 0 & 1 \\ \rho(-\alpha u + v) & 0 & \alpha & 0 & -1 & -\alpha \\ 0 & \rho(-\alpha u + v) & 0 & -1 & \alpha & 1 \\ -\alpha & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We deduce that $Y_\alpha = (y_1, y_2, y_3, y_4, y_5, y_6)$ where

$$y_1 = \frac{-\alpha}{-\alpha u + v}, \quad y_2 = \frac{1-\alpha^2}{-\alpha u + v}, \quad y_3 = -y_1, \quad y_4 = 1, \quad y_5 = \alpha,$$

$$y_6 = \frac{1}{-\alpha u + v}\left[2\alpha(a\gamma - \tau\alpha) + \alpha\mu(1+\alpha^2) + \alpha(1-\alpha^2)\left[\frac{\gamma-\alpha}{2} - a\frac{\gamma+\sigma}{2}\right]\right]$$

where α is one of the two real roots of

$$\rho(-\alpha u+v)^2+\left[\tfrac{1}{2}(\gamma-\sigma)\right](\alpha^2-1)+2\tau\alpha-(\alpha^2+1)\left[\mu+\tfrac{1}{2}a(\gamma+\sigma)\right]=0.$$

6. Vorticity in plane flow

In plane flow, there is one nonzero component of vorticity satisfying

$$\rho\frac{\partial^2\zeta}{\partial t^2}+2\rho(\boldsymbol{u}\cdot\nabla)\frac{\partial\zeta}{\partial t}-A\frac{\partial^2\zeta}{\partial x^2}-2B\frac{\partial^2\zeta}{\partial x\partial y}-C\frac{\partial^2\zeta}{\partial y^2}+l=0, \tag{6.1}$$

where l is of lower order and A, B, C are defined by (5.4), i.e.,

$$\begin{aligned} A &= -\rho u^2+\mu+\tfrac{1}{2}\sigma(1+a)-\tfrac{1}{2}\gamma(1-a),\\ C &= -\rho v^2+\mu-\tfrac{1}{2}\sigma(1-a)+\tfrac{1}{2}\gamma(a+1),\\ B &= \tau-\rho uv. \end{aligned} \tag{6.2}$$

The analysis of evolution is most easily framed relative to (6.1). Let us start with a general definition.

A linear partial differential operator of second order

$$L\zeta=P\left(\boldsymbol{x},\ t,\ \zeta_{tt},\ \tau_{tx_1},\dots,\zeta_{tx_n},\ \zeta_{x_1x_1},\ \zeta_{x_1x_2},\dots,\zeta_{x_nx_n}\right)+\text{lower order terms}$$

is evolutionary with respect to t is some domain D of $\mathbb{R}^n$ if for every unit vector $\boldsymbol{k}=(k_1,\dots,k_n)$ in $\mathbb{R}^n$, for any $t\in\mathbb{R}$ and any $\boldsymbol{x}\in D$, the quadratic polynominal in α

$$P\left(\boldsymbol{x},\ t,\ -\alpha^2,\ -\alpha k_1,\dots,-\alpha k_n,\ -k_1^2,\ -k_1k_2,\dots,-k_n^2\right)=0$$

has only *real zeros*. In the case of constant coefficients, this definition implies that there are no plane wave solutions with arbitrarily large amplitude, i.e., there are no Hadamard instabilities.

The polynominal $P=0$ evaluated for (6.1) becomes

$$\rho\alpha^2+2\alpha\lambda(uk_1+vk_2)-Ak_1^2-2Bk_1k_2-Ck_2^2=0.$$

This must have real zeroes for every unit vector $\boldsymbol{k}=(k_1,\ k_2)$. This leads to

$$(A+\rho u^2)k_1^2+2(B+\rho uv)k_1k_2+(C+\rho v^2)k_2^2>0,\quad \forall\, k_1,k_2.$$

which implies

$$A+\rho u^2>0 \quad\text{and}\quad (B+\rho uv)^2-(C+\rho v^2)(A+\rho u^2)<0.$$

Using (6.2), this is equivalent to (5.3).

The same relationship arises from analysis of stability to short waves. We fix $\boldsymbol{u}$, $\boldsymbol{\tau}$ (hence A, B, C) and their values at x_0 and put $l=0$ and introduce $\chi=x-x_0$, writing

$$\zeta(x,\ y,\ t)=\hat{\zeta}(x_0,\ y_0)\exp \mathrm{i}\left[k_1(x-x_0)+k_2(y-y_0)-\omega t\right]. \tag{6.3}$$

We find that, with $k^2 = k_1^2 + k_2^2$

$$c^2 = \left[\mu + (\sigma+\gamma)\tfrac{1}{2}a\right]k^2 + \tfrac{1}{2}(\sigma-\gamma)\left(k_1^2 - k_2^2\right) + 2\tau k_1 k_2, \tag{6.4}$$

where

$$c^2\text{-}\omega^2 + 2\omega(uk_1 + vk_2) + u^2k_1^2 + v^2k_2^2 + 2uvk_1k_2.$$

For evolution it is necessary that $c^2 > 0$ for all $k_1,\ k_2 \in \mathbb{R}$ when $k = (k_1^2 + k_1^2)^{1/2} + \infty$. If $c^2 < 0$, then

$$\operatorname{Im}(\omega/k) = \pm\text{positive constant}$$

and we have a nasty instability to short waves. The criterion $c^2 > 0$ is exactly (5.3). When expressed in principal coordinates $\sigma \geq \gamma$ and $\tau = 0$, we find that the vorticity is of evolution type provided that

$$\mu + \tfrac{1}{2}\frac{a(\sigma+\gamma)}{2} + \tfrac{1}{2}(\gamma - \sigma) > 0. \tag{6.5}$$

We may relate the criterion for a change of type in steady flow to the criterion for loss of evolution. Consider $\Delta_{\text{def}} = B^2 - AC$ when $\rho = 0$. Therefore,

$$\begin{aligned} \Delta &= -f_1 f_2 + \tau^2, \\ f_1 &= \mu + a\tfrac{1}{2}(\gamma+\sigma) - \tfrac{1}{2}(\sigma-\gamma), \\ f_2 &= \mu + a\tfrac{1}{2}(\gamma+\sigma) - \tfrac{1}{2}(\sigma-\gamma). \end{aligned} \tag{6.6}$$

It we suppose that the system (5.1) (or equivalently the unsteady vorticity equation (6.1)) is of evolution type, then we have (5.3) which clearly implies $B^2 - AC < 0$ for $\rho = 0$, i.e., the steady vorticity equation with $\rho = 0$ is elliptic. Thus hyperbolicity of the steady vorticity equation with $\rho = 0$ implies that the full quasilinear system (5.1) is not evolutionary.

The converse is not true. The equation for vorticity with $\rho = 0$ in the steady case is elliptic when

$$\tau^2 - \left[\mu + \tfrac{1}{2}(\sigma-\gamma) + \tfrac{1}{2}a(\sigma+\gamma)\right]\left[\mu + \tfrac{1}{2}(\gamma-\sigma) + \tfrac{1}{2}a(\sigma+\gamma)\right] < 0.$$

This inequality with $a \neq 0$ does not imply the condition (5.3) for evolution when viewed in principal coordinates $\sigma \geq \gamma$, $\tau = 0$; for example, (5.3) is violated when $a < 0$; $\tau = 0$; $\gamma \gg 1$; $\sigma \gg 1$; $0 \geqslant \sigma - \gamma \gg 1$. To understand this, consider the equation

$$\frac{\partial^2\phi}{\partial t^2} - A\frac{\partial^2\phi}{\partial x^2} - C\frac{\partial^2\phi}{\partial y^2} = 0. \tag{6.7}$$

The steady equation is elliptic when $AC > 0$. But the unsteady equation is evolutionary (hyperbolic) with respect to t, if and only if $AC > 0$ and $A > 0$. If $AC > 0$ and $A < 0$, (6.7) is an elliptic equation!

We recall that the quasilinear system (5.1) is evolutionary if and only if (6.1) is evolutionary. We can study loss of evolution by using results from the study of change of type in steady inertialess flow. It is perhaps useful to remark that *we must have a loss of evolution, instability to short waves, whenever the vorticity of an inertialess steady flow becomes hyperbolic. Conversely, if the vorticity of an inertialess steady flow is elliptic and* $A > 0$, *where* $A = \mu + \frac{1}{2}(\sigma - \gamma) + \frac{1}{2}a(\sigma + \gamma)$, *then the system (5.1) is of evolution type.*

All of the models considered here, except the upper and lower convected Maxwell models may change type in an inertialess steady flow.

7. Examples taken from linear theory

In fact, the theory of evolution is based on equations linearized on an arbitrary flow, called basic. We may evaluate the criteria for evolution and change of type on the basic flow. Many examples of this procedure were given in [3] and by Yoo et al. [23] for the study of change of type in steady flow.

It is of interest to examine the relationship of change of type in steady flow to the study of short wave instability in unsteady flow. Section 11 of [3] gives analysis for change of type in motions for an upper convected Maxwell model perturbing *shear flow, extensional flow, sink flow and circular Couette flow. All these problems are elliptic when* $\rho = 0$ *and all undergo a change of type for* $\rho \neq 0$.

A similar type of analysis, using an upper convected Maxwell model, was given by Yoo et al. [23] of the three-dimensional sink flow and by Yoo and Joseph [24] for Poiseuille flow in a channel with wavy walls. These flows also change type when $\rho \neq 0$ and are always evolutionary.

It is of interest to study these problems in cases in which it is possible to lose evolution. We shall examine the examples treated in [3] for Oldroyd models ($a \neq \pm 1$, $\boldsymbol{l} = -\boldsymbol{\tau}$ in (3.1)) and some new examples. The corotational Maxwell model ($a = 0$) seems to lose evolution at the lowest levels of stress (the smallest Weissenberg numbers).

7.1 Simpler shear flow

(a) Oldroyd models ([3, p. 244]).

For simple shear flows of Oldroyd models we find that $u = \kappa y$, $v = 0$, $\tau = \eta\kappa/D$, $D = 1 + \kappa^2\lambda^2(1 - a^2)$, $\sigma = \tau\lambda\kappa(a + 1)$, $\gamma = \tau\lambda\kappa(a - 1)$. The steady vorticity equation for the linear perturbation is hyperbolic in a strip outside the origin defined by

$$\rho\kappa^2 y^2 > \frac{\eta}{\lambda}\frac{1 + \lambda^2\kappa^2}{1 + \lambda^2\kappa^2(1 + a^2)}. \tag{7.1}$$

When $\rho = 0$ we cannot satisfy this inequality. The steady vorticity equation with $\rho = 0$ is always elliptic. Moreover $A = \mu + (\sigma - \gamma)/2 + a(\sigma + \gamma)/2 = \mu + \lambda\kappa(1 + a^2) > 0$. From Section 6, the linear systems perturbing shear flows are evolutionary.

(b) A Giesekus model ($a = 1$, $\boldsymbol{l} = -(\tau + (\alpha/\mu)\tau^2)$).
The system (5.1) has the form:

$$\begin{aligned}
&\sigma_t + u\sigma_x + v\sigma_y - 2\sigma u_x - 2\tau u_y - 2\mu u_x = -\frac{\alpha}{\eta}(\sigma^2 + \tau^2) - \frac{\sigma}{\lambda},\\
&\tau_t + u\tau_x + v\tau_y - \gamma u_y - \sigma v_x - \mu(u_y + v_x) = -\frac{\alpha}{\eta}(\sigma\tau + \gamma\tau) - \frac{\tau}{\lambda},\\
&\gamma_t + u\gamma_x + v\gamma_y - 2\tau v_x - 2\gamma v_y - 2\mu v_y = -\frac{\alpha}{\eta}(\tau^2 + \gamma^2) - \frac{\gamma}{\lambda},\\
&\rho(u_t + uu_x + vu_y) + p_x - \sigma_x - \tau_y = 0,\\
&\rho(v_t - uv_x + vv_y) + p_y - \tau_x - \gamma_y = 0,\\
&u_x + v_y = 0.
\end{aligned} \tag{7.2}$$

In simple shear flow, we find $u = \kappa y$, $v = 0$, $\tau = \eta\kappa/(1 + \kappa^2\lambda^2)$, $\sigma = \lambda\kappa\tau$, $\gamma = -\sigma$ when $\alpha = 1$.

The steady vorticity equation for the linearized flow is hyperbolic in a strip outside the origin defined by

$$\rho\kappa^2 y^2 > \mu.$$

This inequality cannot be satisfied for $\rho = 0$ and the steady equation for vorticity is always elliptic in inertialess flows. Moreover, $A = \mu + \sigma = \mu + \lambda\eta\kappa^2/(1 + \kappa^2\lambda^2) > 0$, and the linear system perturbing shear flow is of evolution type.

7.2 Poiseuille flow of an Oldroyd Model

In this example, $p_x = -K$, $K > 0$ is a prescribed constant and

$$\begin{aligned}
&\tau_y = -K,\\
&\tau(1 + a)u_y = \sigma/\lambda,\\
&\tau(1 - a)u_y = -\gamma/\lambda,\\
&\left[\tfrac{1}{2}(\sigma - \gamma) - \tfrac{1}{2}a(\sigma + \gamma) - \frac{\eta}{\lambda}\right]u_y = -\frac{\tau}{\lambda}.
\end{aligned} \tag{7.3}$$

Putting $u_y = \kappa = \kappa(y)$, we find that

$$\tau = \frac{\eta\kappa}{\lambda^2(1-a^2)\kappa^2+1},$$
$$\sigma = \frac{\lambda\eta\kappa^2(1+a)}{\lambda^2(1-a^2)\kappa^2+1}, \tag{7.4}$$
$$\gamma = \frac{\lambda\eta\kappa^2(a-1)}{\lambda^2(1-a^2)\kappa^2+1}.$$

Flows perturbing Poiseuille flow of an Oldroyd fluid will become unstable to short waves and lose evolutionarity wherever the steady vorticity equation with $\rho = 0$ is hyperbolic. This condition may be expressed using (5.8) as

$$B^2 - AC = -\frac{\eta^2}{\lambda^2} + \tau^2 + \tfrac{1}{4}(\sigma^2+\gamma^2)(1-a^2) - \frac{\eta}{\lambda}a(\sigma+\gamma) > 0. \tag{7.5}$$

In the present case, this condition reads

$$\tau^2\left[1+\tfrac{1}{2}\lambda^2\kappa^2(1-a^4)\right] - 2a^2\eta\kappa\tau - \frac{\eta^2}{\lambda^2} > 0, \tag{7.6}$$

where τ is given by (7.4). After a short computation, the left-hand side of (7.6) is evaluated as a positive coefficient times $-\tfrac{1}{2}\lambda^2\kappa^4(1-a^4) - \lambda^2\kappa^2 - 1$, hence the inequality (7.6) cannot be satisfied for any $a \in (-1, 1)$.

The flows perturbing Poiseuille flow of an Oldroyd model are always evolutionary, it is only necessary to verify that

$$A = \left[\frac{\eta}{\lambda}\right]\left[\frac{1+2\lambda^2\kappa^2}{1+\lambda^2(1-a^2)\kappa^2}\right] > 0.$$

7.3 Extensional flow

(a) Oldroyd models [3, p. 244].

We find that

$$[u, v, \tau, \sigma, \gamma] = \left[sx, -sy, 0, \frac{2\eta s}{p}, -\frac{2\eta s}{q}\right]$$

where $p = 1 - 2a\lambda s$, $q = 1 + 2a\lambda s$. We take $s \geqslant 0$ and small enough so that p and q are bounded from above, $s^2 < 1/4a^2\lambda^2$.

The steady vorticity of motions perturbing extensional flow is hyperbolic when

$$\rho s^2x^2\left[\frac{\eta}{\lambda} + \frac{2\eta s(2a^2\lambda s - 1)}{pq}\right] + \rho s^2y^2\left[\frac{\eta}{\lambda} + \frac{2\eta s(1+2a^2\lambda s)}{pq}\right]$$
$$> \left[\frac{\eta}{\lambda} + \frac{2\eta s(1+2a^2\lambda s)}{pq}\right]\left[\frac{\eta}{\lambda} + \frac{2\eta s(2a^2\lambda s - 1)}{pq}\right]. \tag{7.7}$$

If the second factor on the right is positive, the region outside an ellipse is hyperbolic. We put $\rho = 0$ for inertialess flow. Then we get hyperbolicity for the vorticity with $\rho = 0$ when

$$0 > \left[\frac{\eta}{\lambda} + \frac{2\eta s(1 + 2a^2\lambda s)}{(pq)^2}\right]\left[\frac{\eta}{\lambda} - 2\eta s\right]/pq.$$

Therefore we lose evolution when

$$pq + 2\lambda s(2a^2\lambda s - 1) < 0.$$

That is, when $s > 1/2\lambda$. This value of s is in the allowed range $s^2 < 1/4a^2\lambda^2$ provided $a \in (-1, 1)$. The linear systems governing flows which perturb extensional flow are unstable to short waves whenever $s > 1/2\lambda$. Steady flows with inertia change type in the manner specified by (5.4) and are evolutionary when $s < 1/2\lambda$. (It is easily verified that A with $\rho = 0$ is equal to $\mu + 2\eta s/pq + 4a^2\eta\lambda s^2/pq > 0$.)

(b) A model of Phan-Thien and Tanner

The analysis given under (a) above applies here also because tr $\tau = 0$ for extensional flow. This model is nonevolutionary when $s > 1/2\lambda$.

(c) A model of Giesekus

The system (7.2) is satisfied by the following extensional flow: $u = sx$, $v = -sy$, $\tau = 0$ where σ and γ are given by

$$\frac{\alpha}{\eta}\sigma^2 + \sigma\left(\frac{1}{\lambda} - 2s\right) - 2s\mu = 0,$$

$$\frac{\alpha}{\eta}\gamma^2 + \gamma\left(\frac{1}{\lambda} + 2s\right) + 2s\mu = 0.$$

It follows that the stresses in extensional flow are given by

$$\sigma_{\pm} = \frac{\eta}{\alpha}\left[\frac{2s\lambda - 1}{\lambda} \pm \left[\left[\frac{1 - 2s\lambda}{\lambda}\right]^2 + \frac{8s\alpha}{\lambda}\right]^{1/2}\right],$$

$$\gamma_{\pm} = \frac{\eta}{\alpha}\left[-\frac{2s\lambda + 1}{\lambda} \pm \left[\left[\frac{1 + 2s\lambda}{\lambda}\right]^2 - \frac{8s\alpha}{\lambda}\right]^{1/2}\right].$$

Since $0 \leqslant \alpha \leqslant 1$, $[(1 + 2s\lambda)/\lambda]^2 - 8s\alpha/\lambda$ is positive and the stresses double valued.

Let us note now, following Giesekus [19, p. 79], that the configuration tensor $\boldsymbol{C} = \boldsymbol{1} + \tau/\mu$ is positive definite. Then $1 + \sigma/\omega > 0$ and $1 + \gamma/\mu > 0$. These inequalities cannot be satisfied for the negative roots of (7.9) when $s < 1/2\lambda$. To show this we set

$$\alpha\left[1 + \frac{\sigma_-}{\mu}\right] = \alpha + 2s\lambda - 1 - \left[(1 - 2s\lambda)^2 + 8s\alpha\lambda\right]^{1/2} \underset{\text{def}}{=} f(\alpha).$$

One has $f(0) = 2(2s\lambda - 1) < 0$ and $f(1) = -1$. On the other hand

$$f'(\alpha) = 1 - \frac{4s\lambda}{\left[(1-2s\lambda)^2 + 8s\alpha\lambda\right]^{1/2}}$$

and

$$f'(0) = \frac{1-6s\lambda}{1-2s\lambda}, \qquad f'(1) = \frac{1-2s\lambda}{1+2s\lambda} > 0.$$

Since $f'(\alpha) = 0$ has one and only one zero, we see that $f(\alpha) < 0$ for $\alpha \in [0, 1]$.

It is easy to verify that $\alpha[1 + \sigma_+/\mu] > 0$ for $\alpha \in [0, 1]$ and σ_+ is admissible under the restriction on the eigenvalues of the configuration tensor.

Similarly the reader can easily verify that $\alpha[1 + \gamma_-/\mu] < 0$ for $\alpha \in [0, 1]$, and that $\alpha[1 + \gamma_+/\mu] > 0$ for $\alpha \in [0, 1]$ provided $s < 1/4\lambda$. It follows that under the assumed restrictions $\sigma = \sigma_+$ and $\gamma = \gamma_+$ for extensional flow.

The steady vorticity of motions perturbing extensional flow is hyperbolic when

$$\rho s^2x^2(\gamma_+ + \mu) + \rho s^2y^2(\sigma_+ + \mu) - (\sigma_+ + \mu)(\gamma_+ + \mu) > 0. \tag{7.10}$$

Since $\mu + \gamma_+$, $\mu + \sigma_+$ are positive, (7.10) describes the exterior of an ellipse.

When $\rho = 0$, (7.10) reduces to

$$(\sigma_+ + \mu)(\gamma_+ + \mu) < 0,$$

which is impossible for $\alpha \in [0, 1]$ and $0 \leqq s < 1/4\lambda$: the linear system perturbing extensional flow is always evolutionary in the Giesekus model whenever the configuration tensor is positive. It is perhaps necessary to note that unlike the upper and lower convected Maxwell models, the Giesekus model does not restrict the range of stresses to an evolutionary domain; an extra condition, perhaps inconvenient for numerical analysis, has to be imposed.

7.4 Sink flow in three dimensions [23]

Maxwell models allow an irrotational solution for flow into a sink:

$$u = -Q/r^2,\ v = 0,$$

where u is the radial component of velocity, $Q > 0$ is the sink strength, v is the other component of velocity. The stresses

$$\begin{bmatrix} \sigma & \tau \\ \tau & \sigma \end{bmatrix}$$

in polar spherical coordinates are given by $\tau = 0$.

$$\frac{\sigma}{\mu} = 4\exp(r^3/3\lambda Q)r^{-4a}\int_r^\infty s^{4a-1}\exp(-s^3/3\lambda Q)\,\mathrm{d}s,$$

$$\frac{\gamma}{\mu} = -2\exp(r^2/3\lambda Q)r^{2a}\int_r^\infty s^{-2a-1}\exp(-s^3/3\lambda Q)\,\mathrm{d}s.$$

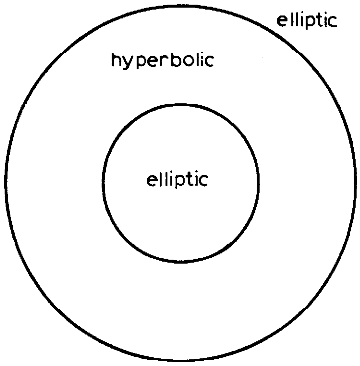

Fig. 1. Region of ellipticity and hyperbolicity for the vorticity of flow perturbing sink flow of a corotational Maxwell model.

The vorticity of all steady axisymmetric flows perturbing this solution is elliptic at large r where the stresses are weak and the steady flow changes from elliptic to hyperbolic as the radius is decreased past certain critical values. The flows for $a = \pm 1$ are always evolutionary. For $a = 0$, Yoo et al. [23] found that the vorticity of steady flows perturbing sink flow becomes elliptic again near the origin as in Fig. 1.

In the elliptic region near the origin we have (see (5.7))

$$0 > B^2 - AC = -\mu^2 + \rho\left[\mu + \tfrac{1}{2}(\gamma - \sigma)\right]u^2 + \tfrac{1}{4}(\sigma^2 + \gamma^2).$$

Moreover,

$$\gamma - \sigma = 6\mu \ln r < 0 \qquad \text{as } r \to 0.$$

Hence, as $r \to 0$,

$$0 > \tfrac{1}{2}\rho u^2(\gamma - \sigma) + \tfrac{1}{4}(\sigma^2 + \gamma^2)$$

and $\gamma - \sigma$ is a large negative number and the criterion (6.4)

$$\mu + \tfrac{1}{2}(\gamma - \sigma) < 0$$

for the loss of evolution is satisfied in the elliptic region near the origin. This region is obviously hyperbolic for the steady vorticity with $\rho = 0$. Smooth, stable (i.e., evolutionary) sink flows of corotational Maxwell fluids are impossible.

8. How to compute a Newtonian viscosity from stress relaxation or sinusoidal oscillations even when it is zero

We have seen that fluids with instantaneous elasticity may undergo Hadamard instabilities to short waves at high levels of stress (high Weissenberg numbers). We already noted in Section 2 that these short wave instabilities may be avoided by introducing various regularizing terms. One effective method for regularization which is also natural for viscoelastic fluids is to add a viscosity term to the constitutive equation (for an example, see [25]). Many popular models of fluids have a Newtonian viscosity. The models of Jeffreys, Oldroyd, Rouse and Zimm and molecular models of solutions with Newtonian solvents lead to Newtonian contributions to the stress. The make this method useful it is necessary that the viscosity used should be appropriate to the fluid under study.

To understand this problem, and the method we propose for approximating the solution of the problem, it is convenient to consider problems linearized around a state of rest. These problems may be considered to be governed by a generalization of Boltzmann's equation for the excess stress

$$\tau = 2\mu \mathbf{D}[\boldsymbol{u}] + 2\int_0^\infty G(s)\mathbf{D}[\boldsymbol{u}(\boldsymbol{x}, t-s)]\, \mathrm{d}s, \tag{8.1}$$

where μ is the Newtonian viscosity, $\mathbf{D}[\boldsymbol{u}]$ is the symmetric part of the velocity gradient $\nabla \boldsymbol{u}$ and $G(s)$ is a smooth relaxation function, $G(s) > 0$, $G'(s) < 0$ for $0 \leqslant s = t - \tau < \infty$ and τ is the past time. Equation (8.1) gives rise to Jeffreys' model when

$$G(s) = (\eta/\lambda)\exp(-s/\lambda),$$

where λ is the relaxation time and η is the elastic viscosity, defined below.

Suppose that we are in the case of steady shearing with one component of velocity $u(x)$ depending on one variable x. The shear stress $\tau(x) = \tau_{12}$ of τ depends then on the rate of shear $\kappa(x) = 2D_{12}$ of $\boldsymbol{D}$ and (8.1) reduces to $\tau = (\mu + \eta)\kappa$ where

$$\bar{\mu} = \mu + \eta \tag{8.2}$$

is the zero shear viscosity and

$$\eta = \int_0^\infty G(s)\, \mathrm{d}s \tag{8.3}$$

is the elastic viscosity. Newtonian fluids have $\eta = 0$, $\bar{\mu} = \mu$. Elastic fluids have $\mu = 0$, $\bar{\mu} = \eta$. In general

$$\bar{\mu} \geqslant \eta, \tag{8.4}$$

with equality for elastic fluids.

To decide about elasticity and viscosity we could consider even more dilute solutions of polymer chains of large molecules in solvents which might be thought to be Newtonian. What happens when we reduce the amount of polymer? There are two good ideas which are in contradiction. The first idea says that there is always a viscosity and some elasticity with an even greater viscous contribution as the amount of polymer is reduced. On the other hand, we may suppose that liquid is elastic so that $\mu = 0$ and the viscosity η is the area under the graph of the relaxation function. Since η is finite in all liquids, we have $\eta = G(0)\bar{\gamma}$, where $\bar{\gamma}$ is a mean relaxation time. Maxwell's idea is that the limit of extreme dilution is such that the rigidity $G(0)$ tends to infinity and $\bar{\gamma}$ to zero in such a way that their product η is finite. Ultimately, when the polymer is gone, we are left with an elastic liquid with an enormously high rigidity. This idea apparently requires anomalous behavior because $G(0)$ appears to decrease with polymer concentration when the concentration is finite.

The contradiction between the two foregoing ideas and the apparent anomaly can be resolved by replacing the notion of a single mean relaxation time with a distribution of relaxation times. This notion is well grounded in structural theories of liquids in which different times of relaxation correspond to different modes of molecular relaxation. It is convenient again to think of polymers in a solvent, but now we can imagine that the solvent is elastic, but with an enormously high rigidity. In fact many of the so-called Newtonian solvents have a rigidity of the order 10^9 Pascals, which is characteristic of glass, independent of variations of the chemical characteristics among the different liquids (for example, see [26]). To find this glassy modulus it is necessary to use very ingenious high frequency devices operating in the range 10^9 Hertz and to supercool the liquids to temperatures near the glassy state. In these circumstances the liquid acts like a glassy solid, the molecular configurations cannot follow the rapid oscillations of stress; the liquid cannot flow. For slower processes it is possible for the liquid to flow and if the relaxation is sufficiently fast the liquid will appear to be Newtonian in more normal flows. For practical purposes there is no difference between Newtonian liquids and liquids with rigidities of order 10^9 and mean relaxation times of 10^{-10} s or so. In fact it is convenient to regard such liquids as Newtonian, even though $\mu = 0$ and $\bar{\mu} = \eta$.

The presence of polymer would not allow the liquid to enter the region of viscous relaxation at such early times. Instead much slower relaxation processes associated with the polymers would be induced. The second epoch of relaxation occurs in a neighbourhood of very early time $t = t_1$. An effective modulus $G(t_1)$ may be defined at $t = t_1$. The effective modulus is well defined when $G(t)$ is not rapidly varying in small neighborhoods of $t = t_1$.

The relaxation function may be measured on standard cone and plate rheometers, using, for example, stress relaxation after a sudden strain. Examples of such stress relaxation, taken on a Rheometric System 4 rheometer is shown in Figs. 2 and 3. The rise time of this instrument is roughly 0.01 s and the more rapid part of the stress relaxation cannot be obtained with

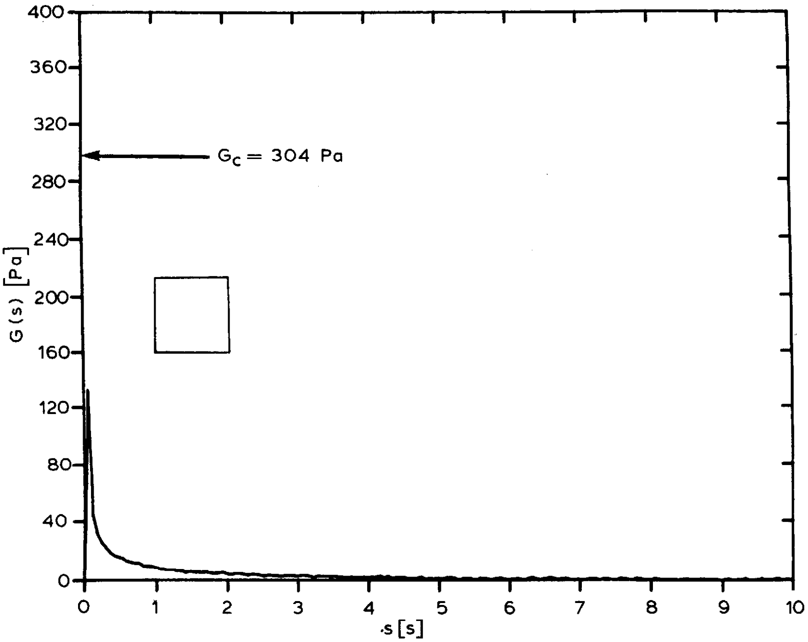

Fig. 2. (Joseph et al. [27]) Shear modulus $G_c = 304$ Pa and relaxation function $G(s)$ for an observed wave speed $c = 50.6$ cm/s in a 1% carboxymethyl cellulose (CMC) solution in 49% water and 50% glycerin. E.H. Lieb [28] photographed shear waves in the same solution using tracers. He estimated $c > 8$ cm/s.

such devices. The modulus G_c was measuied by Joseph et al. [27] using a wave speed meter. They measure transit times of impulsively generated shear waves into a viscoelastic liquid at rest. A Couette apparatus is used; the outer cylinder is moved impulsively; the time of transit of the shear wave from the outer to inner cylinder is measured. They set up criteria to distinguish between shear waves and diffusion. One criterion is that transit times δt should be reproducible without large standard deviations and such that $d = c\delta t$, where d is gap size, and c, the wave speed is a constant independent of d. In other words, transit speeds are independent of gap size. Then, using theoretical results for propagation of shear wave into rest $c = (G_c/\rho)^{1/2}$. We could regard G_c as the effective modulus or the effective rigidity.

It is clear from Figs. 2 and 3 that the rapidly relaxing part of the shear relaxation function, even ignoring the possibility of enormously fast relaxations in times of order 10^{-10} s in the glycerin and water solutions associated with glassy states of the two solutions, is missed out on the data of the Rheometrics four. We may also note that the tail end of $G(t)$ is also not

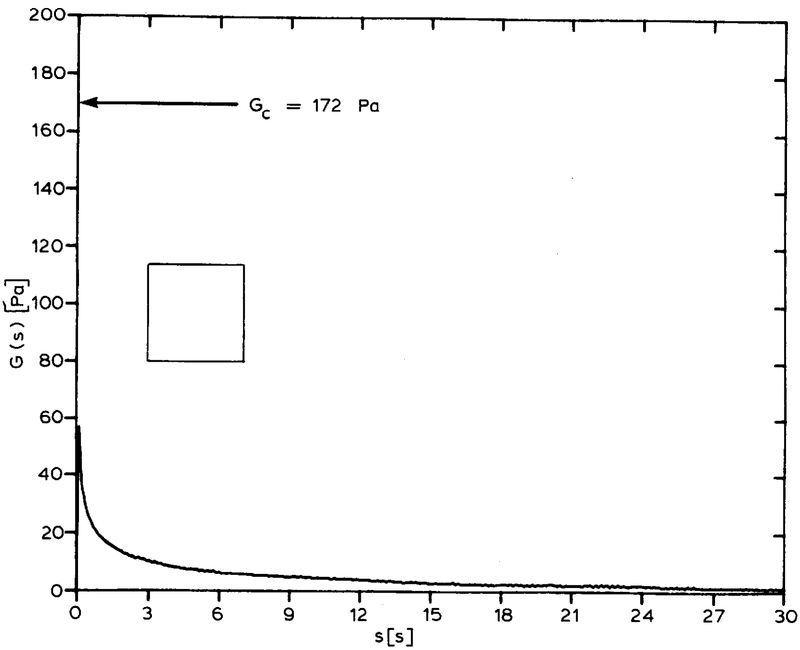

Fig. 3. (Joseph et al. [27]). Shear modulus $G_c = 172$ Pa and relaxation $G(s)$ for an observed wave speed $c = 38.4$ in a 1.5%. Poly(acrylamide) Separan AP30 in 50% glycerin and 48.5% water. The zero shear viscosity, $\bar{\mu} = 160$ Pa-s is the area of the square. Bird et al. [29] exhibit frequency data ($\omega \leq 1000$ rad/s) of Huppler et al. [32] for this solution. They say that the storage modulus $G'(\omega)$ is nearly at its limiting with $G'(\omega) = 140$ Pa at frequencies of $\omega = 100$.

accessible on standard rheometers because the transducers do not work when the levels of stress are too low.

Similar limitations of capacity are characteristic for the gap loading devices used for sinusoidal oscillations in standard cone and plate rheometers. The high frequency devices which are used to determine glassy responses of low molecular weight liquids do not work well for polymeric liquids.

To compute a good value for the Newtonian viscosity even when its zero we need to find a way to put the part of the viscosity which associated with rapid relaxation into a Newtonian viscosity. For this it suffices to have, say, $G(t)$ for $0.01 < t < \hat{t}$. We can get this from any standard rheometer with a stress relaxation capacity. Or, we could use the complex viscosity, computed by standard rheometers, which have upper limits of 500 rad/s. In addition it is necessary to measure the zero shear or static viscosity $\bar{\mu}$, given in Figs. 2 and 3 as the area of the box.

The computation procedure, starting from stress relaxation is as follows: (1) We fit a shear relaxation spectrum to the given relaxation function.

From the measured values of $G(t)$, $t_0 \leqslant t \leqslant t_1$, we get a theoretical function $\hat{G}(t)$, $0 \leqslant t \leqslant \infty$. We should make the curve fitting in an honest way such that

$$\hat{\eta} = \int_0^{\infty} \hat{G}(t)\, \mathrm{d}t$$

is as small as it can honestly be. Of course $\hat{G}(0) < G_c$ ($\leqq G(0)$) may be much less.

(2) Measure $\bar{\mu}$, the area of the box.

(3) $\mu = \bar{\mu} - \hat{\mu}$ is the required value of the Newtonian viscosity, larger than it probably should be, but honest.

We could use small sinusoidal oscillation data instead of stress relaxation to compute $\overline{G}(t)$.

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