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## RECENT RESULTS ON THE STABILITY OF ROTATING FLOWS OF TWO FLUIDS

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### I. MOTIVATIONS AND APPLICATIONS

Flows of two fluids are important and interesting because they are commonplace, they lend themselves to technological application and they introduce new phenomena without counterpart in the flow of one fluid.

Many configurations of flow of two fluids are possible. We see layers, slugs, rollers, sheets, bubbles, drops and dynamic emulsions and foams.

These structures are often topologically different from the rest configurations from which they arise.

The evolution process involves breakup and fracture of liquids which are not included in the usual statements governing dynamics (say, the Navier-Stokes equations).

In some approximate sense the configurations and the flow of two fluids which are ultimately achieved in practice are controlled by the problem of placements and the problems of shapes. In the problem of placements we must describe the massive transport required to transport the two fluids to

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the places they ultimately occupy. This problem is controlled by fingering and fracturing flows and frequently is such that the low viscosity constituent is found in the regions of high shear.

“High viscosity liquids hate to work. Low viscosity liquids are the victims of the laziness of high viscosity liquids because they are easy to push around.”

The problem of shapes has to do with the geometric form of the interfaces between flowing fluids. This problem involves surface tension and other potential terms which enter into each problem. This problem is treated for rotating flows in Chapter II.

Applications which involve the lubrication of one fluid by another are possible because lubricating flows are often stable (Joseph, Nguyen and Beavers, 1984; Joseph, Renardy and Renardy, 1984; Renardy and Joseph, 1985; Rosso, Than and Joseph, 1985). In the flow of two liquids down pipes there is a tendency for the low viscosity liquid to migrate to the pipe walls, lubricating the flow. The addition of water in the pipeline transport of oil, leads to a reduction in the pressure gradient needed for transport of a given amount of oil (Charles and Lilleleht, 1965; Gemmell and Epstein, 1962; Russell and Charles, 1959). There are significant reserves of heavy viscous crude oil in the United States, Canada, Venezuela and Europe which are becoming of increasing commercial importance. Heavy crudes may have viscosities of 1000 poise at room temperature. These viscous crudes cannot be transported by the usual pipeline methods. One proposed scheme would be to lubricate the walls with a second less viscous liquid, say water. The same type of technology could find a rich application for the transport of coal slurries and other granular materials in pipelines. It is known that some solids-liquids mixtures can move through pipes as plug flows in which the solids are concentrated in a large central core or plug which is surrounded by a thin annulus of relatively clear liquid. Investigators have reported that the resistance of certain mixtures moving in plug flow is appreciably less than that of similar mixtures when the solids are dispersed over the whole area of the pipe. It therefore appears that the lubrication of fluid by another may also be of interest in pipeline transport of fluidized solids.

The lubrication of one fluid by another was used by Macosko, Ocansey and Winter (1982) to create a planar extensional flow for application in the field of rheometrical measurements of material properties of fluids. They showed that planar stagnation flow could be achieved using molten polystyrene in a die with walls lubricated by silicone oil.

Polymer processing applications of the flows of two liquids which are of commercial interest are the coextrusion of two polymer melts through a slit die to form a multilayer film having unique optical and mechanical properties (Schrenk and Alfrey, 1972; Radford, Alfrey and Schrenke, 1972; Guillothe, 1972) and in the stratified flow of two polymer melts through a tube, a process essential in the spinning of two-component fibers, which are important for their self-erumping characteristics (Hicks, Ryan, Taylor and Techinor, 1960; Buckley and Phillips, 1969).

The extrusion of metal filaments in molten glass sheaths stabilizes the metal and allows the formation of thin wires of constant diameter which would be destroyed by capillary instabilities if extruded into air (Manfrè, Servi and Ruffino, 1974).

Some interesting studies of compound jets and their potential applications are reported by Hertz and Hermanrud (1983) and Hermanrud (1981). A compound jet consists of a central primary jet surrounded by a sheath of secondary fluid which has been entrained by the primary jet during its passage through the secondary fluid. The compound jet is useful because the outer layer of the jet can be used for lubrication of the tube as it passes through the jet spray orifice. It allows larger diameter orifices whilst maintaining a small diameter for the ink jet.

A different kind of application is associated with the simulation of artificial microgravity environments by matching densities. The principle is that gravity enters into interface dynamics through the relative weight per unit volume, which is equal to the density difference times gravity. Application of density matching have been explored in rheometrical devices using free surfaces. The amplitude of the deformation of the free surface may be amplified through density matching.

Certain geophysical processes, like mantle convection, may be modeled by two flowing liquids. The type of modeling is useful because it reduces geophysical flows to laboratory scale where they can be studied by controlled experiments.

## II. RIGID MOTIONS OF TWO FLUIDS

Rigid motions of a fluid are possible provided that the fluid rotates steadily about a fixed axis. Drops, bubbles, different types of fluids in all types of container may rotate rigidly. Various kinds of perturbations of rigid motion are also of interest. The main point of the analysis is that the configu-

rations which are stable are those which minimize a well defined interfacial potential.

The results which are discussed in this chapter are taken from papers by Joseph, Nguyen and Beavers (1984) called JNB, Joseph, Renardy, Renardy and Nguyen (1985) called JRRN, Guillopè, Joseph, Nguyen and Rosso (1985), called GJNR, and Preziosi and Joseph (1985), called PJ.

### 1. Steady rigid rotation of two fluids.

A single liquid which fills a container rotating steadily around some fixed axis will eventually rotate with the container. But in the case of two fluids it is necessary to determine the places occupied by the two fluids and the shape of the interfaces between the two fluids. We call this generic problem the problem of placements and shapes.

Two liquids occupy the region

$$G = \{x = (r, \theta, x), R_1 < r < R_2, 0 \leq \theta < 2\pi, -L < x < L\}$$

between two coaxial cylinders of radius  $R_1$  and  $R_2$  which rotate with a common angular velocity  $\Omega$ . Liquid one is in  $G_1$  and two is in  $G_2$ ,  $G_1 \cup G_2 = G$ . The interface between  $G_1$  and  $G_2$  is called  $\Sigma$ . It may be of disjoint parts.

Candidates for rigid motions, with gravity neglected, are

$$[u_0, p_0] = (\Omega r e_\theta, \rho \Omega^2 r^2 / 2 + c) \quad (1.1)$$

The velocity is continuous across  $\Sigma$  no matter what  $\Sigma$ , and the excess stress vanishes. We call (1.1) a candidate because it need not satisfy the normal stress condition

$$[[p_0]] = 2HT \text{ on } \Sigma \quad (1.2)$$

$$[[p_0]] = [[\rho]] \frac{1}{2} \Omega R + [[C]] \quad (1.3)$$

where  $R$  is the value of  $r$  at a point on  $\Sigma$  and  $2H$  is the sum of the principal curvatures. An expression for  $2H$  in cylindrical coordinates is given by:

$$2H = \frac{RR_{\theta\theta}(1+R_x^2) + RR_{xx}(R^2+R_\theta^2) - R^2(1+R_x^2) - 2R_\theta^2 - 2RR_\theta R_x R_{x\theta}}{(R^2 + R_\theta^2 + R^2 R_x^2)^{3/2}} \quad (1.4)$$

## 2. Disturbance equation.

Set

$$\underline{u} = \underline{u}_0 + \hat{\underline{u}} \quad , \quad p = p_0 + \hat{p} \quad . \quad (2.1)$$

Then:

$$\rho [\hat{\underline{u}}_t + \hat{\underline{u}} \cdot \nabla \underline{u}_0 + \underline{u}_0 \cdot \nabla \hat{\underline{u}} + \hat{\underline{u}} \cdot \nabla \hat{\underline{u}}] = -\nabla \hat{p} + \text{div } S[\hat{\underline{u}}] \quad (2.2)$$

where  $\hat{\underline{u}}$  is solenoidal and satisfies the no slip condition on cylinder walls.

At the interface  $\Sigma$

$$\begin{aligned} [[\hat{\underline{u}}]] &= \underline{0} \\ -[[\hat{p}]] \cdot \underline{n} + [[\underline{S}]] \cdot \underline{n} &= [[p_0]] \cdot \underline{n} + 2HT\underline{n} \quad . \end{aligned} \quad (2.3)$$

For any integrable function  $f$  which is equal to  $f_1$  in  $G_1$  and  $f_2$  in  $G_2$ , we define:

$$\langle f \rangle \stackrel{\text{define}}{=} \int_{G_1} f_1 dx + \int_{G_2} f_2 dx \quad . \quad (2.4)$$

For any  $g$  defined on  $\Sigma$  we define:

$$\langle g \rangle_{\Sigma} = \int_{\Sigma} g d\Sigma \quad . \quad (2.5)$$

Since the total volume for each incompressible fluid is conserved, we deduce that

$$\langle \underline{u} \cdot \underline{n} \rangle_{\Sigma} = 0 \quad . \quad (2.6)$$

If  $\Sigma$  is given by

$$F(x(t), t) = 0$$

then

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \underline{u} \cdot \nabla F = 0 \quad (2.7)$$

where we have assumed that the normal component of the velocity  $d\underline{x}/dt$  of the surface  $\Sigma$  and the particles of fluid on either side of  $\Sigma$  are the same. In fact, the velocity  $\underline{u}$  is continuous across  $\Sigma$ .

When  $F = r - R(\theta, x, t)$  we get:

$$\underline{u} \cdot \underline{n} = \underline{u} \cdot \frac{\nabla F}{|\nabla F|} = \frac{1}{|\nabla F|} \frac{\partial R}{\partial t} \quad . \quad (2.8)$$

### 3. Energy equation for rigid motion of two fluids.

The disturbance equation given in § 2 imply that:

$$\frac{d\mathcal{E}[\hat{u}]}{dt} + \mathcal{D}[\hat{u}] = \langle \hat{u} \cdot \underline{n}(\llbracket p_0 \rrbracket + 2HT) \rangle_{\Sigma} \quad (3.1)$$

where

$$\mathcal{E}[\hat{u}] = \left\langle \rho \frac{|\hat{u}|^2}{2} \right\rangle \quad (3.2)$$

is the energy and

$$\mathcal{D}[\hat{u}] = \langle 2\mu \underline{D}[\hat{u}] : \underline{D}[\hat{u}] \rangle \quad (3.3)$$

is the dissipation.

Moreover, using Eq. (9.6.11) of Joseph II (1976), we find that:

$$\begin{aligned} \langle \underline{u} \cdot \underline{n}(\llbracket p_0 \rrbracket + 2HT) \rangle_{\Sigma} &= \langle \underline{u} \cdot \underline{n} \llbracket p_0 \rrbracket \rangle_{\Sigma} - \\ &- \langle \underline{u}_0 \cdot \underline{n}(\llbracket p_0 \rrbracket + 2HT) \rangle_{\Sigma} + T \left\{ \frac{d|\Sigma|}{dt} + \int_{\partial\Sigma} \underline{\tau} \cdot \underline{U} dl \right\} \end{aligned}$$

where  $|\Sigma|$  is the area of  $\Sigma$ ,  $\underline{\tau}$  is the outward normal to  $\Sigma$ , in  $\Sigma$ , and  $\underline{U}$  is the velocity of a point of the contact line  $\partial\Sigma$ . Under certain circumstances, specified in 5 we can express (3.4) as the time derivative of some potential  $\mathcal{P}$ ; that is,

$$\langle \hat{u} \cdot \underline{n}(\llbracket p_0 \rrbracket + 2HT) \rangle_{\Sigma} = - \frac{d\mathcal{P}}{dt} \quad (3.5)$$

We may then write (3.1) as

$$\frac{d(\mathcal{E} + \mathcal{P})}{dt} = - \mathcal{D} \quad (3.6)$$

### 4. Assumptions about the interface $\Sigma$ .

We assume that the interface  $\Sigma$  between the two fluids has a finite number of components; each of them is represented locally by a finite number of equations  $r = R(\theta, x, t)$ , where  $R$  is a continuously differentiable function, periodic in  $\theta$ . These local charts also satisfy suitable continuity conditions at common points.

We assume also that the boundary  $\partial\Sigma$  of the interface has the following properties: either  $\partial\Sigma$  has measure zero as in the case of bubbles, drops and

emulsions or  $\partial\Sigma$  has a finite number of components, say  $\partial\Sigma = \partial\Sigma_L \cup \partial\Sigma_c$  where  $\partial\Sigma_L$  lies on the side walls at  $x = \pm L$  and can be represented by a graph  $r = R(\theta, \pm L, t)$  and where  $\partial\Sigma_c$  lies on one of the cylinders and is composed of a finite number of contact lines, each of them with a graph of the form  $x = X(R_i, \theta, t)$ ,  $i = 1$  or  $2$ .

### 5. Reduction of interface terms.

We proceed now to the terms on the left of (3.4). These were computed by JRRN for the periodic problem without contact lines and by GJNR when there are such lines. We assume that  $\Sigma$  has an equation  $r = R(\theta, x, t)$  for  $x_1(\theta, t) < x < L$ ,  $0 \leq \theta < 2\pi$ ,  $R(\theta, x_1(\theta, t), t) = R_1$  for  $0 \leq \theta < 2\pi$ . The function  $R$  is continuously differentiable and  $R(0) = R(2\pi)$ . Here  $\partial\Sigma_c$  is given by  $x = X(R_1, \theta, t) = x(\theta, t)$  ( $= x_1(\theta)$ , for short) where  $X$  is continuously differentiable and periodic in  $\theta$  and  $\partial\Sigma_L$  is given by the curve  $r = R(\theta, L, t)$ .

Since  $\langle \underline{u}_0 \cdot \underline{n} \rangle_\Sigma = 0$ , we find that

$$\langle \underline{u}_0 \cdot \underline{n} \llbracket p_0 \rrbracket \rangle_\Sigma = \langle \underline{u}_0 \cdot \underline{n} R^2 \rangle_\Sigma \frac{1}{2} \llbracket \rho \rrbracket \Omega^2 .$$

We define

$$I_1 = -\langle \underline{u}_0 \cdot \underline{n} R^2 \rangle_\Sigma = \Omega \int_0^{2\pi} \int_{x_1(\theta)}^L R_\theta R^3 dx d\theta$$

where  $\underline{u}_0 = \Omega R e_\theta$ . Using Leibnitz's rule we find that

$$\frac{d}{d\theta} \int_{x_1(\theta)}^L R R^3 dx = 4 \int_{x_1(\theta)}^L R^3 R_\theta dx - \frac{\partial x_1}{\partial \theta} R^4(\theta, x_1(\theta))$$

and after integrating, using periodicity, find that  $I_1 = 0$ ,

$$\langle \underline{u}_0 \cdot \underline{n} \llbracket p_0 \rrbracket \rangle_\Sigma = 0 \tag{5.1}$$

The calculation of  $\langle \underline{u} \cdot \underline{n} \llbracket p_0 \rrbracket \rangle_\Sigma$  is similar.

Since  $\langle \underline{u} \cdot \underline{n} \rangle_\Sigma = 0$ , we have

$$\langle \underline{u} \cdot \underline{n} \llbracket p_0 \rrbracket \rangle_\Sigma = \frac{1}{2} \Omega^2 \llbracket \rho \rrbracket \langle \underline{u} \cdot \underline{n} R^2 \rangle_\Sigma .$$

Then, using (2.8) and  $d\Sigma = R |\nabla F| d\theta dx$ , we get

$$I_2 = \langle \underline{u} \cdot \underline{n} R^2 \rangle_{\Sigma} = \int_0^{2\pi} \int_{x_1(\theta, t)}^L R_t R^3 dx d\theta .$$

Since

$$\begin{aligned} \frac{d}{dt} \int_0^{2\pi} \int_{x_1(\theta, t)}^L R^4(\theta, x, t) dx d\theta &= 4I_2 - \int_0^{2\pi} \frac{\partial x_1}{\partial t} R^4(\theta, x_1(\theta, t), t) d\theta = \\ &= 4I_2 - R_1^4 \frac{d}{dt} \int_0^{2\pi} x_1(\theta, t) d\theta . \end{aligned} \quad (5.2)$$

It follows now, from (5.2), that  $I_2$  is the time derivative of some function and

$$\langle \underline{u} \cdot \underline{n} \llbracket p_0 \rrbracket \rangle_{\Sigma} = - \frac{\Omega^2 \llbracket \rho \rrbracket}{8} \cdot \frac{d}{dt} \left\{ \langle R^3 |\nabla F|^{-1} \rangle_{\Sigma} + \int \Phi dl \right\} \quad (5.3)$$

where

$$\Phi = \pm R_i^3 X_{|r=R_i} \underline{t} \cdot \underline{e}_{\theta} , \quad i = 1, 2$$

and  $\underline{t}$  is the unit tangent vector to  $\partial\Sigma_c$  (see Exercise 1.2).

In the expression for  $\Phi$  the sign is + (resp. -) on the parts of  $\partial\Sigma_c$  being on the right (resp. left) side of a component of fluid  $s$ .

We turn next to the reduction of:

$$\langle \underline{u}_0 \cdot \underline{n} 2HT \rangle_{\Sigma} = - T\Omega \int_0^{2\pi} \int_{x_1(\theta, t)}^L 2HR R_{\theta} dx d\theta . \quad (5.4)$$

Using the formula:

$$R R_{\theta} 2H = \frac{\partial}{\partial \theta} \left[ \frac{R(1 + R_x^2)}{|\nabla F|} \right] + \frac{\partial}{\partial x} \left[ \frac{R R_x R_{\theta}}{|\nabla F|} \right]$$

derived in JRRN. We may integrate this expression. The second term leads to:

$$\int_0^{2\pi} \int_{x_1(\theta)}^L \frac{\partial}{\partial x} \left[ \frac{R R_x R_{\theta}}{|\nabla F|} \right] dx d\theta = \int_0^{2\pi} \left[ \frac{R R_x R_{\theta}}{|\nabla F|} \Big|_L - \frac{R R_x R_{\theta}}{|\nabla F|} \Big|_{x_1} \right] d\theta . \quad (5.5)$$

For the first term we use Leibnitz's rule:



$$\frac{d}{d\theta} \int_x^L \frac{-R(1+R_x^2)}{|\nabla F|} dx = \int_{x_1}^L \frac{\partial}{\partial \theta} \left[ \frac{-R(1+R_x^2)}{|\nabla F|} \right] dx + \frac{\partial x_1}{\partial \theta} \left[ \frac{R(1+R_x^2)}{|\nabla F|} \right]_{x_1}$$

Since  $R_1 = R(\theta, x_1(\theta))$ ,  $\partial x_1 / \partial \theta = -R_\theta / R_x$ . Hence:

$$\int_0^{2\pi} \int_{x_1}^L \frac{\partial}{\partial \theta} \left[ \frac{-R(1+R_x^2)}{|\nabla F|} \right] dx d\theta = R_1 \int_0^{2\pi} \left[ \frac{R_\theta}{R_x} \frac{1+R_x^2}{|\nabla F|} \right]_{x_1} d\theta. \quad (5.6)$$

Collecting results from (5.4), (5.5) and (5.6), we find that

$$\langle \underline{u}_0 \cdot \underline{n} \ 2HT \rangle_\Sigma = -T\Omega \int_0^{2\pi} \left\{ \frac{R_x R_\theta R}{|\nabla F|} \Big|_L + \frac{R_1 R_\theta}{R_x |\nabla F|} \Big|_{x_1} \right\} d\theta. \quad (5.7)$$

We may reexpress the second term on the right in terms of  $X$ ,

$$R_1 \frac{R_\theta}{R_x} \frac{1}{|\nabla F|} \Big|_{x_1} = - \frac{R_1 X_\theta X_r}{|\nabla F|} \Big|_{R_1} \quad (5.8)$$

where  $|\nabla F| = \left(1 + \frac{R_\theta^2}{R^2} + R_x^2\right)^{1/2}$  in the first term on the right of (5.7) and  $|\nabla F| = \left(1 + X_\theta^2/R^2 + X_r^2\right)^{1/2}$  in the second term, using (5.8).

Finally the calculation of  $\underline{\tau} \cdot \underline{U}$  on  $\Sigma$ , given as Exercise I.2, implies that:

$$\int_{\partial \Sigma} \underline{\tau} \cdot \underline{U} dl = \int_0^{2\pi} \left\{ \frac{R R_x R_t}{|\nabla F|} \Big|_L - \frac{R_1 X_r X_t}{|\nabla F|} \Big|_{R_1} \right\} d\theta. \quad (5.9)$$

Let  $\alpha_L$  be the angle between the interface  $\Sigma$  and the end walls and  $\alpha_c$  the angle between the interface  $\Sigma$  and the cylinders. Then:

$$\left. \begin{aligned} \cos \alpha_L &= \underline{n} \cdot \underline{e}_x = -R_x / |\nabla F|, \\ \cos \alpha_c &= \underline{n} \cdot \underline{e}_r = X_r / |\nabla F|. \end{aligned} \right\} \quad (5.10)$$

After introducing (5.10) into (5.7) and (5.9) and collecting all the previous results we find that:

$$\begin{aligned}
& -\langle \hat{u} \cdot \underline{n} (\llbracket p_0 \rrbracket + 2HT) \rangle_{\Sigma} = \\
& = T \int_0^{2\pi} \left\{ (R_t + \Omega R_{\theta}) R \cos \alpha_L \Big|_{x=L} + (X_t + \Omega X_{\theta}) R_1 \cos \alpha_c \Big|_{r=R_1} \right\} d\theta + \\
& + \frac{d}{dt} \left( T |\Sigma| + \frac{\Omega^2}{8} \llbracket \rho \rrbracket \left\{ \left\langle \frac{R^3}{|\nabla F|} \right\rangle + \int_{\partial \Sigma_c} \Phi dl \right\} \right). \quad (5.11)
\end{aligned}$$

## 6. The interface potential (GJNR).

In order to reduce the interface terms (5.11) to potential form we assume that:

- (\*) The contact angle at the interface on the end walls depends only on the distance  $r$  from the contact line to the axis of the cylinder. (At two different points  $\theta_1$  and  $\theta_2$  at which  $r_1 = r_2$ , the contact angle will be the same.)
- (\*\*) The contact angle at the interface on the cylinders depends only on the distance  $L - x$  of the contact line to the end wall at  $x = L$ .

The form of the functional dependence will be explicitly discussed in Section 10. The assumption (\*) and (\*\*) imply the existence of two functions  $\psi_L(R)$  and  $\psi_C(X)$  such that

$$\left. \begin{aligned} R \cos \alpha_L(R) &= \psi'_L(R) \\ R_1 \cos \alpha_C(X) &= \psi'_C(X) \end{aligned} \right\} \quad (6.1)$$

The reduction of (5.11) to a time derivative of a potential, using (6.1), is straightforward. We write:

$$\begin{aligned}
& \int_0^{2\pi} \{ (R_t + \Omega R_{\theta}) \psi'_L(R) + (X_t + \Omega X_{\theta}) \psi'_C(R) \} d\theta = \\
& = \int_0^{2\pi} \frac{D(\psi_L + \psi_C)}{Dt} d\theta = \frac{d}{dt} \int_0^{2\pi} [\psi_L(R) + \psi_C(X)] d\theta. \quad (6.2)
\end{aligned}$$

Where  $\frac{D}{Dt} = \frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial \theta}$  is a derivative following rigid motion.

The motivations in (6.2) are slightly misleading: the integration over  $\theta$  is to be carried out on each and every contact line.

We may write (6.2) as

$$\frac{d}{dt} \int_{\partial \Sigma} \psi dl$$

where  $\psi = \underline{t} \cdot \underline{\ell}_\theta$   $\left\{ \begin{array}{ll} \pm \psi_{\pm L}(R) \Big|_{x = \pm L} & \text{on } \partial \Sigma_L \\ \pm \psi_C(X) \Big|_{r = R_i (i = 1, 2)} & \text{on } \partial \Sigma_C \end{array} \right.$

It now follows that  $\mathcal{P}$  in (3.6) is given by

$$\mathcal{P} = T \left\{ |\Sigma| - \int_{\Sigma} \psi dl \right\} + \frac{[\rho] \Omega^2}{8} \left\{ \left\langle \frac{R_3}{|\nabla F|} \right\rangle + \int_{\partial \Sigma} \Phi dl \right\}. \quad (6.3)$$

## 7. Poincaré's inequality and the energy inequality.

Let  $\underline{v}$  belong to a space  $\underline{X}$  of square integrable solenoidal vectors defined in  $G$  which vanish on the solid parts of the boundary of  $G$ , or are periodic in  $x$ , with period  $2L$ , if the cylinders are infinitely long. Suppose further that the gradients of such functions are also square integrable in  $G$  where integration is in the sense (2.4). Such functions are said to lie in  $H^1(G)$  and they are automatically continuous in  $G$ , even across  $\Sigma$ ,  $[\underline{v}] = 0$ .

Each such  $\underline{v}$  satisfies Korn's inequality:

$$\langle |\underline{v}|^2 \rangle \leq 2k \langle \underline{D}[\underline{v}]^2 \rangle \quad (7.1)$$

for some positive constant  $k$ . Since

$$2 \langle |\underline{D}^2[\underline{v}]| \rangle = \langle |\nabla \underline{v}|^2 \rangle + \langle (\operatorname{div} \underline{v})^2 \rangle$$

and  $\operatorname{div} \underline{v} = 0$ , the constant  $k$  is Poincaré's constant.

Using (7.1) we may establish that

$$\mathcal{D}[\underline{v}] \geq 2\tilde{\lambda} \mathcal{E}[\underline{v}] \quad \forall \underline{v} \in \underline{X} \quad (7.2)$$

where

$$\tilde{\lambda} = \min[\mu_1, \mu_2] / k \max[\rho_1, \rho_2]. \quad (7.3)$$

The inequality (7.2) holds for connected configurations as well as for bubbles, drops and emulsions.

It now follows from (7.2) and (2.6) that:

$$\frac{d(\mathcal{E} + \mathcal{P})}{dt} \leq -2\tilde{\lambda} \mathcal{E}. \quad (7.4)$$

### 8. Integrability of the energy.

Integrating (15.3) from  $t = 0$  to  $t$ , we find that

$$\mathcal{E}(t) + \mathcal{P}(t) = \mathcal{E}(0) + \mathcal{P}(0) - \int_0^t \mathcal{D}(\tau) d\tau - \leq \mathcal{E}(0) + \mathcal{P}(0) - \lambda \int_0^t \mathcal{E}(\tau) d\tau .$$

It follows that:

$$\lambda \int_0^t \mathcal{E}(\tau) d\tau \leq \mathcal{E}(0) + \mathcal{P}(0) - \mathcal{E}(t) - \mathcal{P}(t) . \quad (8.1)$$

Let us suppose that  $\mathcal{P}$  is bounded below on the set of allowed interfaces. In fact, if  $G$  is a bounded region,  $\mathcal{P}$  is bounded from below. In unbounded domains  $\mathcal{P}$  need not be bounded below. In a bounded domain we could centrifuge all the light fluids to the outer cylinder wall. In an unbounded domain we would centrifuge a certain amount of liquid off the inner rod before reaching some equilibrium in which the potential is bounded.

If  $\mathcal{P}(0)$  is bounded below as a functional on the set of interfaces,  $\mathcal{E}(t)$  and  $\mathcal{D}(t)$  are integrable and

$$\lim_{t \rightarrow \infty} [\mathcal{E}(t) + \mathcal{P}(t)] < \infty .$$

### 9. Minimum of the potential.

Let us consider the limit configuration  $[\mathcal{E}(\infty), \mathcal{P}(\infty)]$ ; since  $\mathcal{E}(\infty) = 0$ , this is a rigid motion and

$$\mathcal{P}(\infty) - \mathcal{P}(0) = \mathcal{E}(0) - \int_0^\infty \mathcal{D}(t) dt . \quad (9.1)$$

Clearly  $\mathcal{P}$  decreases in every transformation for which the right side of (9.1) is negative. We may find disturbances  $\hat{u}$  at  $t = 0$ , for any  $\Sigma$ , such that  $\mathcal{E}(0)/\mathcal{D}(0)$  is arbitrarily small. For these:

$$\mathcal{P}(\infty) - \mathcal{P}(0) < 0 . \quad (9.2)$$

It follows that all configurations which give rise to different than  $\mathcal{P}(\infty)$  are unstable and that:

$$\mathcal{P}(\infty) = \lim_{t \rightarrow \infty} \mathcal{P}(t) = \lim_{\Sigma \in \mathcal{J}} \mathcal{P}[\Sigma] . \quad (9.3)$$

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Finally we note that critical points of  $\mathcal{P}$  correspond to steady rigid motions for which the normal stress equation (1.2) holds:

$$[[p_0]] + 2HT = 0$$

and the boundary conditions (6.1) are Euler equations for the minimization problem defined by (9.3) subject to the constraints of constant volumes for the two liquids.

**THEOREM 9.1.** Assume that the contact angle assumptions (\*) and (\*\*) hold. Then

- (i) Rigid motions are almost stable in the sense that the energy disturbances is integrable on  $(0, \infty)$ .
- (ii) The stable configurations associated with rigid motions minimize  $\mathcal{P}$  in  $\mathcal{S}$ .

## 10. Relation between the contact line and the contact angle.

Assumption (\*) and (\*\*) are equivalent to assuming a functional relation between the contact angle determining it by its position on the contact line. Precisely, such a functional relation is a differentiable map  $\mathcal{F}$  between the set  $\mathcal{C}$  of contact lines and the set  $\mathcal{A}$  of contact angles. For instance, in the case of side wall ( $x = \pm L$ ),  $\mathcal{C}$  is the space of  $2\pi$  periodic functions  $R = R(\theta)$ , which are continuously differentiable with values in  $[R_1, R_2]$ ;  $\mathcal{A}$  is the space of  $2\pi$  periodic functions, which are continuous with values in  $[-R_2, R_2]$ ; and:

$$\mathcal{F} : \left\{ \begin{array}{ll} \mathcal{C} \rightarrow & \mathcal{A} \\ R \mapsto & R \cos \alpha_L \\ \text{contact angle} & \text{contact line} \end{array} \right.$$

The equation of Young and Dupré is given by

$$\mathcal{F}(R) = CR$$

where  $C (= \cos \alpha_L)$  is a fixed constant in  $[-1, 1]$ .

Assumption (\*) is equivalent to the more general equation  $\mathcal{F}(R)(\theta) = \Phi_{\pm L}[R(\theta)]$  where  $\Phi_{\pm L}$  is a given absolutely continuous function from  $[R_1, R_2]$  into  $(-R_1, R_2)$ .

## 11. Spatially periodic connected interfaces.

In this section we shall assume that the interface is a graph:

$$r = R(\theta, x)$$

periodic, with period  $2\pi$  in  $\theta$ , and period  $\frac{2\pi}{\alpha} = 2L$  in  $x$ . Following Preziosi and Joseph (1985), we show that either  $R = d$ , where  $d$  is the mean radius of  $R$  or the minimizing solution crosses the axis at  $r = 0$ . This means that we get periodic arrays of drops and bubbles which have contact lines on the inner rod. The analysis neglects the effects of these lines on the potentials but is in good agreement with experiments away from these lines.

The analysis of stable configurations starts from the expression (6.3) for  $\mathcal{P}$ . It is assumed that there are no end plates and that  $R(\theta, x) \geq a$ , with possibly flat tangents at  $R = a$ . The contact line potentials  $\Phi$  and  $\psi$  are put to zero. Then we may write (6.3) as:

$$\mathcal{P} = \int_0^{2\pi/\alpha} \int_0^{2\pi} \{ T [R^2 + R_\theta^2 + R^2 R_x^2]^{1/2} - 1/8 [\rho] \Omega^2 R^4 \} d\theta dx \quad (11.1)$$

where:

$$d^2 = \int_0^{2\pi/\alpha} \int_0^{2\pi} R^2 d\theta dx. \quad (11.2)$$

In the analysis which follows we will work with a potential  $M$ , differing from  $\mathcal{P}$  by terms which are independent of  $R$ .

## 12. Mathematical formulation (PJ, 1985).

Joseph, Renardy, Renardy and Nguyen (1985), showed that rigid motions of two liquids between concentric cylinders of radius  $R_1$  and  $R_2$  are stable to spatially periodic disturbances of arbitrary amplitude and that the stable interface  $r = R(\theta, x)$  minimizes the potential.

$$M = T([R^2 + R_\theta^2 + R^2 R_x^2]^{1/2}) - \frac{1}{8} [\rho] \Omega^2 ([R^2 - d^2]^2) \quad (12.1)$$

where  $T$  is the interfacial tension,  $[\rho] = \rho_1 - \rho_2$  where  $\rho_1$  is the density of the inner fluid,  $\Omega$  is the angular velocity of the two fluids,  $d^2$  is the spatial average of  $R^2$

$$\langle\langle R^2 \rangle\rangle = \langle\langle d^2 \rangle\rangle \quad (12.2)$$

where

$$\langle\langle \quad \rangle\rangle = \int_0^{2\pi/\alpha} dx \int_0^{2\pi} (\quad) d\theta$$

and  $2\pi/\alpha$  is the wave length in the direction  $x$ . When the heavy fluid is outside,  $\llbracket \rho \rrbracket < 0$ ,  $M$  is minimized by  $R(\theta, x) = d$  whenever

$$J \stackrel{\text{def}}{=} - \frac{\llbracket \rho \rrbracket \Omega^2 d^3}{T} > 4. \quad (12.3)$$

If  $J < 4$  the minimizing solution is not of constant radius. The volume constraint (12.2) eliminates solutions of constant radius other than  $d$ . When  $J = 0$ , the interface is a surface of constant mean curvature, spherical, independent of  $\theta$ . We are going to study the  $\theta$  independent solutions following the work of Preziosi and Joseph (1985).

We measure all length in units  $d$ , setting  $r = R(x)/d$  where  $x$  and  $\alpha$  are dimensionless. Then there is a new  $M$  which is the old one divided by  $Td$  and such that

$$M = \langle\langle r[1 + r_x^2]^{1/2} \rangle\rangle + \frac{J}{8} \langle\langle [r^2 - 1]^2 \rangle\rangle, \quad (12.4)$$

where

$$\langle\langle r^2 \rangle\rangle = \langle\langle 1 \rangle\rangle. \quad (12.5)$$

We seek to minimize  $M$  among periodic functions  $r(x)$ , in the class  $C'(x)$ , satisfying (12.5). To do this, we introduce a Lagrange multiplier  $\lambda$  and seek the minimum of  $M + 2\lambda \langle\langle r^2 - 1 \rangle\rangle$  among periodic  $C'(x)$  functions  $r(x)$ . The Euler equations for this problem are

$$\frac{1 + r'^2 - r''r}{(1 + r'^2)^{3/2}} + \left[ \frac{J}{2} r(r^2 - 1) - \lambda r \right] = 0. \quad (12.6)$$

We may find a first integral of (2.6) by following a change of variables first introduced by Beer (1869). Consider the interface curve formed in the intersection of the axisymmetric interface and the axis  $r = 0$  of revolution. The coordinates in this plane are  $(x, r)$  and the angle between the interface curve  $r = r(x)$  and  $x$  is  $\psi$ . We define

$$v = \cos \psi, \quad 0 \leq v^2 \leq 1. \quad (12.7)$$

Then

$$r' = \tan \psi = \frac{\sqrt{1-v^2}}{v}$$

and

$$r'' = \frac{dr'}{dr} r' = \frac{d \tan \psi}{dr} \tan \psi = -\frac{1}{v^3} \frac{dv}{dr}.$$

The Euler equations (12.6) become

$$\frac{d}{dr}(rv) + \frac{J}{2} r^3 - \mu r = 0 \quad (12.8)$$

where  $\mu = \frac{J}{2}(1 + \lambda)$  is as yet undetermined and

$$v(r) = -\frac{J}{8} r^3 + \frac{\mu r}{2} - \frac{c}{r} \quad (12.9)$$

where  $c$  is a constant of integration.

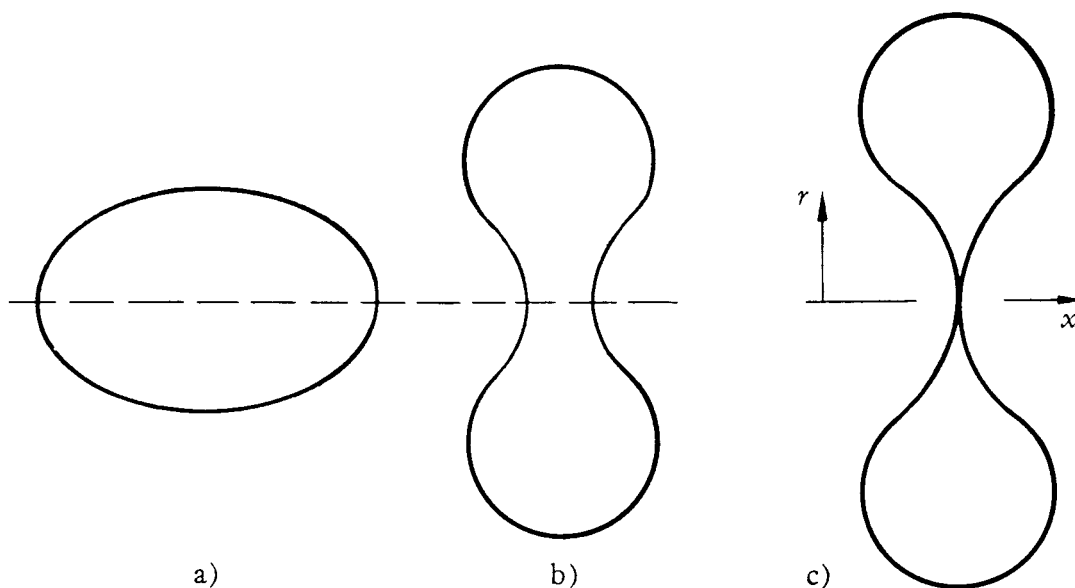


Fig. 1 — Schematic drawing of the minimizing solutions. Minimizing solutions touch the axis with a perpendicular tangent.

(a) Solutions of unduloid type are convex,  $4 \geq J \geq -5.42285$ .

(b) Solutions of nodoid type have a point of inflection  $-5.42285 \geq J \geq -8.18834$ .

(c) Limiting (toroidal) form the solution for  $J = -7.553908$ .

The solution (12.9) is to be associated with an interface profile satisfying  $r' = \sqrt{1-v^2}/v$  and the volume constraint



$$0 = ((r^2 - 1)) = 4\pi \int_{r_1}^{r_2} \frac{(r^2 - 1)v}{\sqrt{1 - v^2}} dr . \quad (12.10)$$

We may find all the axisymmetric solutions of our problem  $v(r(x))$  governed by (12.9). There are solutions (I) of unduloid type,  $v = \cos \psi$  (see Fig. 1)

$$\text{for any } r \quad 0 \leq v(r) \leq 1 \quad (\text{I})$$

and solutions (II) of nodoid type (see Fig. 1)

$$\text{there exists } \hat{r} / v(\hat{r}) < 0 . \quad (\text{II})$$

The angle between the interface curve and the  $x$ -axis is  $\psi$  and  $r' = \tan \psi$ . An unduloid is a surface of constant mean curvature,  $J = 0$ , which is generated by the focus of a rolling ellipse. A nodoid is a surface of constant mean curvature,  $J = 0$ , which is generated by the focus of a rolling hyperbola.

### 13. Preliminary analysis of the minimum problem.

It is convenient to replace the parameters  $(\mu, c)$  with  $(r_1, r_2)$ , the minimum and maximum values of  $r(x)$ . Since  $r'(r_1) = r'(r_2) = 0$  we have  $v(r_1) = v(r_2) = 1$  and, using (12.9), we find that

$$\begin{aligned} \frac{J}{8} r_1^4 - \frac{\mu r_1^2}{2} + r_1 + c &= 0 , \\ \frac{J}{8} r_2^4 - \frac{\mu r_2^2}{2} + r_2 + c &= 0 , \\ c &= r_1 r_2 \left[ \frac{J}{8} r_1 r_2 - \frac{1}{r_1 + r_2} \right] , \\ \frac{\mu}{2} &= \frac{J}{8} (r_1^2 + r_2^2) + \frac{1}{r_1 + r_2} , \\ r v &= \frac{J}{8} (r^2 - r_1^2)(r_2^2 - r^2) + \frac{r^2 + r_1 r_2}{r_1 + r_2} . \end{aligned} \quad (13.1)$$

Moreover, since  $r' = \sqrt{1-v^2}/v$

$$x = \int_{r_1}^r \frac{v(\xi)}{\sqrt{1-v^2(\xi)}} d\xi. \quad (13.2)$$

The period of periodic solutions is given  $x = \lambda$  where

$$\lambda = 2 \int_{r_1}^{r_2} \frac{v(\xi)}{\sqrt{1-v^2(\xi)}} d\xi.$$

Solutions (I) of the unduloid type have  $0 \leq v \leq 1$ . The constraint from above leads us to

$$(r-r_1)(r_2-r) \left[ \frac{J}{8} (r+r_1)(r+r_2) - \frac{1}{r_1+r_2} \right] \leq 0.$$

Hence

$$J \leq \frac{8}{(r+r_1)(r+r_2)(r_1+r_2)} \quad (13.3)$$

for all  $r \in [r_1, r_2]$ . When  $r = r_2$ , (13.3) reduces to

$$J \leq \frac{4}{r_2(r_1+r_2)^2}. \quad (13.4)$$

In the problem treated by JRRN the fluid is confined by cylinders of radius  $R_1$  and  $R_2$ . Hence  $r_1 \geq R_1/R_2$  and  $r_2 \geq 1$ , so that if

$$J \leq \frac{4}{\left[1 + \frac{R_1}{d}\right]^2}, \quad (13.5)$$

then  $J$  satisfies (13.4). The largest possible  $J$  for which a solution of unduloid type is possible is obviously  $J = 4$ . When  $J \geq 4$  the only solution of our minimum problem is the interface of constant radius  $r(x) = 1$ .

The other condition  $v \geq 0$  for an unduloid leads us to the inequality

$$J \geq \frac{-8(r^2 + r_1 r_2)}{(r_1 + r_2)(r_2^2 - r^2)(r^2 - r_1^2)} \quad (13.6)$$

for all  $r \in [r_1, r_2]$ . Let us choose  $r \in [r_1, r_2]$  so as to make the right side of (13.6) as large as possible; i.e.,  $r^2 = \sqrt{r_1 r_2} (r_1 + r_2 - \sqrt{r_1 r_2})$

$$J \geq \frac{-8}{(r_1 + r_2)^2 (\sqrt{r_2} - \sqrt{r_1})^2} \stackrel{\text{def}}{=} -\theta. \quad (13.7)$$

Solutions of type (I) are possible only when  $J$  satisfies the inequalities (13.4) and (13.7).

If  $J < -\theta$ , solutions of type (I) are not possible and we get solutions (II) of nodoid type.

$PJ$  showed that minimizing solutions of both types cross the axis,  $r_1 = 0$ . These solutions have  $r'(0) = \infty$  at the axis. To see this we note that  $r'(x) = \sqrt{1 - v^2}/v$  where  $v(r(x)) = v(0)$  is evaluated at  $r(x) = r_1 = 0$  and, from (13.1),  $v(0) = 0$ . It follows that solutions which minimize are singular in that they are limits of solutions for which  $r'(0) = 0$ , for ever larger curvatures. Solutions of type (I) which cross the axis satisfy the inequality (13.7) with  $r_1 = 0$ , where  $r_2$  must be found from the volume constraint (12.10). We shall show that this inequality cannot be satisfied if  $J < -5.42285$ . In this case the minimizing solution is of type (II). It has a point of inflection at some  $r > 0$  and  $r' = \infty$  at the points  $r_1 = 0$  which cross the axis.

#### 14. Numerical analysis of the minimum problem.

We turn now to the problem of minimization. After introducing the new variables into (12.4) and (12.5), we seek the minimum of

$$M = 4\pi \int_{r_1}^{r_2} \frac{\left[ r + \frac{J}{8} (r^2 - 1) v \right]}{\sqrt{1 - v^2}} dr \quad (14.1)$$

subject to the volume constraint (12.10). We note that  $v(r_1) = v(r_2) = 1$  so that the integrands in (14.1) and in (12.10) are singular at end points of integration. This singularity is integrable because

$$r^2 (1 - v^2) = (r - r_1)(r_2 - r) g(r) \quad (14.2)$$

where

$$g(r) = (r + r_1)(r + r_2) \left[ \left[ \frac{1}{r_1 + r_2} - \frac{J}{8} (r^2 + r_1 r_2) \right]^2 - \frac{J^2}{64} r^2 (r_1 + r_2)^2 \right]$$

is positive for all  $r \in [r_1, r_2]$ . This singularity may be removed by the following change of variables

$$r = \frac{r_1 + r_2}{2} + \frac{r_2 - r_1}{2} \sin \theta \stackrel{\text{def}}{=} f(\theta). \quad (14.3)$$

We find then that

$$M = 4\pi \int_{-\pi/2}^{\pi/2} \left[ \frac{r^2 + \frac{J}{8} (r^2 - 1)^2 r v(r)}{\sqrt{g(r)}} \right] d\theta, \quad (14.4)$$

$$0 = 4\pi \int_{-\pi/2}^{\pi/2} \left[ \frac{(r^2 - 1) r v(r)}{\sqrt{g(r)}} \right] d\theta. \quad (14.5)$$

These integrals are nice because the limits are fixed and the integrands are not singular if  $r_1 \neq 0$ . The volume constraint gives  $r_2$  as a function of  $r_1$ ,  $0 \leq r_1 \leq 1$ . The stable configuration is the one which minimizes  $M$  as  $r_1$  varies.

Renardy (see JRRN) proved that when  $J \geq 4$  the minimizing interface is of constant radius.  $PJ$  find that when  $J < -8.18834$  there are no axisymmetric minimizers. Exactly the same criterion of non-existence was given by Chandrasekhar (1965) in his study of drops rotating in a vacuum. He mentions the possibility of toroidal figures of equilibrium (see Fig. 1). In the context of this paper a toroidal figure of equilibrium at large negative values of  $J$  might be interpreted to mean that there are no locally stable flows with heavy fluid inside; all of the heavy liquid has been centrifuged to the outer cylinder giving rise to robustly stable flows with  $J > 4$ .

When  $-8.18834 \leq J < 4$  the minimum of  $M$  is taken by interfaces which cross the axis,  $r_1 = 0$  at periodic points on the axis  $x$  of revolution. When  $-8.18834 < J < -5.42285$  the minimizing solutions are of nodoid type (II). When  $-5.42285 \leq J < 4$  the solutions are of unduloid type (I) (see table 1). When  $r_1 = 0$ ,  $v(r)$  reduces to

$$v(r) = \frac{Jr}{8} (r_2^2 - r^2) + \frac{r}{r_2} \quad (14.6)$$

and, using (13.2)

$$x = \int_{-\pi/2}^{\phi} \frac{v(y) \sqrt{y} d\theta}{\left[ (r_2 + y) \left[ \frac{J}{8} y^2 - \frac{1}{r_2} \right]^2 - \frac{J^2}{64} r_2^2 y^2 \right]^{1/2}} \quad (14.7)$$

where  $y = \frac{r_2}{2} (1 + \sin \theta)$  and  $\phi = \arcsin \left[ \frac{2r}{r_2} - 1 \right]$ . The period  $\lambda$  of these

periodic solutions is given (14.7) with  $r = r_2$  and  $\phi = \pi/2$  and  $x = \lambda/2$  where  $\lambda$  is the period. The function  $\lambda(J)$ ,  $J > 0$  giving the period of the minimizing solution is a monotonically increasing function and  $\lambda(J) = \infty$  for  $J \geq 4$ . This means that the wave length gets larger and as  $J < 4$  is increased.

TABLE 1. Parameter values for minimizing solutions.

$J$	$r_2$	$\lambda$	$\lambda/r_2$	$[[\rho]] \Omega^2 a^3 / ST$
<i>Solutions of unduloid type I</i>				
0	1.22	2.45	2	0
-1	1.22	1.98	1.63	0.227
-2	1.20	1.65	1.36	0.439
-2.1	1.20	1.62	1.35	0.859
-3	1.89	1.39	1.17	0.625
-4	1.17	1.20	1.02	0.798
-5	1.15	1.03	0.91	0.942
-5.42285	1.14	0.98	0.865	1.00
-6	1.13	0.91	0.807	1.07
-7	1.12	0.78	0.692	1.23
-8	1.15	0.58	0.505	1.52
-8.18839	1.19	0.446	0.374	1.73
<i>Solutions of nodoid type II</i>				
-8.1880	1.19	0.442	0.370	1.73
-8.0	1.26	0.246	0.145	2.02
-7.55395	1.35	0	0	2.32

### 15. Periodic solutions, drops and bubbles.

Solutions which touch the axis may be regarded as limiting cases of periodic solutions. Then we get a periodic array of drops ( $J < 0$ ) or bubble ( $J > 0$ ) lined up with their centers on the axis of rotation. If we put a rod of radius  $a$  at the center of the array the resulting configuration would assume the form of an array of rollers. When the periodic solution is viewed as an array of drops or bubble,  $r_2$  is one radius of the drop and  $\lambda/2$  is the other. The sphere with  $r_2 = \sqrt{3/2}$  corresponds to a volume of  $\pi\sqrt{6}$  which is the volume of a right circular cylinder of unit radius and height  $2r_2$ . Graphs of  $\lambda$  and the ratio  $r_2/\lambda$  which gives the aspect ratio of the drop or bubble is shown in Fig. 2 and Table 1. We may compute  $J$  from this aspect ratio. It is also of interest to briefly consider the mean curvature  $H(r) = [v(r)r]'/2r = \frac{J}{8}(r_2^2 - 2r^2) + \frac{1}{r_2}$  at the poles  $r = 0$ ,  $H(0) = \frac{J}{8} + \frac{1}{r_2}$ , or at the equator  $H(r_2) = -\frac{J}{8}r_2^2 + \frac{1}{r_2}$  or the rotating drop or bubble. The curvature at the equator is negative when  $J \leq -8/r_2^3$  corresponding to solutions (II) of nodoid type.

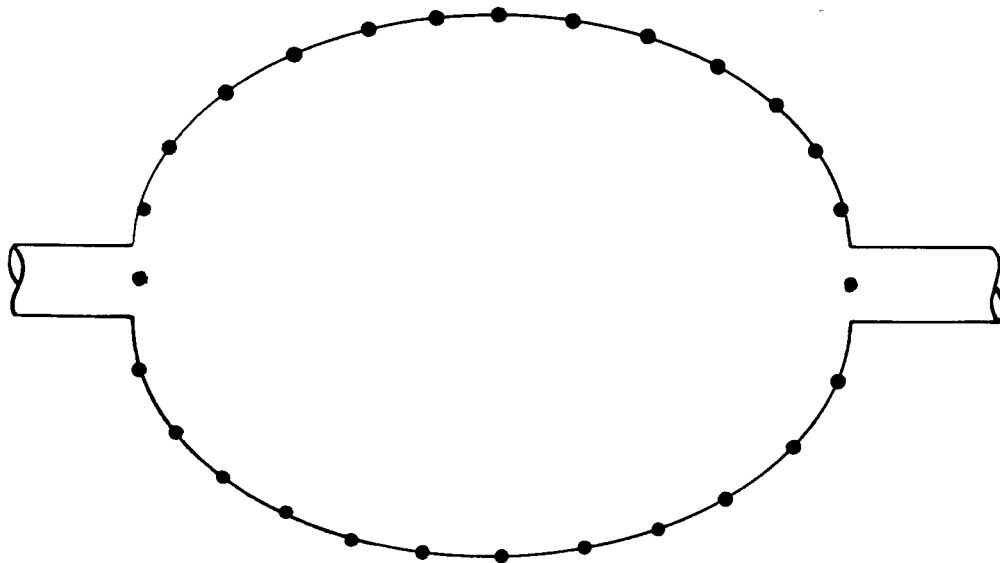


Fig. 2 — Rotating bubble with  $J = 1.32$ . The dots are values given by the minimizing solution which neglects capillarity at the contact line.

Solutions which touch the axis of rotation will certainly touch the inner cylinder. In a strict formulation such cylinder touching solutions would bring in contact line energies which were neglected in the analysis of § 12-14. These solutions are same as the ones given by Rosenthal (1962) for rotating bubbles ( $J > 0$ ) and by Chandrasekhar (1965) for rotating drops ( $J < 0$ ).

For rotating drops and bubbles, the volume, rather than the mean radius  $d$  is prescribed. The appropriate parameter, replacing  $J$ , is  $\tilde{\Sigma} = -Jr_2^3/8$ . The parameter is used by Chandrasekhar and it is listed in Tabel 1. In the case of bubbles  $J > 4$  cannot be achieved because the bubble will elongate as  $\Omega$  is increased in such a way that the effective mean radius

$$d = [4T/[\rho]\Omega^2]^{1/3}$$

is a decreasing function of  $\Omega$ . This property of rigidly rotating bubbles has been used as a basis for a device (Princin, Zia and Mason, 1967) for measuring interface tension. In our experiments, using end plates, we may suppress the elongation and achieve a solution with constant radius  $R - d$  when  $J > 4$ .

Spatially periodic configurations do appear in the experiments of  $PJ$ , but periodic solutions with heavy fluid outside ( $J > 0$ ) occur only as transients.

Periodic arrays of bubbles of permanent form rarely appear. Instead one finds isolated bubbles pierced by a shaft, as in the experiments of Plateau (1863) on rotating fluid drops (see Fig. 2). In the experiments of  $PJ$  with silicone oils, castor oil and paratac bubbles rotating in water, the form of the bubbles near the piercing center cylinder was not very strongly influenced by capillarity at the line of contact (see Fig. 2).

Periodic arrays of drops, even periodic solutions, can be seen, even as solutions of permanent form. We may regard the periodic arrays of rollers shown in the paper of Joseph, Nguyen and Beavers (1980) as a manifestation of minimization of the interface potential subject to certain unilateral constraints. Rollers may be regarded as a nearly rigid motion perturbed by shear due the drag of a second liquid and to gravity. It is necessary to establish that there are realizable conditions under which the perturbations can be suppressed.

Photographs of periodic (or nearly periodic) films or layers of liquids on rotating rods are exhibited in the papers Moffatt (1977) and Preziosi and Joseph (1985). In § 19 of this paper we argue that the effects of gravity on the periodic structures in the axial direction are not important. These solutions do not exhibit contact lines and they cannot be minimizing solutions which cross the axis. It is probable that those periodic structures are described by solutions in which the presence of the rod is acknowledged by the prescription of flat contact angles at points where  $R = R_1$ .

F. Leslie (1985) has solved the problem of rotating bubble shapes in a low gravity environment. His drop can contact end walls which are perpen-

dicular to the axis of rotation. He applies the condition of constant contact angle given by the Young-Dupré formula at the side walls. He does not use density matching to achieve micro-gravity but instead does the experiment in a free falling aircraft.

C.S. Yih (1960) considered the problem of stability of a film of liquid on a cylinder rotating in air. He treats this problem in the linearized approximation. He studies the stability of rigid motions with a free surface of constant radius with gravity neglected. Naturally these are unstable because  $J < 0$ . The rigid motion is stable, but not the free surface of constant radius.

Renardy and Joseph (1984) studied the stability of Couette flow of two fluids between concentric cylinders with gravity neglected. The linear theory is used and the interface is assumed to have a constant radius. A thin layer of less viscous fluid next to the cylinder is stable. There are surely other stable motions in which the interface does not have a constant radius.

## 16. Perturbation of rigid motions due to the drag of a second liquid.

To fix our topic we may think of a drop of oil rotating in vast expanse of water at rest at infinity. This configuration is like the one used by Plateau (1863) in his study of rotating drops and like another used by Joseph, Nguyen and Beavers (1984) to study rollers.

Brown and Scriven (1980) note that:

“J.A.F. Plateau though blind was far — seeing when he began experimenting on the shapes of rotating liquid drops, for he intended his centimetre — sized drops hold together by surface tension to be models for immense liquid masses hold together by self gravitation... .

“Plateau drop was pierced by a shaft and immersed in a tank of liquid having almost the density of the drop. The shaft was mounted vertically and by turning it Plateau... could bring the drop into rotation: not rigid rotation, because of the drag of the surrounding liquid, but some sort of rotation that Plateau presumed to approximate rigid-body rotation”.

The dynamics governing the drops in Plateau's experiments is not well understood. The working fluids for those experiments were olive oil and water-alcohol, with water-alcohol outside (a rotating bubble). There is no doubt that if the viscosity of the fluid on the rotating rod is sufficiently large, the fluid on the rod may rotate as a rigid body. Consider laminar flow viscous fluids filling the plane outside of a cylinder of radius  $R$  rotating at an angular velocity  $\Omega_1$  with gravity neglected. The viscosity ratio of the



two fluids is  $m = \mu_1/\mu_2$  where  $\mu_1$  is the viscosity of the fluid on the rod. We seek and find a solution in circles with

$$\begin{aligned} v_1 &= \frac{\Omega_1 R_1}{g} \left\{ \frac{R_1}{D^2} (1-m)r - \frac{R_1}{r} \right\}, & R \leq r \leq D; \\ v_2 &= -\frac{m R_1^2 \Omega_1}{gr}, & D \leq r \leq \infty \end{aligned} \quad (16.1)$$

where  $r = D$  is the position of the interface,

$$g = (1-m) \frac{R_1^2}{D^2} - 1$$

across which

$$\left[ \left[ \frac{d(v/r)}{dt} \right] \right] = 0.$$

The pressure may be obtained by integrating:

$$\frac{dp_i}{dr} = \frac{\rho_i v_i^2}{r}, \quad i = 1, 2$$

where the constant of integration may be selected to satisfy the normal stress equation

$$\llbracket p \rrbracket = \frac{T}{D}.$$

For fixed values of  $R_1$  and  $D$ , we get:

$$\begin{aligned} v_1 &= \Omega_1 r & (16.2) \\ v_2 &= \Omega_1 D^2/r, \\ p_1 &= \frac{\rho_1 \Omega_1^2 r^2}{2} + C_1, \\ p_2 &= \frac{-\rho_2 \Omega_1^2 D^4}{2r^2} + \llbracket C \rrbracket - C_1 \\ \llbracket p \rrbracket &= T/D \end{aligned}$$

when  $m = \mu_1/\mu_2$  is sufficiently large. The velocity field  $v_1 = \Omega_1 r$ ,  $R_1 \leq r \leq D$  is a rigid rotation.

The solution (16.1) and (16.2) would not be expected to be stable for

all values of the parameters. Indeed the analysis of stability of rigid motions given in Section 12 to 13 suggests that a constant radius with  $R = D$  could not be maintained for small  $\Omega_1$  (recall that  $J \geq 4$  for stability) or for any  $\Omega_1$  when the inner fluid is heavy (drops).

In the case of water outside oil, the water could be expected to undergo ordinary hydrodynamics transitions leading to turbulence Reynolds numbers well below those required for instability in the viscous drop. What might be the effect of the turbulence in the water? It is possible that the turbulent solutions could lead to a thin boundary layer of water on the oil across which pressure is effectively constant and equal to the pressure in the water (which could be atmospheric plus the gravitational head) at large distances. In this case the dynamical stabilizing effects of the water outside, which in rigid motions keeps the light fluid from centrifuging, would be lost and the bubble would act like a drop.

The experiments of JNB, JNNR and *PJ* also involve the rotation of oil masses called rollers in water. These rollers are lubricated everywhere by water, and if the viscosity of the oil is large enough, the rotation of the oil mass is rigid. The 1000 poise and 6000 poise silicone oil rollers in water (0.01) rotate rigidly. The 200 poise polymeric oils (STP) also rotate in water rigidly, or nearly so with some evidence for slight effects of shearing against water. Viscous liquids of smaller viscosity begin to show marked relative motion under the influence of shearing in water. In these fluids we get sheet lubrication and dynamic emulsions. The formation of these interesting questions is related more to the problem of placements than to the problem of shapes. The problem of placements arises whenever the effects of shearing are important.

### 17. Physical mechanism for the instability of rigid rotation of fluid. Bifurcation to non-axisymmetric shapes.

The problem of shapes on fluids, which rotate rigidly without shear when gravity is neglected, is determined by a balance of the capillary force against pressure forces associated with centripetal accelerations. When there is no rotation, and no other constraints, surface tension will pull the interface into a sphere. Of course, the sphere is an axisymmetric figure for any axis.

Plateau, in the (1863) work already cited, found that as  $\Omega$  increased the drop passed through a sequence of shapes, axisymmetric, ellipsoidal and two lobed. Then the majority of the liquid broke free of the shaft, forming a toroidal ring which was stable for a short time. Brown and Scriven (1980)

$R > d$ ,  $\Delta > 0$ ,  $\Delta_x = 0$ , because, on the average (17.3) requires  $\Delta < 0$ . Near these points we get larger values of  $(1 + \Delta_x^2)(1 + \Delta)^2$ . As  $\Omega$  is small, it increases the maximum values at which  $\delta = R - d > 0$  get larger, and  $\Delta_x$  grows near such points. When the positive values of  $\delta$  are sufficiently large we pay a greater price by increasing  $\Delta_x$  than by increasing  $\Delta_\theta$ . Then bifurcation occurs. This argument suggests that we should expect to see non axisymmetric solutions form on rings  $R - d > 0$  of relatively short length. Such bifurcation on narrow rings have been observed in the experiments of *PJ* and can be inferred from inspecting the paper of Moffat (1977).

### 18. Some effects of gravity. Intrinsically unsteady and steady solutions.

Up to now we have neglected the effects of gravity. Gravity can have a big effect, depending on parameters. It is certain that gravity is most important when the density difference is largest; gravity enters through the normal stress equation and it appears there as  $[[\rho]] g$  and it has no effect when  $[[\rho]] = 0$ . We can achieve microgravity environments by density matching. In this section we want to understand what the main effects of gravity may be; so we emphasize its effect by considering the dynamics of a viscous liquid which coats a horizontal rod rotating in air.

It is impossible to maintain an interface of constant radius on a film or layer coating a rotating rod. The effects of gravity and the interface potential lead to wavy surfaces and drops which are forced to rotate by adherence to the rotating rod. If the layer of oil is thick, gravity can be very effective in creating a large secondary motion with gravity opposing the direction of motion on one side of the rod and supporting on the other, leading to a film cross-section with a first mode ( $\sin \theta$ ) azimuthal variation, as in Fig. 3.

On the other hand even though there is secondary motion due to gravity, there is also the competition between centripetal acceleration and surface tension, tending to minimize the potential. At low speeds  $\Omega$  of rotation, this minimum tends to be achieved by axisymmetric solutions, which are then superimposed on the first mode azimuthal variation due to gravity. This leads to shapes of the type shown in Fig. 3.

The effect of gravity are greater when there is more liquid on the rod. For fixed volume of liquid, the effects of gravity are diminished when the viscosity is increased; however, more liquid will remain on the rod at given speed when the liquid is more viscous. The two effects compete. At low speeds an equilibrium is established with a "lop-sided" configuration as in Fig. 3, which is steady in laboratory coordinates.

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As the speed of rotation is increased the out of roundness shown in Fig. 3 begins to rotate relative to laboratory coordinates. At the same time the crest of the waves of grow and rings develop, in a manner suggested in the sketch of Fig. 4.

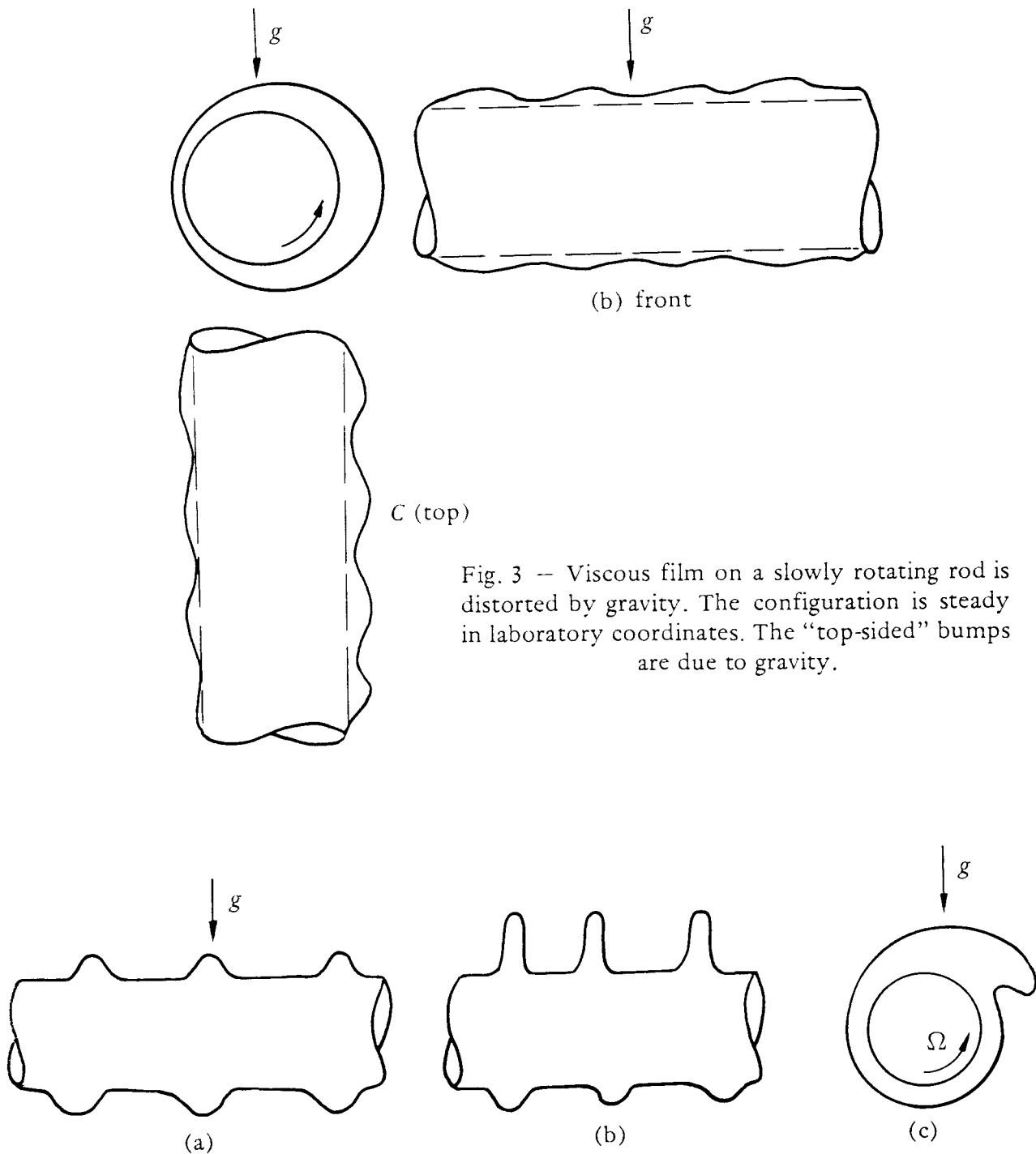


Fig. 3 — Viscous film on a slowly rotating rod is distorted by gravity. The configuration is steady in laboratory coordinates. The “top-sided” bumps are due to gravity.

Fig. 4 — Liquid film on a rod rotating at a higher angular velocity than in Fig. 3. (a) Front view, (b) top view, (c) side view. This configuration is intrinsically unsteady. The lobe in (c) undergoes a differential rotation and also rotates as a whole with the rod.

Configurations like the one shown in Fig. 4 were studied by Moffatt (1977). He did some experiments with 80 poise golden syrup solutions. In his experiments gravity and secondary motion were important.

If the speed of rotation is now increased liquid will be flung off the rod, reducing the volume of fluid on the rod. In some circumstances it is possible to see bifurcation to many lobed configurations, more or less axisymmetric as in Fig. 5.

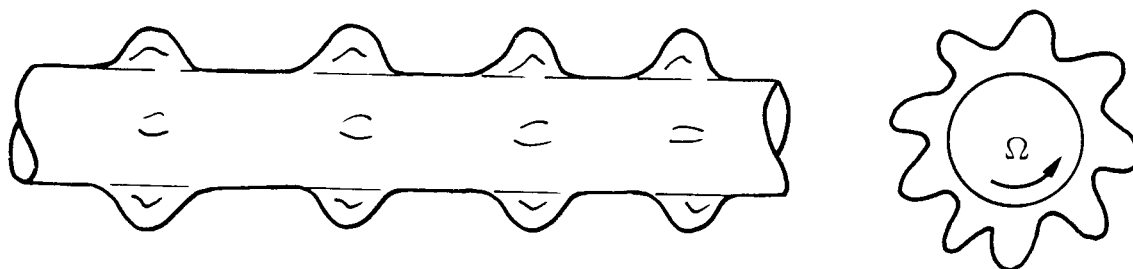


Fig. 5 — Nearly rigid bifurcated figures on a rotating rod. These figures are almost steady in a coordinate system rotating with angular velocity  $\Omega$ .

At very high speeds, most of the liquid is thrown off the rod. Gravity has nothing to do with throwing off because ejected particles of fluid are flung out radially. An equilibrium is reached in which there are pendant drops on a rotating rod. There is a tendency for these drops to form with a diamond plan form as in Fig. 6.

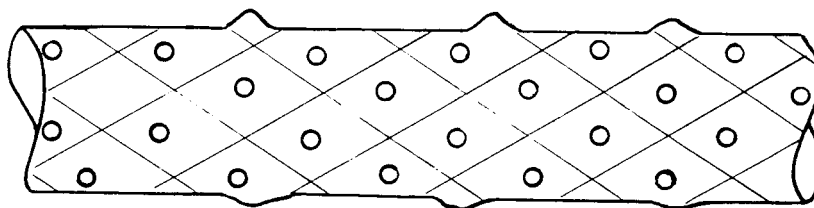


Fig. 6 — Pendant drops on a rapidly rotating rod. This configuration rotates rigidly and is perfectly steady in a coordinate system which rotates with angular velocity  $\Omega$ .

The pendant drops are exactly the same as would form on a wet ceiling under the influence of gravity. The effective gravity is  $\Omega^2 R_1$ , where  $R_1$  is the radius of the rotating rod.

At low speeds, the coating flows are strongly influenced by gravity and the interfacial potential. They are stationary in coordinates rotating with angular velocity  $\Omega$ . At high speeds, gravity is unimportant. The film must bifurcate into various regular patterns of increasing complexity as the speed

of rotation is increased.

Let us consider rimming flows. These are flows which coat the interior wall of a rotating annular cylinder. At high speeds all the liquid will be centrifuged to the outside wall. At lower speeds there will be some effect of gravity. The effects of gravity will be small (see Exercise II.6) when the ratio

$$\frac{2g}{R\Omega^2} \ll 1.$$

Note that this criterion is independent of density.

We suppress gravity by rapid rotation, as is well known.

### Exercises.

*Exercise II.1* - A container is filled with an incompressible fluid and it is rotating with angular velocity  $\underline{\Omega}(t)$ . Show that rigid motions of the fluid in the container are possible only when  $\underline{\Omega}$  is independent of  $t$ .

*Exercise II.2* (GJNR) - Calculation of  $\underline{\tau} \cdot \underline{U}$  on  $\partial\Sigma$ .

The interface  $\Sigma$  is described locally by the equation

$$r = R(\theta, x, t), \quad x_1(\theta) \leq x_2(\theta), \quad 0 \leq \theta < 2\pi$$

or by

$$x = X(r, \theta, t), \quad r_1(\theta) \leq r \leq r_2(\theta), \quad 0 \leq \theta \leq 2\pi.$$

Assume that  $\partial\Sigma$  has a part on the side walls:

$$\partial\Sigma_L = \{x = R(\theta, \pm L, t)\underline{e}_r\}$$

and a part on the cylinders

$$\partial\Sigma_C = \{x = R_i \underline{e}_r + X(R_i, \theta, t)\underline{e}_x\}.$$

Define:

$\underline{n} = \nabla F / |\nabla F|$ , the unit vector normal to  $\Sigma$ , from fluid 1 to fluid 2, with

$$F(x) = r - R(\theta, x) \text{ or } F(\underline{x}) = X(r, \theta) - x;$$

$\underline{t} = d\underline{x}/dl$ , a unit tangent vector to  $\partial\Sigma$ ;

$\underline{\tau} = \underline{n} \wedge \underline{t}$ , a unit vector normal to  $\partial\Sigma$  in  $\Sigma$ ;

$\underline{U} = d\underline{x}/dt$ , the velocity of a point on the contact line  $\partial\Sigma$ .

The function  $\underline{\tau} \cdot \underline{U}$  is calculated as follows

$$\underline{\tau} \cdot \underline{U} = \underline{U} \cdot (\underline{n} \wedge \underline{t}) = \underline{n} \cdot (\underline{t} \wedge \underline{U}).$$

On the side walls at  $x = \pm L$ , we have

$$\begin{aligned}\nabla F &= \underline{e}_r + \underline{e}_\theta \frac{R_\theta}{R} - R_x \underline{e}_x, \\ \underline{t} &= (\underline{e}_r R_\theta - \underline{e}_\theta R) d\theta/dl, \\ \underline{U} &= \underline{e}_r R_t + (\underline{e}_r R_\theta + \underline{e}_\theta R) d\theta/dt.\end{aligned}$$

On the cylinders at  $r = R_i$ , we have

$$\begin{aligned}\nabla F &= \underline{e} X_r + \underline{e}_\theta X_\theta / R_i - \underline{e}_x, \\ \underline{t} &= [\underline{e}_\theta R_i + \underline{e}_x X_\theta] d\theta/dl, \\ \underline{U} &= [\underline{e}_\theta R_i + \underline{e}_x X_\theta] d\theta/dt + \underline{e}_x X_t, \\ \underline{\tau} \cdot \underline{U} &= R_i X_r X_t |\nabla F|^{-1} d\theta/dl.\end{aligned}$$

**Exercise II.3** - Show that (6.1) and (1.2) are Euler equations for (9.3).

**Exercise II.4** (the inequality (2.2)) - Consider the functional

$$\lambda[\theta] = \int_0^1 \mu \theta'^2 dx / \int_0^1 \theta' dx$$

where  $\mu, \mu_1, \mu_2$  are positive constants:

$$\mu = \begin{cases} \mu_1, & 0 \leq x \leq l \\ \mu_2, & l < x \leq 1 \end{cases} \quad \mu_2 \neq \mu_1,$$

$\theta(x)$  is continuous across  $x = l$ ,  $\theta(0) = \theta(1) = 0$  and  $\theta$  is twice differentiable above and below  $l$ . Show that  $\lambda[\theta]$  is positive and bounded from below,

$$\tilde{\lambda} = \lambda(\tilde{\theta}) = \min \lambda[\theta] > 0.$$

Show that

$$\tilde{\theta}'' + \lambda^2 \tilde{\theta} = 0$$

holds above and below 1 and

$$\mu_2 \tilde{\theta}'|_{x \downarrow l} = \mu \tilde{\theta}'|_{x \uparrow l}.$$

Find  $\tilde{\lambda}$ .

*Exercise II.5* (the inequality (7.2)) - Let

$$\tilde{\lambda} = \lambda[\tilde{v}] = \min_{\underline{v} \in X} \lambda[\underline{v}]$$

where

$$\lambda[\underline{v}] = \frac{\mathcal{D}[\underline{v}]}{2 \mathcal{E}[\underline{v}]} .$$

Find Euler's equations and the natural boundary conditions for  $\tilde{\lambda}$  and  $\tilde{v}$ .

#### REFERENCES

- Bear, A.: *Einleitung in die Mathematische Theorie der Elastizität und Capillarität*, A. Gresen Verlag, Leipzig, 1869.
- Brown, R.A. and Scriven, L.E.: *The Shape and Stability of Rotating Liquid Drops*, Proc. Roy. Soc. A, 331-357 (1980).
- Buckley, R.A. and Phillips, R.J.: Chem. Eng. Proc. 66, 41 (1962).
- Chandrasekhar, S.: *The Stability of a Rotating Liquid Drop*. Proc. Roy. Soc. London 286, 1-26 (1965).
- Charles, M.E. and Lillelekt, L.U.: Can. J. Chem. Eng., 43, 110 (1965).
- Gemmel, A.R. and Epstein, N.: Can. J. Chem. Eng. 40, 215 (1962).
- Guillope', C., Joseph, D.D., Nguyen, K. and Rosso, F.: *Nonlinear Stability of Rotating Flow of Two Fluids* (to appear).
- Hermanrud, B.: *The Compound Jet - A New Method to Generate Fluid Jets for Ink Jet Printing*. Report 1-1981. Dept. of Electrical Measurements Lund Institute of Technology, Sweden.
- Hertz, C.H. and Hermanrud, B.: *A Liquid Compound Jet*, J. Fluid Mech. 131, 271-287 (1983).
- Hicks, E.M., Ryan, J.F., Taylor, R.B. and Techinor, R.L.: *Textile*, Res. J. 30, 675 (1968).
- Joseph, D.D.: *Stability of Fluid Motions II*, Springer, 1976.
- Joseph, D.D., Renardy, Y., Renardy, M. and Nguyen, K.: *Stability of Rigid Motions and Rollers in Bicomponent Flows of Immiscible Liquids*, J. Fluid Mech. 153, 151-165 (1985).
- Joseph, D.D., Nguyen, K. and Beavers, G.S.: *Non-uniqueness and Stability of the Configuration of Flow of Immiscible Fluids with Different Viscosities*, J. Fluid Mech. 141, 319-345 (1984).



- Leslie, F.: *Measurements of Rotating Bubble Shapes in a Low Gravity Environment*, J. Fluid Mech. (1985), to appear.
- Macosko, C., Ocansey, M.A. and Winter, H.H.: *Steady Planar Extension with Lubricated Dies.*, J. Non Newtonian Fluid Mech. 11, 301-316 (1982).
- Manfrè, G., Servi, G., Ruffino, C.: J. Materials Sci. 9, 74 (1974).
- Moffatt, K.: *Behaviour of a Viscous Film on the Outer Surface of a Rotating Cylinder*, J. Mec. 16, 651-673 (1977).
- Plateau, J.A.F.: *Experimental and theoretical researches on the figures of equilibrium of a rotating liquid mass withdrawn from the action of gravity*, Annual Report of the Board of Regents and Smithsonian Institution Washington DC, 270-285 (1863).
- Preziosi, L. and Joseph, D.D.: *Stability of Rigid Motions of Two Fluids* (to appear on the J. Fluid Mech.).
- Primar, H.M., Zia, I.Y.Z. and Mason, S.G.: *Measurements of Interfacial Tension from the Shape of a Rotating Liquid Drop*. Journal of Colloid and Interface Science, 23, 99-107 (1967).
- Rosenthal, D.K.: *The Shape and Stability of a Bubble at the Axis of a Rotating Liquid*, J. Fluid Mech. 12, 358-366 (1962).
- Rosso, F., Than, P.T. and Joseph, D.D.: *Instability of Poiseuille Flow of Two Immiscible Liquids with Different Viscosities in a Channel* (1985, to appear).
- Russell, T.W. and Charles, M.E.: Can. J. Chem. Eng., 18, 215 (1959).
- Yih, C.S.: *Instability of a Rotating Liquid Film with a Free Surface*, Proc. R. Soc. London A 258, 63-86 (1960).

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