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Two fluids heated from below

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Summary.

We consider the problem of stability of the conduction solution of two fluids in two layers heated from below. This is the classical Bénard problem, but for two fluids. The two fluid problem is never self-adjoint. We compute the adjoint. The computation is not of a familiar type. The onset of convection can occur as a Hopf bifurcation. We formulate the problem for non-linear studies. In doing this we are obliged to extend the conduction solution. The problem is reduced to local form suitable for the study of bifurcation. Energy integrals are formulated and we obtain some estimates of the usual type, but for two fluids. These are easy estimates, but they are new. We pay attention to a new nasty interface term which is not bounded in H^1. This term frustrates the method of energy. So our study is incomplete and the estimation of the interface term, or a proof that this term can always lead to instability is posed as a challenge for future work.

In this paper we shall consider the classical problem of the instability of a fluid layer heated from below when there are two fluids, one of which is floated on another. This problem was first studied by Zeren and Reynolds (1972). They include the effects of surface tension gradients due to temperature giving rise to Marangoni convection. Sternling and Scriven (1959) showed that surface tension induced instability of heat conduction of one fluid could occur with purely imaginary eigenvalues, corresponding to time-periodic convection. Renardy and Joseph (1985) showed that even if Marangoni convection is suppressed, the onset of convection may occur as a Hopf bifurcation with time periodic convection replacing conduction. Renardy (1986) showed that there are situations in which conduction, with a thin layer of heavy fluid above is stable. These features also appear in other problems of stratified fluids, as in the thermohaline problem.

There are technological possibilities for using two fluid systems to promote or inhibit heat transfer. These systems may also find application in modeling a geophysical process. Busse (1981) argues that the horizontal scale of the motion in the lower mantle may determine the horizontal scale of flow in the upper mantle.

An energy stability theory for the Bénard problem for two fluids is given in Section 5. There are two energy identities and one of them contains a boundary term, not bounded in H^1. This boundary term prevents one from proving global stability, or even linear stability, of the conduction solution. In other problems, like the stability of layered Couette flows, the corresponding boundary term actually can lead to instability, even at zero Reynolds numbers (see Hooper and Boyd [1983]).

The proof or negation of some form of unconditional stability for the two-fluid Bénard problem is an outstanding open problem in the energy theory of stability for two fluids.

1. The conduction solution and the extended conduction solution.

Two fluids fill a rectangular container shown as Figure 1. The box has a height $\ell_1+\ell_2$ and a base of size $L_x \times L_y$. Gravity $\mathbf{g} = \mathbf{e}_z g$ acts downward. The interface Σ is given by $z = \delta(x,y,t)$. The fluid above $z = \delta$ lies in the region

$$V_1 = \{x,y,z \mid x \in [-L_x, L_x],\ y \in [-L_y, L_y],\ \ell_1 \geq z \geq \delta\}. \quad (1.1)$$

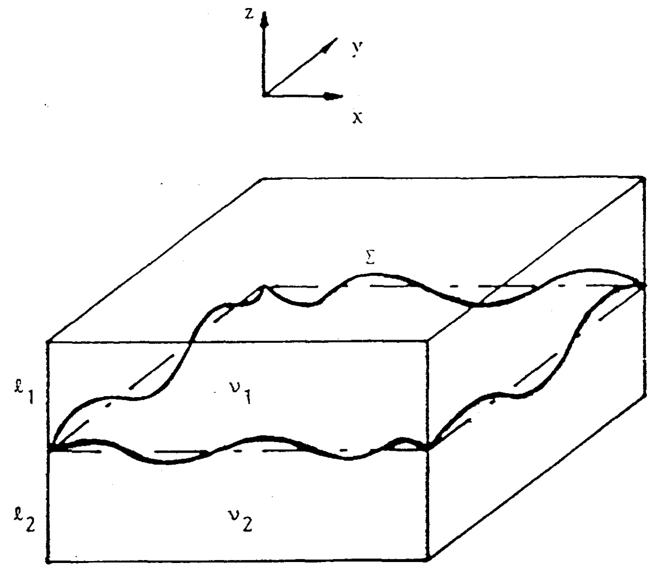

Figure 1. Two fluids in a box. The origin of z is on the plane of mean height $\bar{\delta} = 0$. $T(x,y,\ell_1, t) = T_1$, $T(x,y,-\ell_2,t) = T_2$, $T_2 - T_1 = \Delta T > 0$.

The fluid below is in V_2 where $-\ell_2 \leq z \leq \delta$. The volume of each of the two fluids is fixed; hence,

$$\bar{\delta}(t) = \int_{-L_x}^{L_x} \int_{-L_y}^{L_y} \delta(x,y,t)\, dx\, dy = 0. \tag{1.2}$$

The top, $z = \ell_1$, and bottom, $z = -\ell_2$, of the box are rigid so that the velocity **V** vanishes there. We could also consider shear free boundaries on which $\mathbf{V} \neq 0$. The temperatures at the top and bottom plates are T_1 and T_2. These are constants, independent of (x,y,t). The side walls are insulated. Either $\mathbf{V} = 0$ on the side walls or the box is a period cell with

$$L_x = \pi/\alpha, \qquad L_y = \pi/\beta \tag{1.3}$$

and periodic conditions in x and y. We confine our attention to the periodic case unless otherwise noted.

The fluid is heated from below

$$T_2 - T_1 = \Delta T > 0 .$$

The reference temperature T_0 in the OB equation of state is constant.

We are going to study the stability of conduction, a motionless solution with $\mathbf{V} = \mathbf{0}$. A perturbation of $\mathbf{V} = \mathbf{0}$ is called $\mathbf{u}$; hence, $\mathbf{V} = \mathbf{u}$. The components of $\mathbf{u}$ relative to (x,y,z) are

$$\mathbf{u} = (u,v,w) .$$

2. Governing equations.

The two fluids each satisfy the Oberbeck-Boussinesq equations.

$$\rho[\mathbf{V}_t + (\mathbf{V}.\nabla)\mathbf{V}] = -\nabla P + \operatorname{div} \mathbf{S}[\mathbf{V}] - \rho g \mathbf{e}_z[1-\alpha(T-T_0)], \quad (2.1)$$

$$\rho c[T_t + (\mathbf{V}.\nabla)T] = k\nabla^2 T \quad (2.2)$$

where

$$\mathbf{S}[\mathbf{V}] = 2\mu \mathbf{D}[\mathbf{V}] , \quad \operatorname{div} \mathbf{V} = 0 . \quad (2.3)$$

The equations (2.1, 2 and 3) hold in V_1 and V_2; they have different coefficients, corresponding to the different fluids, in each region.

The interface conditions are that the velocity, temperature and heat flux are continuous across $F = z - \delta(x,y,t) = 0$, the jump in the traction is balanced by surface tension and $dF/dt = 0$ on $F = 0$, respectively,

$$[\mathbf{V}] = 0 \ ,$$

$$[T] = 0 \ ,$$

$$[k\nabla T].\mathbf{n} = 0 \ , \qquad (2.4)$$

$$[S[V]].n - [P]\mathbf{n} + 2H\sigma\,\mathbf{n} + \nabla_{II}\sigma = 0 \ ,$$

$$\frac{dF}{dt} = F_t + u.\nabla F = -\delta_t + w - u\delta_x - v\delta_y = 0$$

where $\mathbf{n} = n_{21}$ and ∇_{II} is a gradient on Σ, $\mathbf{n}.\nabla_{II}\sigma = 0$. In general $\sigma = \sigma(T)$ so that $\nabla_{II}\sigma = \sigma'(T)\nabla_{II}T$. The side walls are insulated (even in the periodic case) and

$$T(x,y,\ell_1,t) = T_1 \ , \qquad T(x,y,-\ell_2,t) = T_2 \ . \qquad (2.5)$$

The velocity vanishes on $z = \ell_1$, $-\ell_2$ and it is not forced at the side walls.

3. The conduction solution and the extended conduction solution.

The data has been arranged to admit a conduction solution. Let $\mathbf{V} = \mathbf{0}$,

$$\nabla P + \rho g e_z[1 - \alpha(T - T_0)] = 0 \ .$$

This can be solved for P only if $\text{curl}\,\nabla P = 0$; that is, $T = T(z,t)$ and T is independent of t because the data is steady. We have

$$T''(z) = 0 \quad \text{in} \quad \mathcal{V}_1 \quad \text{and} \quad \mathcal{V}_2$$

$$T(\ell_1) = T_1 \qquad T(-\ell_2) = T_2$$

$$[T] = [kT'] = 0 \ .$$

Hence

$$T - T_0 = \begin{cases} -A_1 z , & z > \delta \\ -A_2 z , & z < \delta \end{cases} \tag{3.1}$$

$$k_1 A_1 = k_2 A_2 ,$$

$$A_1 = (T_0 - T_1)/\ell_1 ,$$

$$A_2 = (T_2 - T_0)/\ell_2 ,$$

and since $[T] = -[A]\delta = 0$, $\delta = 0$.

$$\Delta T = T_2 - T_1 = A_1 \ell_1 + A_2 \ell_2 \geq 0$$

is prescribed. Since $\delta = 0$, $\nabla_{II} T = 0$, $(2.4)_5$ reduces to $[P] = 0$ on $\delta = 0$ and

$$\frac{dP}{dz} + \rho g[1 - \alpha(T - T_0)] = 0 .$$

Hence

$$P - P_0 = -g \begin{cases} \rho_1 z + \rho_1 \alpha_1 A_1 z^2/2 , & z > 0 \\ \rho_2 z + \rho_2 \alpha_2 A_2 z^2/2 , & z < 0 \end{cases} \tag{3.2}$$

where P_0 is some constant, $[P_0] = 0$.

A conduction solution is given by (2.1) with $\delta = 0$ and (3.2). This solution is not unique in the sense that we could study solutions in many layers. Even with two layers, heavy fluid may be above or below. There are even situations in which heavy fluid above is stable.

We call

$$(V, T, P, \delta) = (0, \tilde{T}, \tilde{P}, \tilde{\delta}) \tag{3.3}$$

an extended conduction solution when for $\delta = 0$, $\tilde{\delta} = 0$

$$\tilde{T}(z) - T_0 = \begin{cases} A_1 z , & z > \delta \\ A_2 z , & z < \delta \end{cases} \tag{3.4}$$

$$\tilde{P}(z) - P_0 = -g \begin{cases} \rho_1 z + \rho_1 \alpha_1 A_1 z^2/2 , & z > \delta \\ \rho_2 z + \rho_2 \alpha_2 A_2 z^2/2 , & z < \delta \end{cases} \tag{3.5}$$

with the same constants. The extended solution is not even a solution.

4. Reduction to local form.

We can decompose the motion any way we like. Set $\mathbf{V} = \mathbf{u}$ and

$$T = \tilde{T} + \theta \qquad P = \tilde{P} + p .$$

We say that $(\mathbf{u},\theta,\delta) \in H$ if $\mathbf{u}$ and θ vanish on the top and bottom, $\operatorname{div}\mathbf{u} = 0$ in V_1 and V_2, $\mathbf{n}.\nabla\theta = 0$ on side walls, $\underline{\mathbf{u}} = \mathbf{0}$ as $\mathbf{u}$ is periodic with period $2\pi/\alpha$, $2\pi/\beta$ in x, y and $\tilde{\delta} = 0$. We find that

$$\begin{aligned} &(\mathbf{u},\theta,\delta) \in H, \\ &\rho\left[\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}.\nabla\mathbf{u} \right] = -\nabla p + \operatorname{div} S[\mathbf{u}] + \rho\alpha g\theta \mathbf{e}_z , \\ &\rho c\left[\frac{\partial\theta}{\partial t} + \mathbf{u}.\nabla\theta - wA\right] = k\nabla^2\theta , \\ &S[\mathbf{u}] = 2\mu D[\mathbf{u}] \end{aligned} \tag{4.1}$$

where (ℓ,c,μ,k,α,A) are different constants in V_1 and V_2. On the interface at $z = \delta$ we have, for example,

$$[k\nabla T].\mathbf{n} = [k\tilde{T}']\mathbf{e}_z.\mathbf{n} + [k\nabla\theta].\mathbf{n} = 0 .$$

Since $[k\tilde{T}'] = [kA] = 0$ we get

$$[k\theta_n] = 0 \quad \text{on} \quad z = \delta$$

where

$$\theta_n = \mathbf{n}.\nabla\theta \ .$$

The interface conditions are

$$[\mathbf{u}] = \mathbf{0} \ ,$$

$$[\theta] - [A]\delta = 0 \ ,$$

$$[k\theta_n] = 0 \ , \tag{4.2}$$

$$[\mathbf{S}[\mathbf{u}] - p\mathbf{1}]\mathbf{n} = -g\mathbf{n}\{[\rho]\delta + [\rho\alpha A]\tfrac{1}{2}\delta^2\} + 2H\sigma\mathbf{n} + \nabla_{II}\sigma \ ,$$

$$\delta_t - w + u\delta_x + v\delta_y = 0$$

where

$$2H = \nabla_2 \cdot \left(\frac{\nabla_2\delta}{(1 + |\nabla_2\delta|^2)^{\frac{1}{2}}}\right)$$

$$\nabla_2 = \mathbf{e}_x\partial_y + \mathbf{e}_y\partial_x$$

$$\nabla_{II} = \nabla - \mathbf{n}.(\mathbf{n}.\nabla) =$$

$$= \mathbf{e}_z\partial_z + \nabla_2 - \frac{[\mathbf{e}_z - \nabla_2\delta]}{1 + |\nabla\delta|^2}\,[\partial_z - \delta_x\partial_x - \delta_y\partial_y] \ . \tag{4.3}$$

It is of interest to consider the term $\nabla_{II}\sigma$ which gives rise to the Marangoni effect.

We may suppose that $\sigma = \sigma(T)$ depends on the temperature. Hence

$$\nabla_{II}\sigma = \sigma'(T)\nabla_{II}T$$

where $T(x,y,z,t)$ is evaluated on $z = \delta(x,y,t)$. This formula makes sense because $[T] = 0$, so $\sigma'(T)\nabla_{II}T$ is unique. Since

$$T(x,y,z,t) = \theta(x,y,z,t) - Az$$

we find, using (4.3), that

$$\nabla_{II}T = \mathbf{e}_z(\theta_z - A) + \nabla_2\theta - \frac{\mathbf{e}_z - \nabla_2\delta}{1+|\nabla_2\delta|^2}[\theta_z - A - \delta_x\theta_x - \delta_y\theta_y] \qquad (4.4)$$

with $z = \delta(x,y,t)$ and $\theta_z(x,y,\zeta(x,y,t),t)$, etc.

It is convenient, and conventional, to treat σ' as a constant and

$$\sigma'\nabla_{II}T = \sigma'(\nabla_{II}\theta - A\nabla_{II}\delta) \ . \qquad (4.5)$$

Since $\delta = [\theta]/[A]$, we can eliminate δ from all the equations.

5. Energy identities.

The following two energy identities may be derived, without approximation

$$\frac{d}{dt}\left(E + \psi + \int_\Sigma \sigma d\Sigma\right) + \int_\Sigma \frac{D\sigma}{Dt} d\Sigma + \oint \sigma \mathbf{U}.\boldsymbol{\tau}\, d\ell$$

$$+ \mathcal{D} + g\langle\rho\alpha\theta w\rangle = 0 \ , \qquad (5.1)$$

$$\frac{d}{dt}\left\langle\rho c \frac{\theta^2}{2}\right\rangle - \langle w\theta A\rangle + \langle k|\nabla\theta|^2\rangle = \int_\Sigma [k\theta\theta_n] d\Sigma \qquad (5.2)$$

where

$$\psi = g\int_{-L_x}^{L_x}\int_{-L_y}^{L_y} [[\rho]\delta^2/2 + [\rho\alpha A]\delta^3/3]\, dx\, dy \ ,$$

$$E = \tfrac{1}{2}\langle\rho|\mathbf{u}|^2\rangle = \int_{-L_x}^{L_x}\int_{-L_y}^{L_y} \{ \int_{-\ell_2}^{\delta} \tfrac{1}{2}\rho_2|\mathbf{u}|^2 dz + \int_{\delta}^{\ell_1} \tfrac{1}{2}\rho_1|\mathbf{u}|^2 dz \} \, dx\, dy \quad , \tag{5.3}$$

is the total energy and

$$D = \langle 2\mu|D[\mathbf{u}]|^2\rangle$$

is the total dissipation. Equations (5.1,2) follow from standard calculations using (4.1,2), $\mathbf{u}.\mathbf{n}\, d\Sigma = \delta_t\, dx\, dy$ and the transport theorem for surface areas (see (96.11) and (96.12) in Joseph (1976)).

If the surface tension is constant and the contact line has a potential $\hat{\psi}$ (see Guillope, Joseph, Nguyen and Rosso [1986]), then

$$\frac{d}{dt}\{ E + \psi + \hat{\psi} + \sigma|\Sigma| \} + D + g\langle\rho\alpha\theta w\rangle = 0 \ . \tag{5.4}$$

It is not possible to represent the interface terms in (5.2) as a potential. The term

$$\int_{\Sigma} [k\theta\theta_n] \, d\Sigma \tag{5.5}$$

is particularly nasty from a mathematical point of view. This term is not bounded in the space of square integrable functions, with square integrable gradients, which vanish at the top and bottom plate. We cannot estimate this integral in terms of $\langle|\nabla\theta|^2\rangle$ and $\langle\theta^2\rangle$. The problem is that θ is discontinuous across $z = \delta$.

The energy identities have a slightly simpler form when linearized. The δ^3 term in ψ may be neglected and Σ is reduced to a plane on $\delta = 0$, for example, put $\delta = 0$ in (5.3) and (5.5) is replaced by

$$\int_{-L_x}^{L_x}\int_{-L_y}^{L_y} [k\theta \frac{\partial\theta}{\partial z}]\, dx\, dy \quad . \tag{5.6}$$

The term (5.6) is just as nasty mathematically as (5.5). We cannot give *a priori* estimates of (5.6) in terms of $<|\nabla\theta|^2>$ and $<\theta^2>$.

6. Energy inequalities, stability estimates.

Since $[\mathbf{u}] = 0$, the Poincaré inequality $\mathcal{D} \geq 2\bar{\lambda}E$ may be applied to (5.4):

$$\frac{d}{dt}\,[E + \psi + \hat{\psi} + \sigma|\Sigma|] + g<\rho\alpha\theta w> \leq -2\bar{\lambda}E \quad . \tag{6.1}$$

This inequality holds for all kinds of interfaces, not just the ones for which there is a graph.

An energy inequality for (5.2) may also be derived, but it has a more restricted range of validity. Suppose that $z = \delta(x,y,z,t)$ is a smooth graph. Each vertical line perpendicular to the (x,y) plane pierces the interface once and only once. Let $\theta_1(z)$ stand for $\theta(x,y,z,t)$ for $z > 0$, $\theta_2(z)$ when $z < 0$.

$$\theta_1(z) = \int_z^{\ell_1} \frac{\partial\theta_1}{\partial z}\, dz \leq (\ell_1 - z)^{\frac{1}{2}}\left(\int_z^{\ell_1} \left(\frac{\partial\theta_1}{\partial z}\right)^2 dz\right)^{\frac{1}{2}}$$

$$\leq (\ell_1 - z)^{\frac{1}{2}}\left(\int_\delta^{\ell_1} |\nabla\theta_1|^2 dz\right)^{\frac{1}{2}} ,$$

$$\theta_2(z) = \int_{-\ell_2}^{z} \frac{\partial\theta_2}{\partial z}\, dz \leq (\ell_2 + z)^{\frac{1}{2}}\left(\int_{-\ell_2}^{\delta} |\nabla\theta_2|^2 dz\right)^{\frac{1}{2}} \tag{6.2}$$

$$\int_\delta^{\ell_1}\int_{-L_x}^{L_x}\int_{-L_y}^{L_y} \theta_1^2\, dydxdz \leq \tfrac{1}{2}(\ell_1 - \delta)^2 \int_\delta^{\ell_1}\int_{-L_x}^{L_x}\int_{-L_y}^{L_y} |\nabla\theta_1|^2\, dxdydz.$$

Hence

$$\int_{V_1} \theta_1^2 \, dV \le \tfrac{1}{2}(\ell_1 - \delta)^2 \int_{V_1} |\nabla\theta_1|^2 \, dV \ ,$$

$$\int_{V_2} \theta_2^2 \, dV \le \tfrac{1}{2}(\ell_2 + \delta)^2 \int_{V_2} |\nabla\theta_2|^2 \, dV \ . \tag{6.3}$$

Let $D^2(\delta) = \min\,[(\ell_1-\delta)^2, (\ell_2+\delta)^2]$. Then

$$\langle|\nabla\theta|^2\rangle \ge \frac{2}{D^2(\delta)} \langle\theta^2\rangle \ . \tag{6.4}$$

It may also be of interest that

$$[A]\delta = [\theta] = -\int_\delta^{\ell_1} \frac{\partial u_1}{\partial z}\, dz - \int_{-\ell_2}^{\delta} \frac{\partial u_2}{\partial z}\, dz$$

$$\le (\ell_1-\delta)^{\frac{1}{2}} \left[\int_\delta^{\ell_1} \left(\frac{\partial u_1}{\partial z}\right)^2 dz\right]^{\frac{1}{2}} + (\ell_2+\delta)^{\frac{1}{2}} \left[\int_{-\ell_2}^{\delta} \left(\frac{\partial u_2}{\partial z}\right)^2 dz\right]^{\frac{1}{2}} \ .$$

Using (6.4) in (5.2) we find that

$$\frac{d}{dt} \langle\rho c \frac{\theta^2}{2}\rangle - \langle w\theta A\rangle - \int [k\theta\theta_n]\, d\Sigma \le - \frac{2}{D^2(\delta)} \langle k\theta^2\rangle \ . \tag{6.5}$$

The bilinear integrals $\langle\rho\alpha\theta w\rangle$ and $\langle Aw\theta\rangle$ are easily estimated in terms of E and $\tfrac{1}{2}\langle\rho c\theta^2\rangle$.

We could regard (6.1) and (6.5) as our basic energy inequalities. We cannot use them to prove a condition for global (unconditional) stability because we cannot estimate the interface term (5.5), even in the linearized case (5.6). In the problem of the stability of plane-Couette flow of two fluids in two layers there is also a nasty interface term which cannot be estimated *a priori* in terms of integrals over the squares of velocity and temperature and the square of the gradients. This interface term leads to instability at zero Reynolds numbers when the volume ratio of thin to thick fluid is large (Yih [1967]) or when the wave length of the disturbance and surface tension tend to zero (Hooper and Boyd [1983]).

The question is: can anything definite be proved about the stability of the conduction solution in two layers of two fluids heated from below using the method of energy?

7. The spectral problem and the adjoint spectral problem.

We linearize (4.1, 2) and replace

$$[\mathbf{u}(x,y,z,t),\ \theta(x,y,z,t),\ p(x,y,z,t),\ \delta(x,y,t)]$$

with

$$e^{st}[\mathbf{u}(x,y,z),\ \theta(x,y,z),\ p(x,y,z),\ \delta(x,y)] \tag{7.1}$$

where

$$s = \xi + i\eta \quad .$$

The linearized equations are rigorously expressed relative to the reference domains V_1^o and V_2^o above and below $\delta = 0$. All jumps are evaluated across $z = 0$ and $\delta(x,y) = [\theta](x,y,0)/[A]$.

The spectral equations are obtained by substitution of (7.1) into the linearized equation

$(\mathbf{u},\theta) \in \mathbb{H}$ where $\mathbb{H}$ is the extension of H into complex values.

The spectral equations are (7.2, 3) below when C_ℓ's on the right side are put to zero

$$\begin{aligned} &\rho s\mathbf{u} + \nabla p - \rho\alpha g\theta \mathbf{e}_z - 2\mu\,\mathrm{div}\,\mathbf{D}[\mathbf{u}] = \mathbf{C}_1 \\ &\rho c(s\theta - Aw) - k\nabla^2\theta = C_2 \end{aligned} \tag{7.2}$$

hold in V_1^o and V_2^o. Across $z = 0$

$$[w] = C_3 ,$$

$$[u] = C_4 ,$$

$$[v] = C_5 ,$$

$$[k\theta_z] = C_6 ,$$

$$[\mu(u_z+w_x)] - \sigma'(\theta_x - \frac{A}{[A]}[\theta_x]) = C_7 , \tag{7.3}$$

$$[\mu(v_z+w_y)] - \sigma'(\theta_y - \frac{A}{[A]}[\theta_y]) = C_8 ,$$

$$[2\mu w_z - p] - \{\sigma\nabla_2^2[\theta] - [\rho]g[\theta]\}/[A] = C_9 ,$$

$$[\theta]s - [A]w = C_{10}[A]$$

where $[C_n] = 0$, $n = 1,\ldots,9$ and $-[A]C_3 = [C_{10}]$ and $\sigma(T_o)$, $\sigma'(T_o)$ are constants.

The adjoint spectral problem is

$$(\mathbf{u}^*,\theta^*) \in \mathbb{H}$$

$$\rho(\bar{s}\mathbf{u}^* - CA\theta^*\mathbf{e}_z) + \nabla p^* - \operatorname{div} 2\mu\mathbf{D}[\mathbf{u}^*] = \mathbf{0} , \tag{7.4}$$

$$\rho(\bar{s}c\theta^* - \alpha g w^*) - k\nabla^2\theta^* = 0 ,$$

in V_1^o and V_2^o. On $z = 0$ we find that

$$[\theta^*] = 0 ,$$

$$[w^*] = 0 , \tag{7.5}$$

$$[u^*] = 0 ,$$

$$[v^*] = 0 ,$$

$$[\mu(u_z^* + w_x^*)] = 0 \ ,$$

$$[\mu(v_z^* + w_y^*)] = 0 \ , \qquad (7.5)$$

$$[k\theta_z^*] + \sigma'(u_x^* + v_y^*) = 0 \ ,$$

$$\sigma\nabla_2^2 w^* - [\rho]gw^* + \bar{s}[p^* - 2\mu w_z^*] - k_2\theta_z^*[A] + A_1\sigma'(u_x^* + v_y^*) = 0 \ .$$

To derive the adjoint problem we multiply $(7.2)_1$ by $\bar{\mathbf{u}}^*$, the conjugate of $\mathbf{u}^*$ and $(7.2)_2$ by $\bar{\theta}^*$. First we integrate over V_1^o, then over V_2^o, we integrate by parts, putting some terms on the interface and add the result.

$$\langle\rho(s\mathbf{u} - \alpha g\theta \mathbf{e}_z).\bar{\mathbf{u}}^* + \rho c(s\theta - Aw)\bar{\theta}^* - \mathbf{u}.\operatorname{div} 2\mu\mathbf{D}[\bar{\mathbf{u}}^*] - k\theta\nabla^2\bar{\theta}^*\rangle$$

$$+ \int_{-L_x}^{L_x} dx \int_{-L_y}^{L_y} dy \ \{[(-p+2\mu w_z)\bar{w}^* + \mu(u_z+w_x)\bar{u}^* + \mu(v_z+w_y)\bar{v}^*$$

$$- 2\mu\bar{w}_z^* w - \mu(\bar{u}_z^*+\bar{w}_x^*)u - \mu(\bar{v}_z^*+\bar{w}_y^*)v + k\theta_z\bar{\theta}^* - k\theta\bar{\theta}_z^*]\}$$

$$= \langle \mathbf{C}_1.\bar{\mathbf{u}}^*\rangle + \langle C_2\bar{\theta}^*\rangle \ . \qquad (7.6)$$

We put C_1, c_2 to zero and choose $\mathbf{u}$, θ, p to vanish first outside V_1^o, then outside V_2^o. For this special choice of $(\mathbf{u},\theta) \in \mathbb{H}$ we get

$$\langle(\rho s\bar{\mathbf{u}}^* - A\rho c\bar{\theta}^*\mathbf{e}_z - \operatorname{div} 2\mu\mathbf{D}[\bar{\mathbf{u}}^*]).\mathbf{u}$$

$$+ (\rho c s\bar{\theta}^* - \rho\alpha g\bar{w}^* - k\nabla^2\bar{\theta}^*)\theta\rangle = 0.$$

Since the special θ is unrestricted we may conclude that

$$\rho c\bar{s}\theta^* - \rho\alpha g w^* - k\nabla^2\theta^* = 0 \ . \qquad (7.7)$$

The special $\mathbf{u}$ has $\operatorname{div}\mathbf{u} = 0$ so that the most we can conclude is that $\mathbf{u}$ is orthogonal to gradients

$$\rho\bar{s}\mathbf{u}^* - A\rho c\theta^*\mathbf{e}_z - \operatorname{div} 2\mu D[\mathbf{u}^*] = -\nabla p^* . \tag{7.8}$$

Returning now to (7.6) with a general $\mathbf{u}$ we may replace the volume integral term on the right with

$$-\langle\nabla\bar{p}^*.\mathbf{u}\rangle = \int_{-L_x}^{L_x} dx \int_{-L_y}^{L_y} dy\ p^*\bar{w}^* .$$

Then

$$\int_{-L_x}^{L_x} dx \int_{-L_y}^{L_y} dy\ [(-p+2\mu w_z)\bar{w}^* - (-\bar{p}^* + 2\mu\bar{w}_z^*)w$$

$$+ \mu(u_z+w_x)\bar{u}^* - \mu(\bar{u}_z^*+\bar{w}_x^*)u$$

$$+ \mu(v_z+w_y)\bar{v}^* - \mu(\bar{v}_z^*+\bar{w}_y^*)v$$

$$+ k\theta_z\bar{\theta}^* - k\theta\bar{\theta}_z^*]$$

$$= \langle C_1.\bar{u}^*\rangle + \langle c_2\bar{\theta}^*\rangle . \tag{7.9}$$

We put C_1 and c_2 to zero, make some special choices of u, v, w, θ and deduce that

$$[\theta^*] = [u^*] = [v^*] = [w^*] = [\mu(u_z^* + w_x^*)] =$$

$$= [\mu(v_z^* + w_y^*)] = 0 . \tag{7.10}$$

Returning to (7.9) with (7.10) we find that

$$\int_{-L_x}^{L_x} dx \int_{-L_y}^{L_y} dy \; \{\bar{w}^*[-p + 2\mu w_z] - w[-\bar{p}^* + 2\mu\bar{w}_z^*]$$

$$+ \bar{u}^*[\mu(u_z + w_x)] + \bar{v}^*[\mu(v_z + w_y)] - [k\theta\bar{\theta}_z^*]$$

$$- \mu(\bar{u}_z^* + \bar{w}_x^*)[u] - \mu(\bar{v}_z^* + \bar{w}_y^*)[v] - \bar{\theta}^*[k\theta_z]$$

$$= \langle \mathbf{C}_1 . \bar{u}^* \rangle + \langle c_2 \bar{\theta}^* \rangle \; .$$

After applying (7.3) to the above we get

$$\int_{-L_x}^{L_x} dx \int_{-L_y}^{L_y} dy \; \{ \frac{\bar{w}^*}{[A]} (\sigma\nabla_2[\theta] - [\rho]g[\theta])$$

$$- s\frac{[\theta]}{[A]} [-\bar{p}^* + 2\mu\bar{w}_z^*] + \bar{u}^*\sigma'(\theta_x - \frac{A}{[A]}[\theta_x])$$

$$+ \bar{v}^*\sigma'(\theta_y - \frac{A}{[A]}[\theta_y]) - [k\theta\bar{\theta}_z^*]$$

$$- C_4\mu(\bar{u}_z^* + \bar{w}_x^*) - C_5\mu(\bar{v}_z^* + \bar{w}_y^*) - C_6\bar{\theta}^* + C_7\bar{u}^*$$

$$+ C_8\bar{v}^* + C_9\bar{w}^* + C_{10}[-\bar{p}^* + 2\mu\bar{w}_z^*]$$

$$= \langle C_1 . \bar{u}^* \rangle + \langle c_2 \bar{\theta}^* \rangle \; . \qquad (7.11)$$

Now we put the C_ℓ's to zero and integrate by parts

$$\int_{-L_x}^{L_x} dx \int_{-L_y}^{L_y} dy \; \{[\theta](\sigma\nabla_2^2\bar{w}^* - [\rho]g\bar{w}^* - s[-\bar{p}^* + 2\mu\bar{w}_z^*])$$

$$- (\bar{u}_x^* + \bar{v}_y^*)(\theta[A] - A[\theta])\sigma' - [k\theta\bar{\theta}_z^*][A]\} = 0 \; . \qquad (7.12)$$

The integrand in (7.12) may be set to zero

$$[\theta\{\sigma\nabla_2^2\bar{w}^* - [\rho]g\bar{w}^* - s[-\bar{p}^* + 2\mu\bar{w}_z^*] + A_1\sigma_1'(\bar{u}_x^*+\bar{v}_y^*)$$

$$- k\bar{\theta}_z^*[A]\}] - \theta_1(\bar{u}_x^*+\bar{v}_y^*)[A]\sigma' = 0 \quad (7.13)$$

provided that

$$\sigma\nabla_2^2 w^* - [\rho]gw^* - \bar{s}[-p^* + 2\mu w_z^*]$$

$$+ A_2(u_x^* + v_y^*) - k\theta_{1z}^*[A] = 0 \quad (7.14)$$

and

$$\sigma\nabla_2^2 w^* - [\rho]gw^* - \bar{s}[-p^* + 2\mu w_z^*]$$

$$+ A_1\sigma'(u_x^* + v_y^*) - k\bar{\theta}_{2z}^*[A] = 0 . \quad (7.15)$$

Equations (7.14, 15) imply that

$$\sigma'(u_x^* + v_y^*) + [k\theta_z^*] = 0 .$$

8. Solvability conditions.

Equations (7.2) and (7.3) cannot be solved unless the inhomogeneous terms satisfy the following conditions of orthogonality with eigenfunctions in the null space of the adjoint. The conditions required follow directly from (7.11) and (7.13). Thus

$$\int_{-L_x}^{L_x} dx \int_{-L_y}^{L_y} dy \,\{C_7\bar{u}^* + C_8\bar{v}^* + C_9\bar{w}^* + C_{10}[-\bar{p}^* + 2\mu\bar{w}^*_z]$$

$$- C_4\mu(\bar{u}_z^* + \bar{w}_x^*) - C_5\mu(\bar{v}_z^* + \bar{w}_y^*) - C_6\bar{\theta}^*\}$$

$$= \langle \mathbf{C}_1\cdot\bar{\mathbf{u}}^*\rangle + \langle C_2\bar{\theta}^*\rangle . \quad (8.1)$$

Equation (8.1) is a fundamental tool for the construction and study of bifurcating solutions.

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