HYPERBOLIC PHENOMENA IN THE FLOW OF VISCOELASTIC FLUIDS

Daniel D. Joseph

University of Minnesota
Department of Aerospace Engineering and Mechanics
Minneapolis, Minnesota

ABSTRACT

This paper treats the problem of hyperbolicity, change of type and nonlinear wave propagation in the flow of viscoelastic fluids. Rate equations for fluids with and without instantaneous elasticity are derived and discussed. The equations of fluids with instantaneous elasticity are hyperbolic in unsteady flow and can change type in steady flow. The wave speeds depend on velocities and stresses. Some estimates of wave speeds into states of rest are given. For many of the popular models of fluids the vorticity is the field variable which changes type. The vorticity of all fluids with instantaneous elasticity can change type in motions which perturb rigid ones. Experiments and analysis exhibiting vorticity of changing type are exhibited. The linearized viscoelastic problem is governed by equations having the properties of a telegraph equation. The damping is small when the fluid is very elastic. Elastic fluids have a long memory, a large time (Weissenberg number) for relaxation. The damping is rapid when the relaxation time is small even when the flow is very supercritical. It is shown that steady flow around a body is of "transonic" type. The linearized problem for flow over a flat plate is reduced to an integral equation for the vorticity distribution on the plate. The problem of nonlinear wave propagation is discussed and the problems of nonlinear smoothing and shocking are considered. It is shown (by M. Slemrod) that the shocks of vorticity can arise from smooth data in some models and shocks of velocity in other models.

1. INTRODUCTION

The equations of gas dynamics have a hyperbolic structure which support waves of compression and rarefaction. The linearized theory of gas dynamics leads to the wave equation in which compression and rarefaction waves are on an equal footing in that both types of waves may propagate without change of form. The nonlinear theory of gas dynamics leads to striking new qualitative understandings of wave propagation. Initially smooth waves of compression must shock up. Initial shocks of rarefaction must smooth out. The equations of steady gas dynamics change type from subsonic to supersonic when the speed of the fluid at some point exceeds the speed of sound. The field of flow can be partitioned into subsonic parts, governed by an elliptic equation, and a supersonic part, governed by a hyperbolic equation. The problem of change of type in gas dynamics is called transonic flow.

Quasilinear systems of equations governing the flow of incompressible viscoelastic fluids are also of mixed type when the constitutive equation admits an instantaneous elastic response. As in gas dynamics, steady flows of such fluids can change type.

I am interested in the possibility that many effects in the flow of viscoelastic liquids, as well as difficulties in numerical simulation are associated with wave propagation, the nonlinear smoothing of shocks, the shock up of smooth solutions and with the appearance of real characteristics and a change of type analogous to the sonic transition in gas dynamics.

The notes for this lecture are a compendium for hyperbolic things arising in the theory of flow of viscoelastic fluids with instantaneous elasticity.

Fluids with instantaneous elasticity have no instantaneous viscosity in the same way that Maxwell models have no instantaneous viscosity and Jeffrey's models do. For example

$$\lambda \frac{D\underline{\tau}}{Dt} + \underline{\tau} = 2\eta \underline{D}[\underline{u}] \tag{1.1}$$

is an Oldroyd rate equation for the determinate part of stress $\underline{\tau}$, D/Dt is an invariant time derivative, $\underline{D}[\underline{u}] = \frac{1}{2}(\nabla\underline{u} + \nabla\underline{u}^T)$, where $\underline{u}(\underline{x},t)$ is the velocity, λ is a relaxation time and η is the "viscosity". Equation (1.1) is a Maxwell model. It has a purely elastic instantaneous response. On the other hand

$$\lambda \frac{D\underline{\tau}}{Dt} + \underline{\tau} = 2\eta \underline{D}[\underline{u}] + \Lambda \frac{D\underline{D}[\underline{u}]}{Dt} \tag{1.2}$$

is an Oldroyd model with a retardation time Λ. This equation is of the Jeffrey's type. The retardation term produces an instantaneous viscous response superimposed on the elastic response. Λ, rather than η, is the measure of the viscous response to impulsive motion. The most general class of simple fluids with instantaneous elasticity are those whose histories are convergent in the weighted L_2 spaces of Coleman and Noll. Joseph, Renardy and Saut (1984), hereafter called JRS, derived the general form of the rate equation which is implied by the fading memory theory of Coleman and Noll. More general theories of fading memory based on different topologies containing, for example, models like (1.2), have been presented by Saut and Joseph (1983).

It is probable that most polymer solutions have some instantaneous viscous response following, say, a model like (1.2) with $\Lambda \neq 0$, but perhaps small. If the instantaneous viscous response is small, say Λ/λ is small, the smoothing effects of viscosity will also be small leading to shock structure of the type which is well known in gas dynamics. A theory for such viscous smoothing of shocks in viscoelastic fluids has been discussed by Saut and Joseph (1983) and Narain and Joseph (1983). It follows that the study of fluids with instantaneous elasticity ought to apply to many real fluids which have a small viscous response.

The visco part of viscoelasticity does not mean that the fluid has a viscous response. It means that the amplitude of impulsive, shock solutions will decay but that the shocks will not smooth. This property of flow is not shared by gas dynamics where there is no damping. When the damping is very large (e.g; when λ is small), the elastic response is short. When the damping is small (e.g.; when λ is large), the elastic response is long. In most problems with damping, even the simple ones which arise in the one dimensional theory of kinematic waves (Whitham, 1974, p. 62) it is not possible to form a shock without first satisfying a criterion for a critical amplitude. Such critical amplitude results are now well known in the theory of nonlinear wave propagation in viscoelastic fluids (see Section 13).

There are possibly many works on wave propagation in viscoelastic fluids. Of the ones known to me, the works of Coleman and Gurtin (1968) and Slemrod (1978) seem to me to be the most important. Problems which perturb uniform flows of Oldroyd fluids have been studied by Giesekus (1970), Ultmann and Denn (1970) and Luskin (1984). In these problems the entire field of flow is either subcritical or supercritical. There is no problem of transonic type. The possibility of transonic type in the flow of Oldroyd fluids was first mentioned by Rutkevich (1970), Joseph, Renardy and Saut (1984) treated the problem of hyperbolicity and change generally. They introduced the notion that in many models and in all models on motions perturbing uniform ones, the vorticity is the hyperbolic variable which changes type. Yoo, Ahrens and Joseph (1984) tried to explain the striking results of Metzner, Uebler & Fong (1969) with an analysis of the vorticity perturbing irrotational sink flow. This type of hyperbolic flow with zero vorticity in a cone like region and nonzero vorticity outside was observed in the flow into a hole.

The first detailed solution of flow exhibiting a change of type has recently been given by Yoo and Joseph (1984). They consider the linearized problem which arises when Poiseuille flow of an upper convected Maxwell model is perturbed with wavy walls. The vorticity of this flow will change type when the velocity in the center of the pipe is larger than a critical value defined by the propagation of shear waves. There is then a region around the pipe axis in which the vorticity equation is hyperbolic and a low speed region near the walls where the vorticity equation is elliptic. They linearize the problem for small amplitude waviness and the linearized problem is solved in detail. The characteristic nets depend on the viscoelastic "Mach" number, which is the ratio ($M = U/C$) of the unperturbed maximum velocity U to the speed of shear waves C into the fluid at rest and the elasticity number E. There is a supercritical (hyperbolic) region around the axis of the pipe when $M > 1$. When $M \gg 1$, the diameter of this hyperbolic region is small when E is large, and large when E is small. Regions of positive and negative vorticity are swept out along forward facing characteristics in the hyperbolic region. There is rapid damping of vorticity through a narrow layer to small vorticity in the core when $M \gg 1$ and the Weissenberg number $W = M\sqrt{E}$ tends to zero. (The Weissenberg number is proportional to the relaxation time of the fluid.)

The rate of damping of vorticity decreases as W is increased. Flows with high M appear to be more "elastic" when W is large in the sense that the damping is suppressed as the relaxation time of the fluid is increased.

I am going to discuss some selected topics from the aforementioned papers. I have introduced some new thoughts in these notes. The problem of flow around bodies treated in Section 11 and the flat plate problem of Section 12 seem not to have been considered as transonic type problems before. In Section 12 I have reduced the linearized supercritical problem for flow over a flat plate to a single integral equation. I would like to know if this problem can be solved. Various parts of the discussion of nonlinear wave propagation in Section 13 are new.

I wish to express my gratitude to my colleagues and collaborators in different aspects of this work; these are Michael Renardy, Jean Claude Saut and, more recently, John Yoo. I also want to acknowledge the help which I have received from my students Mark Ahrens, Edmond O'Donovan and Oliver Riccius. Marshal Slemrod has got a fine result, right in this ballpark, which appears as Appendix A to this work. Mark Ahrens prepared Appendix B.

2. RATE EQUATIONS FOR FLUIDS WITH INSTANTANEOUS ELASTICITY

It is perhaps useful to try to understand hyperbolicity in fluids with instantaneous elasticity without choosing special models. We want to know which phenomena are intrinsic and which are accidental arising only for one or the other of the special models.

A general theory of rate equations for fluids with instantaneous elasticity was given by JRS. Here we give a slightly different derivation leading to the general form of quasilinear problems associated with motions of fluids with instantaneous elasticity.

Let $\underline{C}_t(\underline{x},t) = \nabla\underline{\chi}_t^T \nabla\underline{\chi}_t$ be the relative Cauchy strain where $\underline{\chi}_t(\underline{x},t)$ is the position of $\underline{x} = \underline{\chi}_t(\underline{x},t)$ at $\tau=t$. The determinate stress $\underline{\tau}$ may be expressed by a functional of the history of the Cauchy strain,

$$\underline{\tau} = \underline{\tilde{F}}\,[t,\underline{C}_t(\underline{x},\tau)]_{\tau=-\infty}^{t}. \tag{2.1}$$

For the moment I state only that the domain of $\underline{\tilde{F}}$ is equipped with the topology of a Hilbert space. Saut and Joseph (1983) show how different choices of the topological domain of $\underline{\tilde{F}}$ lead to different types of constitutive equations. I want to make this point again, under (2.4), in the context of this discussion of the general form of a rate equation. $\underline{\tilde{F}}$ depends on the history of $\underline{C}_t(\underline{x},\tau)$ and explicitly on the present time, perhaps through a kernel function and in other ways.

It follows from differentiating (2.1) that

$$\frac{d\underline{\tau}}{dt} = \frac{\partial}{\partial t}\,\underline{\tilde{F}} + \underline{\tilde{F}}_1\Big[t,\underline{C}_t \Big| \frac{d\underline{C}_t}{dt}\Big] \tag{2.2}$$

where $\underline{\tilde{F}}_1[t,\underline{C}_t|\cdot]$ is a Frechet derivative, evaluated on $\underline{C}_t(\underline{x},t)$. The representation theorem for a linear functional on a Hilbert space leads us to

$$\underline{\tilde{F}}_1\left[t,\underline{C}_t \Big| \frac{d\underline{C}_t}{dt}\right] = \int_{-\infty}^{t} \underline{K}(t-\tau,\underline{C}_t(\underline{x},\tau))\,\frac{d\underline{C}_t}{dt}\,(\underline{x},\tau)dt \tag{2.3}$$

where $\underline{K}(t-\tau,\underline{C}_t(\underline{x},\tau))$ is a fourth order, tensor valued kernel function, not too singular, satisfying some invariance conditions arising from isotropy (see JRS). The term $\frac{\partial \underline{\tilde{F}}}{\partial t}$ is a derivative holding the history of $\underline{C}_t$ fixed. This term is again a functional of the form (2.1); in the case of single integral models with a kernel proportional to $e^{-(t-\tau)/\lambda}$ we have $\partial\underline{\tilde{F}}/\partial t = -\underline{\tau}/\lambda$ (cf. (2.19)). Motivated by this last observation we introduce the suggestive notation

$$\underline{\tilde{\tau}} \overset{\text{def}}{=} -\lambda\,\frac{\partial \underline{\tilde{F}}}{\partial t} \tag{2.4}$$

for a constant λ. $\underline{\tilde{\tau}}$ is stress-like and λ is a "relaxation time".

Saut and Joseph (1983) have shown that the smoothness of the kernel $\underline{K}$ in (2.3) depends on the choice of the topology of convergence for the history $\underline{C}_t(\underline{x},\tau)$. If $\frac{d\underline{C}_t}{dt}\,(\underline{x},\cdot)$ lies in the weighted $L_h^2(0,\infty)$ (fading memory) spaces of Coleman and Noll, then the kernel $\underline{K}$ must have at least the smoothness which is required for the validity of the application of the Schwarz inequality to the integral in (2.3). If $\frac{d\underline{C}_t}{dt}$ lies in the (fading memory) Sobolev spaces used by Saut and Joseph (1983), then various derivatives of Dirac measures can appear in the kernel $\underline{K}$. These give rise to "viscosity" terms like the retardation term in (1.2). The fading memory spaces of Coleman and Noll allow only those constitutive equations which exhibit instantaneous elasticity. The $L_h^2(0,\infty)$ fading memory spaces of Coleman and Noll therefore do not form a general basis for discussion of problems of rheology. They do form a general basis

for discussion of problems of instantaneous elasticity. In the work below dom $\underline{\tilde{F}} = L_h^2(0,\infty)$.

Since the position at $\tau < t$ of all particles $\underline{x}$ on the same path line is the same

$$\frac{d\chi_t}{dt} = \frac{\partial \chi_t}{\partial t} + \underline{u}(x,t)\cdot\nabla\chi_t = 0 \tag{2.5}$$

we have

$$\frac{d\underline{F}_t}{dt} = \nabla\frac{\partial\chi_t}{\partial t} + (\underline{u}\cdot\nabla)\nabla\chi_t = -\nabla(\underline{u}\cdot\nabla\chi_t) + (\underline{u}\cdot\nabla)\nabla\chi_t = -\underline{F}_t\underline{L}$$

where $\underline{L} = \underline{L}(\underline{x},t) = \nabla\underline{u}(\underline{x},t)$ and $\underline{F}_t = \nabla\chi_t(\underline{x},t)$ Hence,

$$\frac{d\underline{C}_t(x,\tau)}{dt} = \frac{d\underline{F}_t^T}{dt}\,\underline{F}_t + \underline{F}_t^T\,\frac{d\underline{F}_t}{dt} = -\underline{L}^T(\underline{x},t)\underline{C}_t(\underline{x},\tau) - \underline{C}_t(\underline{x},\tau)\underline{L}(\underline{x},t) \tag{2.6}$$

and

$$\left(\int_{\infty}^{t} \underline{K}(t-\tau,\underline{C}_t)\frac{d\underline{C}_t}{dt}\,d\tau\right)_{ij} = \int_{-\infty}^{t} K_{ijk\ell}\,\frac{dC_{k\ell}}{dt}\,dt = M_{ijkp}(\underline{x},t)L_{pk}(\underline{x},t) \tag{2.7}$$

where

$$M_{ijkp}(\underline{x},t) = -\int_{-\infty}^{t} (K_{ijk\ell} + K_{ij\ell k})\,\underline{C}_{p\ell}\,(\underline{x},\tau)d\tau \tag{2.8}$$

is symmetric in the first pair of indices and $L_{pk} = \frac{\partial u_p}{\partial x_k}$. We may write (2.8) as

$$\frac{d\underline{\tau}}{dt} = \underline{M}(\underline{x},t)\,\underline{L}(\underline{x},t) - \frac{1}{\lambda}\,\underline{\tilde{\tau}} \tag{2.9}$$

where $\underline{M}$ is the fourth order tensor valued functional defined by (2.8). Together with the equations of motion

$$\rho\frac{du}{dt} = -\nabla p + \text{div}\ \underline{\tau}\ ,\ \text{div}\ \underline{u}=0, \tag{2.10}$$

(2.9) and (2.10) are a quasilinear system in derivatives of $\underline{u}$ and $\underline{\tau}$. The stress-like term $\tilde{\underline{\tau}}(\underline{x},t)$ and the coefficients $\underline{M}(\underline{x},t)$ are of lower order (see JRS) since they do not involve derivatives of $\underline{u}$ or $\underline{\tau}$, and have other required properties. Hence, the system

$$\frac{\partial\tau_{ij}}{\partial t} + u_e\frac{\partial\tau_{ij}}{\partial x_e} - M_{ijke}\frac{\partial u_e}{\partial x_k} + \frac{1}{\lambda}\tilde{\tau}_{ij} = 0,$$

$$\rho\left[\frac{\partial u_i}{\partial t} + u_e\frac{\partial u_i}{\partial x_e}\right] + \frac{\partial p}{\partial x_i} - \frac{\partial}{\partial x_j}\tau_{ij} = 0,$$

$$\frac{\partial u_e}{\partial x_e} = 0 \tag{2.11}$$

is linear in the derivatives of u_i, τ_{ij} and p.

We may decompose the fourth order tensor $\underline{M}$ into symmetric and skew symmetric parts,

$$M_{ijke} = S_{ijke} + A_{ijke}$$

where $\underline{S}$ is symmetric and $\underline{A}$ is axisymmetric in k and e. We note that

$$S_{ijke}\frac{\partial u_e}{\partial x_k} = S_{ijke}D_{ek},$$

$$A_{ijke}\frac{\partial u_e}{\partial x_k} = A_{ijke}\Omega_{ek}$$

where

$$D_{ek} = \frac{1}{2}\left[\frac{\partial u_e}{\partial x_k} + \frac{\partial u_k}{\partial x_e}\right]$$

and

$$\Omega_{ek} = \frac{1}{2}\left[\frac{\partial u_e}{\partial x_k} - \frac{\partial u_k}{\partial x_e}\right] = \varepsilon_{ekm}\zeta_m$$

can be expressed in terms of the vorticity $\underline{\zeta}$ = curl $\underline{u}$. Bernard Coleman has shown that his thermodynamics imply that the fourth order tensor $\underline{S}$ is derivable from an energy and, as a consequence

$$S_{ijke} = S_{keij}$$

3. WAVE SPEEDS I, THEORETICAL

The quasilinear system is called evolutionary if the Cauchy initial value problem for perturbation of arbitrary motion is well posed (see Gelfand, 1963). We look at periodic initial data whose period is so small that $\underline{\tau},\underline{u}$,p are essentially constant over the period $2\pi/k$, $k=|\underline{k}|$ over which

$$(\delta\underline{\tau},\delta\underline{u},\delta p) = (\sigma,v,\pi)\exp i\underline{k}\cdot\underline{x}$$

are rapidly varying, $k\to\infty$. We seek propagating waves of the form

$$(\delta\underline{\tau},\delta\underline{u},\delta p) = (\underline{\sigma},\underline{v},\underline{\pi})\exp i(\underline{k}\cdot\underline{x}+\omega t) \tag{3.1}$$

and $(\underline{\sigma},\underline{v},\pi)$ depend only on the components of $\underline{u}$ and $\underline{M}$ which appear as coefficients of derivatives in the quasilinear system (2.11). The resulting equations are then divided by k and the right side of (2.11) vanishes as $k\to\infty$. This leads to the homogeneous equations for the components of $(\underline{\sigma},\underline{v},\pi)$ with $\underline{k}/k \overset{\text{def}}{=} \underline{n}$ fixed

$$\rho\left[\frac{\omega}{k} + \underline{u}\cdot\underline{n}\right]v_i + n_i\pi - n_j\sigma_{ij} = O(\tfrac{1}{k})$$

$$\left[\frac{\omega}{k} + \underline{u}\cdot\underline{n}\right]\sigma_{ij} - M_{ijke}v_kn_e = O(\tfrac{1}{k}) \tag{3.2}$$

$$v_en_e = 0$$

The homogeneous system corresponding to (3.2) can be solved if and only if the determinant of the coefficient vanishes. The system is evolutionary if there are real wave speeds

$$C = \lim_{k\to\infty} \frac{\omega}{k} .$$

There may of course be more than one wave speed. If C is not real there will be rapidly growing solutions. The reader may see more details of this type of analysis in the studies by Rutkevich (1969, 1972) of Oldroyd models.

4. WAVE SPEEDS II, PHYSICAL

The subject of gas dynamics would be in deep trouble if it were not possible to know the speed of sound. This type of deep trouble should be a concern in the study of hyperbolicity in the flow of viscoelastic fluids. In fact the way to know if a fluid has instantaneous elasticity is to show it has a wave speed. Now showing that a certain fluid has a wave speed is not easy. First of all, there are many fluids for which there are no wave speeds. These common cases are for fluids like Newtonian ones, or even viscoelastic ones with a big viscous response (say, Λ/λ is not small), which do not have an instantaneous elastic response, or a predominantly elastic response to impulsive data. Secondly, the fluids which admit a wave speed admit infinitely many wave speeds. Just as the speed of sound in a gas at a point depends on the thermodynamic condition there, a wave in a viscoelastic fluid at a point depends on the velocity and the state of stress there. So each problem gives rise to its own field of wave speeds.

The simplest type of wave propagation is into a region at rest. The wave speed for this is

$$C = \sqrt{\frac{G(0)}{\rho}} \tag{4.1}$$

where G(0) is the instantaneous value of the shear relaxation modulus G(s). In fact all kinds of small amplitude discontinuities will propagate with this speed, jumps in acceleration (Coleman and Gurtin, 1968), jumps in velocity and even in displacement (Narain and Joseph, 1982). The same propagation speed (4.1) holds for waves propagating into regions undergoing rigid motions. We shall show that flows which perturb rigid motions go supercritical when $U/C > 1$. This wave speed is a fundamental material parameter. There are at present no rheometers for measuring G(0) (or C) and there are no tables of values. E. H. Lieb, (1975), in his Ph.D. thesis, measured $C = 8.0$ cm/sec in one fluid (carbomethylcellulose in 50% glycerin and 49% water solution) using a rotating cylinder apparatus. For Maxwell models with a single relaxation time we have $G(0) = \eta/\lambda$, where η is the viscosity and λ is the relaxation time. There are some papers which report viscosities and relaxation times which are obtained by fitting data with some impulse experiments (for example, see Papanastasiou, Scriven and Macosko (1983), Dodson, Townsend and Walters (1971), Laun (1978). This type of rheometry suffers from two defects. Usually a special form for the kernel in the linear theory is assumed. The kernel is represented by an exponential or by a sum of exponentials. A more serious defect is that it is usually assumed, wrongly, that the early time response to impulse which gives G(0) satisfies an overly simplified asymptotic (for large time) theory (see Narain and Joseph (1984) for a full discussion and remedy for this defect). From the impulse data reported in many different papers we have estimated that wave speeds for water-based polymer solutions are of the order of 10 cm/sec. The wave speeds of polymer melts are of the order of 100 cm/sec. At the same time, the large viscosities of melts reduce the possibility of achieving large velocities. Hence for the flow of melts we expect that the flow is usually subcritical $M < 1$.

We have developed a wave speed meter which appears to give direct, reproducible values for C. A patent has been applied for.

5. VORTICITY

Some important special models give rise to a hyperbolic vorticity equation. It is of interest to derive the equation for the vorticity without making assumptions other than that the fluid has instantaneous elasticity, i.e., the quasilinear system (2.11) governs.

The vorticity $\underline{\zeta}$ satisfies

$$\rho\left(\frac{\partial}{\partial t}\underline{\zeta} + \underline{u}\cdot\nabla\underline{\zeta}\right) = \rho\underline{\zeta}\cdot\nabla\underline{u} + \text{curl div } \underline{\tau}. \tag{5.1}$$

Equation (5.1) is satisfied by $\underline{\zeta} = 0$ if and only if curl div $\underline{\tau} = 0$. This condition is always satisfied by Newtonian fluids for which $\underline{\tau} = \eta(\nabla\underline{u} + \nabla\underline{u}^T)$ for a constant viscosity η giving rise to curl div$\underline{\tau} = \eta\nabla^2\underline{\zeta}$. In this case potential flow is always a solution of (5.1). More generally we may verify that <u>potential flow</u> $\underline{u} = \nabla\Phi$, $\forall\, \Phi\ \ C^1(\underline{x})$ is a solution of (5.1) <u>if and only if div $\underline{\tau}$ is the gradient of</u>

a potential. Given the potential Φ we may obtain the velocity as a gradient $\underline{u} = \nabla\Phi$. Then $\underline{\zeta} = \text{curl } \underline{u} = 0$. The velocity $\underline{u}$ determines the path lines $\chi_t(\underline{x},\tau)$ and the past values of the relative Cauchy strain $\underline{C}_t(\underline{x},\tau)$. This gives rise, through (2.11), to the values $\underline{\tau}(\underline{x},t)$ of the extra stress at the present time t. It is not guaranteed that this computation will lead to a $\underline{\tau}(\underline{x},t)$ such that curl div $\underline{\tau} = 0$ at each and every point at which $\underline{\zeta} = 0$.

Whenever $\underline{u} = \nabla\Phi$ there are two more equations than unknowns. Evidently only certain special potential flows are compatible with (5.1). Uniform motions, say $\underline{u} = \underline{e}_x U$, for constant U, have $\underline{\tau} = 0$ and are always potential flows.

Another form of the vorticity equation is fundamental in the study of hyperbolicity and change of type. We first apply the substantial derivative $\frac{d}{dt}$ to (5.1). Then we apply $\frac{d}{dt}$ curl to (2.10), eliminate curl div $\frac{d\tau}{dt}$ and find that

$$\rho\frac{d^2\underline{\zeta}}{dt^2} - \text{curl div } (\underline{M}\underline{L}_T) = \underline{\Theta}_1 + \underline{\Theta}_2 + \underline{\Theta}_3 \tag{5.2}$$

where

$$\underline{L} = \nabla\underline{u},\ (\nabla u)_{ij} = \partial_j u_i$$

$$\underline{\zeta} = \text{curl } \underline{u},$$

$$\underline{\Theta}_1 = -\frac{1}{\lambda}\text{ curl div } \underline{\tilde{\tau}},$$

$$\underline{\Theta}_2 = \frac{d}{dt}(\text{curl div } \underline{\tau}) - \text{curl div } \frac{d\tau}{dt}, \tag{5.3}$$

$$\underline{\Theta}_3 = -\rho\frac{d}{dt}\left(\text{curl}\frac{du}{dt} - \frac{d}{dt}\text{ curl } \underline{u}\right)$$

Third derivatives of $\underline{u}$ are all on the left side of (5.2). The right side contains second derivatives of $\underline{u}$ and $\underline{\tau}$, at most. ($\underline{\Theta}_2$ is essentially a second derivative of $\underline{\tau}$.) Equation (2.9) shows that $\underline{\tau}$ and $\underline{u}$ are of the same differential order, so that the left side has third order derivatives of $\underline{u}$ and the left side has second order derivatives of $\underline{u}$, at most.

$$\rho\left[\partial_{tt}\zeta_k + 2\underline{u}\cdot\nabla\partial_t\zeta_k + u_e u_j \zeta_{k,ej}\right] - \varepsilon_{k\ell j}M_{jmpq}u_{q,pm\ell}$$

$$= \rho\left[(\partial_t + \underline{u}\cdot\nabla)(\underline{\zeta}\cdot\nabla)u_k - (\partial_t\underline{u}\cdot\nabla)\zeta_k - u_e u_{j,e}\zeta_{k,j}\right]$$

$$+ \varepsilon_{kej}[M_{jmpq,e}u_{q,pm} + (M_{kjmpq,m}u_{q,p})_{,e}$$

$$-u_{p,e}\tau_{jq,pq} + (u_{p,q}\tau_{jq,p})_{,e} + \frac{1}{\lambda}\tilde{\tau}_{jm,me}] \tag{5.4}$$

where $(\cdot)_{,m} \overset{\text{def}}{=} \frac{\partial}{\partial x_m}(\cdot)$. Equation (5.4) reduces to a second order equation for the vorticity plus lower order terms whenever the last term on the left of (5.4) can be expressed in terms of second order derivatives of $\underline{\zeta}$ (see (6.3)). Potential flow is possible only when (5.4) is satisfied with $\zeta=0$, $\underline{u}=\nabla\Phi$.

6. SPECIAL MODELS

Special constitutive models of fluids with instantaneous elasticity arise from (2.9) by choosing special forms for the fourth order tensor $\underline{M}$ and the stress-like term $\underline{\tilde{\tau}}$. In fact $\underline{\tilde{\tau}} = \underline{\tau}$ for all single integral models with separable kernels in exponential form with time constant λ.

We will consider four special families of constitutive models.

(1) JRS models (introduced by JRS (1984)).

(2) Three parameter family of "Maxwell" models introduced by Oldroyd.

(3) Models of the Giesekus type.

(4) BKZ models.

The JRS models arise when the fourth-order tensor $\underline{M}$ is expressed by the most general expression which involves second order tensors. This leads to

$$\frac{d\tau}{dt} = \underline{PD} + \underline{DP}^T + \frac{1}{2}(\underline{A\Omega} + (\underline{A\Omega})^T) + \frac{1}{\lambda}\tilde{\tau} \tag{6.1}$$

for an arbitrary second order tensor $\underline{P}$; the fourth order tensor $\underline{A}$ has already been introduced. Coleman's thermodynamics implies that $\underline{P} = \underline{P}^T$.

The Oldroyd models with three constants arises from (6.1) when
The Oldroyd models with three constants arises from (6.1) when
$\underline{P} = a\underline{\tau} + \frac{\eta}{\lambda}\underline{1}$, $-1 \le a \le 1$, $\underline{\tilde{\tau}} = \underline{\tau}$ and $\underline{A}$ chosen so that

$$\lambda\left[\frac{d\underline{\tau}}{dt} - a(\underline{D}\,\underline{\tau} + \underline{\tau}\,\underline{D}) + \underline{\tau}\,\underline{\Omega} - \underline{\Omega}\,\underline{\tau}\right] + \underline{\tau} = 2\eta\underline{D} \tag{6.2}$$

The upper, lower and corotational Maxwell model correspond to a=1,-1,0, respectively.

We may say that the JRS models are of the Giesekus type when

$$\tilde{\underline{\tau}} = \underline{\tau} + \alpha\underline{\tau}^2$$

with mobility constant α. The model actually proposed by Giesekus (1982) has retardation time on the right side which we put to zero for instantaneous elasticity, $\alpha\underline{\tau}^2$ is added to the left side and a = 1

$$\lambda\left[\frac{d\underline{\tau}}{dt} - \nabla\underline{u}\,\underline{\tau} - \underline{\tau}\,\nabla\underline{u}^T\right] + \underline{\tau} + \alpha\underline{\tau}^2 = 2\eta\underline{D} \tag{6.3}$$

Equations of the form (6.1), (6.2) and (6.3) permit hyperbolic waves of vorticity. The last term on the left of (5.4) may be expressed as a second order derivative of the vorticity

$$\varepsilon_{kej}M_{jmpq}u_{q,pme} = -\frac{1}{2}\varepsilon_{kej}P_{jq}\,(\text{curl }\underline{\zeta})_{q,e}$$
$$+ \frac{1}{2}P_{mp}\zeta_{k,mp} + \frac{1}{2}\varepsilon_{kej}\varepsilon_{pqr}A_{jmpq}\zeta_{r,me}$$

Then (5.4) may be regarded as a second order equation for the vorticity.

The vorticity equation for the three parameter family of Oldroyd models can be written as follows.

$$\rho\left[\partial_{tt}\zeta_k + 2\underline{u}\cdot\nabla\partial_t\zeta_k + u_eu_j\zeta_{k,ej}\right] + \frac{1}{2}(a-1)\varepsilon_{kej}\tau_{jq}[\text{curl }\underline{\zeta}]_{q,e}$$
$$-\frac{1}{2}(a+1)\tau_{mp}\zeta_{k,mp} - \frac{\eta}{\lambda}\nabla^2\zeta_k = -\frac{1}{\lambda}\varepsilon_{kej}\tau_{jm,me}$$
$$+ \rho\left[\left(\frac{\partial}{\partial t} + \underline{u}\cdot\nabla\right)(\underline{\zeta}\cdot\nabla u_k) - (\partial_t\underline{u}\cdot\nabla)\zeta_k - u_eu_{j,e}\zeta_{k,j}\right]$$
$$+ \varepsilon_{kej}\left[\frac{a+1}{2}\,\tau_{mp,m}\,\zeta_{k,p} + \frac{1-a}{2}\,\tau_{jq,e}\,(\text{curl }\underline{\zeta})_q\right]$$
$$+ \varepsilon_{kej}\Big[\frac{a+1}{2}(\tau_{mp,e}u_{j,mp} + \tau_{mp,me}u_{j,p} - \tau_{jm,mp}u_{p,e})$$
$$+ \frac{a-1}{2}\,(\tau_{mp,e}u_{p,jm} + \tau_{mp,me}u_{p,j} + \tau_{jp,me}u_{p,m}$$
$$+ \tau_{jp,m}u_{p,me} + \tau_{jp,me}u_{m,p} + \tau_{jp,m}u_{m,pe})\Big] \tag{6.4}$$

Moreover

$$\varepsilon_{kej}\tau_{jm,me} = (\text{curl div }\underline{\tau})_k = \rho\left[\text{curl }\frac{d\underline{u}}{dt}\right]_k = \rho\left[\frac{d\zeta_k}{dt} + (\zeta\cdot\nabla)u_k\right] \tag{6.5}$$

Potential flow with $\zeta=0$ can be a solution of (6.4) only if the last bracket in (6.4) vanishes $\underline{u}=\nabla\Phi$.

The BKZ model is a single integral model which is motivated by the nonlinear theory of elasticity. Let $\underline{C}(\tau)$ and $\underline{C}^{-1}(\tau)$ be the right relative Cauchy strain tensor with invariants

$$(I,II) = (\text{tr}\underline{C}(\tau),\text{tr}\underline{C}^{-1}(\tau))$$

and "strain energy"

$$W(t-\tau,I,II).$$

The determinate stress is given by

$$\underline{\tau} = \int_{-\infty}^{t} \{-W_{,II}(t-\tau,I,II)\ \underline{C}^{-1}(\tau) + W_{,I}(t-\tau,I,II)\underline{C}(\tau)\}d\tau \tag{6.6}$$

where $W_{,I} = \partial W/\partial I$, etc. The determinate stress vanishes on the rest state ($\underline{C}=\underline{C}^{-1}=\underline{1}$); hence

$$W_{,II}(t-\tau,3,3) = W_{,I}(t-\tau,3,3). \tag{6.7}$$

The rate equation satisfied by $\underline{\tau}$ may be obtained from (6.4) by differentiating with respect to t, using

$$\frac{d\underline{C}}{dt} = -\underline{L}^T(t)\underline{C}(\tau) - \underline{C}(\tau)\underline{L}(t),$$

$$\frac{d\underline{C}^{-1}}{dt} = \underline{L}(t)\underline{C}^{-1}(\tau) + \underline{C}^{-1}(\tau)\underline{L}^T(t)$$

$$\frac{dI}{dt} = tr\ \frac{d\underline{C}}{dt} = -2\ tr[\underline{C}(\tau)\underline{L}(t)]$$

$$\frac{dII}{dt} = tr\frac{d\underline{C}^{-1}}{dt} = 2\ tr[\underline{C}^{-1}(\tau)\underline{L}^T(t)] \tag{6.8}$$

We find that

$$\frac{d\underline{\tau}}{dt} + \frac{1}{\lambda}\ \tilde{\underline{\tau}} = \underline{L}\ \underline{P}_1 + \underline{P}_1\underline{L}^T + \underline{P}_2\underline{L}$$

$$+ \underline{L}^T\underline{P}_2 + \int_{-\infty}^{t} \{\underline{F}_1(\tau)tr[\underline{C}(\tau)\underline{L}(t)]$$

$$+ \underline{F}_2(\tau)tr[\underline{C}^{-1}(\tau)\underline{L}^T(t)]\}d\tau \tag{6.9}$$

where

$$\tilde{\underline{\tau}} = \int_{-\infty}^{t}\left[\frac{\partial W_{,II}}{\partial t}\ \underline{C}^{-1} - \frac{\partial W_{,I}}{\partial t}\underline{C}\right]d\tau$$

$$\underline{P}_1 = -\int_{-\infty}^{T} W_{,II}(t-\tau,I,II)\underline{C}^{-1}(\tau)d\tau$$

$$\underline{P}_2 = -\int W_{,I}(t-\tau,I,II)\underline{C}(\tau)d\tau$$

$$\underline{F}_1(\tau) = 2\left[W_{,I\ II}\underline{C}^{-1}(\tau) - W_{,II}\underline{C}(\tau)\right]$$

$$\underline{F}_2(\tau) = 2\left[-W_{,II\ II}\underline{C}^{-1}(\tau) + W_{,I\ II}\underline{C}(\tau)\right]$$

The last terms of (6.9) (under the integral) are of leading order in hyperbolic analysis because they contain first derivatives $L_{ij} = \partial_j u_i(\underline{x},t)$ of $\underline{u}_i(\underline{x},t)$ at the present time. These terms are spoilers because they lead to third derivatives of $\underline{u}$, as in (5.4), which do not reduce to second derivatives of $\underline{\zeta}$. This shows that not every popular fluid model has a vorticity of changing type.

It may be helpful to draw a distinction between two types of rate equations. The first type among our special examples depends only on the velocity and the stress. We may regard the stress components as unknown, to-be-determined, field variables, Equations (6.2) and (6.3). The other type, like (6.1) and (6.9), has coefficients which are functionals on the history of the Cauchy tensor. One could say that these equations are not "true" rate equations because the coefficients do not depend directly on the field variables and problems based on these equations must be posed as initial history rather than initial value problems. Rate equations of the first type are clearly convenient because the balance equations of mechanics are also relations among the velocity and stresses and the whole system of balance and constitutive euqations is then closed. Points of convenience are not necessarily points of principle.

7. CLASSIFICATION OF TYPE IN STEADY PLANE FLOW

In plane flows there are six equations for six unknowns; u_1, u_2, τ_{22}, τ_{11}, $\tau_{21}=\tau_{12}$, p. The quasilinear equations which governs the flow of fluids with instantaneous elasticity may be written as

$$\rho\frac{\partial u_1}{\partial t} + \rho u_1\frac{\partial u_1}{\partial x} + \rho u_2\frac{\partial u_1}{\partial y} + \frac{\partial p}{\partial x} - \frac{\partial \tau_{11}}{\partial x} - \frac{\partial \tau_{12}}{\partial y} = 0$$

$$\rho\frac{\partial u_2}{\partial t} + \rho u_1\frac{\partial u_2}{\partial x} + \rho u_2\frac{\partial u_2}{\partial y} + \frac{\partial p}{\partial y} - \frac{\partial \tau_{12}}{\partial x} - \frac{\partial \tau_{22}}{\partial y} = 0$$

$$\frac{\partial \tau_{11}}{\partial t} + u_1\frac{\partial \tau_{11}}{\partial x} + u_2\frac{\partial \tau_{11}}{\partial y} - \left[M_{1111}\frac{\partial u_1}{\partial x} + M_{1112}\frac{\partial u_2}{\partial x} + M_{1121}\frac{\partial u_1}{\partial y} + M_{1122}\frac{\partial u_2}{\partial y}\right] = \tilde{\tau}_{11}/\lambda$$

$$\frac{\partial \tau_{12}}{\partial t} + u_1\frac{\partial \tau_{12}}{\partial x} + u_2\frac{\partial \tau_{12}}{\partial y} - \left[M_{1211}\frac{\partial u_1}{\partial x} + M_{1212}\frac{\partial u_2}{\partial y} + M_{1221}\frac{\partial u_1}{\partial y} + M_{1222}\frac{\partial u_2}{\partial y}\right] = \tilde{\tau}_{12}/\lambda$$

$$\frac{\partial \tau_{22}}{\partial t} + u_1\frac{\partial \tau_{22}}{\partial x} + u_2\frac{\partial \tau_{22}}{\partial y} - \left[M_{2211}\frac{\partial u_1}{\partial x} + M_{2212}\frac{\partial u_2}{\partial x} + M_{2221}\frac{\partial u_1}{\partial y} + M_{2222}\frac{\partial u_2}{\partial y}\right] = \tilde{\tau}_{22}/\lambda$$

$$\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} = 0 \qquad (7.1)$$

where $(x_1,x_2) = (x,y)$. We may write this system as

$$\frac{\partial \tilde{q}}{\partial t} + \underline{A}\frac{\partial \tilde{q}}{\partial x} + \underline{B}\frac{\partial \tilde{q}}{\partial y} = \tilde{q} \qquad (7.2)$$

where $\tilde{q}$ stands for the column vector whose components are $[u_1,u_2,p,\tau_{11},\tau_{21},\tau_{22}]$. The characteristic directions $\alpha = dy/dx$ for steady flows are determined as the roots of

$$\det\left|\underline{A}\alpha - \underline{B}\right| = 0.$$

This may be written as

$$\begin{bmatrix} \rho(\alpha u_1 - u_2) & 0 & \alpha & -\alpha & 1 & 0 \\ 0 & \rho(\alpha u_1 - u_2) & -1 & 0 & -\alpha & 1 \\ -\alpha M_{1111} + M_{1121} & -\alpha M_{1112} + M_{1122} & 0 & \alpha u_1 - \alpha u_2 & 0 & 0 \\ -\alpha M_{1211} + M_{1221} & -\alpha M_{1212} + M_{1222} & 0 & 0 & \alpha u_1 - u_2 & 0 \\ -\alpha M_{2211} + M_{2221} & -\alpha M_{2212} + M_{2222} & 0 & 0 & 0 & \alpha u_1 - u_2 \\ \alpha & -1 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad (7.3)$$

$$= (\alpha u_1 - u_2)^2[a_4\alpha^4 + a_3\alpha^3 + a_2\alpha^2 + a_1\alpha + a_0] = 0$$

where

$$a_4 = M_{1212} - \rho u_1^2,$$

$$a_3 = -M_{2212} + M_{1211} - M_{1222} + M_{1112} + 2\rho u_1 u_2,$$

$$a_2 = -M_{2211} + M_{2222} - M_{1212} - M_{1221} + M_{1111} - M_{1122} - \rho(u_1^2 + u_2^2)$$

$$a_1 = M_{2221} - M_{1211} + M_{1222} - M_{1121} + 2\rho u_1 u_2,$$

$$a_0 = M_{1221} - \rho u_2 .$$

In general the roots of the 6th order polynomial (7.3) are neither all real nor all complex. The real roots correspond to real characteristic lines across which discontinuities in certain of the variables may propagate.

It is necessary to identify the real characteristic directions and the variables to which they associate. It is clear from (7.3) that streamlines are always doubly characteristic,

$$\left[\frac{dy}{dx} - \frac{u_2}{u_1}\right]^2 = 0. \tag{7.4}$$

The streamlines do not change type, from hyperbolic to elliptic because dy/dx is always real and never complex.

For problems of changing type we have to analyze the roots of the quartic

$$\alpha^4 a_4 + \alpha^3 a_3 + \alpha^2 a_2 + \alpha a_1 + a_0 = 0 \tag{7.5}$$

The roots of a quartic may be factored into the roots of two quadratics. The roots of the quadratics are real or complex. Real roots correspond to hyperbolicity with real characteristic directions.

For JRS fluids one of the quadratics has imaginary roots and the other can have complex or real roots depending on the values of parameter, velocities and stresses at a point. In this situation we encounter a change of type with real characteristic directions only in certain regions of the flow. The roots which govern a change of type are associated with the vorticity. The imaginary roots are associated with the relation $\nabla^2\psi = -\zeta$, where ζ is the vorticity and ψ is the streamfunction. The streamfunction is associated with an elliptic operator. Discontinuous initial data for the streamfunction will be smoothed.

The analysis of hyperbolicity in plane steady flows is easiest when formed in terms of the vorticity $\zeta = \zeta_3$, where ζ is governed by (6.4) in the plane 1,2. After a little calculation we find that

$$\rho u_j u_k \frac{\partial^2\zeta}{\partial x_j \partial x_k} - \frac{1}{2} P_{jj} \frac{\partial^2\zeta}{\partial x_k \partial x_k} - \frac{1}{2} A_{kjke} \frac{\partial^2\zeta}{\partial x_j \partial x_e} + \frac{1}{2} A_{ejke} \frac{\partial^2\zeta}{\partial x_j \partial x_k}$$

$$= \text{L.O.T.}, \qquad j,k,e = 1,2 \tag{7.6}$$

where L.O.T. stands for lower order terms and $A_{ijk\ell}$ is symmetric in ij and skew symmetric in $k\ell$. Eq (7.6) may be written as

$$\left[\rho u_1^2 - A_{1221} - \frac{1}{2}(P_{11} + P_{22})\right] \frac{\partial^2\zeta}{\partial x_1^2} + [2\rho u_1 u_2 - (A_{1112} + A_{2221})] \frac{\partial^2\zeta}{\partial x_1 \partial x_2}$$

$$+ \left[\rho u_2^2 + A_{1221} - \frac{1}{2}(P_{11} + P_{22})\right] \frac{\partial^2\zeta}{\partial x_2^2} = \text{L.O.T.} \tag{7.7}$$

We may apply the theory of 2nd order quasilinear equations to (7.7). We look for real characteristic surfaces $\phi(x_1,x_2)$ = const. Let $\nabla\phi = (\phi_1,\phi_2,0)$, then (7.7) generates real characteristic surfaces given by

$$\left[\rho u_1^2 - A_{1221} - \frac{1}{2}(P_{11} + P_{22})\right]\phi_1^2 + [2\rho u_1 u_2 - (A_{1112} + A_{2221})]\phi_1\phi_2$$

$$+ \left[\rho u_2^2 + A_{1221} - \frac{1}{2}(P_{11} + P_{22})\right]\phi_2^2 = 0 \tag{7.8}$$

We note that if $\phi(x_1,x_2)$ = const. gives a plane characteristic curve $x_2(x_1)$, then $\phi_1 + \phi_2 \frac{dx_2}{dx_1} = 0$, $\frac{dx_2}{dx_1} = -\frac{\phi_1}{\phi_2}$. It follows from (7.8) that we have real characteristics given by

$$\frac{dx_2}{dx_1} = \frac{B \pm \sqrt{B^2 - AC}}{A} \tag{7.9}$$

provided that

$$B^2 - AC \geq 0 \tag{7.10}$$

where

$$A = \rho u_1^2 - A_{1221} - \frac{1}{2}(P_{11} + P_{22}),$$

$$2B = 2\rho u_1 u_2 - (A_{1112} + A_{2221}),$$

$$C = \rho u_2^2 + A_{1221} - \frac{1}{2}(P_{11} + P_{22})$$

The condition (7.10) for real characteristics may be written as

$$\rho u_1^2\left[\frac{P_{11}+P_{22}}{2}-A_{1221}\right]+\rho u_2^2\left[\frac{P_{11}+P_{22}}{2}+A_{1221}\right]-2\rho u_1u_2[A_{1112}+A_{2221}]$$

$$+(A_{1112}+A_{2221})^2+A_{1221}^2-\left[\frac{P_{11}+P_{22}}{2}\right]^2\geq 0 \qquad 7.11)$$

Equations governing change of type in Oldroyd models arise from those just given under the substitutions mentioned in (2.19). In particular the characteristic surface equation becomes

$$\left[\rho u_1^2-\left[\frac{a+1}{2}\tau_{11}+\frac{a-1}{2}\tau_{22}+\frac{\eta}{\lambda}\right]\right]\phi_1^2$$

$$+2[\rho u_1u_2-\tau_{12}]\phi_1\phi_2 \qquad (7.12)$$

$$+\left[\rho u_2^2-\left(\frac{a-1}{2}\tau_{11}+\frac{a+1}{2}\tau_{22}+\frac{\eta}{\lambda}\right)\right]\phi_2^2=0$$

where we have put $[P_{11},P_{22},A_{1112},A_{2221},A_{1221}]=[a\tau_{11},a\tau_{22},\tau_{12},\tau_{12},\frac{1}{2}(\tau_{11}-\tau_{22})]$.

8. CONDITIONS FOR A CHANGE OF TYPE. PROBLEMS OF NUMERICAL SIMULATION.

In general the condition (7.11) for hyperbolicity appears to be easiest to satisfy when the speed is large and the normal stresses small, or large and positive. For the Oldroyd models (7.11) may be written as

$$\left[\rho u_1^2+\frac{\tau_{22}}{2}(1-a)-\frac{\tau_{11}}{2}(1+a)-\frac{\eta}{\lambda}\right]\left[(1+a)\frac{\tau_{22}}{2}+(a-1)\frac{\tau_{11}}{2}+\frac{\eta}{\lambda}-\rho u_2^2\right]$$

$$+(\rho u_1u_2-\tau_{12})^2>0 \qquad (8.1)$$

We may write (8.1) in dimensionless form by using a scale length and a scale velocity U. Then dimensionless variables related to dimensional variables by

$$[u,v,\tau,\sigma,\Upsilon,M]=\left[\frac{u_1}{U},\frac{u_2}{U},\frac{\tau_{12}}{\tau_0},\frac{\tau_{11}}{\tau_0},\frac{\tau_{22}}{\tau_0},\frac{U}{C}\right]$$

where $\tau_0=\eta/\lambda$ and

$$C=\sqrt{\frac{\eta}{\rho\lambda}}=\sqrt{\frac{\tau_0}{\rho}} \qquad (8.2)$$

is the speed of a wave of vorticity into a fluid in uniform motion. The criterion (8.1) for hyperbolicity

$$\left[M^2u^2+\frac{\Upsilon}{2}(1-a)-\frac{\sigma}{2}(1+a)-1\right]\left[(1+a)\frac{\Upsilon}{2}+(a-1)\frac{\sigma}{2}+1-M^2v^2\right]$$

$$+[M^2uv-\tau]^2>0 \qquad (8.3)$$

depends only on the viscoelastic Mach number M (and the parameter $a\varepsilon[-1,1]$). When a = 1 (upper convected Maxwell model), the criterion (8.3) can be written as

$$M^2[v^2(1+\sigma)+u^2(1+\Upsilon)-2\tau uv]$$

$$-(1+\Upsilon)(1+\sigma)+\tau^2>0 \qquad (8.4)$$

This criterion is satisfied for high speed flow if the stresses σ,Υ,τ are not large and of the wrong sign.

The vorticity equation changes type at points at which the right side of (8.1) or (8.3) vanishes. Numerical simulations of problems of changing type can be very difficult. For example numerical simulations of transonic flows have only become satisfactory in recent years and they are still being improved. Problems of the transonic type can be expected in numerical simulations of the flow of viscoelastic fluid.

Steady flow of the Oldroyd fluids (6.2) is governed by the following system of dimensionless equations

$$\operatorname{div}\underline{u}=0,$$

$$M^2(\underline{u}\cdot\nabla)\underline{u}+\nabla p-\operatorname{div}\underline{\tau}=0,$$

$$\underline{u}\cdot\nabla\underline{\tau}-a(\underline{D\tau}+\underline{\tau D})+\underline{\tau\Omega}-\underline{\Omega\tau}-\underline{D}=-\underline{\tau}/W \qquad (8.5)$$

where

$$W=U\lambda/d$$

is the Weissenberg number, which may be regarded as a dimensionless relaxation time measured in units of d/U. Stresses relax slowly when W is large. The fluid has a long memory.

The high Weissenberg number problem is that people doing numerical analysis can't get answers when the stress levels are high. This problem is evidently independent of the "transonic" problem of change of type. Most of the computations in which one encounters this problem are for flows without inertia, $\rho = 0$. Yoo and Joseph (1984) also encountered this problem in high speed flows involving change of type. They found that regions of positive and negative vorticity decay rapidly when W is small (even if the "Mach" number is large). When W is large the supercritical regions are more "purely" hyperbolic in that the damping of the vorticity is suppressed. We have something like a telegraph equation with damping proportional to 1/W. (See (12.19).

Though the high Weissenberg number problem and problems of change of type are independent they may be related problems in certain special cases. Though flows without inertia will not ordinarily change type, the criterion for change of type wherein $B^2 - AC$ changes sign can be interesting even when $\rho = 0$.

For the upper convected Maxwell model (a = 1) we find from (8.1) that when $\rho = 0$

$$B^2 - AC = \tau_{12}^2 - \left[\tau_{11} + \frac{\eta}{\lambda}\right]\left[\tau_{22} + \frac{\eta}{\lambda}\right]. \tag{8.6}$$

The equations for steady flow are elliptic when this is negative. In fact, this quantity appears to approach zero very rapidly as λ (or W) is increased in numerical integrations of steady flows. In fact, $B^2 - AC$ is always negative for the upper convected Maxwell model

$$\underline{\tau} + \frac{\eta}{\lambda}\underline{1} = \frac{\eta}{\lambda^2}\int_{-\infty}^{t} e^{-(t-\tau)/\lambda}\, \underline{C}_t^{-1}(\tau)d\tau. \tag{8.7}$$

Since $\underline{C}_t^{-1}$ has positive eigenvalues, the principal values $\tau_{11} + \frac{\eta}{\lambda}$, $\tau_{22} + \frac{\eta}{\lambda}$ are positive and therefore

$$\det\begin{bmatrix}\tau_{11} + \frac{\eta}{\lambda} & \tau_{12} \\ \tau_{12} & \tau_{22}+\frac{\eta}{\lambda}\end{bmatrix} \geq 0. \tag{8.8}$$

In numerical integrations using finite elements or finite differences, values of τ_{ij} and u_i are obtained from discrete steps in which the criterion (8.8) may be violated. In doing numerical integrations one should verify that the condition (8.8) is not violated.

Some constitutive models, like the corotational one with a = 0 may undergo real change of type even when $\rho = 0$.

People doing flow computations for viscoelastic fluids are able to go to higher Weissenberg numbers when they have constitutive equations with more Newtonian viscosity (non-zero retardation times). This observation suggests that viscosity methods for dealing with problem of change of type and shocks could also be useful for solving the high Weissenberg number problem. So far however the people doing numerical works have not used the viscosity method in the limits of small viscosity.

9. LINEARIZED PROBLEMS OF CHANGE OF TYPE

Up to now our study has been exact and fully nonlinear. We may advance our understanding of the problems of vorticity of changing type by considering simpler problems which arise under linearization.

We first identify a class of motions, say motions in the plane or axisymmetric motions. We find some special exact solution of all the equations which fall in the given class. Usually this special solution is featureless, like uniform flow, flow into a sink, flow in a channel, flow between cylinders, extensional flow and so on. The featureless solution does not exhibit the unusual features of change of type. We then perturb all the equations around the special one with perturbations in the given class, and we linearize. This leads us to linearized problems with variable coefficients depending alone on the special solution and not on the perturbation. We analyze the linear perturbed problem for hyperbolicity and change of type. The equations for the characteristics are given by (7.10) where A,B,C are evaluated on the unperturbed special solution.

The procedures of linearization are such that the characteristic directions are the same for each and every linear problem perturbing the special one. The characteristic surfaces for linearized problems are a gift, since nothing beyond the special solution is needed to compute them. We can know which are the elliptic regions and which the hyperbolic regions of flow cheaply, but to find other properties of the perturbed flow, say isovorticity and streamlines, we are obliged to solve linear PDE's.

JRS (1984) treated the linearized problem for a change of type in shear flow and extensional flow for the family of Oldroyd models characterized by the parameter a [1,1]. They also treated sink flow in the plane and circular Couette flow of an upper convected Maxwell model (a=1). They identified the regions of subcritical (elliptic) and supercritical (hyperbolic) flow, but they did not compute the characteristics or solve some boundary value problem. Shear flow $u = \kappa y$, κ is the rate of shear, is hyperbolic outside a strip centered on $y = 0$. Extensional flow $(u,v) = s(x,-y)$, s is the rate of extension, is either hyperbolic outside an ellipse (where u and v are large) or inside a region bounded by branches of a hyperbola $(x^2/A^2) - (y^2/B^2) < 1$ (where the velocity need not be large. Sink flow with radial velocity $u_r = -Q/r$ (potential flow) is hyperbolic when the radius r is small

$$r < \left[\lambda Q\left(\frac{\rho}{\eta}Q-2\right)\right]^{1/2}$$

provided that the source strength $Q > 2\eta/\rho$, where η is the viscosity. Couette flow outside a rotating cylinder of radius a is hyperbolic in an annulus whose inner radius is either a or is greater than a, depending on conditions. When the inner radius of the annulus of hyperbolicity is greater than a we have another example where the region of high speed flow is elliptic and regions of lower speed are hyperbolic. The variation of the stresses is important.

Yoo, Ahrens and Joseph (1984) have tried to explain some striking experimental results of Metzner, Uebler and Fong (1969) with an analysis of the vorticity perturbing irrotational sink flow. The main point at issue in the experiments of Metzner, et al., is that they observe a conical region of zero vorticity. Outside this region the flow is rotational. If we accept the experimental results at face value, we must conclude that there are surfaces across which some derivatives of the vorticity are discontinuous. This type of behavior says "look for hyperbolicity and change of type." Such a field could not be supported by an elliptic vorticity field.

In the experiments of Metzner, Uebler & Fong (1969) a fluid is sucked from a pipe of large diameter through a sudden contraction. If the hole into which the flow goes is small the problem may be thought to be a form of sink flow. Because there are boundary walls, the flow through a sudden contraction is not a sink flow in a strict sense. We shall imagine first that the flow into the hole is not strongly influenced by the walls of the large pipe. We then have a hole in the semi infinite region above a plane. This flow is then regarded as an axisymmetric perturbation of sink flow without boundaries. The characteristic surfaces for the vorticity of all axisymmetric linearized problems perturbing sink flow can then be obtained by integrating the differential equations for the characteristics. Yoo, Ahrens and Joseph then tried to determine if the characteristic surfaces computed in this way could be the locus for the discontinuity in the derivatives of the vorticity observed in the experiments. It is important to verify that the characteristics are cone like in the region where potential sink flow was observed and that the region of potential flow is in the spherical annulus of hyperbolicity. These issues are addressed below.

Metzner, Uebler and Fong (1969) consider high speed flow of viscoelastic fluids into a sudden contraction. They say that "A tentative analysis of the observed velocity field suggests the flow upstream of the small duct to be radially directed toward the origin of the spherical coordinate system. If this is so the continuity equation gives

$$ur^2 = f(\theta). \qquad (1)"$$

They actually measure velocities in the cone and they report that their measurements were accurate and that $f(\theta)$ may be taken as constant when $0 \le \theta \le 10°$. They also write that $u_\theta = u_\varnothing = 0$ in the cone. Outside of the cone there is secondary motion and nonzero vorticity.

The nature of the comparison of theory and experiment, explained in the captions of Figs. 9.1 and 9.2, is discussed in greater detail in the paper of Yoo, Ahrens and Joseph (1984). It turns out that the regions of hyperbolicity are model sensitive. Using the measured values of physical parameters in the equation one finds that the observed potential flow does lie in the hyperbolic region (some points are outside the hyperbolic region in Fig. 9.1(b)). The characteristic surfaces are cone like in the regions where potential flow was observed.

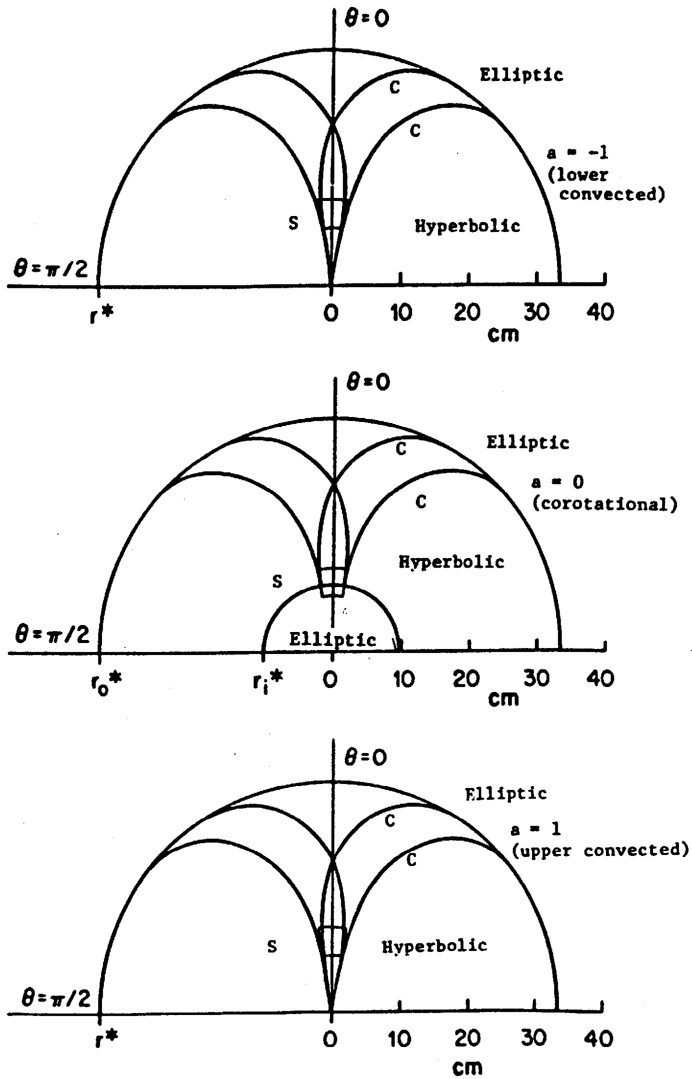

Fig. 9.1. Metzner, Uebler and Fong (1969) measured potential flow in the sectorial box designated S. The cross-sections of characteristic surfaces of revolution which are tangent to the cone of semi vertex angle 10° at the origin are called C. There are two such surfaces of revolution.

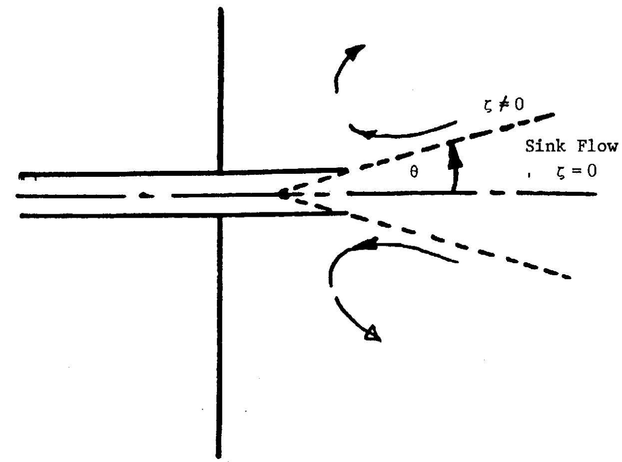

Fig. 9.2. Schematic diagram of flow into a sudden contraction (Metzner, Uebler and Fong (1969)). The vorticity appears to vanish in a cone with nonzero vorticity outside. Accepting this, a jump in some derivative of the vorticity on the cone is required. The measurements verifying potential flow were taken in a cone with $\theta = 10°$ and at a certain value in and out of the pipe. The regions where potential flow was measured are in the sectorial boxes shown in Fig. 9.2.

We regard the comparisons of theory and experiment in Fig. 9.2 as exploratory and not definitive. It is of course striking that the experiments of Metzner, Uebler and Fong (1969) do appear to involve a vorticity of changing type. It would be interesting to see if this striking type of experimental result could be repeated by other investigators using different fluids and experimental arrangements. We hasten to add that the Separan solution used in the experiment is not an Oldroyd model and surely cannot be characterized by a viscosity and relaxation time. In fact only special models give the vorticity precisely as the quantity which changes type. We have already remarked that models with true viscosity; e.g., retardation times, will smooth discontinuities, with only a little smoothing if the retardation "viscosity" parameter is small. Probably all the polymer solutions used in experiments have some small smoothing. In view of all these uncertainties in theory and experiments it would be premature to make strong claims.

10. CHANNEL FLOWS WITH WAVY WALLS

Yoo and Joseph (1984) solved the problem of flow of an upper convected Maxwell model through a channel with wavy walls linearized for small waviness. It is easy to find an exact solution of the equations when the walls are flat. The velocity profile is the same quadratic one that one finds in Newtonian fluids. There are some normal stresses which are absent in the Newtonian case. This Poiseuille flow is perturbed and linearized. The characteristics and regions of hyperbolicity can be computed without specifying the nature of the perturbation. Yoo and Joseph went further. They used a specific perturbation, the amplitude of the waviness of the walls. They defined the linearized problem for small waviness and solved it numerically. The Yoo-Joseph paper gives the first actual computation of a flow with change of type.

We are going to outline the analysis and some of the main results of Yoo and Joseph. We shall express the equations in terms of the "Mach" number $M = U/C$ and the elasticity number $E = \eta\lambda/\rho d^2$ (this is independent of U). The Weissenberg number $U_0\lambda/d$ is given by $W = M\sqrt{E}$.

$$u_x + v_y = 0, \tag{10.1a}$$

$$uu_x + vu_y + \frac{1}{M^2}(p_x - \sigma_x - \tau_y) = 0, \tag{10.1b}$$

$$uv_x + vv_y + \frac{1}{M^2}(p_y - \tau_x - \gamma_y) = 0, \tag{10.1c}$$

$$u\sigma_x + v\sigma_y - 2(\sigma+1)u_x - 2\tau u_y = -\frac{\sigma}{M\sqrt{E}}, \tag{10.1d}$$

$$u\tau_x + v\tau_y - (\gamma+1)u_y - (\sigma+1)v_x = -\frac{\tau}{M\sqrt{E}}, \tag{10.1e}$$

$$u\gamma_x + v\gamma_y - 2\tau v_x - 2(\gamma+1)v_y = -\frac{\gamma}{M\sqrt{E}}. \tag{10.1f}$$

We shall seek and find a solution of these equations satisfying no slip conditions at the walls

$$u = v = 0 \text{ at } y = \pm(1 + \varepsilon \sin nx) \tag{10.2}$$

with symmetric streamlines

$$\frac{\partial u}{\partial y} = v = 0 \text{ at } y = 0. \tag{10.3}$$

It is noteworthy that our solution is completely determined by data (10.2) and (10.3) on the velocity alone. It is not necessary, and it would be wrong to prescribe more about velocity or stresses. The vorticity $\underline{\omega} = \text{curl } \underline{u}$ is related to the streamfunction

$$\underline{\omega} = \text{curl } \underline{u} = \omega\underline{e}_z = -\left[\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right]\underline{e}_z \tag{10.4}$$

where $\omega(x,y)$ satisfies

$$\begin{aligned}(M^2u^2-\sigma-1)\omega_{xx} &+ 2(M^2uv-\tau)\omega_{xy} + (M^2v^2-\gamma-1)\omega_{yy} \\ &+ \left[-P_x + \frac{M}{\sqrt{E}}u\right]\omega_x + \left[-P_y + \frac{M}{\sqrt{E}}v\right]\omega_y \\ &+ (u_x-v_y)(\sigma_{xy}+\gamma_{xy}+\tau_{xx}+\tau_{yy}) + (u_y+v_x)(\gamma_{yy}-\sigma_{xx}) \\ &+ \sigma_y u_{xx} + 2\tau_y u_{xy} + \gamma_y u_{yy} - \sigma_x v_{xx} - 2\tau_x v_{xy} - \gamma_x v_{yy} = 0.\end{aligned} \tag{10.5}$$

Equation (10.5) may be written as

$$A\omega_{xx} + 2B\omega_{xy} + C\omega_{yy} + \text{L.O.T.} = 0 \tag{10.6}$$

where the terms L.O.T. are of lower order for hyperbolic analysis (see JRS, 1984). Characteristic directions

$$\frac{dy}{dx} = \frac{B}{A} \pm \frac{\sqrt{B^2-AC}}{A},$$

$$A = M^2u^2 - \sigma - 1,$$

$$B = M^2uv - \tau,$$

$$C = M^2v^2 - \gamma - 1 \qquad (10.7)$$

for the vorticity exist whenever

$$\Sigma \overset{\text{def}}{=} \tau^2 - 2M^2\tau uv - (1+\gamma)(1+\sigma) + M^2v^2(1+\sigma) + M^2u^2(1+\gamma) > 0. \qquad (10.8)$$

The expression (10.8) is expressed in terms of unknown velocity and stress fields. The criterion (10.8) for hyperbolicity can be satisfied in some regions of flow and not in others. The border $\Sigma = 0$ between the elliptic and hyperbolic regions of flow is like the sonic line in gas dynamics. Across this line the equations are said to change type.

Equation (10.1) and all the equations of this section are general in that they apply to every plane problem, not just the channel flow problem introduced in Section 1.

Now we shall solve the governing equations for flow in a channel with straight walls $\varepsilon = 0$.

$$(u_0, v_0) = (1-y^2, 0),$$

$$(p_0, \tau_0) = -2M(x,y)\sqrt{E},$$

$$(\sigma_0, \gamma_0) = (8M^2Ey^2, 0),$$

$$\omega_0 = 2y \qquad (10.9)$$

The basic motion depends exclusively on the Weissenberg number $W = M\sqrt{E}$ measuring the size of stresses. The solution (10.9) is relatively featureless and, in particular, it gives no indication of hyperbolicity.

Now we consider any plane perturbation of (10.9). The problem with wavy walls is one such perturbation, but there are infinitely many

others. We may linearize the formula (10.3) for the characteristics of any flow slightly perturbing the Poiseuille flow (10.9). The characteristics for all these perturbations

$$\frac{dy}{dx} = \frac{1}{M}\left[\frac{-2y\sqrt{E} \pm \sqrt{(y^2-1)^2 - 4Ey^2 - \frac{1}{M^2}}}{-(y^2-1)^2 + 8Ey^2 + \frac{1}{M^2}}\right] \qquad (10.10)$$

are defined in terms of quantities defined for the basic flow (10.9) and are given once and for all, independent of the perturbation.

Equation (10.10) shows that flows perturbing plane Poiseuille flow can exhibit a change of type with a "sonic" line $\Sigma = 0$ given by

$$\Sigma(y^2) = (y^2-1)^2 - 2Ey^2 - \frac{1}{M^2} = 0 \qquad (-1 \le y \le 1).$$

Since $\Sigma(y^2)$ is monotonically decreasing, it has a maximum at $y^2 = 0$ and

$$\Sigma(0) = 1 - \frac{1}{M^2} > 0$$

if and only if the viscoelastic "Mach" number $M > 1$. The "sonic" line across which the flow changes type is $y = y^*$ where $\Sigma(y^{*2}) = 0$

$$y^* = \pm\left[1 + 2E - 2\left[E^2 + E + \frac{1}{4M^2}\right]^{1/2}\right]^{1/2}$$

The linearized problem for small ε is

$$u_x + v_y = 0, \qquad (10.11a)$$

$$(1-y^2)u_x - 2yv + \frac{1}{M^2}(p_x - \sigma_x - \tau_y) = 0, \qquad (10.11b)$$

$$(1-y^2)v_x + \frac{1}{M^2}(p_y - \tau_x - \gamma_y) = 0, \qquad (10.11c)$$

$$(1-y^2)\sigma_x + 16M^2Eyv - 2(8M^2Ey^2+1)u_x + 4y\tau + 4Myu_y\sqrt{E} + \sigma/M\sqrt{E} = 0, \qquad (10.11d)$$

$$(1-y^2)\gamma_x + 4Myv_x\sqrt{E} - 2v_y + \gamma/M\sqrt{E} = 0, \tag{10.11e}$$

$$(1-y^2)\tau_x - 2Mv\sqrt{E} + 2y\gamma - (8M^2Ey^2+1)v_x - u_y + \tau/M\sqrt{E} = 0 \tag{10.11f}$$

where

$$u(x,\pm1) = 2 \sin nx$$

$$v(x,\pm1) = 0,$$

$$v(x,0) = u_y(x,0) = 0. \tag{10.12}$$

Yoo and Joseph solved the equations (10.11) using only the velocity data (10.12) and the method of separation to reduce these equations to ordinary differential equations in y. The ordinary differential equations were solved numerically. The vorticity

$$\omega \overset{\text{def}}{=} v_x - u_y = -\nabla^2\psi \tag{10.13}$$

satisfies an equation

$$\left[(1-y^2)^2 - 8Ey^2 - \frac{1}{M^2}\right]\omega_{xx} + 4\frac{\sqrt{E}}{M}\, y\omega_{xy} - \frac{\omega_{yy}}{M^2} + \left[\frac{2\sqrt{E}}{M} + \frac{1}{M\sqrt{E}}(1-y^2)\right]\omega_x$$

$$= -\frac{v}{M\sqrt{E}} - 16Eyu_{xx} + \frac{4\sqrt{E}}{M}u_{xy} - 2(1-y^2)v_x$$

$$+ \frac{1}{M^2}[2\tau_x + 2y\gamma_{yy} - 2y\sigma_{xx} + 4\gamma_y] \tag{10.14}$$

of changing type whose characteristics are given by (10.7).

In Figs. 10.1, 10.2 and 10.3 we have graphed vorticity and streamlines for M = 10. In 10.1a and 10.2a we superimposed characteristics (light) on zero vorticity lines. We first observe that when W is small the decay of the vorticity is rapid. When W is large the vorticity decays only very slowly with oscillations all the way to the center.

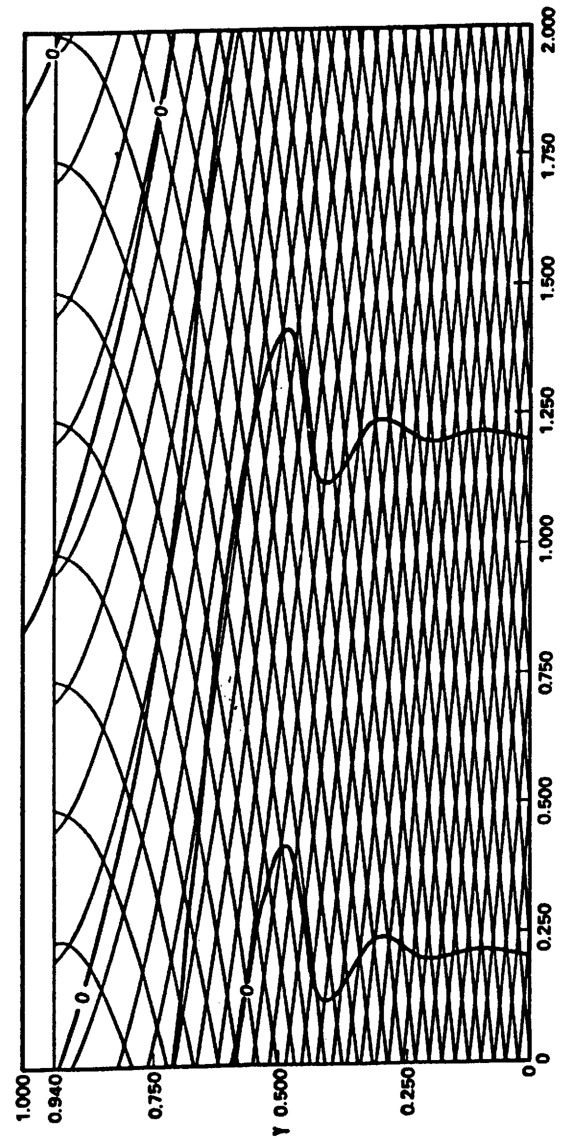

Fig. 10.1(a): Zero vorticity curves for (E,M,R,W) = (0.001,10,316,0.316)

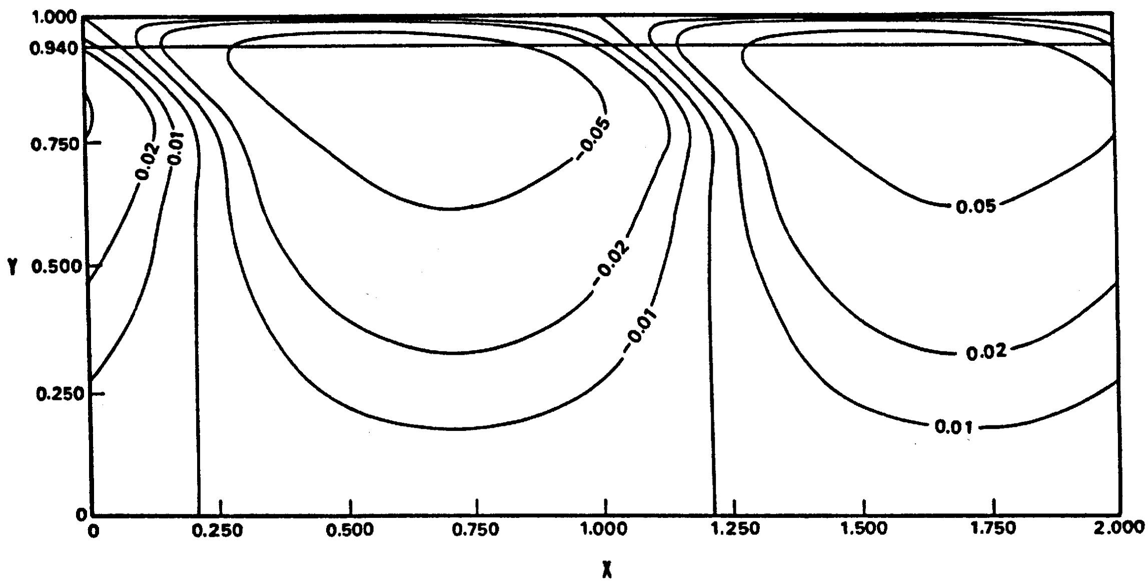

Fig. 10.1(b): Streamlines

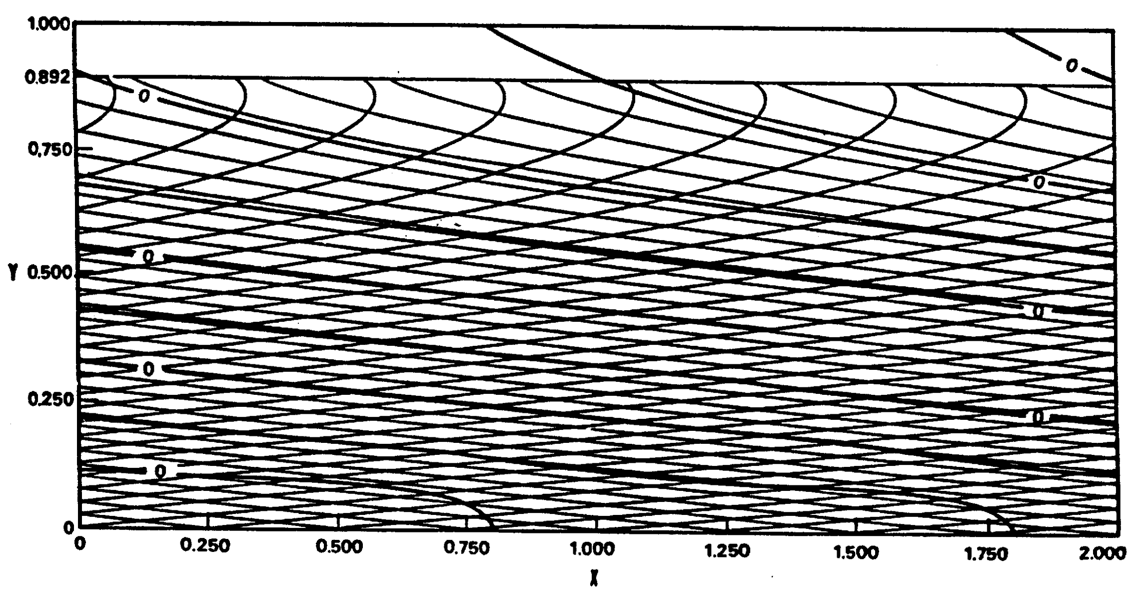

Fig. 10.2(a): Zero vorticity curves for (E,M,R,W) = (0.01,10,100,1)

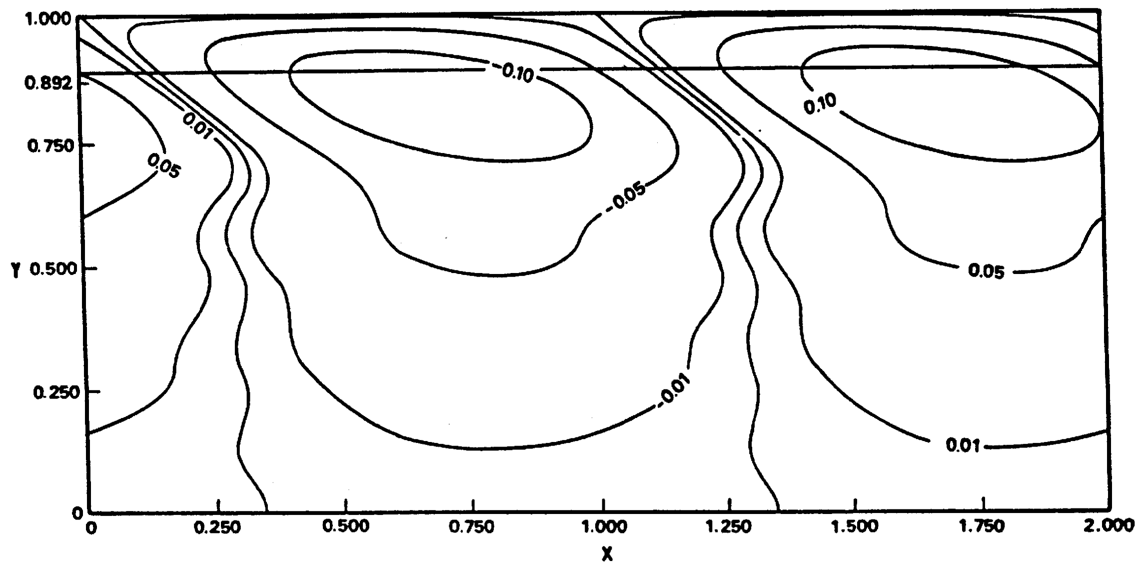

Fig. 10.2(b): Streamlines

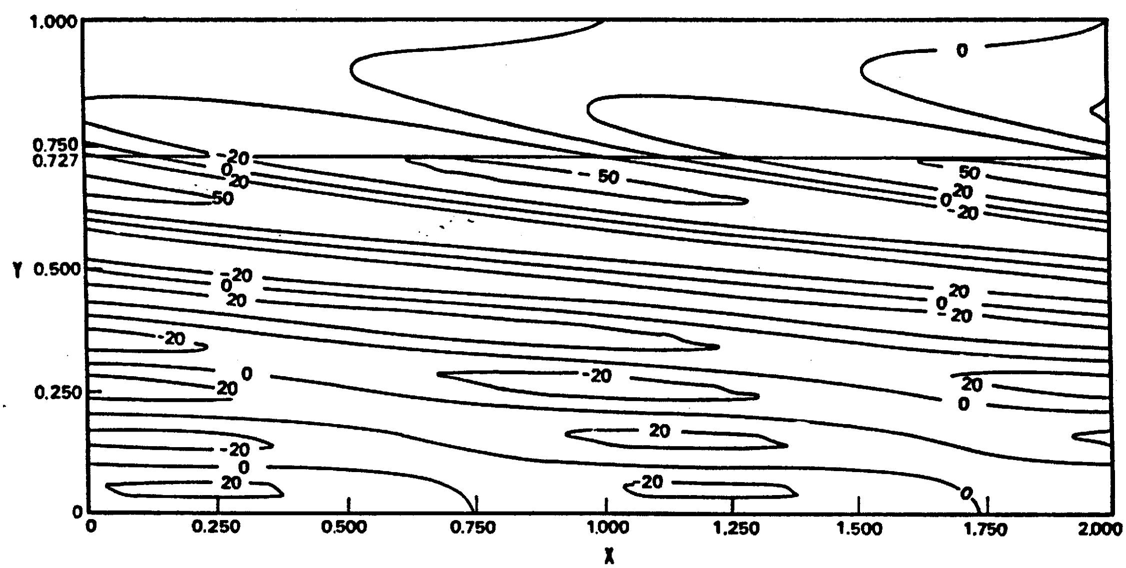

Fig. 10.3(a): Iso-vorticity curves for (E,M,W,R) = (0.1,10,31.6,3.16)

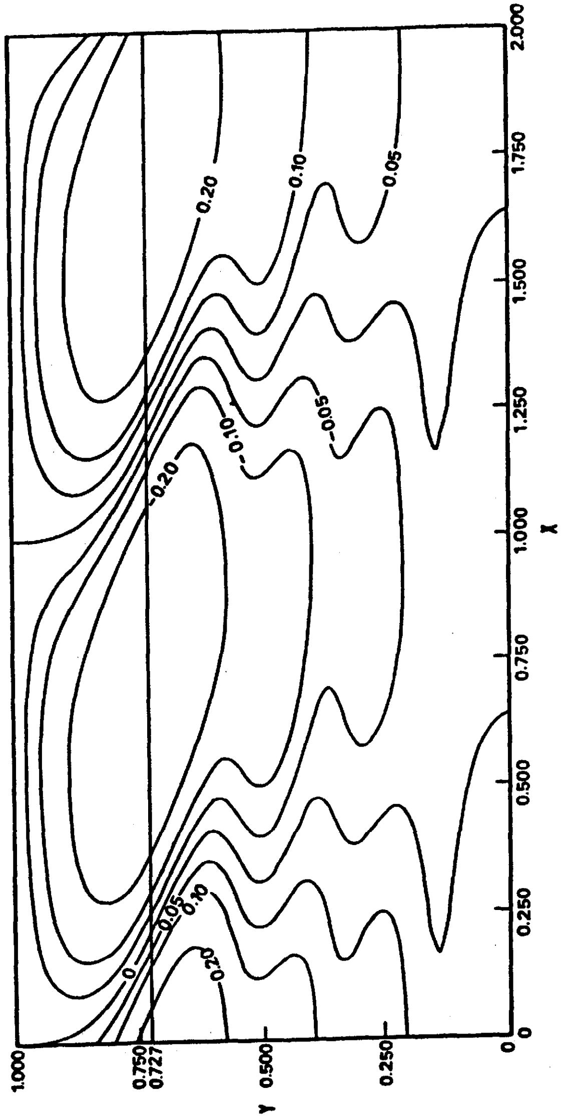

Fig. 10.3(b): Streamlines

(a) (b) (c) (d)

Fig. 10.4. (After Brenschede and Klein, 1970). Delayed die swell in polyisobutilene solution is toluol.
$D = 4\mathring{Q}/\pi R^3$ where $\mathring{Q}$ is the volume flow rate and R is the pipe radius. The critical value D is 14×10^4 reciprocal seconds.
(a) $D = 8 \times 10^4$ (b) $D = 15.2 \times 10^4$
(c) $D = 22.8 \times 10^4$ (d) $D = 29 \times 10^4$

a b c

Fig. 10.5. (After H. Giesekus, 1968). Delayed die swell in a .5% polyacrylamide solution in water.
(a) slow speed, (b) post critical speed,
(c) high speed

Yoo and Joseph did an asymptotic analysis which give the frequency of the oscillations and other properties of the solution. The second property is that regions of positive and negative vorticity which are not damped are swept out along characteristics. It would be hard to understand these solutions from the streamline plots.

The same type of hyperbolic dynamics which Yoo and Joseph found in channels will occur in pipes. Mark Ahrens is working on this problem. The change of type which occurs in the center of a pipe may have applications in the problem of delayed die swell. The phenomenon of delayed die swell is not well known. At low speeds the jet will spread near the exit of the jet, as in Fig. 10.4(a) and 10.5(a). At yet higher speeds the jet does not spread near the exit, the swell is delayed, as in Fig. 10.4(b-d). Fig. 10.5 shows a form of delayed die swell which we have seen repeatedly in our own experiments. The delayed swell seems to occur at a critical speed, not so different than what one might expect from a change of type. Of course, the reason for the delayed swell is not understood. The form of the jet reminds one of a hydraulic jump, which is the shock phenomenon corresponding to shocks in gas dynamics.

11. PROBLEMS ASSOCIATED WITH THE FLOW OF VISCOELASTIC FLUIDS AROUND BODIES

It is perhaps not unreasonable to think that far from the body we have only a small perturbation of uniform motion with constant velocity $U\underline{e}_x$ in the direction x. The body is stationary.

The vorticity of all steady flows of viscoelastic fluids with instantaneous elasticity which perturb uniform flow can change type.

To be precise, the linearized equations for the vorticity of flows perturbing uniform flow vanish on states of constant vorticity and the type of this equation in steady flow changes when the ratio of the free stream velocity of the wave speed increases through unity. This criterion for a change in type may be expressed in terms of a viscoelastic Mach number.

Ultman and Denn (1970, 1971) consider the equations for two-dimensional steady flow of an upper convected Maxwell fluid. They linearize at a motion with uniform velocity and zero stress, and they show that these linearized equations change type when a viscoelastic "Mach" number

$$M = \frac{U}{C} = U/\sqrt{\eta/\lambda\rho}$$

exceeds one (see Eq. (2.4)). Here U is the velocity of the unperturbed uniform flow, and C is the wave velocity for propagation of shear waves in a Maxwell fluid. Ultman and Denn (1970) did not notice that it is precisely the vorticity which changes type. They refer to their linearization as an Oseen approximation. Oseen introduced his linearization around uniform flow for slow viscous flow, because Stokes equations have no solution for flow around bodies in two dimensions. Oseen's equations do not change type. Ultman and Denn (1971) use Oseens approximate method to compute subcritical flow and they say that their calculations agree with their experiments when the fluid parameters are properly chosen.

In a second paper, Ultman and Denn (1970) consider the supercritical flow but they do not give experimental results and they do not discuss or try to solve their equations. They attempt to correlate some experimental observations of D. F. James (1967) with the change of type. James observes a sudden change in the slope of the heat transfer curve as a function of velocity. This happens at a critical velocity which for the Polyox solution used by James, was about 1 cm/sec. It is not clear from the graphs how abrupt this change of slope is, but there is a change of slope. Ultman and Denn (1970) also suggest that the transition from subcritical to supercritical flow might explain abrupt changes in the drag coefficient they say was observed by A. Fabula (1966). Again, the idea is that the critical velocity at transition is the wave speed C. They make an estimate of C from a molecular theory and correlate this prediction with the data of James. Of course, any such estimate can at best be expected to give an order of magnitude, since the molecular theory is coarse and the fluids used in the experiment are not Maxwell fluids.

Recently Ambari, Deslouis and Tibollet (1984) have considered mass transfer in the flow of a viscoelastic fluid around a cylinder. They also find a critical value at which the mass transfer undergoes an anomolous transition.

Oseen's methods work because slow flow around bodies is in some sense a perturbation of uniform motion. The case of fast flow is different because the uniform stream is not small, but the velocity on the body must vanish. In supercritical flow the ratio of the stream velocity to the wave speed is greater than one, and the wave velocity which is a property of the fluid, cannot be controlled. It follows that even if the flow away from the body is supercritical the flow near the body will always be subcritical, and the underlying problem is really one of "transonic" type.

We are going to show that the equations which govern perturbations of uniform flow have a simple form, depending on the density ρ and relaxation modulus $G(s)$ alone, and are model independent. The flow near the body depends on the constitutive model.

The stress in a simple incompressible fluid may be decomposed into an isotropic and a determinate part. The determinate part is the constitutive equation which relates stress and deformation. In linearization at uniform motion, the determinate stress in fluids with instantaneous elasticity is given by an integral (see JRS, 1984)

$$\underline{\tau} = \int_{-\infty}^{t} G(t-\tau)\underline{A}[\underline{u}(\underline{\chi},\tau)]\, d\tau \tag{11.1}$$

where

$$\underline{\chi} = \begin{bmatrix} x - U(t-\tau) \\ y \\ z \end{bmatrix},$$

$$\underline{A}[\underline{u}] = \nabla\underline{u} + \nabla\underline{u}^T, \quad \underline{A}_{ij} = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}$$

and G is smooth, positive and monotone decreasing. The speed of various kinds of shear waves (Coleman and Gurtin, 1968, Narain and Joseph, 1982) into a fluid of density ρ at rest is

$$C = \sqrt{\frac{G(0)}{\rho}}. \tag{11.2}$$

The "Mach" number is U/C. By differentiating (2.1) with respect to t, holding $\underline{x}$ fixed, we find that

$$\frac{\partial\tau}{\partial t} + U\frac{\partial\tau}{\partial x} = G(0)\underline{A}[\underline{u}(\underline{x},t)] + \int_{-\infty}^{t} G'(t-\tau)\underline{A}[\underline{u}(\underline{\chi},\tau)]\, d\tau \tag{11.3}$$

where the last term is of lower differential order for hyperbolic analysis.

For all the Oldroyd models the perturbation of uniform motion leads to the linearized Maxell model with

$$G(s) = \frac{\eta}{\lambda}\exp(-s/\lambda) \tag{11.4}$$

The rate equation for a Maxwell model in a flow perturbing uniform flow is

$$\frac{\partial\tau}{\partial t} + U\frac{\partial\tau}{\partial x} = \frac{\eta}{\lambda}\underline{A}[\underline{u}(x,t)] - \frac{\tau}{\lambda}. \tag{11.5}$$

The equations of motion are

$$\rho\left[\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x}\right] = -\nabla p + \operatorname{div}\underline{\tau}.$$

The vorticity $\underline{\zeta} = \operatorname{curl}\underline{u}$ satisfies

$$\rho\left[\frac{\partial\underline{\zeta}}{\partial t} + U\frac{\partial\underline{\zeta}}{\partial x}\right] = \operatorname{curl}\operatorname{div}\underline{\tau}. \tag{11.6}$$

Using (2.1) and $\operatorname{curl}\operatorname{div}\underline{A}[\underline{u}(\underline{x},\tau)] = \Delta\underline{\zeta}(\underline{\chi},\tau)$

where

$$\Delta = \frac{\partial^2}{\partial x^2} + \nabla_2^2, \quad \nabla_2^2 = \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2},$$

we get

$$\rho\left[\frac{\partial \underline{\zeta}}{\partial t} + U\,\frac{\partial \underline{\zeta}}{\partial x}\right] = \int_{-\infty}^{t} G(t-\tau)\Delta\underline{\zeta}(\underline{\chi},\tau)\,d\tau. \tag{11.7}$$

The x derivatives under the integral may be replaced with χ derivatives. Equation (11.7) shows that states of uniform vorticity are solutions of the equations of motion of fluids with instantaneous elasticity which perturb uniform flow. Potential flow is possible.

For steady flows it is convenient to change variables in the integrals. We write

$$\chi = x - U(t-\tau),\quad d\chi = U\,d\tau \tag{11.8}$$

Equation (11.3) may be written as

$$U\,\frac{\partial \underline{\tau}}{\partial x} = G(0)\underline{A}[\underline{u}(\underline{x})] + \frac{1}{U}\int_{-\infty}^{x} G'\left[\frac{x-\chi}{U}\right]\underline{A}[\underline{u}(\underline{\chi})]\,d\chi \tag{11.9}$$

Equation (11.6) is now

$$\rho U\,\frac{\partial \underline{\zeta}}{\partial x} = \operatorname{curl}\operatorname{div}\underline{\tau} \tag{11.10}$$

and (11.7) becomes

$$\rho U^2\,\frac{\partial \underline{\zeta}}{\partial x} = \int_{-\infty}^{x} G\left[\frac{x-\chi}{U}\right]\left[\frac{\partial^2}{\partial \chi^2} + \nabla_2^2\right]\underline{\zeta}(\underline{\chi})\,d\underline{\chi} \tag{11.11}$$

which, after integrating the first term under the integral by parts, becomes

$$\rho U^2\,\frac{\partial \underline{\zeta}}{\partial x} = G(0)\,\frac{\partial \underline{\zeta}}{\partial x} + \frac{G'(0)}{U}\,\underline{\zeta} + \frac{1}{U^2}\int_{-\infty}^{x} G''\left[\frac{x-\chi}{U}\right]\underline{\zeta}(\underline{\chi})\,d\chi$$

$$+ \nabla_2^2\int_{-\infty}^{x} G\left[\frac{x-\chi}{U}\right]\underline{\zeta}(\underline{\chi})\,d\chi \tag{11.12}$$

The analysis of supercritical flow is conveniently framed for equations of second order. We differentiate (2.12) with respect to x and find that

$$(M^2-1)\,\frac{\partial^2 \underline{\zeta}}{\partial x^2} - \frac{G''(0)}{UG(0)}\,\frac{\partial \underline{\zeta}}{\partial x} = \nabla_2^2\underline{\zeta} + \frac{G''(0)}{U^2G(0)}\,\underline{\zeta}$$

$$- \frac{1}{UG(0)}\int_{-\infty}^{x}\left[G'\left[\frac{x-\chi}{U}\right]\nabla_2^2 + \frac{1}{U^2}\,G'''\left[\frac{x-\chi}{U}\right]\right]\underline{\zeta}(\underline{\chi})\,d\chi. \tag{11.13}$$

This equation (11.13) changes type from elliptic to hyperbolic when

$$M^2 = \rho U^2/G(0) = U^2/C^2$$

increases through one.

For Maxwell models (11.13) reduces, using (11.4), to

$$\left[\rho U^2 - \frac{\eta}{\lambda}\right]\frac{\partial^2 \underline{\zeta}}{\partial x^2} - \nabla_2^2\underline{\zeta} + \frac{\rho U}{\lambda}\,\frac{\partial \underline{\zeta}}{\partial x} = 0 \tag{11.15}$$

this equation changes type when $M^2 = U^2/C^2$, $C^2 = \eta/\rho\lambda$ increases through one.

The vorticity of steady flows with instantaneous elasticity perturbing uniform flow will change type from elliptic to hyperbolic when the ratio of the velocity of the free stream to the velocity of propagation of shear waves

into a fluid at rest exceeds unity. Characteristics for vorticity in plane flow are given by

$$y-d = \frac{\pm x}{\sqrt{M^2-1}}$$

where d is a constant and U is in the direction x. The characteristics form a net of straight lines.

If we suppose that a small two dimensional body perturbs uniform flow when U > C then there would be an undisturbed region in front of the body which could not be reached by disturbances of the vorticity traveling at velocity C. In a linear theory the first changes in the vorticity would occur across the leading characteristics which form an angle like the Mach angle of gas dynamics (see Fig. 11.1). We could call the undisturbed region of uniform flow a "region of silence".

No one has yet solved a problem of supercritical flow over a body, even in the linearized case and no one has looked for a region of silence in experiments.

An air bubble rising in a liquid will reach a steady terminal velocity. The larger the bubble the larger is the terminal speed. Astarita and Apuzzo (1965) were the first to notice that the terminal velocity of the bubble is not a smooth function of the bubble volume. There is a critical volume which is associated with a jump in the terminal velocity, as in Fig. 11.2.

This phenomenon has also been studied by Calderbank, Johnson & Loudon (1970). The fluids used in these experiments are water-based polymers. The lower value of the critical terminal velocity ranges from 0.1 to 10 cm per sec. These values are not unlike values which we think are typical for wave speeds in these polymers (see Section 4). The reasons for the abrupt rise in the terminal velocity are not understood. Maybe the abrupt change is associated with a change of type. Then we might expect to see an abrupt change near M = 1. For Maxwell models M = RW where R is the Reynolds number and W is the Weissenberg number. Zana and Leal (1978) give values of RW which vary by decades at criticality. Their method of calculating was not clear. They seemed to have used some normal stress data which could give very inaccurate values for W. If they are nearly right it will be hard to support the idea that the abrupt rise is associated with a change of type. The problem is open. To settle this problem we need to have accurate values for the wave velocity used in experiments.

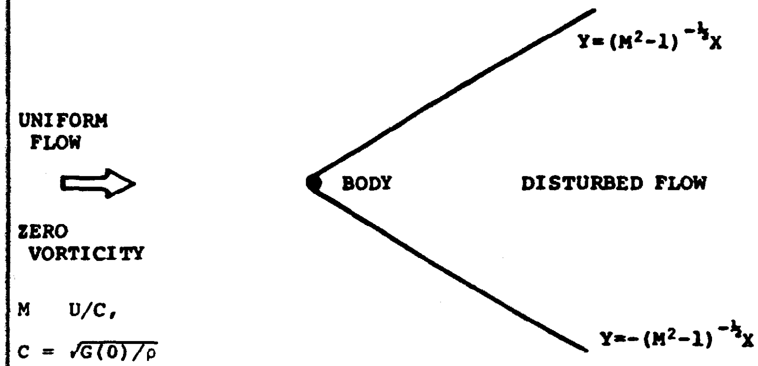

Fig. 11.1. Leading characteristics for vorticity in a plane uniform flow.

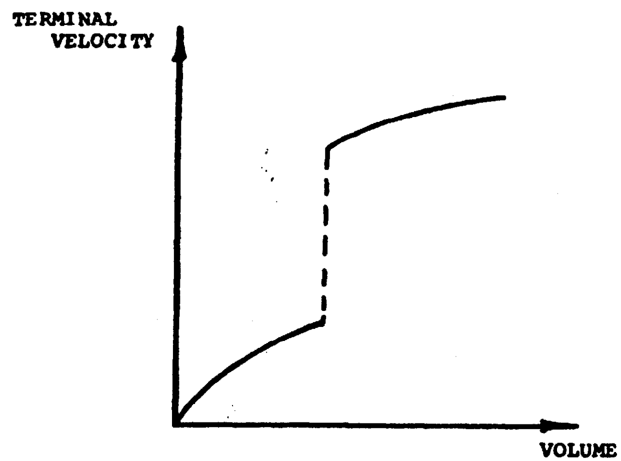

Fig. 11.2. The jump in the terminal velocity for an air bubble rising in a liquid. The jump can be large, say a five-fold increase.

12. FLOW OVER A FLAT PLATE

The problem of flow around a flat plate could be thought to be closer than flow around a body to a perturbation of uniform flow. In fact, the theory of slender bodies in aerodynamics is a perturbation of uniform flow which is perturbed less by slender bodies than by fat ones. The aerodynamic theory works well, but only because of the flow following condition in which the fluid is required to slip along the slender body. In this chapter I am going to give an exact supercritical theory, valid for all fluids with instantaneous elasticity, such that the vorticity is prescribed arbitrarily along the flat plate, with a vanishing normal component of velocity on the along the flat plate, with a vanishing normal component of velocity on the plate surface. Presumably the no slip condition on the plate can also be satisfied by choosing the correct vorticity distribution on the plate. In the nonlinear problem there will be a diffusive subcritical region near the plate and a "transonic" surface outside where the viscoelastic Mach number passes through one. This type of important problem has yet to be considered and solved in the theory of flow of viscoelastic fluids.

We now consider two dimensional flow past a flat plate. The plate is on the half line $y = 0$, $x \geq 0$. The velocity components corresponding to (x,y) are (u,v) and there is only one component

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \tag{12.1}$$

of vorticity. The velocity components

$$(u,v) = \left[-\frac{\partial \Psi}{\partial y}, \frac{\partial \Psi}{\partial x}\right] \tag{12.2}$$

may be obtained from a stream function Ψ,. We suppose that the velocity (u,v) vanishes at infinity and that

$$(u,v) = (-U,0) \text{ on } y = 0,\ x \geq 0. \tag{12.3}$$

The vorticity of the perturbed flow satisfies

$$(M^2-1)\frac{\partial^2\zeta}{\partial x^2} - \frac{G'(0)}{UG(0)}\frac{\partial\zeta}{\partial x} = \frac{\partial^2\zeta}{\partial y^2} + \frac{G''(0)}{U^2G(0)}\zeta$$

$$- \frac{1}{UG(0)}\int_{-\infty}^{x}\left[G'\left[\frac{x-\chi}{U}\right]\frac{\partial^2}{\partial y^2} + \frac{1}{U}G''\left[\frac{x-\chi}{U}\right]\right]\zeta(\chi)\,d\chi. \tag{12.4}$$

For the Maxwell model (11.4) we get a telegraph equation

$$(M^2-1)\frac{\partial^2\zeta}{\partial x^2} - \frac{\partial^2\zeta}{\partial y^2} + \frac{\rho U}{\eta}\frac{\partial\zeta}{\partial x} = 0.$$

We may write this telegraph equation in dimensionless form

$$\frac{\partial^2\zeta}{\partial x^2} + \frac{M^2}{\beta^2 W}\frac{\partial\zeta}{\partial x} - \frac{1}{\beta^2}\frac{\partial^2\zeta}{\partial y^2} = 0 \tag{12.5}$$

where $(x,y) \sim (x/\ell, y/\ell)$, ℓ is a scale length

$$\beta^2 = M^2 - 1 > 0$$

and

$$W = U\lambda/\ell$$

is a Weissenberg number based on ℓ.

Equation (12.5) can be approximated by a wave equation without damping when $W \to \infty$ and $M > 1$ is fixed. For this same M we may expect rapid damping of the backward heat equation when W is very small.

The other prescribed conditions are that the vorticity must vanish far from the plate and since $v = 0$ on the plate,

$$\zeta = -\frac{\partial u}{\partial y} \text{ on } y = 0,\ x \geq 0 \tag{12.6}$$

We need vorticity fields satisfying (12.4) or (12.5) over R^2 and (12.6) and (12.3) on the plate.

The stream function and vorticity are related by a second order equation

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right]\Psi = -\zeta(y,x). \tag{12.7}$$

We may therefore expect that corresponding to any good field $\zeta(x,y)$ we may solve (12.7) subject to the conditions that Ψ vanishes far from the plate and

$$v = \frac{\partial \Psi}{\partial x} = 0 \text{ on } y = 0,\ x \geq 0. \tag{12.8}$$

In general, we cannot expect that for any field $\zeta(y,x)$

$$U = \frac{\partial \Psi}{\partial y} \text{ on } y = 0,\ x \geq 0. \tag{12.9}$$

It follows that the existence of a solution of the linearized flat plate problem, in both the subcritical $M < 1$ and supercritical $M > 1$ case depends on finding a special prescription of the vorticity ζ on the plate (12.6) such that (12.9) is verified.

I wish now to consider the supercritical case and to indicate the method by which our linearized problem may be resolved. It will be convenient to start this discussion with an analysis of the telegraph equation (12.5) for $M > 1$. To put this problem into canonical form we change variables setting

$$(x,y) = \left[\frac{\beta^2 W}{M^2}t, \frac{\beta W}{M^2}z\right] \tag{12.10}$$

and find that $\tilde{\zeta}(z,t) = \zeta(y,x)$ satisfies

$$\frac{\partial^2\tilde{\zeta}}{\partial t^2} + \frac{\partial\tilde{\zeta}}{\partial t} - \frac{\partial^2\tilde{\zeta}}{\partial z^2} = 0 \tag{12.11}$$

We may expect that this hyperbolic problem will not allow the plate to influence the flow upstream. We therefore seek solutions $\tilde{\zeta}(z,t)$ such that the upstream vorticity vanishes;

$$\tilde{\zeta}(z,t) = 0 \text{ for } t < 0,\ -\infty < z < \infty$$

This implies that "initially"

$$\tilde{\zeta}(z,0) = \tilde{\zeta}_t(z,0) = 0 \tag{12.12}$$

whereas

$$\tilde{\zeta}(0,t) = g(t) \tag{12.13}$$

is prescribed for $t > 0$. Asymptotically, for large $|z|$ we require that $\tilde{\zeta} \to 0$.

The problem (12.11), (12.12) and (12.13) can be solved by Laplace transform techniques. A well known solution of this problem with uniform vorticity on the plate

$$f(t) = H(t) = \begin{matrix} 0, t<0 \\ 1, t>0 \end{matrix} \tag{12.14}$$

has been given by Carlslaw and Jaeger (1963) in the form

$$\tilde{\zeta}(z,t) = \left[e^{-z/2} + \frac{z}{2}\int_z^t \frac{e^{-\sigma/2}}{\sqrt{\sigma^2-z^2}} I_1\left(\frac{1}{2}\sqrt{\sigma^2-z^2}\right) d\sigma\right] H(t-z) \tag{12.15}$$

where I_1 is the modified Bessel function of the first kind. The vorticity $\tilde{\zeta}(-z,t)$ on the bottom of the plate is given by (12.15) with $-z$ replacing z. This solution for flow over a flat plate was given by B. Caswell (1976) in a study of the effects of a leading edge singularity.

We next note that the field $\zeta(y,x) = \tilde{\zeta}(z,t)$ can now be inserted into (12.7), which is an elliptic problem leading to a nearly everywhere differentiable Ψ, even when ζ has simple discontinuities, as in Figure 1. We solve (12.7) subject to (12.8), but the solution will not satisfy (12.9). We can hope to satisfy (12.8) and (12.9) simultaneously by prescribing the perfect vorticity distribution $f(t)$ on the plate.

To get the $\tilde{\zeta}(t,z,g)$ corresponding to different prescriptions (12.13) of the vorticity $f(t)$ on the plate, we could use the method of Duhamel type integrals introduced for start up problems by Narain and Joseph (1983). For these integrals we need to superpose using the fundamental singular solution of (12.11), (12.12) and (12.13) when

$$g(t) = \delta(t-\tau) \tag{12.16}$$

is a Dirac function. It is easy to see and not hard to prove that the required solution is the time derivative of the step function problem just derived when

$$\zeta(0,t) = \begin{matrix} 1 \text{ for } t > \tau \\ 0 \text{ for } t \leq \tau \end{matrix} . \tag{12.17}$$

Using the aforementioned method I find that the solution of (12.11), (12,12) and (12.13) is

$$\tilde{\zeta}(t,z) = \int_0^{t-z} g(\tau)\ \frac{\partial f}{\partial t}(z,t-\tau)d\tau + e^{-z/2}\ g(t-z) \tag{12.18}$$

where

$$f(z,t) = e^{-z/2} + \frac{z}{2}\int_z^t e^{-\sigma/2}\ \frac{I_1(\frac{1}{2}\sqrt{\sigma^2-z^2})}{\sqrt{\sigma^2-z^2}}\ d\sigma .$$

The solution $\zeta(y,x)$of

$$\frac{\partial^2\zeta}{\partial x^2} + \frac{M^2}{\beta^2 W}\ \frac{\partial\zeta}{\partial x} - \frac{1}{\beta^2}\ \frac{\partial^2\zeta}{\partial y^2} = 0 ,$$

$$\zeta(0,x) = g(x), \tag{12.19}$$

$$\zeta(y,0) = 0$$

is

$$\zeta(y,x) = \int_0^{x-\beta y} g(\tau)\ \frac{\partial f}{\partial x}(y,x-\tau)d\tau + \exp\left[-\frac{yM^2}{2\beta W}\right]g(x-\beta y) \tag{12.20}$$

where

$$\frac{\partial f}{\partial x} = \frac{yM^2}{2\beta W}\ \frac{e^{-x/2}I_1\left[\frac{1}{2}\left[x^2 - \frac{M^4y^2}{4\beta^2W^2}\right]^{1/2}\right]}{\left[x^2 - \frac{M^4y^2}{4\beta^2W^2}\right]^{1/2}}$$

The amplitude of ζ on $x = \beta y$ is given by $\exp[-\frac{yM^2}{2\beta W}]$. The amplitude decays rapidly when W is small (see Fig. 12.1).

Turning next to the stream function, we find that the solution of (12.7) with $\zeta(y,x)$ given by (12.20) and

$$\psi(x,0) = 0 \quad \text{for } x > 0 \tag{12.21}$$

is

$$\psi = -\int\int_{R_-} G(y,y_0|x,x_0)\zeta(y_0,x_0)dy_0\ dx_0 \tag{12.22}$$

where R_- is the complement to the line $x > 0$, $y = 0$. The Green function for this domain, vanishing on the plate is given by (10.1.40) on p. 1208 of Morse & Feshbach (1953). We have to divide their solution by 4π.

$$G(y,y_0|x,x_0) = \overline{G}(r,r_0|\phi,\phi_0) ,$$

$$= \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left[\frac{n\phi_0}{2}\right]\sin\left[\frac{n\phi}{2}\right]\left[\frac{r}{r_0}\right]^{n/2}, \quad r < r_0$$

$$= \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left[\frac{n\phi_0}{2}\right]\sin\left[\frac{n\phi}{2}\right]\left[\frac{r_0}{r}\right]^{n/2}, \quad r > r_0 \tag{12.23}$$

where $(x,y) = r(\cos\phi,\sin\phi)$

The solution (12.7), satisfying (12.21) is given by (12.22) with $\zeta(y,x)$ given by (2.20). The fluid will not slip on the plate if $U = \frac{\partial\psi}{\partial y}(x,0)$ when $x > 0$; that is,

$$-U = \int \int_{R_-} \frac{\partial}{\partial y}\Big|_{\substack{y=0\\x>0}} G(y,y_0|x,x_0)\zeta(y_0,x_0)dy_0\,dx_0 \tag{12.24}$$

We have to choose the prescribed plate vorticity g(x) to satisfy (12.24). I have no guarantee that there is a g(x) satisfying (12.24). Edmond O'Donovan is trying to find a numerical approximation for the g(x) which satisfies (12.24).

It is probable that the approach which I have taken in this problem can be generalized to fluids satisfying (12.4). This equation applies to all fluids with instantaneous elasticity. Such a study would show how the flow depends on the kernel G(s). The Laplace transform methods used by Narain and Joseph (1982, 1983) to study the linearized dynamics of shearing motions perturbing rest in viscoelastic fluids are appropriate for this study too. As in their work I get the following problem for the transform

$$\bar{\zeta}(y,\omega) = \int_0^\infty e^{\omega x}\,\zeta(x,y)\,dx \tag{12.25}$$

where $\bar{\zeta}(y,\omega)$ vanishes for large y and satisfies

$$k(\omega)\bar{\zeta} = \bar{g}(\omega)\,\frac{d^2\bar{\zeta}}{dy^2}$$

$$\bar{\zeta}(0,\omega) = \int_0^\infty e^{-\omega x}\,f(x)\,dx \tag{12.26}$$

and

$$k(\omega) = \omega[\rho U^2 - G(0)] - \frac{G'(0)}{U} - \frac{\bar{h}(\omega)}{U},$$

$$\bar{h}(\omega) = \int_0^\infty e^{-\omega s}\,G''\left[\frac{s}{U}\right] ds,$$

$$\bar{g}(\omega) = \int_0^\infty e^{-\omega s}\,G\left[\frac{s}{U}\right] ds.$$

One finds that

$$\bar{h}(\omega) = \frac{G''(0)}{\omega} + O\left[\frac{1}{\omega}\right],$$

$$\bar{g}(\omega) = \frac{G(0)}{\omega} + \frac{G'(0)}{\omega^2 U} + O\left[\frac{1}{\omega^2}\right]$$

if G" is integrable. The character of the solutions is determined by the symbol of the operator. We should therefore look at the problem which arises at large ω

$$\left[\omega[\rho U^2 - G(0)] - \frac{G'(0)}{U}\right]\bar{\zeta} = \left[\frac{G(0)}{\omega} + \frac{G'(0)}{\omega^2 U}\right]\frac{d^2\bar{\zeta}}{dy^2}$$

or

$$\left[\beta^2\omega^2 - \frac{\rho U G'}{G^2(0)}\,\omega\right]\bar{\zeta} = \frac{d^2\bar{\zeta}}{dy^2}$$

which is the transform of the equation

$$\beta^2\zeta_{xx} - \frac{\rho U G'(0)}{G^2(0)}\,\zeta_x = \zeta_{xx}. \tag{12.27}$$

From this equation we expect that the solution of the present problem is a simple wave as in Figure 12.1 with $\zeta = 0$ to the left of the line $x = \beta y$ on the top of the plate, and $x = -\beta y$ on the bottom plate. The amplitude of the wave on $x = \beta y$ is given by

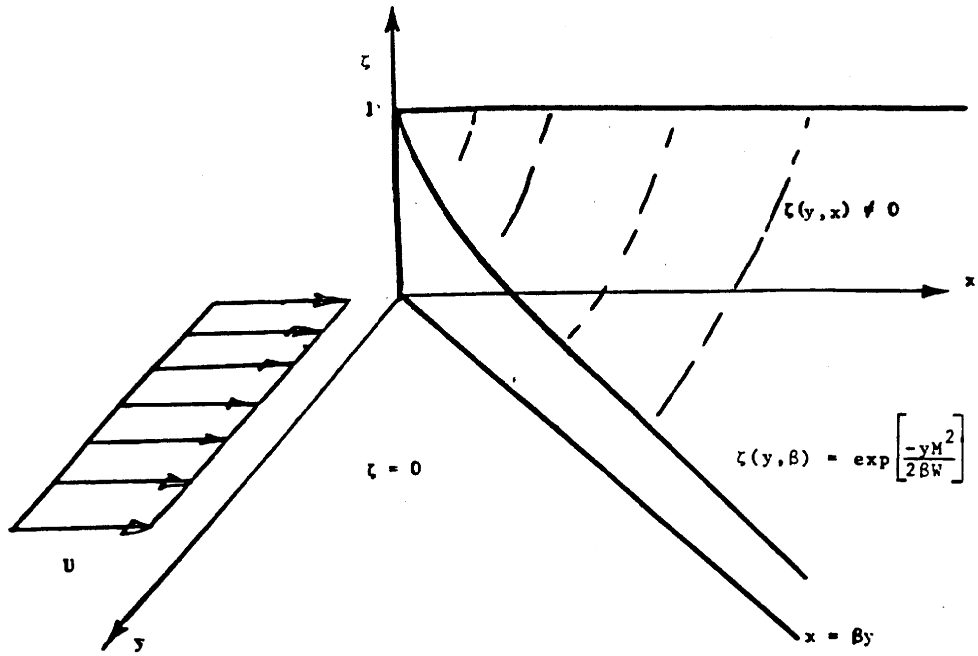

Fig. 12.1. Solution (12.20) for the problem (12.19) for a unit jump of vorticity. The solution decays rapidly if 1/W is large.

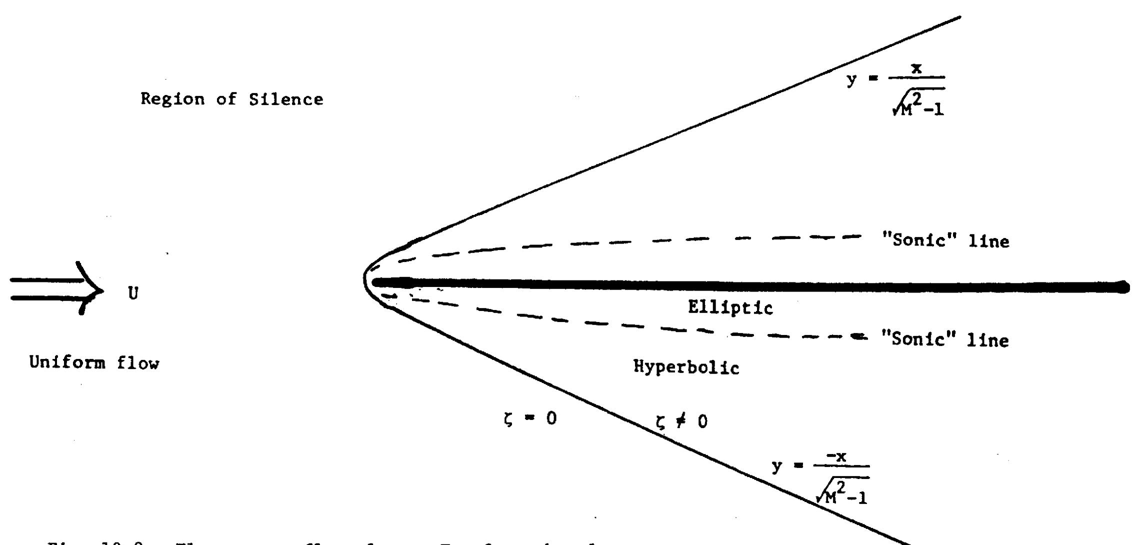

Fig. 12.2. Flow over a flat plate. Far from the plate the linear theory is valid and the vorticity will be confined to the region behind the "Mach" cone. Near the plate the flow must be subcritical and the details will depend on the constitutive equation.

$$\exp\left[\frac{x\rho UG'(0)}{\beta^2 G^2(0)}\right]$$

This amplitude tends to zero as $G'(0) \to -\infty$. When $G(0) > 0$ and $G'(0) = -\infty$ we have the case of singular kernels. Renardy (1982) showed that for some special singular kernels shear waves would propagate into a fluid at rest with the wave velocity $C = \sqrt{G(0)/\rho}$, with a zero amplitude at the front and C^∞ connection with non-zero solution across the front. These properties, as well as the analytic smoothing of sharp fronts with small viscous terms (small retardation times) occur in the present steady problem as well as in the theory of propagation of waves.

Supercritical flow past a flat plate cannot be considered to be a perturbation of uniform flow. If it is supercritical, the free stream velocity is finite, perhaps large, but the fluid must come to rest on the plate. Near the plate, the velocity will be small and the local "Mach" number less than one even when $M_\infty^2 = \rho U^2/G(0) > 1$. The governing problem near the plate is therefore elliptic, or at least not hyperbolic. It may be true that when M is large the solution of the nonlinear problem is close to the linear solution (if it exists) except in regions immediately near the plate. This might suggest boundary layers, but that thought should be eschewed. We should instead think of a narrow subcritical region near the plate which goes supercritical at the "sonic" line which could also be close to the plate when $M >> 1$. We have therefore to consider a "transonic" type of problem and not a boundary layer type of problem. We know almost nothing about such problems for viscoelastic fluids.

13. NONLINEAR WAVE PROPAGATION AND SHOCKS

It is well to motivate this chapter by reminding the reader of the huge differences between linear and nonlinear theories of gas dynamics. In the linearized theory there are no essential differences between rarefaction and compression waves. These waves propagate according to the wave equation without change of form. In the nonlinear theory an impulsive rarefaction will be smoothed by nonlinear effects and a smooth compression will shock up.

Compression waves are impossible in incompressible fluids. Instead we may perhaps speak of waves of shear or of waves of vorticity. Very little is known about the nonlinear effects in the flow of viscoelastic fluids. We should like to know the answers to the following questions. We are given a constitutive equation in the class with instantaneous elasticity: Different subclasses in this class can lead to different results. We want some classification of results.

(i) Suppose that we are given smooth data. Is there a shock up?

(ii) When there is a shock, what variables (vorticity, velocity, displacement) become discontinuous?

(iii) We are given impulsive data. Is it possible that the nonlinear terms smooth discontinuities?

We first consider the problem of wave propagation in fluids undergoing rectilinear shear flow. The formulation, due to Coleman and Noll, is embodied in the representations for shear flow shown in (1.2) of the Appendix, by M. Slemrod to this paper. Coleman and Gurtin (1968), following earlier work on longitudinal acceleration waves in compressible materials (1965), showed that the amplitude of a jump discontinuity in the fluid acceleration satisfies a simple nonlinear differential equation of Bernoulli type with coefficients determined by the instantaneous value of the relaxation kernel at the wave and second order instantaneous modulus evaluated at the wave. The instantaneous value of the relaxation kernel is designated as

$$G(\kappa,0)$$

where κ is the shear rate at the wave and $G(\kappa,s)$ arises as the kernel of the integral representation implied by the Riesz theorem for first functional derivatives of the functional t (·) of (1.2) in Appendix A evaluated in the weighted $L_h^2[0,\infty]$ spaces of Coleman and Noll. This kernel reduces to G(s) when the wave advances into a region at rest. The speed of the wave

$$C = \sqrt{\frac{G(\kappa,0)}{\rho}}$$

depends on the rate of shear where the first derivative is evaluated. In general problems the speed of waves depends on the motion. The second order general problems the speed of waves depends on the motion. The second order modulus is an instantaneous evaluation the second Frechet derivative at the wave.

The amplitude equation of Coleman and Gurtin is notable because it is simple, general, rigorous and implies interesting physical results. They showed that an initial jump discontinuity may either decay or grow depending on the sign and magnitude of the initial discontinuity. The assumed jump discontinuity may blow up in finite time. The cause of this blow up is associated with the nonlinearity. They showed that an acceleration wave entering into a region at rest would always decay. This is a type of nonlinear result of category (ii) which shows that nonlinear terms can lead to blow up. It is generally assumed that the loss of C^n regularity (blow up) implies the formation of a jump in the n-1 derivative of velocity (shock up). The decay of acceleration waves is a result of category (iii) which shows that nonlinear terms can force the decay of initial discontinuities.

It was not clear how discontinuities in acceleration, which are equivalent to shocks of the vorticity, would appear in the fluid. The results discussed in the appendix by M. Slemrod help to clarify this issue.

Slemrod (1978) showed that the equation of motion for the shearing perturbation $\hat{v}(x,t)$ of a shearing motion will admit a differentiable (in x,t) solution for only a finite time for appropriate smooth initial velocity histories when the constitutive relation is

$$T^{\langle xy\rangle} = \sigma\left(\int_0^\infty e^{-\tau/\lambda}\, v_x(x,t-\tau)\,d\tau\right). \tag{13.1}$$

where σ is a nonlinear odd function. The loss of differentiability is assumed to imply the appearance of a discontinuity in v, as shock up of the velocity, a vortex sheet. This result is like the one proved by Coleman and Gurtin, but implies the shock of smooth data.

In the appendix of this paper Slemrod shows that when the constitutive relation is

$$T^{\langle xy\rangle} = -\int_0^\infty e^{\tau/\lambda}\, \sigma(v_x(x,t-\tau)\,d\tau \tag{13.2}$$

for a nonlinear $\sigma(\cdot)$, the second derivatives of v(x,t) can blow-up when the smooth data is given in a certain way. The shock up assumption

which one is obliged to associate with this blow-up of second derivatives is the appearance of a jump in v_x, a vortex shock or an acceleration discontinuity. It is interesting to note that $T^{\langle xy\rangle}$ given by (13.2) satisfies a rate equation

$$\frac{d}{dt} T^{\langle xy\rangle} + \frac{1}{\lambda} T^{\langle xy\rangle} = \sigma(v_x(x,t)) \tag{13.3}$$

vaguely resembling some popular models.

The results just reviewed show the remarkable effects of the choice of constitutive relations.

It is possible to entertain the notion of successive shock ups from smooth data. First we get a vortex shock from nonlinear effects associated with (13.2). This gives an acceleration discontinuity which will lead to blow up of the vorticity if the amplitude of the vortex shock is larger than the critical amplitude of Coleman & Gurtin. In reservation, I wish to note that recent calculations with popular models show that the critical amplitude can be infinite.

I think it would be interesting to see what sort of blow up results could be obtained for shear flows of Oldroyd models with instantaneous elasticity.

The shear flow of the Oldroyd models (6.2) are governed by the system of first order quasilinear equations.

$$\sigma_t - (a+1)\tau v_x + \frac{\sigma}{\lambda} = 0,$$

$$\tau_t + \left[(1-a)\sigma - \frac{\eta}{\lambda}\right]v_x + \frac{\tau}{\lambda} = 0,$$

$$\rho v_t - \tau_x = 0 \tag{13.4}$$

where v(x,t) is the rectilinear velocity in the direction y and $(\sigma,\gamma,\tau) = (T^{\langle yy\rangle}, T^{\langle xx\rangle}, T^{\langle xy\rangle})$. This can be written as

$$\underline{A}\, \underline{q}_t + \underline{B}\, \underline{q}_x + \underline{D} = 0$$

$$\underline{q} = \begin{bmatrix} \sigma \\ \tau \\ v \end{bmatrix}, \quad \underline{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \rho \end{bmatrix}, \quad \underline{D} = \begin{bmatrix} \sigma/\lambda \\ \tau/\lambda \\ 0 \end{bmatrix}$$

$$\underline{B} = \begin{bmatrix} 0 & 0 & -(a+1)\tau \\ 0 & 0 & (1-a)\sigma-\eta/\lambda \\ 0 & -1 & 0 \end{bmatrix}$$

The characteristic directions are given by

$$\det\left|\underline{A}\dot{x} - \underline{B}\dot{x}\right| = \left[\rho^2\dot{x}^2 + (1-a)\sigma - \eta/\lambda\right] = 0$$

where $\overset{\circ}{x}$ is the wave speed. The streamlines are characteristic; $\dot{x} = 0$, because streamlines don't move. The two waves

$$\overset{\circ}{x} = \pm\sqrt{\frac{(a-1)\sigma + \eta/\lambda}{\rho}}$$

can be identified with waves of vorticity. It is clear that the speed of the wave depends on how the liquid is stressed.

We should like to know if smooth initial data, given to (13.4) can blow up. This depends on the constitutive equation, as we have already seen. For example, the problem for the upper and lower convected Maxwell models ($a^2=1$) cannot lead to blow up of smooth data because the governing problems are linear.

Johnson and Segalman (1977) have shown that $(13.4)_1$ and $(13.4)_2$ can be integrated once:

$$\tau = \frac{\eta}{\lambda}\int_{-\infty}^{t} e^{-(t-s)/\lambda}\,\kappa(s)\cos\left(\sqrt{1-a^2}\int_s^t \kappa(s')ds'\right)ds,$$

$$\sigma = \frac{\eta}{\lambda}\sqrt{\frac{1+a}{1-a}}\int_{-\infty}^{t} e^{-(t-s)/\lambda}\,\kappa(s)\sin\left(\sqrt{1-a^2}\int_s^t \kappa(s')ds'\right)ds$$

where

$$\kappa(s) = v_x(x,s)$$

is the shear rate at the layer x at time $s \le t$. It follows that

$$\tau_t - \frac{\eta}{\lambda}\kappa(t)\left[1 - \sqrt{1-a^2}\int_{-\infty}^{t} e^{-(t-s)/\lambda}\,\kappa(s)\sin\left(\sqrt{1-a^2}\int_s^t \kappa(s')ds'\right)ds\right] = -\frac{\tau}{\lambda}.$$

We may consider fluids with short memories, small λ. After integrating by parts we find that

$$\tau_t - \frac{\eta}{\lambda}\kappa(t)[1 - (1 - a^2)\lambda^2\kappa^2(t) + O(\lambda^3)] + \frac{\tau}{\lambda} = 0 \tag{13.5}$$

This is in the form

$$\frac{d}{dt}T^{\langle xy\rangle} + \frac{1}{\lambda}T^{\langle xy\rangle} = \frac{\eta}{\lambda}v_x(x,t)\left[1 - (1 - a^2)\lambda^2 v_x^2(x,t) + O(\lambda^3)\right]$$

which, up to terms $O(\lambda^2)$ is like (13.3), leading to the blow up of vorticity when $1 - a^2 < 0$.

We turn next to a still more complicated flow involving wave propagation, which is originally set in two space dimensions.

N. Phan Thien and R. I. Tanner (1983), hereafter called PT&T, have considered the problem of flow of an upper convected Maxwell model which is induced by squeezing the fluid between infinite parallel planes. They do not linearize or neglect terms. They call their solution exact because they find a separable solution (13.6) which reduces the governing three dimensional quasilinear system to a nonlinear two-dimensional system which they integrate numerically. They have impulsive initial data in the following sense. Initially an irrotational plug flow is prescribed. The velocity v normal to the plates increases linearly from zero at the bottom to the prescribed value at the top. At this same initial instant the velocity u parallel to the plate is squeezed out as a plug flow. Since u must vanish at the plates, this component is prescribed as discontinuous. In linear problems this discontinuity would propagate and decay along characteristics. PT&T do note that their numerical work shows wave propagation but their graphs are confusing; it is hard to tell what is propagating and how it is propagating.

I have analyzed their problem using some elementary hyperbolic theory. I work with their reduced equations. The reduced equations given by (13.10) are a consequence of representation.

$$[u,v] = V[-xf_y(y,t), f(y,t)] \tag{13.6}$$

Here, subscripts denote differentiation. We treat the rest of the reduction later. For now, consider Eq. (6.4) with a = 1 for the vorticity

$$\rho \frac{\partial^2 \zeta}{\partial t^2} + \rho u \frac{\partial^2 \zeta}{\partial x \partial t} + 2\rho v \frac{\partial^2 \zeta}{\partial y \partial t} + (\rho u^2 - \sigma - \frac{\eta}{\lambda}) \frac{\partial^2 \zeta}{\partial x^2}$$

$$+ 2(\rho u v + \tau) \frac{\partial^2 \zeta}{\partial x \partial y} + (\rho v^2 - \Upsilon - \frac{\eta}{\lambda}) \frac{\partial^2 \zeta}{\partial y^2} = \text{L.O.T.} \tag{13.7}$$

The vorticity compatible with (13.6) satisfies

$$\zeta = -\frac{1}{2}\frac{\partial u}{\partial y} = \frac{V}{2} x f_{yy} \overset{\text{def}}{=} x\omega,$$

$$\zeta_x = \omega, \quad \zeta_{xx} = 0.$$

These relations imply that

$$\left[\frac{\partial^2 \zeta}{\partial x \partial t}, \frac{\partial^2 \zeta}{\partial x \partial y}\right] = \left[\frac{\partial \omega}{\partial t}, \frac{\partial \omega}{\partial y}\right]$$

are first derivatives of ω and hence of lower order. We may then write (13.7) as

$$A \frac{\partial^2 \zeta}{\partial t^2} + 2B \frac{\partial^2 \zeta}{\partial y \partial t} + C \frac{\partial^2 \zeta}{\partial y^2} = \text{L.O.T.}$$

where $A = \rho$, $B = \rho v$, $C = \rho v^2 - \Upsilon - \frac{\eta}{\lambda}$. The characteristic equation for this second order problem are given by

$$\frac{dy}{dt} = \frac{B}{A} \pm \sqrt{\frac{B^2 - AC}{A}}$$

$$= v \pm \sqrt{\frac{\Upsilon}{\rho} + C^2} \tag{13.8}$$

where $C = \sqrt{\eta/\lambda\rho}$ is the speed of small amplitude vorticity waves into regions of uniform motion. The wavespeed formula (13.8) shows how the speed of vorticity for solutions of the type (13.6) depends on the pointwise values of the normal component of velocity and the tensile stress $\Upsilon = \tau_{yy}$. Every problem which can be represented in (13.6) will have wave speeds for the vorticity which will satisfy (13.8). Numerical results of PT&T suggest that for their problem $\Upsilon/\rho + C^2 > 0$.

In principle, it should be possible to determine some of the effects of nonlinearity on propagation by more careful studies of the problem treated by PT&T.

Let h_0 be the initial distance between the plates and let v be the constant squeezing velocity

$$(y,t) = \frac{1}{h_0}(y, Vt)$$

are dimensionless variables, also called y and t. The configuration of flow is shown as Fig. 13.1. The dimensionless velocity components are given by

$$(u,v) = (xf_y(y,t), f(y,t))$$

and have been made dimensionless with V. The stresses are made dimensionless with $\eta V/h_0$.

The dimensionless stress and pressure π are then represented by

$$[\tau^{\langle xx \rangle}, \tau^{\langle yy \rangle}, T^{\langle xy \rangle}, P] = \left[X_1 + x^2 X_2, xT, Y, \frac{1}{2}x^2 p_1 + p_2\right] \tag{13.9}$$

where X_1, X_2, T, Y depend on (y,t) and p_2 and

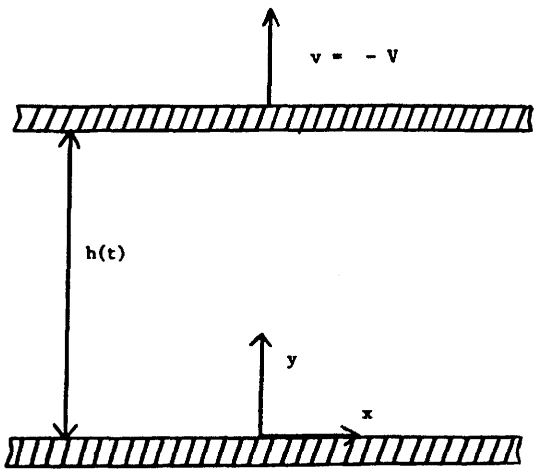

Fig. 13.1 Squeeze Flow, $V > 0$, $h(0) = h_o$.

$$p_1(t) = 2X_2 + T_y + R\left[f_{ty} + f_y^2 + ff_{yy}\right]$$

depend only on t, and R is the Reynolds number.

We may write the reduced problem as

$$\underline{A}\, \underline{q}_y + \underline{B}\, \underline{q}_t = \underline{D} \qquad (13.10)$$

where $\underline{q}$ is the column vector whose components are $[f,p,X_1,Y,X_2,T,g]$ where $g = f_y$,

$$\underline{A} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & fW & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & fW & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & fW & 0 & 2WT \\ 0 & 0 & 0 & 0 & 0 & fW & 1+WY \\ 0 & 0 & 0 & 0 & 0 & 1 & fR \end{bmatrix}$$

$W = V\lambda/h_0$ is the Weissenberg number, $\underline{B}$ is diagonal with entries $[0,0,W,W,W,W,R]$ and $\underline{D}$ depends on $\underline{q}$ but not on its derivatives. The wave speeds for this system $\mathring{y} = dy/dt$ are given as the real roots of

$$\det[\underline{A} - \mathring{y}\underline{B}] = W^3(f-\mathring{y})^3[M^2(f-\mathring{y})^2 - 1 - WY] = 0$$

where $M^2 = RW$. The normal component of velocity $v = f$ is triply characteristic. The wave speeds for the vorticity are given by the dimensionless form of (13.8).

$$\mathring{y} = f \pm \frac{1}{M}(1+WY)^{1/2}$$

Numerical results of PT&T show that $Y \to 0$ as $W \to \infty$ in such a way that $1 + WY > 0$. *There is no Hadamard instability.*

An initial boundary value problem for the reduced quasilinear system requires that one set of boundary and initial conditions. At the boundaries $y = 0$ and $y = H(t)$ we have adherence

$$u = v = 0 \quad \text{at } y = 0,$$

$$v = 0, \quad v = -V \quad \text{at } y = H(t).$$

To specify initial conditions it is enough to give the fields u and v, that is, $f(y,t)$ at $t = 0$. PT&T set $f(y,0) = -y$. Hence

$$(u,v)\big|_{t=0} = (x,-y) \qquad (13.11)$$

which is an irrotational squeeze flow with vanishing stresses and an outflow independent of y at each x. The wave speeds (13.8) are

$$dy/dt\big|_{t=0} = C \quad \text{at the bottom plate}$$

and

$$dy/dt\big|_{t=0} = -1 - C \quad \text{at the top plate.}$$

The velocity $v = -y$ at $t = 0$ is compatible with the prescribed boundary conditions but $u = 0$ on $y = 0$ and on $y = H(t)$ is incompatible with $u = x$. The velocity v and all the stresses are continuous as $t \to 0$ but the velocity $u = 0$ at the walls is incompatible with (13.9). In a linear theory we could expect this initial discontinuity to propagate into the interior with a decay in the magnitude of the jump in u. This does not appear to be true in the results presented by PT&T.

In Fig. 13.2 I have reproduced their results for the evolution of $-f_y(y,t) = u/x$ for $R = 1$ and increasing W. The graphs of the evolution of $f = v$ do not show interesting features and are not presented here.

I cannot find evidence for propagating discontinuities of u/x near the top wall. The wave in u/x propagating from the top plate seems to have been smoothed. On the other hand the points near the bottom plate which could show the development of discontinuous u/x for large time and large W. Perhaps it is these points at which we may see the development of shocks of u/x are only what PT&T call "unavoidable noise." It is possible to find pictures, say $\tau = 0.5$, $W = 1$, in which one can imagine a point of discontinuity of the slope of u/x which can be identified with a shock in the vorticity.

I *think that the results shown support the notion, already explored* in Sections 10 and 12, that the Weissenberg number is a measure of the fluids elasticity and in hyperbolic problems with damping, the damping is large when the Weissenberg number is small and vice versa. The effect of the waves is almost immediately damped in the case $W = 0.1$ and is extraordinarily persistent when $W = 500$.

It would be interesting to see if and what discontinuities develop in the squeeze flow problem when all of the prescribed data is smooth at $t = 0$.

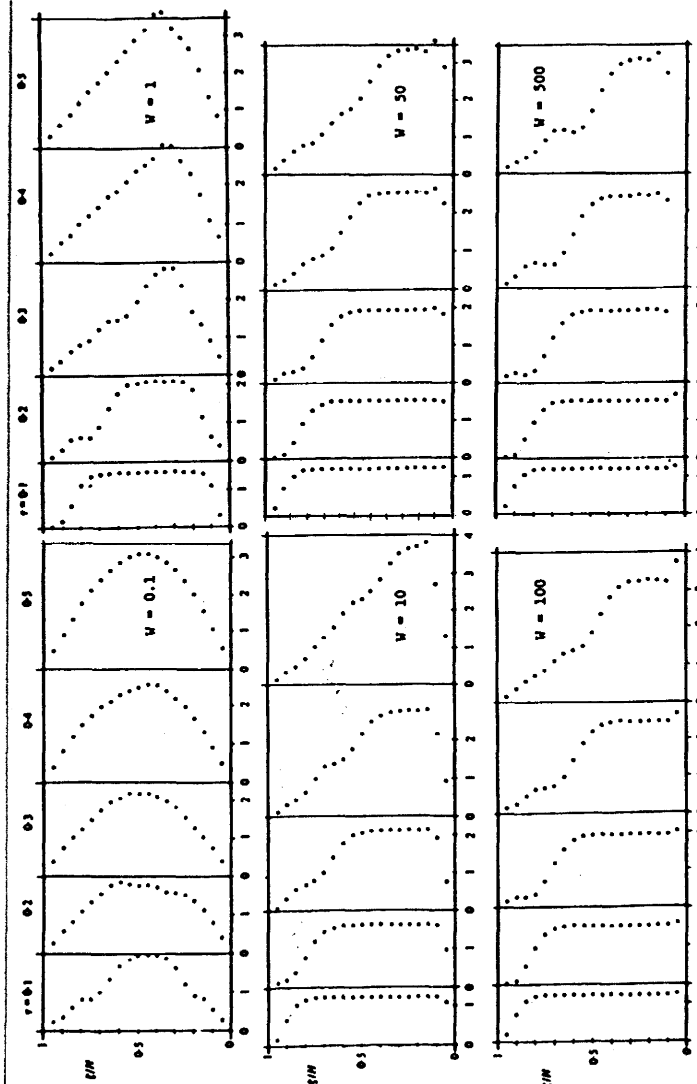

Fig. 13.2. (After Phan Thien and Tanner (1983)). Evolution of the vertical component of velocity $u_x - f_y = u/x$ is plotted, in squeeze flow. The Reynolds number is unity. W is the Weissenberg number.

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APPENDIX A: BREAKDOWN OF SMOOTH SHEARING FLOW IN VISCOELASTIC FLUIDS FOR TWO CONSTITUTIVE RELATIONS: THE VORTEX SHEET VS. THE VORTEX SHOCK.

A0. INTRODUCTION

The purpose of this note is to study the effect of choice of constitutive relations on shearing perturbations of steady shearing flows in a non-linear, isotropic, incompressible, viscoelastic fluid. In an earlier paper [1], I showed that the simple constitutive relation relating the shearing stress to the shear rate v_x

$$T^{xy}(t) = \sigma\left(\int_0^\infty e^{-\alpha\tau} v_x(x,t-\tau)d\tau\right). \qquad (0.1)$$

yields the following result:

The equation of motion for the shearing perturbation $\hat{v}(x,t)$ of a non-trivial steady shearing motion will admit a C^1 in (x,t) solution for only a finite time for appropriately chosen smooth initial velocity histories. Here σ is a non-linear, odd, real analytic function, $\sigma' > 0$, α is a positive constant.

The proof of the result hinged on the fact that equations for the perturbed fluid motion $\hat{v}(x,t)$ can be written as a non-linear hyperbolic conservation law with linear damping. In this form the equations of motion are amenable to study via the use of Riemann invariants and an argument of Lax [2]. As a side benefit from this formulation continuation theory for hyperbolic equations shows that loss of C^1 regularity implies $|\hat{v}_t| + |\hat{v}_x| \to \infty$ in finite time. ; This suggests the appearance of a jump discontinuity in $\hat{v}$, i.e. formation of a vortex sheet.

Sometime after appearance of my result C.J.S. Petrie remarked to me that a similar result could be obtained for the choice of constitutive relation

$$T^{xy}(t) = \int_0^\infty e^{-\alpha\tau}\sigma(v_x(x,t-\tau))d\tau \qquad (0.2)$$

for σ,α as above. Upon working out Petrie's idea I noticed the results are similar but not the same. Specifically the above mentioned result holds with C^1 replaced by C^2. D. D. Joseph observed that this may have important physical consequences as now the appearance of jump discontinuity in $\hat{v}_x$ may be expected, i.e. formation of a vortex shock.

At the request of D. D. Joseph I have prepared this short note analyzing the different effects of (1.1) and (1.2). The technical details of the breakdown proof are omitted and may be found [1].

A1. RECTILINEAR SHEARING FLOWS

If in a fixed Cartesian co-ordinate system x,y,z, the velocity fields of a flowing fluid body has the form

$$v^x = 0,\ v^y = v(x,t),\ v^z = 0 \tag{1.1}$$

we say that the motion is a rectilinear shearing flow. For such a flow the condition of incompressibility div $\underline{v} = 0$ is automatically satisfied. Coleman & Noll [3] have shown that if the fluid is a simple fluid, then the components of stress obey the relations

$$T^{xy}(t) = \overset{\infty}{\underset{s=0}{\mathbf{t}}}\,(\Lambda^t(s))\ ,$$

$$T^{xx}(t) - T^{zz}(t) = \overset{\infty}{\underset{s=0}{\mathbf{s}_1}}\,(\Lambda^t(s))\ ,$$

$$T^{yy}(t) - T^{zz}(t) = \overset{\infty}{\underset{s=0}{\mathbf{s}_2}}\,(\Lambda^t(s))\ ,$$

$$T^{xz} = T^{yz} = 0\ , \tag{1.2}$$

where Λ^t is the relative shearing history defined by

$$\Lambda^t(s) = -\int_{t-s}^{t} v_x(x,\tau)d\tau \qquad (0 \le s \le \infty)\ ,$$

and $\mathbf{t},\mathbf{s}_1,\mathbf{s}_2$ are real valued functionals obeying the identities

$$\overset{\infty}{\underset{s=0}{\mathbf{t}}}\,(-\Lambda^t(s)) = -\overset{\infty}{\underset{s=0}{\mathbf{t}}}\,(\Lambda^t(s))\ ,$$

$$\overset{\infty}{\underset{s=0}{\mathbf{s}_i}}\,(-\Lambda^t(s)) = \overset{\infty}{\underset{s=0}{\mathbf{s}_i}}\,(\Lambda^t(s))\ , \quad (i = 1,2). \tag{1.3}$$

Here subscripts x,t denote partial derivatives with respect to x and t respectively.

A2. CONSTITUTIVE ASSUMPTIONS

To proceed further it is necessary to make some mathematical assumptions as to the nature of the functional $\mathbf{t}$. For this analysis we assume $\mathbf{t}$ has two particularly simple forms, i.e.,

$$\overset{\infty}{\underset{s=0}{\mathbf{t}}}\,(\Lambda_t(s)) = \sigma\left[\int_0^\infty e^{-\alpha s}v_x(x,t-s)ds\right]\ , \tag{2.1}$$

and

$$\overset{\infty}{\underset{s=0}{\mathbf{t}}}\,(\Lambda_t(s)) = \int_0^\infty e^{-\alpha s}\sigma(v_x(x,t-s))ds\ , \tag{2.2}$$

where σ is a real valued, odd, analytic function defined on the real line and α is a positive constant. In (2.1) we see $\mathbf{t}$ is a (generally) nonlinear function of the linear functional

$$\int_0^\infty e^{-\alpha s}v_x(x,t-s)ds\ .$$

In (2.2) we see that $\mathbf{t}$ is a linear functional of the nonlinear function $\sigma(v_x(x,t-s))$.

A motivation for the choice (2.1) may be based on the multiple integral expansions for $\mathbf{t}$ originally presented by Green & Rivlin [4] and Chacon & Rivlin [5] for general viscoelastic materials. In [5] the authors showed that for $\mathbf{t}$ continuous on an appropriate function space, $\mathbf{t}$

may be approximated by a finite series of multiple integrals:

$$\mathop{\mathbf{t}}_{s=0}^{\infty}(\Lambda^t(s)) \approx \sum_{n\ \mathrm{odd}} \int_0^\infty \cdots \int_0^\infty K_n(s_1,\ldots,s_n)\,\Lambda^t(s_1)\ldots\Lambda^t(s_n)ds_1\ldots ds_n \tag{2.3}$$

(The restriction to odd order follows from the isotropy condition (1.3).) If we assume more is true, namely that in a manner similar to the Taylor series expansion for an analytic function we take (2.3) to be an equality where the sum may be infinite. Choose

$$K_n(s_1,\ldots,s_n) = -\alpha^n\sigma_n e^{-\alpha(s_1+\ldots+s_n)} \quad n = 1,3,5\ldots$$

where α is a positive constant. (This is consistent with fluid of fading memory type [3], [6].) Finally if we define

$$\sigma(\xi) = \sum_{n\ \mathrm{odd}} \sigma_n\xi^n$$

and integrate (2.3) by parts we obtain (2.1).

Constitutive equation (2.2) is equivalent to the choice

$$\mathop{\mathbf{t}}_{s=0}^{\infty}(\Lambda^t(s)) = -\int_0^\infty e^{-\alpha s}\sigma((\Lambda^t(s))_s)ds\ .$$

An alternative motivation arises from consideration of a material of rate type where $T^{xy}(t)$ satisfies the ordinary differential equation

$$\frac{d}{dt}T^{xy}(t) + \alpha T^{xy}(t) = \sigma(v_x(x,t)) \tag{2.4}$$

for all t, $-\infty < t < \infty$, and all x; $0 < x < h$, where α as before is a positive constant. Integration of (2.4) in the usual manner for first order ordinary differential equations yields (2.2).

A3. SHEARING PERTURBATION OF A STEADY SHEARING FLOW

Let us assume a viscoelastic fluid satisfies either the constitutive relation (2.1) or (2.2). Consider the problem when the fluid is confined between two parallel walls of infinite extent at $x = 0$ and $x = h$. The top wall at $x = h$ moves with velocity V. In the absence of body and driving forces the equation of conservation of linear momentum (see Coleman & Gurtin [7]) becomes

$$\rho v_t(x,t) = \sigma\left(\int_0^\infty e^{-\alpha s}v_x(x,t-s))ds\right)_x \tag{3.1}$$

for constitutive relation (2.1) and

$$\rho v_t(x,t) = \int_0^\infty e^{-\alpha s}\sigma(v_x(x,t-s))_x\,ds \tag{3.2}$$

for constitutive relation (2.2). Here ρ is an (assumed) constant mass density, $\rho > 0$. Also we consider the case of no-slip boundary conditions

$$v(0,t) = 0\ , \qquad v(h,t) = V\ . \tag{3.3}$$

Systems (3.1), (3.3) and (3.2), (3.3) admit the steady rectilinear flow ($v_t(x,t) = 0$) solution

$$v(x) = \frac{V_x}{h}\ .$$

To study stability of the flow against shearing perturbations we set

$$\hat{v}(x,t) = v(x,t) - \frac{Vx}{h}\ .$$

We observe that for relation (2.1) this implies

$$\rho\hat{v}_t(x,t) = \sigma\left(\int_0^\infty e^{-\alpha s}\hat{v}_x(x,t-s)ds + \frac{V}{\alpha h}\right)_x \tag{3.4}$$

with boundary condition

$$\hat{v}(0,t) = \hat{v}(h,t) = 0 \tag{3.5}$$

On the other hand for relation (2.2) we see that the perturbation $\hat{v}(x,t)$ satisfies

$$\rho\hat{v}_t(x,t) = \int_0^\infty e^{-\alpha s}\sigma\left(\hat{v}_x(x,t-s) + \frac{V}{h}\right)_x ds \tag{3.6}$$

along with boundary condition (3.5).

In either case we prescribe a smooth velocity history

$$\hat{v}(x,t) = \hat{v}_0(x,t) \quad -\infty < \tau \le 0$$

consistent with either (3.4), (3.5) or (3.5), (3.6). Thus the two fluid cases are governed by non-linear boundary initial history value problems.

A4. ANALYSIS OF FLOW WITH FIRST CONSTITUTIVE ASSUMPTION: STRESS NON-LINEAR FUNCTION OF A LINEAR FUNCTIONAL OF SHEAR RATE.

We consider the perturbed flow governed by constitutive relations (2.1). In this case the time evolution is governed by (3.4), (3.5). To simplify matters we write

$$\hat{\sigma}(\xi) = \frac{1}{\rho}\left[\sigma\left(\xi + \frac{V}{\alpha h}\right) - \sigma\left(\frac{V}{\alpha h}\right)\right],$$

$$u(x,t) = \int_0^\infty e^{-\alpha s}\,\hat{v}_t(x,t-s)ds,$$

$$w(x,t) = \int_0^\infty e^{-\alpha s}\,\hat{v}_x(x,t-s)ds.$$

An integration by parts shows

$$u(x,t) = \hat{v}(x,t) - \alpha\int_0^\infty e^{-\alpha s}\,\hat{v}(x,t-s)ds$$

and hence

$$u_t = \hat{v}_t - \alpha u. \tag{4.1}$$

If we combine (4.1) with (3.4) we find u,w satisfy the system

$$w_t = u_x,$$
$$u_t = \hat{\sigma}(w)_x - \alpha u, \tag{4.2}$$

with boundary conditions

$$u(o,t) = u(h,t) = 0, \tag{4.3}$$

and initial conditions

$$u(x,0) = u_0(x),$$
$$w(x,0) = w_0(x), \quad 0 \le x \le h.$$

The values of $u_0(x)$ and $w_0(x)$ are obtained from their respective definitions by insertion of the given velocity history $\hat{v}_0(x,\tau)$, $-\infty < \tau \le 0$.

In order that the constitutive relation(2.1) be truly non-linear we must have $\sigma''(\xi^x) \neq 0$ for some real ξ^x. Hence when the speed of top wall is given

$$V = \alpha h\,\xi^k$$

we see $\hat{\sigma}$ satisfies

$$\hat{\sigma}(0) = 0 ,$$

$$\hat{\sigma}''(0) \neq 0 . \tag{4.4}$$

We further impose the condition that (4.2) be strictly hyperbolic, i.e. the matrix

$$\begin{bmatrix} 0 & \hat{\sigma}' \\ 1 & 0 \end{bmatrix}$$

possesses real distinct eigenvalues. Strict hyperbolicity is easily seen to be equivalent to the condition $\sigma' > 0$.

A5. ANALYSIS OF FLOW WITH SECOND CONSTITUTIVE ASSUMPTION: STRESS LINEAR FUNCTIONAL OF A NON-LINEAR FUNCTION OF SHEAR RATE.

We consider the perturbed flow governed by constitutive relation (2.2). In this case the time evolution is governed by (3.5), (3.6). Again to simplify matters we write

$$\sigma(\xi) = \frac{1}{\rho}\left[\sigma(\xi + \frac{V}{h}) - \sigma\left[\frac{V}{h}\right]\right] ,$$

$$\bar{u} = \hat{v}_t ,$$

$$\bar{w} = \hat{v}_x .$$

We note (3.6) can be written as

$$\hat{v}_t = \int_{-\infty}^{t} e^{-\alpha(t-\tau)} \sigma(\hat{v}_x(x,\tau))_x \, d\tau \tag{5.1}$$

which upon differentiating with respect to t yields

$$\hat{v}_{tt} = -\alpha \int_{-\infty}^{t} e^{-\alpha(t-z)} \sigma(\hat{v}_x (x,\tau))_x \, d\tau + \sigma(v_x(x,t))_x \tag{5.2}$$

i.e.

$$\hat{v}_{tt} + \alpha\hat{v}_t = \sigma(\hat{v}_x(x,t))_x \tag{5.3}$$

where we have used (5.1).

From the definitions of $\bar{u}$, $\bar{w}$ above we see

$$\bar{w}_t = \bar{u}_x ,$$

$$\bar{u}_t = \sigma(\bar{w})_x - \alpha\bar{u} , \tag{5.4}$$

with boundary conditions

$$\bar{u}(0,t) = \bar{u}(h,t) = 0 , \tag{5.5}$$

and initial conditions

$$\bar{u}(x,0) = \bar{u}_0(x),$$

$$\bar{w}(x,0) = \bar{w}_0(x), \qquad 0 < x < h .$$

Again the values of $\bar{u}_0(x)$, $\bar{w}_0(x)$ can be obtained from the respective definition of $\bar{u},\bar{w}$ in terms of the given velocity history $\hat{v}_0(x,\tau)$, $-\infty < \tau \leq 0$.

Also we note that if $\sigma''(\xi) \neq 0$ for some ζ real and when the speed of top wall is given by

$$V = h\zeta$$

then

$$\sigma(0) = 0 ,$$

$$\sigma''(0) \neq 0 . \tag{5.6}$$

As before the condition $\sigma' > 0$ implies (5.4) is strictly hyperbolic.

A6. A BREAKDOWN RESULT.

We have shown in sections 4 and 5 that evolution of perturbed flow is governed by a system of the form

$$W_t = U_x$$

$$U_t = K(W)_x - \alpha U\ , \qquad (6.1)$$

with

$$U(0,t) = U(h,t) = 0,$$

$$U(x,0) = U_0(x), \qquad W(x,0) = W_0(x) \qquad (6.2)$$

and

$$K' > 0, \quad \kappa(0) = 0, \quad K''(0) \neq 0.$$

Analysis of (6.1), (6.2) has been given in [1]. We shall not repeat that analysis but only state the relevant breakdown result.

Define Riemann invariants for (6.1) by

$$\begin{matrix} r \\ s \end{matrix} = U \pm \emptyset(W) \qquad (6.3)$$

where

$$\emptyset(W) = \int_0^W \sqrt{K'(s)}\ ds.$$

The transformation given by (6.3) from (U,W) R x R to (r,s) R x R is one-one. Also we assume the initial data $r(0,x) = r_0(x)$, $s(0,x) = s_0(x)$ to be smooth functions.

Our main breakdown result is as follows.

Theorem 6.1: Suppose $|r_0|$, $|s_0|$ are sufficiently small and $K'(0) > 0$, $K''(0) > 0$. If r_0,x or s_0,x is positive and sufficiently large at any point x, then (6.1), (6.2) has a solution (W,U) in $C^1[0,h] \times C^1[0,h]$ for only a finite time. A similar result holds if $K''(0) < 0$ and r_0,x or s_0,x is sufficiently negative at any point x.

Proof: See the proof of Thm. 3.1 in [1].

Standard existence theorems (see for example Section 1.8 of [8] and Chaps. 2 and 3 of [9]) imply that under the hypothesis of Theorem 6.1 we have $|U_t| + |U_x| + |W_t| + |W_x| \to \infty$ in finite time. This suggests but does not prove the occurrence of a jump discontinuity in U and W, i.e. the formation of a shock.

Also we note that since

$$\begin{matrix} r_x \\ s_x \end{matrix} = U_x \pm \phi'(W)W_x$$

we will have r_{0_x} large if U_{0_x} or W_{0_x} is large, s_{0_x} large if U_{0_x} or $-W_{0_x}$ is large, $-r_{0_x}$ large if $-U_{0_x}$ or $-W_{0_x}$ large, $-s_{0_x}$ large if $-U_{0_x}$ or W_{0_x} large.

7. PHYSICAL IMPLICATIONS OF BREAKDOWN OF SMOOTH SOLUTIONS

We examine the implications of the breakdown result of Section 6 with respect to our two constitutive relations.

First we consider constitutive relation (2.1) In this case we see that if $\sigma''(V/\alpha h) \neq 0$ and $u_{0_x}(x)$ or $w_{0_x}(x)$ is appropriately sufficiently large of sufficiently negative u_{0_x} (depending on the sign of $\sigma''(V/\alpha h)$), Theorem 6.1 implies $|u_x| + |w_x| \to \infty$ in finite time.

Since we know from (3.6) that

$$\hat{u}_t = \hat{\sigma}(w)_x$$

and from the definition of u we have

$$\hat{u}_x = u_x + \alpha w,$$

$|u_x| + |w_x| \to \infty$ in finite time implies $|\hat{v}_t| + |\hat{v}_x| \to \infty$ in finite time. Again this suggests but doesn't prove that $\hat{v}$ and hence v forms a jump discontinuity in finite time. In this case the singular surface across which the discontinuity in v is called a vortex sheet.

Next we consider the case of constitutive relation (2.2). In this case we see that if $\bar{u}_{0_x}(x) = v_{tx}(x,0)$ or $\bar{w}_{0_x}(x) = v_{xx}(x,0)$ is appropriately sufficiently large or sufficiently negative (depending on the sign of $\sigma''(V/h)$) Theorem 6.12 implies $|v_{tx}| + |v_{xx}| \to \infty$ in finite time. Once more this suggests but doesn't prove that either v_t or v_x forms a jump discontinuity in finite time. However from the Rankine-Hugoniot jump condition for (5.4)

$$-(\frac{ds}{dt})[\bar{w}] = [\bar{u}]$$

$$-(\frac{ds}{dt})[\bar{u}] = [\sigma(\bar{w})] \qquad (7.1)$$

we see

$$-(\frac{ds}{dt})[v_t] = \left[\sigma(v_x - \frac{V}{h})\right] = \frac{1}{\rho}[\sigma(v_x)] \qquad (7.2)$$

where x = s(t) denotes the surface across which the jump occurs. Hence if $[v_x] \neq 0$ then $[\sigma(v_x)] \neq 0$ (since $\sigma' > 0$) and by (7.2) $[v_t] \neq 0$. Conversely if $[v_t] \neq 0$ then from (7.1) we have

$$-(\frac{ds}{dt})[v_x] = [v_t] \qquad (7.3)$$

and hence $[v_x] \neq 0$. Thus the appearance of a jump in v_t or v_x implies a jump in the other.

We define a propagating singular surface across which the acceleration v_t experiences a jump discontinuity as an <u>acceleration wave</u>. Similarly we define a propagating singular surface across which the vorticity $\underline{w} = \text{curl}(v^z, z^y, y^z) = v_x(x,t)\underline{e}_z$ experiences a jump discontinuity as a <u>vortex shock</u>. Our analysis shows that for our flow the vortex shock and acceleration waves are equivalent and can be expected to form in finite time if constitutive relation (2.2) holds.

We thus see the remarkable effect of choice of constitutive relation. In one case (2.1) appropriately chosen initial data appears to force the formation of a jump in v (a vortex sheet) while in the second case (2.2) appropriately chosen initial data suggests formation of a jump discontinuity in v_x (a vortex shock and equivalently in v_t (an acceleration wave).

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Appendix A by
Marshall Slemrod
Department of Mathematical Sciences
Rensselaer Polytechnic Institute
Troy, New York 12180