

# Effects of quadratic drag on convection in a saturated porous medium

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The effects of inertia (involving a drag which is quadratic in the velocity) on convection in a fluid-saturated porous medium are considered. It is shown that the effect of quadratic drag is physically significant for natural convection, at realistic values of the Rayleigh number, in a thin layer of a medium whose overall Prandtl number is small. The qualitative effect of quadratic drag on the global stability of the conduction regime, and on bifurcation into the convection regime, is reported. Convection in an inclined slab of material is also discussed.

In this brief communication we consider the effects of inertia on convection in a fluid-saturated porous material. In particular, we obtain estimates of the physical significance of inertial effects in natural convection in horizontal and inclined slabs.

There has been a difference of opinion on what is the appropriate form of the equation of motion. In most studies of convection in porous media the authors have assumed that the flow is governed by Darcy's law, so that, if the flow is steady and the Oberbeck-Boussinesq approximation is justified, the equation of motion takes the form

$$0 = -\nabla P - \rho_0 \alpha g(T - T_0) - (\mu/K) \mathbf{U}_m. \quad (1)$$

We have adopted the notation used in Joseph's book.<sup>1</sup> Here  $P$  is the reduced pressure,  $T$  is the temperature,  $\mathbf{U}_m$  is the seepage velocity,  $\rho_0$  is the density of the fluid at some standard temperature  $T_0$ ,  $\alpha$  and  $\mu$  are the volume expansion coefficient and the viscosity of the liquid,  $K$  is the permeability of the medium, and  $\mathbf{g}$  is the gravitational acceleration. There is general agreement that Eq. (1) is appropriate if the motion is sufficiently slow and the porosity  $\eta$  is not close to unity.

If the porosity is large, then it may be appropriate to use a Brinkman equation in which a Laplacian term  $\tilde{\mu} \nabla^2 \mathbf{U}_m$  is added to the right-hand side of Eq. (1). The use of the Brinkman equation has been discussed in detail by Nield.<sup>2,3</sup> The Laplacian term is in any case of importance only in boundary layers whose thickness is of order  $K^{1/2}$ . In this paper we will suppose that the Laplacian term may be neglected.

When the motion is not slow, it is necessary to consider inertial effects. Several authors have included such effects by adding to the left-hand side of Eq. (1) the terms

$$\frac{1}{\eta} \frac{\partial \mathbf{U}_m}{\partial t} + \frac{\rho_0}{\eta^2} (\mathbf{U}_m \cdot \nabla) \mathbf{U}_m.$$

The inclusion of the term in  $\partial \mathbf{U}_m / \partial t$  is not controversial, but it has been realized that the inclusion of the term in  $(\mathbf{U}_m \cdot \nabla) \mathbf{U}_m$  cannot be correct. Beck<sup>4</sup> pointed out that the order of the differential equation (1) is raised by the inclusion of  $(\mathbf{U}_m \cdot \nabla) \mathbf{U}_m$  and no additional boundary condition is available to keep the resulting boundary value problem determinate (unless one includes a Laplacian term as well). A more important objection is that  $(\mathbf{U}_m \cdot \nabla) \mathbf{U}_m$  vanishes identically if the flow is unidirectional and hence cannot represent the known effect (increase in drag) in that case. In some cases it is possible to avoid this embarrassing situation by neglecting the quadratic term  $(\mathbf{U}_m \cdot \nabla) \mathbf{U}_m$ , but in general this let-out is not permissible.

For the sort of porous media which occur naturally, we believe that the appropriate extension of Eq. (1) is

$$\frac{1}{\eta} \frac{\partial \mathbf{U}_m}{\partial t} = -\nabla P - \rho_0 \alpha g(T - T_0) - \frac{\mu}{K} \mathbf{U}_m - c \rho_0 K^{-1/2} |\mathbf{U}_m| \mathbf{U}_m. \quad (2)$$

This is a modification of an equation associated with the names of Dupuit and Forchheimer. The effect of inertia is a drag term which is quadratic in the velocity  $\mathbf{U}_m$ . The coefficient  $c$ , which we call the form drag constant, is independent of the viscosity and other properties of the fluid, but is dependent on the geometry of the medium. Experimental support for this form of the quadratic drag is described by Ward<sup>5</sup> and Beavers, Sparrow, and Rodenz,<sup>6</sup> while the many experimen-

tal results summarized by Macdonald *et al.*<sup>7</sup> are consistent with this form. Further discussion of quadratic drag is contained in a paper by Joseph, Nield, and Papanicolaou,<sup>8</sup> who made a determination of the form drag constant by means of an analysis of a type which Brinkman calls self-consistent.

An equation similar to Eq. (2) has been used by Somerton,<sup>9</sup> but in his paper the quadratic drag term is a vector which is not parallel to the seepage velocity. (Indeed, his term  $U^2\hat{e}_x + V^2\hat{e}_y + W^2\hat{e}_z$  is dependent on the choice of coordinate axes.) It seems to us that on a macroscopic scale, the drag has to be in the direction opposite to the seepage velocity. [For the case of a viscous fluid, a term of the form  $(\mathbf{U}\cdot\nabla)\mathbf{U}$  can produce a transfer of momentum across a shear flow, but in the case of a porous medium this transfer mechanism is upset by the randomness of the geometry.] However, this deficiency does not affect Somerton's major conclusion, namely, that the dependence of the heat transfer on the nature of the medium can be explained in terms of a dependence on Prandtl number arising from a quadratic drag term.

We will now complete the system of governing equations and put them in nondimensionless form, and briefly report on some qualitative effects of quadratic drag on bifurcation of the conduction solution and the energy-stability criterion for a horizontal layer (the Bénard–Darcy problem). Later we estimate the magnitude of the quadratic drag term for the Bénard–Darcy problem, and demonstrate that it is physically significant, in thin layers of a medium for which the overall Prandtl number is small, at realistic values of the Rayleigh number. This is in accord with Somerton's explanation of the results of some heat transfer experiments performed by Combarous.<sup>10</sup> The corresponding problem of convection in an inclined layer will also be discussed briefly. We find that the range of parameter values for which the quadratic drag is physically significant is rather larger for the inclined layer than for the horizontal layer.

For the convection problem, we have in addition to Eq. (2) the equation of continuity

$$\nabla\cdot\mathbf{U}_m = 0, \quad (3)$$

and the energy transport equation

$$(\rho_0 C_0)_m \frac{\partial T}{\partial t} + (\rho_0 C_0)_f \mathbf{U}_m \cdot \nabla T = \nabla \cdot (k_m \nabla T), \quad (4)$$

where  $T$  is the temperature,  $C_0$  is the specific heat at constant pressure, the subscripts  $m$  and  $f$  refer to the fluid–solid mixture and the fluid, respectively, and  $k_m$  is the overall thermal conductivity. We take scales  $l$  for length,  $l^2/\kappa$  for time,  $\Delta T$  for temperature,  $\kappa/l$  for velocity and  $\mu\kappa/K$  for pressure, where  $\kappa = k_m/(\rho_0 C_0)_f$ . Equations (2)–(4) can be written as

$$\frac{B}{\eta} \frac{\partial \mathbf{U}_m}{\partial t} = -\nabla P + \text{Re}_z \left( T - \frac{T_0}{\Delta T} \right) - \mathbf{U}_m - J |\mathbf{U}_m| \mathbf{U}_m, \quad (5)$$

$$A \frac{\partial T}{\partial t} + \mathbf{U}_m \cdot \nabla T = \nabla^2 T, \quad (6)$$

$$\nabla \cdot \mathbf{U}_m = 0. \quad (7)$$

We have confined ourselves to the case where there is no internal heating, and the viscosity is independent of tem-

perature. The more general situation is discussed in Ref. 1. The nondimensional parameters are

$$B = \frac{K\kappa}{l^2\nu}, \quad R = \frac{\alpha g \Delta T l K}{\nu\kappa}, \quad J = c \left( \frac{\kappa}{\nu} \right) \frac{K^{1/2}}{l}, \quad A = \frac{(\rho_0 C_0)_m}{(\rho_0 C_0)_f}. \quad (8)$$

The quantity  $R$  is the Rayleigh–Darcy number. In terms of the Prandtl number  $\text{Pr} = \nu/\kappa$  and the length-scale ratio  $\delta = K^{1/2}/l$ , we have  $B = \delta^2/\text{Pr}$  and  $J = c\delta/\text{Pr}$ . In normal circumstances,  $\delta$  will be very small and  $B$  may be set equal to zero.

The following remarks concern modifications, caused by the effect of quadratic drag, to the theory given in Secs. 71 and 72 of Ref. 1. It is obvious that the quadratic drag will have no effect on the linearized stability problem for the onset of convection, and it follows that the loss of stability of the conduction solution can be framed in terms of the theory of bifurcation at a real simple eigenvalue. With the introduction of quadratic drag, the analytic theory of bifurcation at a simple eigenvalue does not apply because the nonlinearity  $|\mathbf{U}_m| \mathbf{U}_m$  is not analytic. However, first derivatives with respect to the amplitude  $\epsilon$  of the bifurcating solution do make sense, and proceeding as in Sec. 72 of Ref. 1, we may calculate  $\mathcal{R}_1 \equiv d\mathcal{R}/d\epsilon$  at  $\epsilon = 0$ , where  $\mathcal{R} = \sqrt{R}$ . It is found that  $\mathcal{R}_1$  is no longer zero. The bifurcation curve in the  $(\mathcal{R}, \epsilon)$  plane is still symmetric with respect to  $\epsilon$ , but it has a vertex at the bifurcation point; the discontinuity in slope reflects the nonanalyticity of the solution. It is also found that the effect of quadratic drag is to increase the domain of global stability. The global stability criterion  $\mathcal{R} < \mathcal{R}_E$ , where  $\mathcal{R}_E$  is independent of  $\epsilon$ , is replaced by  $\mathcal{R} < \tilde{\mathcal{R}}_E$ , where  $\tilde{\mathcal{R}}_E = \mathcal{R}_E$  when  $\epsilon = 0$ , but  $\tilde{\mathcal{R}}_E > \mathcal{R}_E$  when  $\epsilon \neq 0$ .

The quadratic drag term in Eq. (5) will be negligible if and only if

$$J |\mathbf{U}_m| \ll 1. \quad (9)$$

We recall that, by definition,  $J = c(\kappa/\nu)(K^{1/2}/l)$ . In the Bénard–Darcy problem,  $l$  is the layer depth. For an estimate for  $|\mathbf{U}_m|$ , we can take the root-mean-square average of  $|\mathbf{U}_m|$ , denoted by  $\bar{U}$ . From equations (5.13) and (5.14) of the paper by Palm, Weber, and Kvernfold,<sup>11</sup> we deduce that  $\bar{U}$  is given by

$$\bar{U} = [R(N-1)]^{1/2}, \quad (10)$$

where  $N$  is the Nusselt number. From their Fig. 3, giving the dependence on  $N$  on  $R$  (based on several experiments), we obtain Table I.

As an example, suppose we take the values  $c = 0.1$ ,  $K = 10^{-3} \text{ cm}^2$ ,  $l = 1 \text{ cm}$ , which are appropriate for a 1 cm thick layer of a medium composed of metallic fibers, and the value  $R = 300$ , which is easily attained and which is of interest because of the Hopf bifurcation that appears to occur near that value (see, for example, Horne and Caltagirone<sup>12</sup>). Then the inequality (9) is satisfied if the Prandtl number satisfies  $\text{Pr} \gg 0.1$ . Thus in this situation the quadratic drag term will be significant so long as the Prandtl number is of order 0.1 or smaller. It should be noted that here the Prandtl number is not that of the fluid but is characteristic of the fluid/solid medium, so this criterion does not restrict the significance to liquid metals. (The Prandtl number is the ratio of

the kinematic viscosity of the fluid to the thermal diffusivity of the medium.) The criterion indicates that quadratic drag should be significant in Combarous's<sup>10</sup> experiments with a lead-water system, for which the value of Pr was 0.18 and the layer depth about 1 cm. This is in accord with Somerton's<sup>9</sup> explanation of the heat-transfer results of those experiments; the heat transfer is comparatively small if the Prandtl number is small.

More generally, for media with larger Pr, the effect of quadratic drag will become significant at rather larger values of the Rayleigh number. On analogy with the situation for convection in fluids, one might expect that at sufficiently large  $R$  there would be a transition to a regime involving some sort of "inertial convection" which could be evident in a medium of low Prandtl number; see, for example, Clever and Busse<sup>13</sup> and Fauve and Libchaber.<sup>14</sup>

For the case of convection in an inclined slab, we again choose  $l$  to be the layer thickness over which a temperature difference  $\Delta T$  is imposed. In terms of nondimensional quantities we suppose that the bounding plates  $z = \pm \frac{1}{2}$  are held at temperatures  $(T_0 \pm \Delta T/2)/\Delta T$ , the  $y$  axis is horizontal and the  $x$  axis makes an angle  $\delta$ ,  $-90^\circ \leq \delta \leq 90^\circ$ , with the upwards vertical. Thus  $\delta$  is positive if the slab is heated on its lower side and negative if heated on the upper side. The governing equations are (5)–(7) with  $\mathbf{e}_z$  replaced by  $\mathbf{e}_x \cos \delta + \mathbf{e}_z \sin \delta$ . For the case  $B = 0$  we have

$$\mathbf{U}_m + J|\mathbf{U}_m|\mathbf{U}_m = -\nabla P + R(T - T_0/\Delta T) \times (\mathbf{e}_x \cos \delta + \mathbf{e}_z \sin \delta), \quad (11)$$

$$A \frac{\partial T}{\partial t} + \mathbf{U}_m \cdot \nabla T = \nabla^2 T, \quad (12)$$

$$\nabla \cdot \mathbf{U}_m = 0. \quad (13)$$

We suppose that there are no imposed temperature or pressure gradients parallel to the bounding plates. We examine the steady-state flow given by  $\mathbf{U}_m = U(z)\mathbf{e}_x$  where  $U(-z) = -U(z)$  so that the net mass flux across any plane  $x = \text{const}$  is zero and  $T = (T_0/\Delta T) - z$ . This is a unicellular flow with streamlines in a vertical plane. From Eq. (11) we deduce that  $\partial P/\partial x$  is a constant and, hence, zero. Hence  $U(z)$  is given by

$$J|U|U + U + Rz \cos \delta = 0. \quad (14)$$

The fluid rises along the hotter plate. Hence when  $z \cos \delta \leq 0$ , then  $U \geq 0$ , and hence

$$U = [-1 + (1 - 4JRz \cos \delta)^{1/2}]/2J. \quad (15)$$

When  $z \cos \delta \geq 0$ , then  $U \leq 0$  and

$$U = [1 - (1 + 4JRz \cos \delta)^{1/2}]/2J. \quad (16)$$

We note that  $U$  and its derivative  $U'$  are continuous, but  $U''$  changes sign discontinuously at  $z = 0$ .

Equations (15) and (16) can be combined in the form

$$|U| = [-1 + (1 + 4JR|z \cos \delta|)^{1/2}]/2J. \quad (17)$$

When  $JR$  is small, we have approximately

$$|U| = R|z \cos \delta| - JR|z \cos \delta|^2. \quad (18)$$

As we should expect, the effect of increasing  $J$  is to reduce the speed of the steady motion.

TABLE I. Approximate values showing the dependence of Nusselt number  $N$  and root-mean-square velocity  $\bar{U}$  on the Rayleigh–Darcy number  $R$ , based on Fig. 3 and Eqs. (5.13) and (5.14) of Ref. 11.

$R$	$N$	$\bar{U}$
$10^2$	3	15
$3 \times 10^2$	6	40
$10^3$	10	100
$3 \times 10^3$	14	200
$10^4$	20	450

From Eq. (14) we see that the quadratic drag becomes significant when  $J|U|^2 \sim O(R)$  and thus  $J|U| \sim O((RJ)^{1/2})$ . We conclude that the quadratic drag has negligible effect if

$$JR \ll 1,$$

i.e., if

$$cag\Delta TK^{3/2}/\nu^2 \ll 1. \quad (19)$$

It is noteworthy that this inequality does not depend on the overall thermal diffusivity  $\kappa$ . For the slab used as an example before, this inequality says that the quadratic drag is negligible if  $\text{Pr} \gg 1$ . Alternatively, it says that if  $\text{Pr} = 0.1$ , then the quadratic drag is significant when  $R$  is as low as 30.

We have shown that quadratic drag plays a significant role in natural convection in thin layers of media of small Prandtl number.

It is obvious that in forced convection (or in mixed convection), quadratic drag must play an important role since, no matter how small  $J$  may be, the product  $J|\mathbf{U}_m|$  will become significant as soon as the mean velocity is forced to a sufficiently high value.

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