

Oscillatory instability in a Bénard problem of two fluids

Yuriko Renardy

Mathematics Research Center, University of Wisconsin-Madison, 610 Walnut Street, Madison, Wisconsin 53705

Daniel D. Joseph

Department of Aerospace Engineering, 107 Akerman Hall, 110 Union Street, S. E., University of Minnesota, Minneapolis, Minnesota 55455

(Received 17 April 1984; accepted 26 August 1984)

A linear stability analysis for a Bénard problem with two layers is considered. The equations are not self-adjoint. The system can lose stability to time-periodic disturbances. For example, it is shown numerically that when the viscosities and coefficients of cubical expansion of the fluids are different, a Hopf bifurcation can occur, resulting in a pair of traveling waves or a standing wave. This may have application in the modeling of convection in the Earth's mantle.

I. INTRODUCTION

The flow of two immiscible fluids often exhibits phenomena which are without parallel in the flow of one fluid. An example is the steady shear flow of two fluids with different viscosities but similar densities. Such flows are described in Joseph, Nguyen, and Beavers.¹ In this paper, we consider the Bénard problem with two fluids in layers, lying between infinite parallel plates, heated from below, and we look for new phenomena. In the Bénard problem for one fluid, the "exchange of stabilities" holds, and all the eigenvalues of the linearized problem are real. In the two-fluid problem, we show that we have both a real and a complex spectrum.

Busse² noted that convection in a Bénard problem with two fluids, heated from below, can admit solutions where the fluids lie in layers, as well as more complicated solutions in which there are convection cells of one fluid surrounded by streamlines of the second fluid. Busse³ has examined the onset of linear nonoscillatory instability of the arrangement where the unperturbed fluids are static and lie in two layers with a flat horizontal interface. In particular, he focused on the range of parameters pertinent to a model of mantle convection in which the two layers represent the upper and lower mantles. The depth of the upper mantle is thin compared with that of the lower mantle. He shows that the horizontal scale of convection in the lower mantle may determine a horizontal scale of flow in the upper mantle. Hence, it is possible that the horizontal scale of convection in the upper mantle may not necessarily indicate the depth of convection there. He does not distort the interface, and time-periodic motions are excluded in his analysis. On the other hand, we include the perturbations of the interface since thermal and viscous coupling of the two fluids are coupled with interface position. Busse's numerical observation that the short-wave response decays exponentially away from the interface is consistent with our asymptotic analysis in Sec. V.

Zeren and Reynolds⁴ considered our problem, including the effect of a linear temperature gradient of the surface tension (Marangoni effect). They state that they do not know if there are purely imaginary eigenvalues at criticality. They note that Sternling and Scriven⁵ found purely imaginary eigenvalues in the problem where the upper fluid is inviscid and the convection is induced by surface tension, which de-

pends linearly on the temperature of the free surface (Marangoni effect). Zeren and Reynolds chose to compute neutral stability curves corresponding to zero eigenvalues. We concentrate on the Bénard problem without the Marangoni effect and show that the equations are not self-adjoint. We give an example of a situation when the marginal eigenvalues are a purely imaginary conjugate pair of multiplicity two (the same eigenvalues appear for negative wavenumbers). Marginal eigenvalues of this type are associated with Hopf bifurcations from the motionless state to either a pair of traveling waves or a standing wave (Ruelle⁶). According to Ruelle,⁶ both the traveling and standing waves are solutions to the nonlinear problem. If they are both supercritical, then only one of them can be stable; otherwise, they are both unstable. The possibility of traveling waves on the interface of immiscible fluids may be relevant to the modeling of mantle convection.

II. LINEAR STABILITY ANALYSIS

We consider the linear stability of a two-dimensional (x^*, z^*) problem when the bottom fluid (fluid 1) occupies a layer from $z^* = 0$ to $z^* = l_1^*$ and the top fluid (fluid 2) lies between $z^* = l_1^*$ and $z^* = l^*$. Asterisks denote dimensional variables. The plate at $z^* = 0$ is at temperature $T_0^* + \Delta T^*$, $\Delta T^* > 0$, and the plate at $z^* = l^*$ is at temperature T_0^* . Fluid i ($i = 1, 2$) has a coefficient of cubical expansion $\hat{\alpha}_i$, thermal diffusivity κ_i , thermal conductivity k_i , viscosity μ_i , kinematic viscosity ν_i , and density ρ_i at temperature T_0^* . We define a Rayleigh number $R = g\hat{\alpha}_1\Delta T^*l^{*3}/(\kappa_1\nu_1)$, a Prandtl number $Pr = \nu_1/\kappa_1$, and a surface tension parameter $Tn = Sl^*/(\kappa_1\mu_1)$, where S is the surface tension, all based on fluid 1. There are six dimensionless ratios:

$$m = \mu_1/\mu_2, \quad r = \rho_1/\rho_2, \quad \gamma = \kappa_1/\kappa_2, \\ \zeta = k_1/k_2, \quad \beta = \hat{\alpha}_1/\hat{\alpha}_2, \quad l_1 = l_1^*/l^*.$$

We define l_2 as $l_2 = 1 - l_1$.

We choose the following dimensionless variables (without asterisks): $(x, z) = (x^*, z^*)/l^*$, $t = \kappa_1 t^*/l^{*2}$, $\mathbf{u} = \mathbf{u}^* l^*/\kappa_1$, $T = T^*/\Delta T^*$, $p = p^* l^{*2}/\rho_1 \kappa_1^2$, where \mathbf{u}^* is the velocity (u^*, w^*) , p^* is the pressure, and T^* is the temperature.

The dimensionless unperturbed temperature is

$$\begin{aligned} T_0 + 1 - A_1 z & \text{ for } 0 \leq z \leq l_1, \\ T_0 + A_2(1 - z) & \text{ for } l_1 \leq z \leq 1, \end{aligned} \quad (1)$$

where $A_1 = 1/(l_1 + \zeta l_2)$ and $A_2 = \zeta A_1$, and the unperturbed motion is static. A linear perturbation proportional to $\exp(\sigma t + iax)$ is superposed on the velocity, temperature, and interface position.

The perturbation θ to the temperature satisfies

$$\begin{aligned} \sigma\theta - wA_1 &= \nabla^2\theta, \quad \text{for } 0 \leq z \leq l_1, \\ \sigma\theta - wA_2 &= (1/\gamma)\nabla^2\theta, \quad \text{for } l_1 \leq z \leq 1. \end{aligned} \quad (2)$$

We use the Boussinesq approximation; that is, the fluid parameters are assumed to be constants, taken equal to their values at the temperature T_0^* of the upper wall, except for the density in the gravity term of the Navier-Stokes equations. The density in the buoyancy term is approximated by

$$\rho_i [1 - \hat{\alpha}_i(T^* - T_0^*)], \quad i = 1, 2 \quad (3)$$

to yield

$$\sigma\mathbf{u} = -\nabla p + R \text{Pr} \theta \mathbf{e}_z + \text{Pr} \nabla^2 \mathbf{u} \quad \text{for } 0 \leq z \leq l_1, \quad (4)$$

$$\sigma\mathbf{u} = -r\nabla p + (R \text{Pr}/\beta)\theta \mathbf{e}_z + (r/m)\text{Pr} \nabla^2 \mathbf{u} \quad \text{for } l_1 \leq z \leq 1,$$

where \mathbf{e}_z is the unit vertical vector. Incompressibility yields

$$\nabla \cdot \mathbf{u} = 0. \quad (5)$$

The boundary conditions are $\mathbf{u} = 0$, $\theta = 0$ at $z = 0$, 1. The perturbed free-surface position is $z = l_1 + h(x, t)$ and $h = h_0 \exp(iax + \sigma t)$.

The following linearized interface conditions (see Zeren and Reynolds⁴ for complete derivation) hold at $z = l_1$. Here $[\cdot]$ denotes $\cdot_1 - \cdot_2$.

Continuity of velocity and incompressibility together yield

$$[w] = \left[\frac{\partial w}{\partial z} \right] = 0. \quad (6a)$$

Continuity of shear stress yields

$$\left[\mu \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) \right] = 0,$$

or, in terms of w ,

$$\left[\mu \left(\frac{\partial^2 w}{\partial z^2} + \alpha^2 w \right) \right] = 0. \quad (6b)$$

Continuity of temperature yields

$$[\theta] = h[A], \quad (6c)$$

where the right-hand side arises from a Taylor expansion of the unperturbed temperature profile about $z = l_1$.

Continuity of heat flux requires that $[k\mathbf{n} \cdot \nabla T^*] = 0$, where the normal \mathbf{n} to the interface is $(-dh/dx, 1)/[1 + (dh/dx)^2]^{1/2}$. This implies

$$\left[\frac{k \partial \theta}{\partial z} \right] = 0. \quad (6d)$$

The linearized kinematic free-surface condition is

$$w = \sigma h. \quad (7)$$

The conservation of volume of the incompressible fluids implies that $h(x, t)$ has a zero mean value as a function of x . This is automatic if $\alpha \neq 0$. There is a difference between $\alpha = 0$ and $\alpha \rightarrow 0$; the former is disallowed. The balance of normal stress is

$$\begin{aligned} \frac{1}{m} \frac{\partial^3 w_2}{\partial z^3} - \frac{\partial^3 w_1}{\partial z^3} + 3\alpha^2 \left(1 - \frac{1}{m} \right) \frac{\partial w_1}{\partial z} \\ - h\alpha^2 \left\{ R \left[\frac{(1/r-1)}{\hat{\alpha}_1 \Delta T^*} + l_2 A_2 \left(1 - \frac{1}{r\beta} \right) \right] - \alpha^2 \cdot \text{Tn} \right\} \\ = \frac{\sigma}{\text{Pr}} \left(\frac{1}{r} - 1 \right) \frac{\partial w_1}{\partial z}. \end{aligned} \quad (8)$$

We will show that the above problem is not self-adjoint. Hence, the eigenvalues need not be real. Let Ω be a strip of width one wavelength $2\pi/\alpha$, covering $0 \leq z \leq 1$. Let Ω_1 be the part of Ω in fluid 1 and Ω_2 be in fluid 2. Let $\bar{\mathbf{u}}^*$ and $\bar{\theta}^*$ be the complex conjugates of the adjoints of \mathbf{u} and θ . The asterisks here denote the adjoint and the overbars the complex conjugates. Integration by parts of

$$\begin{aligned} \int_{\Omega_1} \bar{\mathbf{u}}^* \cdot (\sigma\mathbf{u} + \nabla p - \text{Pr} \nabla^2 \mathbf{u} - R \text{Pr} \theta \mathbf{e}_z) \\ + \int_{\Omega_2} \bar{\mathbf{u}}^* \cdot \left(\frac{\sigma}{r} \mathbf{u} + \nabla p - \frac{1}{m} \text{Pr} \nabla^2 \mathbf{u} - \frac{R \text{Pr}}{r\beta} \theta \mathbf{e}_z \right) \\ + \int_{\Omega_1} \bar{\theta}^* (\sigma\theta - wA - \nabla^2\theta) \\ + \int_{\Omega_2} \frac{\bar{\theta}^*}{\xi} [\gamma(\sigma\theta - wA) - \nabla^2\theta] \end{aligned} \quad (9)$$

yields

$$\begin{aligned} \int_{\Omega_1} \mathbf{u} \cdot \{ \sigma \bar{\mathbf{u}}^* - \text{Pr} [\nabla^2 \bar{\mathbf{u}}^* + \nabla(\nabla \cdot \bar{\mathbf{u}}^*)] - A \bar{\theta}^* \mathbf{e}_z \} \\ - \int_{\Omega_1} p \nabla \cdot \bar{\mathbf{u}}^* + \int_{\Omega_2} \mathbf{u} \cdot \left(\frac{\sigma}{r} \bar{\mathbf{u}}^* - \frac{\text{Pr}}{m} [\nabla^2 \bar{\mathbf{u}}^* + \nabla(\nabla \cdot \bar{\mathbf{u}}^*)] \right. \\ \left. - \frac{A\gamma}{\xi} \bar{\theta}^* \mathbf{e}_z \right) + \int_{\Omega_1} \theta (\sigma \bar{\theta}^* - R \text{Pr} \bar{w}^* - \nabla^2 \bar{\theta}^*) \\ + \theta \left(\frac{\gamma\sigma}{\xi} \bar{\theta}^* - \frac{R \text{Pr}}{r\beta} \bar{w}^* - \frac{1}{\xi} \nabla^2 \bar{\theta}^* \right) - B, \end{aligned} \quad (10)$$

where B consists of boundary integrals taken at $z = l_1$ over one wavelength in x . We give the expression for B later. The above integration is facilitated by expressing $\int_{\Omega_1} \nabla^2 \mathbf{u} \cdot \bar{\mathbf{u}}^*$ as $\int_{\Omega_1} \bar{\mathbf{u}}^* \cdot \nabla \cdot [\nabla \mathbf{u} + (\nabla \mathbf{u})^T]$, where superscript T denotes the transpose, taking advantage of $\nabla \cdot \mathbf{u} = 0$, to obtain, for example,

$$\begin{aligned} \int_{\Omega_1} \nabla^2 \mathbf{u} \cdot \bar{\mathbf{u}}^* &= \int_{\Omega_1} \mathbf{u} \cdot \nabla^2 \bar{\mathbf{u}}^* + (\mathbf{u} \cdot \nabla)(\nabla \cdot \bar{\mathbf{u}}^*) \\ &+ \int_{z=0}^{2\pi/\alpha} \left[\bar{\mathbf{u}}^* \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) + 2\bar{w}^* \frac{\partial w}{\partial z} \right. \\ &\left. - u \left(\frac{\partial \bar{w}^*}{\partial z} + \frac{\partial \bar{w}^*}{\partial x} \right) - 2w \frac{\partial \bar{w}^*}{\partial z} \right] dx. \end{aligned} \quad (11)$$

Choosing u , θ , and p and its derivatives to vanish in the neighborhood of the interface, we find $\nabla \cdot \bar{u}^* = 0$ and other adjoint equations. Since $\nabla \cdot u = 0$, the coefficients of u in (10) do not vanish but are the gradients of a function we denote by \bar{p}^* . Hence,

$$\sigma \bar{u}^* - \text{Pr} \nabla^2 \bar{u}^* - A \bar{\theta}^* e_z = -\nabla \bar{p}^*, \quad \text{in fluid 1} \quad (12)$$

$$\sigma \bar{\theta}^* - R \text{Pr} \bar{w}^* - \nabla^2 \bar{\theta}^* = 0,$$

and

$$\begin{aligned} (\sigma/r) \bar{u}^* - (\text{Pr}/m) \nabla^2 \bar{u}^* - (\gamma A / \zeta) \bar{\theta}^* e_z &= -\nabla \bar{p}^*, \\ &\text{in fluid 2} \quad (13) \\ (\gamma/\zeta) \sigma \bar{\theta}^* - (R \text{Pr}/r\beta) \bar{w}^* - (1/\zeta) \nabla^2 \bar{\theta}^* &= 0. \end{aligned}$$

We examine B to find the adjoint interface conditions. The integration is over one wavelength in x at $z = l_1$.

$$\begin{aligned} B &= \int - [p \bar{w}^*] + \text{Pr} \left[\bar{u}_1^* \left(\frac{\partial u_1}{\partial z} + \frac{\partial w_1}{\partial x} \right) - \frac{\bar{u}_2^*}{m} \left(\frac{\partial u_2}{\partial z} + \frac{\partial w_2}{\partial x} \right) + \frac{u^2}{m} \left(\frac{\partial \bar{u}_2^*}{\partial z} + \frac{\partial \bar{w}_2^*}{\partial x} \right) - u_1 \left(\frac{\partial \bar{u}_1^*}{\partial z} + \frac{\partial \bar{w}_1^*}{\partial x} \right) \right. \\ &\quad \left. + 2 \bar{w}_1^* \frac{\partial w_1}{\partial z} - \frac{2}{m} \bar{w}_2^* \frac{\partial w_2}{\partial z} - 2 w_1 \frac{\partial \bar{w}_1^*}{\partial z} + \frac{2}{m} w_2 \frac{\partial \bar{w}_2^*}{\partial z} \right] + \bar{\theta}_1^* \frac{\partial \theta_1}{\partial z} - \frac{\bar{\theta}_2^*}{\zeta} \frac{\partial \theta_2}{\partial z} + \frac{\theta_2}{\zeta} \frac{\partial \bar{\theta}_2^*}{\partial z} + \theta_1 \frac{\partial \bar{\theta}_1^*}{\partial z} dx \\ &= \int - \left(p_1 - 2 \text{Pr} \frac{\partial w_1}{\partial z} \right) \bar{w}_1^* + \left(p_2 - \frac{2 \text{Pr}}{m} \frac{\partial w_2}{\partial z} \right) \bar{w}_2^* - 2 \text{Pr} \left(w_1 \frac{\partial \bar{w}_1^*}{\partial z} - \frac{w_2}{m} \frac{\partial \bar{w}_2^*}{\partial z} \right) \\ &\quad + \text{Pr} \left\{ [\bar{u}^*] \left(\frac{\partial u_1}{\partial z} + \frac{\partial w_1}{\partial x} \right) + u_1 \left[\frac{1}{m} \left(\frac{\partial \bar{u}_2^*}{\partial z} + \frac{\partial \bar{w}_2^*}{\partial x} \right) - \left(\frac{\partial \bar{u}_1^*}{\partial z} + \frac{\partial \bar{w}_1^*}{\partial x} \right) \right] \right\} \\ &\quad + [\bar{\theta}^*] \frac{\partial \theta_1}{\partial z} - \left(\theta_1 \frac{\partial \bar{\theta}_1^*}{\partial z} - \frac{\theta_2}{\zeta} \frac{\partial \bar{\theta}_2^*}{\partial z} \right) dx. \end{aligned}$$

Conditions (6) and (7) yield

$$[\bar{u}^*] = [\bar{\theta}^*] = \left[\mu \left(\frac{\partial \bar{u}^*}{\partial z} + \frac{\partial \bar{w}^*}{\partial x} \right) \right] = 0. \quad (14)$$

We can add $\int_{\Omega} u \cdot \nabla \bar{p}^*$ to (9) which introduces $\int [p^* w]$ dx into B in (10). Hence,

$$\begin{aligned} B &= \int - \bar{w}_1^* \left(p_1 - 2 \text{Pr} \frac{\partial w_1}{\partial z} \right) + \bar{w}_2^* \left(p_2 - \frac{2 \text{Pr}}{m} \frac{\partial w_2}{\partial z} \right) \\ &\quad + \omega_1 \left(\bar{p}_1^* - 2 \text{Pr} \frac{\partial \bar{w}_1^*}{\partial z} \right) - \omega_2 \left(\bar{p}_2^* - \frac{2 \text{Pr}}{m} \frac{\partial \bar{w}_2^*}{\partial z} \right) \\ &\quad - \text{Pr} \left(\theta_1 \frac{\partial \bar{\theta}_1^*}{\partial z} - \frac{\theta_2}{\zeta} \frac{\partial \bar{\theta}_2^*}{\partial z} \right) dx. \end{aligned}$$

Condition (8) can be written as

$$p_2 - p_1 + 2 \text{Pr} \left(\frac{\partial w_1}{\partial z} - \frac{1}{m} \frac{\partial w_2}{\partial z} \right) + h (M_1 + \alpha^2 M_2) = 0, \quad (15)$$

where

$$M_1 = P \text{Pr} \left[\frac{(1-1/r)}{\hat{\alpha}_1 \Delta T^*} + \left(\frac{1}{r\beta} - 1 \right) l_2 A_2 \right], \quad M_2 = \text{Pr} \text{ Tn}.$$

Using (6) and $-\alpha^2 \equiv \partial^2 / \partial x^2 \equiv$, the last term in (15) is

$$(1/[A]) ([\theta] M_1 - [\theta''] M_2). \quad (16)$$

We use (6), (15), and (16) to obtain

$$\begin{aligned} B &= \int w_1 \left[\left(\bar{p}_1^* - 2 \text{Pr} \frac{\partial \bar{w}_1^*}{\partial z} \right) - \left(\bar{p}_2^* - \frac{2 \text{Pr}}{m} \frac{\partial \bar{w}_2^*}{\partial z} \right) \right] \\ &\quad - [\bar{w}^*] \left(p_1 - 2 \text{Pr} \frac{\partial w_1}{\partial z} - \theta_1 \frac{M_1}{[A]} + \theta_1'' \frac{M_2}{[A]} \right) \\ &\quad - \frac{1}{[A]} [\bar{w}^* (\theta M_1 - \theta'' M_2)] - \theta_1 \frac{\partial \bar{\theta}_1^*}{\partial z} + \frac{\theta_2}{\zeta} \frac{\partial \bar{\theta}_2^*}{\partial z} dx. \end{aligned}$$

We choose

$$[\bar{w}^*] = 0, \quad (17)$$

and use (7) to find

$$\begin{aligned} B &= \int \frac{[\theta]}{[A]} \left[\left([\bar{p}^*] - 2 \text{Pr} \frac{\partial \bar{w}_1^*}{\partial z} + \frac{2 \text{Pr}}{m} \frac{\partial \bar{w}_2^*}{\partial z} \right) \right. \\ &\quad \left. - (\bar{w}_1^* M_1 - \bar{w}_1'' M_2) - [A] \frac{\partial \bar{\theta}_1^*}{\partial z} \right] \\ &\quad - \theta_1 \left(\frac{\partial \bar{\theta}_1^*}{\partial z} - \frac{1}{\zeta} \frac{\partial \bar{\theta}_2^*}{\partial z} \right) dx. \end{aligned}$$

We choose

$$\frac{\partial \bar{\theta}_1^*}{\partial z} - \frac{1}{\zeta} \frac{\partial \bar{\theta}_2^*}{\partial z} = 0 \quad (18)$$

and

$$\begin{aligned} \sigma \left([\bar{p}^*] - 2 \text{Pr} \frac{\partial \bar{w}_1^*}{\partial z} + \frac{2 \text{Pr}}{m} \frac{\partial \bar{w}_2^*}{\partial z} \right) \\ - (\bar{w}_1^* M_1 - \bar{w}_1'' M_2) - [A] \frac{\partial \bar{\theta}_1^*}{\partial z} = 0. \quad (19) \end{aligned}$$

Equations (12)–(14) and (17)–(19) are adjoint equations.

III. NUMERICAL SCHEME

We use (5) to eliminate u so that in each fluid, we have the heat equation and one momentum equation, linear in σ :

$$\left. \begin{aligned} \text{Pr} (L^2 w - \alpha^2 R \theta) &= \sigma L w \\ w A_1 + L \theta &= \sigma \theta \end{aligned} \right\} \text{for } 0 < z < l_1,$$

and

$$\left. \begin{aligned} \text{Pr} [r/m L^2 w - (\alpha^2 R / \beta) \theta] &= \sigma L w, \\ w A_2 + (1/\gamma) L \theta &= \sigma \theta \end{aligned} \right\} \text{for } l_1 < z < l,$$

where L is defined to be the operator $\partial^2/\partial z^2 - \alpha^2$. We have replaced the derivatives with respect to x by $i\alpha$.

We use a spectral method, namely the Chebyshev polynomials⁷ $T_m(z)$, $m = 0, 1, \dots$, defined over $-1 < z < 1$, to discretize the equations. Orszag⁷ noted that this method is very efficient for the solution of the Orr-Sommerfeld equations. Here C^∞ functions are approximated with infinite-order accuracy. To facilitate this method, we change the variable z to z_i in fluid i defined by $z_1 = (2/l_1)z - 1$ and $z_2 = (2/l_2)(z - 1) + 1$ so that the z_i range over $[-1, 1]$ in each fluid. We then expand $w(z_i)$ and $\theta(z_i)$ in powers of Chebyshev polynomials $T_m(z_i)$ for $m = 0, \dots, N$ giving a total of $4N + 4$ unknown coefficients. Together with the free-surface variable h_0 , there are $4N + 5$ unknowns. There are six boundary conditions and seven interface conditions. The term of highest differential order in the momentum equation is $\partial^4 w/\partial z^4$. Since we choose w to be an N th degree polynomial, the term $\partial^4 w/\partial z^4$ is of degree $N - 4$ and therefore the momentum equation is truncated at the $N - 4$ th degree, yielding $N - 3$ equations in each fluid. Similarly, since the term of highest differential order in the heat equation is $\partial^2 w/\partial z^2$, we truncate this equation at the $N - 2$ th degree, yielding $n - 1$ equations in each fluid. The eigenvalues of the resulting $4N + 5$ square matrix equation were computed in complex double precision on a VAX11-780 using the IMSL routine EIGZC.

To check the accuracy and convergence of our computer code, we computed the eigenvalues for the Bénard problem in one fluid with $\text{Pr} = 1$, $R = 2177.41$ and 47005.6 , and $\alpha = 2$. The eigenvalues for this problem are real and are given by Reid and Harris.⁸ The eigenvalues at criticality (at which the real part of σ should vanish) are less than 10^{-5} when $N = 15$. A convergence test with $N = 15$ and 20 showed that several other eigenvalues had converged to at least five figures at $N = 15$.

The computations for two fluids were checked against Zeren and Reynolds⁴ by adding an extra term into the shear stress balance at the interface in order to take into account the Marangoni effect. We define a Marangoni number based on fluid 1: $\text{Ma} = (-dS/dT)(\Delta T^* l^*/\mu_1 \kappa_1)$, and our shear stress condition at $z = l_1$ is modified to

$$\alpha^2(m-1)\omega_1 + \frac{m \partial^2 w_1}{\partial z^2} - \frac{\partial^2 w_2}{\partial z^2} + \text{Ma} m \alpha^2 (\theta_1 - A_1 h) = 0.$$

We used their Table 2 for the values of the physical variables at 10°C for benzene lying above water. We checked our eigenvalues against their Table 3 for $l_1 = 0.1$ and 0.6 for heating from below. Note that our definition of the R and Ma are different from theirs. At $l_1 = 0.1$, converting their parameters to ours, they find criticality at $\text{Ma} = 1255.71$, $R = 178.3045$, $\hat{\alpha}_1 \Delta T^* = 0.00032537$, $\text{Pr} = 8.1$, $\alpha = 3.5$, and $\text{Tn} = 460320$. We computed $(\sigma/\text{Pr}) = 0.006186$ using both $N = 15$ and 20 . This yields 0.00175 for the eigenvalue q of Zeren and Reynolds. At $l_1 = 0.6$, their parameters in Table 3 become $\text{Ma} = 4016.7153$, $R = 570.3736$, $\hat{\alpha}_1 \Delta T^* = 0.0010408$, and $\alpha = 2.5$. We computed -0.00436 for their eigenvalue q at $N = 15$ and 20 . In both cases, we also found stable complex conjugate pairs in the spectrum.

IV. NUMERICAL RESULTS

To aid the reader in the interpretation of the numerical results, we recall some results from the Bénard problem with one fluid. In the simplest case, the layer is bounded at $z = 0$ and $z = 1$ by stress-free conducting boundaries and⁸

$$\sigma = -\frac{1}{2}(1 + \text{Pr})(n^2\pi^2 + \alpha^2) \pm \left[\frac{1}{4}(\text{Pr} - 1)^2(n^2\pi^2 + \alpha^2)^2 + \alpha^2 R \text{Pr}/(n^2\pi^2 + \alpha^2)^{1/2} \right],$$

for $n = 1, 2, \dots$. (20)

Hence, for $\alpha \approx 0$,

$$\sigma \approx -\frac{1}{2}(1 + \text{Pr})n^2\pi^2 \pm \frac{1}{2}|\text{Pr} - 1|n^2\pi^2 < 0, \quad (21)$$

and as $\alpha \rightarrow \infty$,

$$\sigma \sim -(\alpha^2/2)[(1 + \text{Pr}) \pm |\text{Pr} - 1|] < 0. \quad (22)$$

In the critical case $\sigma = 0$, the least value of R occurs when $\alpha = \pi/\sqrt{2}$ and $R = (\pi^2 + \alpha^2)^3/\alpha^2$. These formulas are for stress-free surfaces but they give an idea of the variation of $\sigma(R, \alpha^2)$ in the classical case of one fluid between rigid boundaries.

Now we consider the case when there are two fluids with equal properties. This would at first thought appear to be a one-fluid problem. However, it is easy to see that there is a solution with $[\theta] = [A] = 0$, $h_0 \neq 0$, and $\sigma = 0$. We shall use the nomenclature introduced by Yih¹⁰ in a related problem and call this mode, which is important when the properties of the two fluids are different, an interfacial mode.¹¹ We track eigenvalues as we vary parameters. Besides the interfacial eigenfunctions, we have other eigenfunctions which we shall call Bénard modes.

In tracking the eigenvalues, we shall fix all the parameters so that there is a critical α such that $\text{Re } \sigma(\alpha, R, \text{Pr}, \text{Tn}, m, r, \gamma, \zeta, \beta, l_1) = 0$ with $\text{Re } \sigma < 0$ for other α . We shall exhibit parameters for which $\text{Im } \sigma \neq 0$ at criticality. Hence we obtain oscillations in the linear problem at criticality ("exchange of stabilities" does not hold), and the nonlinear problem for Bénard convection in two fluids can have time-periodic solutions near criticality.

Let the two fluids have equal densities at temperature T_0^* and the same thermal diffusivities and conductivities: $r = \gamma = \zeta = 1$. We let $R = 1695.7$, $\text{Pr} = 1$, $\hat{\alpha}_1 \Delta T^* = 0.001$, $\text{Tn} = 0$, $\text{Ma} = 0$, $m = 1.1$, and $\beta = 0.9$. Thus, if fluid 1 occupies the entire flow, the Rayleigh number is lower than the critical one 1708 (see Reid and Harris⁸). If fluid 2 occupies the entire flow, R is 2072.52 and the flow is linearly unstable for a range of α . We choose $l_1 = 0.4$. Figure 1 is a graph of the growth rate $\text{Re } \sigma$ against α .

We are approximately at criticality when $\alpha = 3.1$. In this case, we compute $\sigma = 0.000072 \pm i5.9259$ with $N = 15, 20$.

The five numbers next to the curves in Fig. 1 denote branches which display different features. The interfacial mode is associated with branches 1, 3, and 5. Branch 1 can be obtained from the interfacial mode with $\alpha \rightarrow 0$ when the properties of the fluids are equal ($\beta = m = 1$) by moving β to 0.9 and m to 1.1. This branch is real-valued. Branch 2 is associated with the least stable of the Bénard modes for a single fluid when $\alpha \rightarrow 0$. This branch is approximately

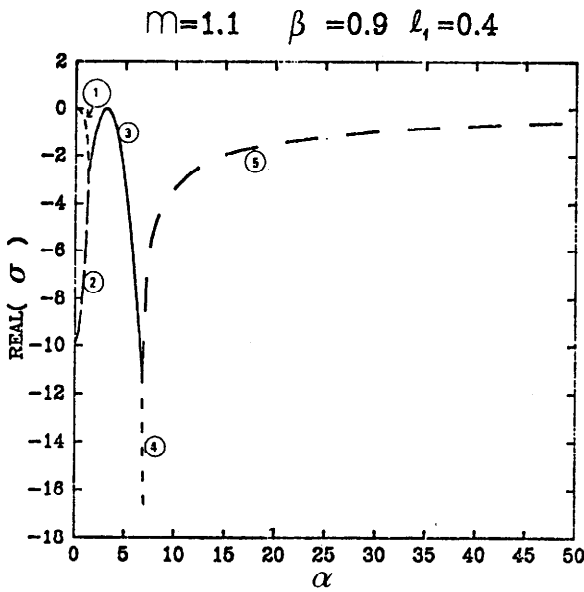


FIG. 1. The growth rate $\text{Re}(\sigma)$ is plotted against the wavenumber α for $R = 1695.7$, $\text{Pr} = 1$, $\hat{\alpha}_1 \Delta T^* = 0.001$, $\text{Tn} = \text{Ma} = 0$, and $r = \gamma = \zeta = 1$. Branches 1, 2, 4, and 5 belong to real-valued eigenvalues. Branch 3 consists of a complex conjugate pair.

– 9.87 as $\alpha \rightarrow 0$ and would correspond to the largest value of (21). Branch 2 is real-valued. Branches 1 and 2 coalesce and split into conjugate pairs at $\alpha = 1.275$. At $\alpha = 6.79$, the conjugate pair again splits into the two real-valued branches 4 and 5. Branch 4 is associated with a Bénard mode and remains real, decreasing rapidly as α is increased, as in the single fluid problem [see (22)].

Branch 5 is an interfacial mode. It is real-valued and negative. The stability for large α which is associated with branches 4 and 5 is explained by our choice of β and the

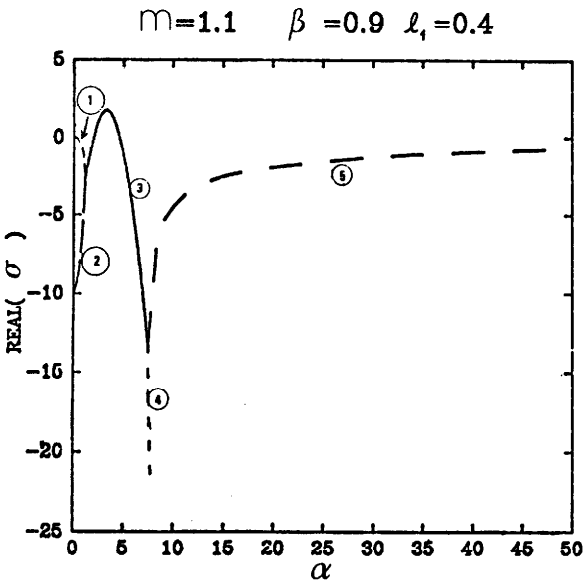


FIG. 2. The growth rate $\text{Re}(\sigma)$ versus wavenumber α for $R = 2177.41$, $\text{Pr} = 1$, $\hat{\alpha}_1 \Delta T^* = 0.001$, $\text{Tn} = \text{Ma} = 0$, and $r = \gamma = \zeta = 1$. Branches 1, 2, 4, and 5 belong to real-valued eigenvalues. Branch 3 consists of a complex conjugate pair.

Boussinesq approximation (3). We consider the densities $\rho_i [1 - \hat{\alpha}_i (T^* - T_0^*)]$ at the unperturbed interface $z = l_1$ when the temperature $T - T_0$ is given by (1). Then with $r = 1$ and $\beta = 0.9$, we find that $\rho_2 (1 - 0.6 \hat{\alpha}_2 \Delta T^*)$ is the density of fluid 2 at $l_1 = 0.4$ and $\rho_2 (1 - 0.54 \hat{\alpha}_2 \Delta T^*)$ is the density of fluid 1. Hence the heavy fluid is below and gravity may be expected to stabilize short (large α) waves. The interfacial eigenvalue on branch 5 is discussed in Sec. V.

Figure 2 shows the growth rates versus α when the Rayleigh number is increased to $R = 2177.41$ while other parameters are fixed as in Fig. 1 and shows that the instability on branch 3 is associated with the complex conjugate pair of branch 3 in Fig. 1.

V. ASYMPTOTIC ANALYSIS OF THE INTERFACIAL EIGENVALUE FOR SHORT WAVES

We now consider disturbances of rapid variation whose length scale of variation is of the same order $O(1/\alpha)$ as the short perturbation wavelength.¹¹

We rescale z to $\eta = \alpha(z - l_1)$ and let η be $O(1)$. The equations in fluid 1 are $\sigma \theta - w A_1 = \alpha^2 L^* \theta$, where $L^* \equiv \partial^2 / \partial \eta^2 - 1$ and $(\sigma - \text{Pr} \alpha^2 L^*) L^* w = -R \text{Pr} \theta$. In fluid 2, $\sigma \theta - w A_2 = (1/\gamma) \alpha^2 L^* \theta$ and $[\sigma - (r/m) \text{Pr} \alpha^2 L^*] L^* w = -(R \text{Pr} / \beta) \theta$. The interface conditions are

$$w_1 = w_2 = \sigma h, \quad (23)$$

$$\left[\frac{\partial w}{\partial \eta} \right] = 0, \quad (24)$$

$$[\theta] = [A] h, \quad (25)$$

$$\frac{m}{\alpha} \frac{\partial^2 w_1}{\partial \eta^2} - \frac{\partial^2 w_2}{\partial \eta^2} + (m-1) w_1 = 0, \quad (26)$$

$$\frac{\zeta}{\alpha} \frac{\partial \theta_1}{\partial \eta} = \frac{\partial \theta_2}{\partial \eta}, \quad (27)$$

$$\begin{aligned} \alpha^2 \left(\frac{1}{m} \frac{\partial^3 w_2}{\partial \eta^3} - \frac{\partial^3 w_1}{\partial \eta^3} \right) + \frac{\partial w_1}{\partial \eta} 3\alpha^2 \left(1 - \frac{1}{m} \right) + h \alpha^3 \text{Tn} \\ - h R \alpha \left[\frac{(1/r - 1)}{\hat{\alpha}_1 \Delta T^*} + l_2 A_2 \left(1 - \frac{1}{r\beta} \right) \right] \\ = \left(\frac{1}{r} - 1 \right) \frac{\sigma}{\text{Pr}} \frac{\partial w_1}{\partial \eta}. \end{aligned} \quad (28)$$

Since the normal stress condition (28) contains both odd and even powers of α , all the variables are formally expanded in powers of $1/\alpha$. To the zeroth and first orders, $L^* \theta = 0$ and $L^* w = 0$ in each fluid. Using conditions (23), (24), and (26), we obtain $w_1 = c_0 (1 - \eta) e^\eta + O(1/\alpha)$ and $w_2 = c_0 (1 + \eta) e^{-\eta} + O(1/\alpha)$ as $\alpha \rightarrow \infty$, which yields to this and the next order, $\partial w / \partial \eta = 0$, at the interface. Hence, the normal stress condition is

$$\begin{aligned} \alpha \left(\frac{1}{m} \frac{\partial^3 w_2}{\partial \eta^3} - \frac{\partial^3 w_1}{\partial \eta^3} \right) \\ - \frac{w}{\sigma} R \left[\frac{(1/r - 1)}{\hat{\alpha}_1 \Delta T^*} + l_2 A_2 \left(1 - \frac{1}{r\beta} \right) \right] = 0, \end{aligned}$$

where, for the moment, surface tension has been neglected. To avoid the trivial solution, we choose $\sigma = \sigma_0 / \alpha + O(1/\alpha^2)$ for large α . The normal stress condition yields

$$\sigma_0 = \frac{R}{2(1/m + 1)} \left(\frac{(1/r - 1)}{\hat{\alpha}_1 \Delta T^*} + l_2 A_2 (1 - 1/r\beta) \right).$$

In the computations for Fig. 1, the asymptotic formula is accurate to 1% for $\alpha > 20$. We computed -1.494 for $\alpha = 20$ using $N = 15$, whereas the asymptotic formula yields -1.48 .

Turning now to a consideration of the effect of surface tension, we find that when $\alpha^2 Tn/R = O(1)$, then

$$\sigma_0 = \frac{R}{2(1/m + 1)} \times \left[\frac{(1/r - 1)}{\hat{\alpha}_1 \Delta T^*} + l_2 A_2 \left(1 - \frac{1}{r\beta} \right) - \frac{\alpha^2 Tn}{R} \right]. \quad (29)$$

We computed the eigenvalue for the parameters of Fig. 1 at $Tn = 1$ and $\alpha = 20$ to be -6.86 and the asymptotic formula yields -6.72 . Equation (29) shows that surface tension is always stabilizing for short wave disturbances. The stabilization of short waves by surface tension, even with adverse density ratios, has been found in other flows such as steady shear flows with two immiscible fluids of different viscosities.^{11,12}

ACKNOWLEDGMENTS

Y. Renardy was supported by the United States Army under Contract No. DAAG29-80-C-0041. D. Joseph was supported by the Fluid Mechanics Division of the National Science Foundation and by the United States Army under Contract No. DAAG29-82-K-0051.

¹D. D. Joseph, K. Nguyen, and G. S. Beavers, *J. Fluid Mech.* **141**, 319 (1984).

²F. H. Busse, *Geophys. Res. Lett.* **9**, 519 (1982).

³F. H. Busse, *Phys. Earth Planet. Int.* **24**, 320 (1981).

⁴R. W. Zeren and W. C. Reynolds, *J. Fluid Mech.* **53**, 305 (1972).

⁵C. V. Sternling and L. W. Scriven, *AIChE J.* **5**, 514 (1959).

⁶D. Ruelle, *Arch. Rat. Mech. Anal.* **51**, 136 (1973).

⁷S. A. Orszag, *J. Fluid Mech.* **50**, 689 (1971).

⁸W. H. Reid and D. L. Harris, *Phys. Fluids* **1**, 102 (1958).

⁹P. G. Drazin and W. H. Reid, *Hydrodynamic Stability* (Cambridge U. P., Cambridge, England, 1982).

¹⁰C.-S. Yih, *J. Fluid Mech.* **27**, 337 (1967).

¹¹A. P. Hooper and W. G. C. Boyd, *J. Fluid Mech.* **128**, 507 (1983).

¹²Y. Renardy and D. D. Joseph, *J. Fluid Mech.* **150**, 381 (1985).