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Dedicated to Walter Noll on the Occasion of his 60^{th} Birthday

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Abstract

The equations governing the flow of viscoelastic liquids are classified according to the symbol of their differential operators. Propagation of singularities is discussed and conditions for a change of type are investigated. The vorticity equation for steady flow can change type when a critical condition involving speed and stresses is satisfied. This leads to a partitioning of the field of flow into subcritical and supercritical regions, as in the problem of transonic flow.

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1. Introduction

The equations of steady gas dynamics change type when the speed of the fluid at some point exceeds the speed of sound. If this happens, then discontinuities can appear in the supersonic region. We are interested in the possibility that many strange effects in the flow of viscoelastic liquids, as well as difficulties in numerical simulation, are also associated with the appearance of real characteristics and a change of type, analogous to the sonic transition.

For a physical interpretation, it is necessary to identify the variables which may propagate and become discontinuous. In gas dynamics, there are compression waves and shock waves of compression. In the present paper, we deal with incompressible materials, so compression is impossible. Instead, we can exhibit cases where singular shear surfaces propagate along characteristics (Chapters 6, 7). In steady flow, the vorticity is the variable which is affected by a change of type and may become discontinuous (Chapters 8–11). The implications of hyperbolicity and change of type for the interpretation of experiments are not yet well understood.

The organization of our paper is shown in the Table of Contents. In § 2, we motivate our study by suggesting that one of the main unsolved practical problems of computation of viscoelastic flow may be partly due to the problem of change of type. We suggest that the solution of this problem is to be found in recently developed switching algorithms of the type used in transonic flow. In § 3 we define some basic concepts needed in our study, including elliptic, hyperbolic, characteristic, symbol of an operator and Hadamard instability. We also give some applications of these concepts which arise in modeling phase changes and may be relevant in analyzing some instabilities in the extrusion of polymers from capillary tubes. Chapter 4 discusses characteristics and classification of type for first order quasilinear systems.

In Chapter 5, we look at constitutive equations for viscoelastic fluids from the point of view of classification of type. For this, we have to maintain a distinction between fluids with and without Newtonian viscosity. In Oldroyd models, the term with Newtonian viscosity is the one associated with a retardation time. The addition of even small amounts of Newtonian viscosity can smooth discontinuities, replacing sharp fronts by thin layers and thus masking the underlying dynamics. To emphasize the effect of hyperbolicity, we confine our attention to models without Newtonian viscosity. In particular, we focus on a three-parameter family of nonlinear Oldroyd models containing the upper and lower convected and corotational Maxwell models. The occurrence of instabilities of the Hadamard type for these models is discussed. These models also form the basis for the discussion of steady flows in Chapters 9–11. We also discuss more general models of integral type. It is shown that the principal part of the linearization at any given motion has the form of a rate equation not involving integrals, provided that the integral kernels have sufficient smoothness. Thus the discussion of change of type does not necessarily require a special constitutive model.

Chapter 6 discusses the linear system of equations for motion perturbing rest. The wave speed along characteristics is given by $\sqrt{G(0)/\varrho}$, where $G(0)$ is the instantaneous value of the relaxation modulus $G(s)$ and ϱ is the density. We review recent results on the propagation of slip surfaces for velocity and displacement, which show in particular the crucial dependence on the nature of the kernel $G(s)$. In particular, consideration is given to the possibility that $G(0)$ or $G'(0)$ may be infinite. In Chapter 7, we discuss the formation and propagation of slip surfaces in nonlinear shearing problems treated by COLEMAN & GURTIN [6], [7] and SLEMROD [43], [44]. We discuss the application of their results to melt fracture.

In Chapter 8 we take up the analysis of change of type in steady problems. This is a natural question from a mathematical point of view, but the first studies of it in the theory of viscoelastic fluids seem to be in the work of RUTKEVICH [40, pp. 44–45], who analyzed the two dimensional equations for an upper convected Maxwell model. ULTMAN & DENN [49] and LUSKIN [27] classified the linearized equations perturbing uniform flow with velocity U of an upper convected Maxwell fluid. Our analysis in Chapter 8 generalizes the results of ULTMAN & DENN and LUSKIN to a wider class of constitutive laws. There is a change of type leading to real characteristics when the viscoelastic Mach number

$$M = U/c, \quad c = \sqrt{G(0)/\varrho}$$

exceeds one. The vorticity is identified as the variable which can become discontinuous along these characteristics. We shall, somewhat loosely, say that "the vorticity changes type". In Chapter 9, we give a complete classification of the quasilinear system describing the upper convected Maxwell model in arbitrary steady two-dimensional motions. The streamlines are double characteristics. The vorticity changes its type when the speeds are great enough. In the supercritical (hyperbolic) case, there are two families of real characteristics for the vorticity, but the formula for the characteristics depends on the solution. There are also complex roots to the characteristic equation associated with the elliptic equation giving the vorticity as the Laplacian of the stream function. In Chapter 10, we discuss a number of specific flows for an upper convected Maxwell fluid. These flows include plane parallel shear flow, steady extensional flow, sink flow in the plane and shear flow outside a rotating cylinder. We discuss characteristics for motions perturbing those flows and characterize the regions of flow where the vorticity equation is hyperbolic. In Chapter 11, we extend our results to a three parameter family of Oldroyd models which contains the upper and lower convected and corotational Maxwell models as special cases. The vorticity is again identified as the variable which changes its type. We compute the characteristic directions for the nonlinear problem without approximation. We exhibit special cases which show that the partitioning of the flow into sub- and supercritical regions is model sensitive. It is therefore desirable to develop this type of theory on a high level of generality, suppressing models. We take some steps in that direction in Chapter 12, where we study fading memory fluids of Coleman-Noll type.

2. Numerical Simulation of Steady Flows of Changing Type

There are some unsolved problems of numerical simulation of the flow of viscoelastic fluids. One problem is that the equations cannot be integrated when the relaxation time is large. Though relaxation times appear expiicitly only in very special models, the concept of a relaxation time is a useful one which can be expressed mathematically in a general context (*e.g.*, C. TRUESDELL [52]). A large relaxation time means that the elastic response of the fluid is persistent; the fluid can be said to have a long memory. The simulation problem associated with highly elastic viscoelastic fluids is sometimes called "the high Weissenberg number problem". Different dimensionless ratios are called "Weissenberg numbers" by different authors. Some authors call the ratio of the first normal stress to the shear stress a "Weissenberg number". This definition leads to a dimensionless function of the rate of shear. Other authors define a different "Weissenberg number" as the ratio of the relaxation time of the fluid to an externally given time which is usually expressed as d/U, where U and d are a typical velocity and length in the flow.

The "high Weissenberg number problem" refers to the failure of numerical simulations when the second of the two "Weissenberg numbers" is large. The problem occurs with different constitutive models and different methods of numerical integration. Maybe there are some underlying mathematical reasons.

Our study here is not framed in terms of a "Weissenberg number". The quantities of interest in our study are values of velocity and stress which in steady flow may lead to a change of type as in the problem of transonic flow. The criterion for change of type may be framed in terms of a viscoelastic. Mach number defined as the pointwise ratio of some speed to a characteristic wave speed. If all other quantities are fixed, this "Mach number" increases with the relaxation time, but our "Mach number" and the "Weissenberg number" are in principle independent. It is probable however that some numerical problems at high "Weissenberg numbers" are actually associated with a change of type, like the transition from subsonic flow to supersonic flow, and that the solution of the problem is to be sought in various hyperbolic algorithms, especially those recently introduced for transonic flow.

To compute subsonic flow you use some central differences. To compute supersonic flow you use the method of characteristics. It would be a disaster to try to do supersonic flow by central differencing of the type used for Laplace's equation.

In the flow over a bump, say an airfoil with the free stream slightly less than $M = 1$,

$$.75 \leqq M < 1,$$

we get a supersonic bubble with unknown boundaries (see Fig. 1).

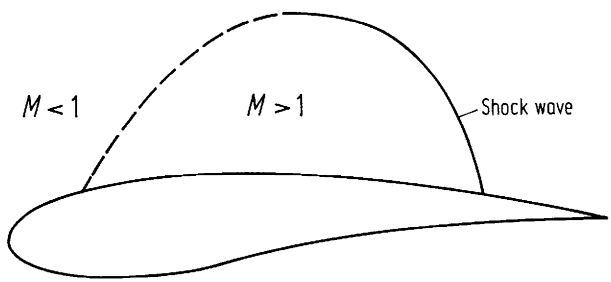

Fig. 1

To solve this problem you have to find the sonic line, the position and strength of the shock wave. This is a very hard free boundary problem. It wasn't solved until 1971 when MURMAN & COLE [32] realized that upwind differencing was necessary in the supersonic part of the flow. In central differencing the nodal point is at the center. In upwind differencing, the information at a nodal point is determined only by the flow upstream (see Fig. 2).

Fig. 2

MURMAN & COLE studied the small disturbance equations. They derived a switching scheme of numerical analysis which tells the computer to use central differencing if the flow is subsonic and upwind differencing if the flow is supersonic. Their upwind differencing equation can be interpreted as approximating a differential equation with an artificial viscosity proportional to the mesh size [31].

MURMAN & COLE'S method was the first success. But this method is too simple for the full nonlinear potential. This more complicated problem was successfully attacked by the artificial viscosity method of JAMESON [19], [20], whose work makes transonic computation possible in a practical sense.

People doing flow computations for viscoelastic fluids are also able to go to higher Weissenberg numbers when they have constitutive equations with more Newtonian viscosity. This procedure masks the problem of dealing with change of type instead of solving it.

3. Concepts and Some Applications of Change of Type

This paper deals with equations which undergo a change of type. To make our notions precise we shall need some classical definitions related to the type of a partial differential equation.

Consider the linear differential operator

$$P\left(\boldsymbol{x}, t, \frac{\partial}{\partial t}, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right), \tag{3.1}$$

where

$$\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \text{ and } t$$

are space and time coordinates. We define the

$$\text{Symbol of } P = P(\boldsymbol{x}, t, i\xi_0, i\xi_1, \ldots, i\xi_n), \tag{3.2}$$

where $i = \sqrt{-1}$. To form the symbol we replace the arguments $\frac{\partial}{\partial t}, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$ of P with the Fourier variables $i\xi_0, i\xi_1, \ldots, i\xi_n$. In this way we obtain a polynomial in the real variables ξ. The symbol of the Laplace operator $-\Delta$ is $\sum_{i=1}^{n} \xi_i^2$; the symbol of the wave operator $\frac{\partial^2}{\partial t^2} - \Delta$ is $-\xi_0^2 + \sum_{i=1}^{n} \xi_i^2$; the symbol of the heat operator $\frac{\partial}{\partial t} - \Delta$ is $i\xi_0 + \sum_{i=1}^{n} \xi_i^2$. The symbol for a system of equations is defined in a similar fashion and is a matrix with polynomial entries.

Characteristic curves are lines along which discontinuous data may propagate. In dimensions higher than two we may speak of characteristic surfaces. Let m be the highest order of the derivatives in P. Then

$$P = \sum_{|\alpha|=m} a_\alpha(\mathbf{x}, t)\, \partial^\alpha + \sum_{|\alpha|<m} a_\alpha(\mathbf{x}, t)\, \partial^\alpha,$$

where $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_n)$ is a multi-index, $|\alpha| = \Sigma\, \alpha_i$ and

$$\partial^\alpha = \frac{\partial^{|\alpha|}}{\partial t^{\alpha_0}\, \partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}.$$

The equation

$$\sum_{|\alpha|=m} a_\alpha(\mathbf{x}, t)\, \sigma^\alpha = 0, \qquad \sigma = (\sigma_0, \ldots, \sigma_n), \tag{3.3}$$

$$\sigma^\alpha = \sigma_0^{\alpha_0} \ldots \sigma_n^{\alpha_n}$$

is called the characteristic equation for P. Only the principal part of P, the terms of highest order, appears in (3.3).

A surface S in $(\mathbf{x}, t)$ space is characteristic for P at a point $s \in S$ if the normal vector to S at s satisfies the characteristic equation. If $\sigma = (\sigma_0, \ldots, \sigma_n)$ is a unit normal vector at s, S is characteristic for P if and only if

$$\sum_{0}^{n} \sigma_k^2 = 1 \quad \text{and} \quad \sum_{|\alpha|=m} a_\alpha(\mathbf{x})\, \sigma^\alpha = 0. \tag{3.4}$$

The characteristic equation for Laplace's equation $\sum_{k=1}^{n} \partial^2 u/\partial x_k^2 = 0$ is $\sum_{k=1}^{n} \sigma_k^2 = 0$. There are no real characteristics because $(3.4)_1$ is not satisfied. More generally, the operators P for which, at every point $(\mathbf{x}, t)$ the equation $(3.4)_2$ has no nontrivial real zeroes are called *elliptic*. For systems, ellipticity means that the only real zeroes of the determinant of the matrix symbol $A(\mathbf{x}, \xi_1, \ldots, \xi_n)$ are $(\xi_1, \xi_1, \ldots, \xi_2) = (0, 0, \ldots, 0)$.

Elliptic problems have existence, uniqueness and continuous dependence on data (are well posed) as boundary value problems [1], [26].

The initial value problem, the Cauchy problem, is not well posed for elliptic equations. For example, a Cauchy problem for Laplace's equation in the domain

$D = \{x, y; x > 0, -\infty < y < \infty)$ is}

$$\begin{aligned} \Delta u &= 0 \quad \text{in } D, \\ u(0, y) &= 0, \\ \frac{\partial u}{\partial x}(0, y) &= U(y), \end{aligned} \tag{3.5}$$

where

$$U(y) = \frac{1}{n^p} \sin ny, \qquad p > 0.$$

The solution of (3.5) is

$$u(x, y) = \frac{1}{n^{1+p}} \sin ny \sinh nx.$$

The mapping $\left(u, \frac{\partial u}{\partial x}\right)\Big|_{x=0} \to u$ for $x > 0$ is not continuous since $U(y)$ is small when n is large and $u(x, y)$ is very big. Small data at $x = 0$ lead to larger and larger oscillations for $x > 0$. This lack of continuous dependence is called *Hadamard instability*. It can be shown that (3.5) has no solution if $U(\cdot)$ is not analytic.

The initial value problem, or mixed initial-boundary value problems are well-posed for hyperbolic equations like the wave equations. For example, the characteristic equation $(3.4)_2$

$$\sigma_0^2 - c^2 \sum_1^n \sigma_k^2 = 0$$

for the n-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \Delta u$$

satisfies the characteristic equation $(3.4)_1$ when $\sigma_0 = \pm c/\sqrt{c^2 + 1}$. Therefore a surface is characteristic for the wave equation if and only if its normal makes an angle β, $\cos \beta = c/\sqrt{c^2 + 1}$, with the t axis. For the one-dimensional wave equation $\Delta = \partial^2/\partial x^2$, this implies that the family of lines $x \pm ct = \text{const}$ are characteristic.

The operator P of (3.1) is called strictly hyperbolic if all the roots ξ_0 of the principal part of its symbol (3.2) are real and distinct for all $(\xi_1, \ldots, \xi_n) \in \mathbb{R}^n \setminus 0$. The Cauchy problem is well posed and the boundary value problem is ill posed for hyperbolic equations. The backward Cauchy problem where t is replaced with $-t$ is also well-posed for hyperbolic equations.

The Cauchy problem is well posed for parabolic problems but the backward Cauchy problem is ill posed. The classic example of a parabolic equation is the heat equation, $\partial u/\partial t = \Delta u$. The characteristic equation $(3.4)_2$ is

$$\sum_{l=1}^n \sigma_l^2 = 0.$$

Hence, from $(3.4)_1$, $\sigma_0^2 = 1$ and the characteristic surfaces are the hyperplanes $t = \text{const}$. The Cauchy problem is not well posed for the backward heat equation $\frac{\partial u}{\partial t} = -\Delta u$. Operators of the form $\frac{\partial u}{\partial t} + Lu$, where L, like $-\Delta$, is a positive definite elliptic operator, are parabolic. These operators are strongly dissipative and lead to diffusion rather than to propagation. Unlike hyperbolic operators, parabolic operators will smooth initially discontinuous Cauchy data.

Two homogeneous scalar operators are said to be of the same type, if up to a transformation of the independent variables, their symbols have the same asymptotic behavior at infinity. If the asymptotic behavior of the symbol changes, then we say that the equation changes type. For example, the Tricomi equation

$$y \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

is hyperbolic when $y < 0$ and elliptic when $y > 0$. Another example is the quasilinear system

$$\frac{\partial u}{\partial t} = \frac{\partial \sigma(v)}{\partial x}, \qquad \frac{\partial v}{\partial t} = \frac{\partial u}{\partial x}, \qquad \left(\text{or} \quad \frac{\partial^2 v}{\partial t^2} = \frac{\partial}{\partial x}\left(\sigma'(v)\frac{\partial v}{\partial x}\right)\right) \tag{3.6}$$

which is hyperbolic for $\sigma'(v) > 0$ and elliptic for $\sigma'(v) < 0$. These problems all involve a change in the sign of the symbol and Hadamard instabilities, which occur if the solution of the Cauchy problem with initial data in the hyperbolic region enters the elliptic region.

Problems of the form (3.6) suggest models for theories of phase changes in solids and fluids. The van der Waals gas is a well-known classical example. In solid mechanics, ideas of this type were introduced by J. L. ERICKSEN [12] in his study of elastic bars. We may suppose that the graph of $\sigma(v)$ is as shown in Fig. 3.

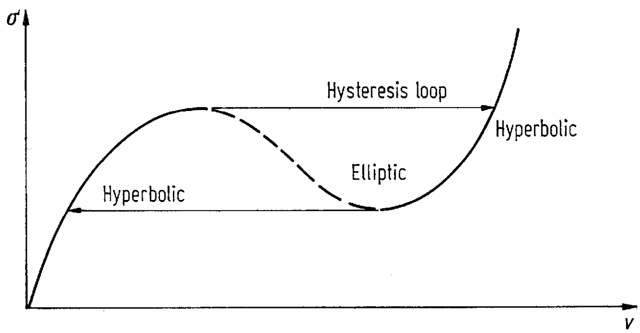

Fig. 3. The system (3.6) is hyperbolic when $\sigma'(v) > 0$. The elliptic branch is unstable in the sense of Hadamard

The solid lines, where $\sigma'(v) > 0$, lead to a hyperbolic equation and the dashed line leads to an elliptic equation. The elliptic portion is rejected because it will exhibit Hadamard instabilities; and actual solutions are required to operate only on the hyperbolic parts of the curve. This leads to spatially segregated solutions, separated by lines of diccontinuity, each part operating on a different hyperbolic branch of the curve. There is hysteresis and abrupt transitions in the response of such models. These features are all present in the recent study of HUNTER & SLEMROD [17], which attempts to explain some observations of TORDELLA [47] of a type of melt fracture called ripple. This phenomenon shows hysteresis loops, double-valued shear rates at certain stresses and spatially segregated flow regimes. Similar ideas have also been used to explain to phenomenon of necking occurring in cold drawing of polymers [8].

REGIRER & RUTKEVICH [36] have considered fluids of the Reiner-Rivlin type which exhibit change of type. Their constitutive law is

$$T = -p\mathbf{1} + \eta f(II)\,D,$$

where $D = \frac{1}{2}(\nabla u + (\nabla u)^T)$, $II = \operatorname{tr} D^2$. Written in terms of a stream function $u = (u, v) = (\psi_y, -\psi_x)$, the equation governing steady two-dimensional flows is as follows.

$$L\psi \overset{\text{def}}{=} a_1\left[\frac{\partial^4\psi}{\partial x^4} + \frac{\partial^4\psi}{\partial y^4}\right] + 2a_2\frac{\partial^4\psi}{\partial x^2\,\partial y^2} + 4a_3\frac{\partial^2}{\partial x\,\partial y}\left(\frac{\partial^2\psi}{\partial y^2} - \frac{\partial^2\psi}{\partial x^2}\right) = H(\psi), \tag{3.7}$$

where $H(\psi)$ is a nonlinear third order operator and the coefficients a are nonlinear functions of the second derivatives of ψ. The characteristic curves $y(x)$ are solutions of

$$a_1y_x^4 + 4a_3y_x^3 + 2a_2y_x^2 - 4a_3y_x + a_1 = 0. \tag{3.8}$$

There are three cases:

(i) $f + 2IIf' > 0$ (no real roots, elliptic),
(ii) $f + 2IIf' = 0$ (parabolic),
(iii) $f + 2IIf' < 0$ (four real roots, hyperbolic).

The hyperbolic regions are those where the stress decreases as a function of shear rate, and the elliptic regions are those where it increases. The unsteady problem corresponding to (3.7) is

$$\varrho\frac{\partial}{\partial t}(\Delta\psi) = L(\psi) - H(\psi).$$

When the right side is elliptic, this problem is parabolic and evolutionary (see GELFAND [14]). When the right-hand side changes type, the problem is neither parabolic nor evolutionary and Hadamard instability occurs. Changes of type and Hadamard instabilities can occur in rheological problems which are not one-dimensional and they need not be associated with non-monotone constitutive equations. An interesting case of this type arises in a stability analysis of plane

Couette flow by AKBAY, BECKER, KROZER & SPONAGEL [3]. In order to obtain a manageable equation, they introduce the "short memory approximation". This means that, in the memory integrals occurring in the equation for the disturbances, only terms of first order in the relaxation time of the fluid are kept. Proceeding thus, they find the following linearized equation for the stream function in two dimensions:

$$\varrho\left(\frac{\partial}{\partial t} + \varkappa x_2 \frac{\partial}{\partial x_1}\right) \Delta\psi = \left(N_1' - \frac{N_1}{\varkappa}\right) \frac{\partial^2}{\partial x_1\, \partial x_2} L\psi + \tau' L^2\psi + \frac{4\tau}{\varkappa} \frac{\partial^4 \psi}{\partial x_1^2\, \partial x_2^2}, \tag{3.9}$$

$$\psi = \frac{\partial\psi}{\partial x_2} = 0 \quad \text{at} \quad x_2 = 0,\ x_2 = h.$$

Here L denotes the operator $\frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_1^2}$. The problem is posed in the strip $-\infty < x_1 < \infty$, $0 < x_2 < h$. $\varkappa$ is the shear rate of the basic Couette flow and $\tau(\varkappa)$, $N_1(\varkappa)$ are the shear stress and the first normal stress difference as functions of the rate of shear. AKBAY *et al.* find that (3.9) admits exponentially growing solutions if

$$\mathrm{We} = \frac{\left(\frac{N_1}{\varkappa}\right)' \varkappa}{\sqrt{\frac{\tau}{\varkappa}\,\tau'}} > 4. \tag{3.10}$$

It was pointed out by AHRENS, JOSEPH, RENARDY & RENARDY [2] that this instability is associated with a change of type. If we consider the symbol of the differential operator, i.e. if we formally set $\frac{\partial}{\partial t} = \sigma$, $\frac{\partial}{\partial x_1} = i\alpha$, $\frac{\partial}{\partial x_2} = i\beta$, then the left-hand side of (3.9) becomes

$$\varrho(\sigma + \varkappa x_2 i\alpha)(-\alpha^2 - \beta^2), \tag{3.11}$$

and the right-hand side becomes

$$-\left(N_1' - \frac{N_1}{\varkappa}\right)(\alpha^2 - \beta^2)\,\alpha\beta + \tau'(\alpha^2 - \beta^2)^2 + \frac{4\tau}{\varkappa}\alpha^2\beta^2. \tag{3.12}$$

This homogeneous polynomials of is fourth degree positive definite for $\mathrm{We} < 4$, but indefinite for $\mathrm{We} > 4$. For $\mathrm{We} > 4$, one thus expects short-wave instabilities of a catastrophic nature, *i.e.* $\mathrm{Re}\,\sigma$ becomes arbitrarily large as the wave length tends to zero. This type of instability seems to occur in some types of melt fracture (see [2] for a more complete discussion).

4. Quasilinear Systems

The analysis of the equations of viscoelastic flow will be framed in terms of systems of equations of first order. We consider linear systems and quasilinear systems. We write the quasilinear system as

$$\sum_{l=0}^{n} A_l \frac{\partial \boldsymbol{u}}{\partial x_l} = f, \qquad \boldsymbol{x} = (t, x_1, x_2, \ldots, x_n), \tag{4.1}$$

where $\boldsymbol{u} = (u_1, u_2, \ldots, u_k)$ is a k vector and A_l are $k \times k$ matrices which like f may depend on x_l and on the components of $\boldsymbol{u}$. If A_l is independent of $\boldsymbol{u}$, and $f = \boldsymbol{B} \cdot \boldsymbol{u}$, then (3.7) is a linear system.

The following definitions apply to both linear and quasilinear systems. A surface S defined by the equation $\phi(t, x_1, \ldots, x_n) = 0$, is characteristic with respect to (4.1) at $\boldsymbol{x} = (t, x_1, \ldots, x_n)$ if

$$\det\left(\sum_{l=0}^{n} A_l \frac{\partial \phi}{\partial x_l}\right)(\boldsymbol{x}) = 0. \tag{4.2}$$

If $\phi = x_n - f(x_0, \ldots, x_{n-1})$, then

$$\det\left(A_n - \sum_{l=0}^{n-1} A_l \frac{\partial f}{\partial x_l}\right) = 0. \tag{4.3}$$

Any one of the $n+1$ quantities $\partial\phi/\partial x_l$ in (4.2) may be regarded as an eigenvalue. We shall say (4.1) is hyperbolic if $\boldsymbol{A} = \boldsymbol{A}_\mu$ is non-singular and for any choice of the real parameters $(\lambda_l, l = 0, 1, \ldots, n; l \neq \mu)$, the roots α of

$$\det\left(\alpha \boldsymbol{A} - \sum_{\substack{l=0 \\ l \neq \mu}}^{n} \lambda_l \boldsymbol{A}_l\right) = 0 \tag{4.4}$$

are real and are associated with k linearly independent characteristic vectors $\boldsymbol{v}$:

$$\alpha \boldsymbol{A} \boldsymbol{v} = \sum_{\substack{l=0 \\ l \neq \mu}}^{n} \lambda_l \boldsymbol{A}_l \boldsymbol{v}. \tag{4.5}$$

First-order systems can be of *mixed type* with real and complex eigenvalues, neither totally elliptic or totally hyperbolic.

We are interested in two-dimensional quasilinear problems of the form

$$\boldsymbol{A} \frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{B} \frac{\partial \boldsymbol{u}}{\partial x} + \boldsymbol{C} \frac{\partial \boldsymbol{u}}{\partial y} = \boldsymbol{f}. \tag{4.6}$$

We consider one-dimensional evolutionary problems in which $\boldsymbol{C} = 0$ and steady problems in which $\boldsymbol{A} = 0$. For evolutionary problems we suppose that $\boldsymbol{A}$ is not singular and write

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{B} \frac{\partial \boldsymbol{u}}{\partial x} = \boldsymbol{f}. \tag{4.7}$$

The characteristic surface is $\phi(x, t) = 0$ and (4.2) becomes

$$\det\left(\frac{\partial \phi}{\partial t} \mathbf{1} + \boldsymbol{B} \frac{\partial \phi}{\partial x}\right) = 0. \tag{4.8}$$

On $\phi(x, t) = 0$, we have

$$d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial t} dt = 0.$$

Hence, (4.8) may be written

$$\det\left(\boldsymbol{B} - \frac{dx}{dt}\mathbf{1}\right) = 0, \tag{4.9}$$

where dx/dt is the slope of the characteristic.

A linear system of the form (4.7) is said to be of *evolutionary* type if $\boldsymbol{B}$ has only real eigenvalues, *hyperbolic* if of evolutionary type and if $\boldsymbol{B}$ can be made diagonal, *strictly hyperbolic* if $\boldsymbol{B}$ has simple real eigenvalues.

If the Cauchy problem for (4.7) with $\boldsymbol{f} = 0$ is well-posed, then (4.7) must be of evolutionary type. Solutions of the form $\boldsymbol{u}(x, t) = \boldsymbol{B}e^{i(\lambda t + \mu x)}$ are bounded for large $|t|$ if and only if the eigenvalues $-\lambda/\mu$ of $\boldsymbol{B}$ are real (GELFAND [14]).

Suppose $\boldsymbol{u}(x, t)$ is given on a curve $\phi(x, t) = 0$. If this line is characteristic, then the equation

$$\frac{d\boldsymbol{u}}{ds} = \frac{\partial \boldsymbol{u}}{\partial t}\frac{dt}{ds} + \frac{\partial \boldsymbol{u}}{\partial x}\frac{dx}{ds},$$

where $t(s)$, $x(s)$ is a parametric representation for the curve $\phi = 0$, and the quasilinear equation (4.7) cannot be uniquely solved for the $2k$ derivatives $\partial \boldsymbol{u}/\partial t$ and $\partial \boldsymbol{u}/\partial x$. This special condition requires that the determinant of the coefficients of the derivatives vanish

$$\det\begin{bmatrix} \mathbf{1} & \boldsymbol{B} \\ \mathbf{1}\, dt & \mathbf{1}\, dx \end{bmatrix} = \det\left[\mathbf{1}\frac{dx}{ds} - \boldsymbol{B}\frac{dt}{ds}\right] = 0.$$

The same considerations apply for the quasilinear steady problem

$$\boldsymbol{B}\frac{\partial \boldsymbol{u}}{\partial x} + \boldsymbol{C}\frac{\partial \boldsymbol{u}}{\partial y} = \boldsymbol{f}.$$

Such problems are frequently associated with a change of type, like transonic flow, in which some regions of flow are subcritical and some supercritical. A typical example is Tricomi's equation. Other, more applicable examples are derived in §§ 8–11.

It is not always possible to assign a definite type to a system of quasilinear equations. There can be both real and complex eigenvalues. Nonlinear problems of mixed type have not been thoroughly studied by mathematicians. Some special results have been given by MOCK [30]. Here it is perhaps useful to give some simple examples from hydrodynamics.

Consider first the Euler equations for the flow of inviscid, incompressible fluids in two dimensions

$$\varrho(\boldsymbol{u} \cdot \nabla)\, \boldsymbol{u} + \nabla p = \boldsymbol{f}, \quad \operatorname{div} \boldsymbol{u} = 0. \tag{4.10}$$

Let (u, v) be the components of $\boldsymbol{u}$ with respect to x and y. Then we can write (4.10) as

$$A_1 q_x + A_2 q_y = f, \tag{4.11}$$

where

$$q = (u, v, p), \quad q_x = \frac{\partial q}{\partial x}, \quad q_y = \frac{\partial q}{\partial y},$$

$$f = (f_1, f_2, 0),$$

$$A_1 = \begin{bmatrix} u & 0 & 1/\varrho \\ 0 & u & 0 \\ \varrho & 0 & 0 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} v & 0 & 0 \\ 0 & v & 1/\varrho \\ 0 & \varrho & 0 \end{bmatrix}.$$

The characteristic equation for (4.11) is

$$\det\left[\frac{dy}{dx}A_1 - A_2\right] = \left(v - \frac{dy}{dx}u\right)\left(\left(\frac{dy}{dx}\right)^2 + 1\right) = 0.$$

Hence

$$\frac{dy}{dx} = \frac{v}{u}, \quad \text{streamlines are characteristic, and}$$

$$\frac{dy}{dx} = \pm i.$$

The presence of imaginary roots means that (4.11) is not hyperbolic. It is not elliptic beause the determinant of the matrix symbol of (4.11)

$$\det\begin{bmatrix} u\xi_1 + v\xi_2 & 0 & \frac{1}{\varrho}\xi_1 \\ 0 & u\xi_1 + v\xi_2 & \frac{1}{\varrho}\xi_2 \\ \varrho\xi_1 & \varrho\xi_2 & 0 \end{bmatrix} = -(u\xi_1 + v\xi_2)(\xi_1^2 + \xi_2^2)$$

vanishes for $u\xi_1 + v\xi_2 = 0$.

Another example is from the theory of irrotational water waves. In this case the velocity potential is elliptic but the height function is governed by a hyperbolic equation giving rise to water waves.

5. Constitutive Equations

A constitutive equation relates stress to deformation. The stress in viscoelastic fluids depends on the history of the deformation. Usually the history is defined on some strain measure. The stress in Newtonian fluids depends on the instantaneous value of the velocity gradient, not on the prior history of the deformation. Viscoelastic fluids have instantaneous elasticity. Elasticity is present also in inviscid compressible fluids. For unsteady problems elasticity is associated with hyperbolic, rather than parabolic response. It is necessary to be more precise about the difference between elastic and viscous responses.

Many constitutive models have been proposed. Each one leads to different answers for the same problem though some groups have similar qualitative properties. In problems of changing type the linearized part is of primary importance. The linear part may be of three types:

1) Constitutive equations with some viscosity. The viscosity which we have in mind is that which rheologists sometimes associate with a retardation time.

2) Constitutive equations without "viscosity". Constitutive equations of integral type with smooth kernels, and various types of rate equations in the class called Maxwell models are of this type. These kind of equations allow propagation of rather than smoothing of discontinuities. In some nonlinear models [16], [28], [43], [44], [51] discontinuities may arise, as do shocks in gas dynamics, from smooth data.

3) Constitutive equations of integral type with singular kernels. These are in a sense intermediate between 1) and 2). Depending on the type of the singularity, the wave speeds may be finite or infinite. However, even if they are finite, *i.e.* real characteristics exist, there is no propagation of discontinuities (see Chapter 6).

The stress in an incompressible fluid is given by

$$T = -p\mathbf{1} + \boldsymbol{\tau}, \tag{5.1}$$

where $\boldsymbol{\tau}$, the determinate stress [50], p. 176 (sometimes called the extra stress [48]), may be related to the deformation, whilst p, the reaction pressure, is determined only through the equations of motion. An example of $\boldsymbol{\tau}$ in the class 1) of constitutive equation with some viscosity is the Jeffreys model with two time constants, a relaxation time λ_1 and a retardation time λ_2. This model may be written in rate form

$$\boldsymbol{\tau} + \lambda_1 \frac{\partial \boldsymbol{\tau}}{\partial t} = \eta A + \eta\lambda_2 \frac{\partial A}{\partial t}, \tag{5.2}$$

where η is a constant, called the zero shear-rate viscosity, $A = \nabla u + \nabla u^T$, $u = u(x, t)$, or in integral form

$$\boldsymbol{\tau} = \frac{\eta\lambda_2}{\lambda_1} A[u(x, t)] + \frac{\eta}{\lambda_1}\left(1 - \frac{\lambda_2}{\lambda_1}\right) \int_{-\infty}^{t} A[u(x, \tau)] \exp\left(\frac{-(t-\tau)}{\lambda_1}\right) d\tau. \tag{5.3}$$

The constant $\eta\lambda_2/\lambda_1$ is a second viscosity which is equal to the zero shear-rate viscosity when $\lambda_2 = \lambda_1$. It is this viscosity that we have in mind when we talk about "with" or "without" viscosity. In § 6 we show that (5.2) and (5.3) enjoy a certain general status when they are regarded as holding only in motions which perturb a state of rest.

The Maxwell model arises from (5.2) and (5.3) when λ_2 is put equal to zero. In § 6 we note that the Maxwell model permits propagation of *waves* with a finite velocity of propagation, but such propagation cannot occur for Jeffreys models; more precisely the viscosity term $\lambda_2\eta/\lambda_1$ smooths discontinuities in the same way that viscosity smooths the discontinuities of solutions of Euler's equations for an inviscid fluid.

Nonlinear models can be classified according to the type of their linearization. Popular models of the rate type include those due to OLDROYD [34], LEONOV [25], GIESEKUS [15]. These models generalize both Maxwell and Jeffreys type fluids, *i.e.* some have "viscosity", some do not. Popular models of integral type include K-BKZ single integral models [4], [22] which may be of type 2) or 3) depending on the nature of the kernel, and the model of CURTISS & BIRD [11], which contains a viscosity term.

We shall do some work with rate equations of Oldroyd type depending upon three constants:

$$\lambda \frac{D\boldsymbol{\tau}}{Dt} + \boldsymbol{\tau} = \eta \boldsymbol{A}. \tag{5.4}$$

Here D/Dt is an invariant time derivative

$$\frac{D\boldsymbol{\tau}}{Dt} = \frac{\partial \boldsymbol{\tau}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\,\boldsymbol{\tau} + \boldsymbol{\tau}\boldsymbol{\Omega} - \boldsymbol{\Omega}\boldsymbol{\tau} - a(\boldsymbol{D}\boldsymbol{\tau} + \boldsymbol{\tau}\boldsymbol{D}), \tag{5.5}$$

where $-1 \leqq a \leqq 1$, $\boldsymbol{D} = \frac{1}{2}(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T)$, $\boldsymbol{\Omega} = \frac{1}{2}(\nabla \boldsymbol{u} - \nabla \boldsymbol{u}^T)$. The upper convected Maxwell model has $a = 1$ and

$$\frac{D\boldsymbol{\tau}}{Dt} = \frac{\partial \boldsymbol{\tau}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\,\boldsymbol{\tau} - \nabla \boldsymbol{u}\boldsymbol{\tau} - \boldsymbol{\tau}\,\nabla \boldsymbol{u}^T, \tag{5.6}$$

where $(\nabla \boldsymbol{u})_{ij} = \partial u_i/\partial x_j$. The lower convected Maxwell model has $a = -1$ and

$$\frac{D\boldsymbol{\tau}}{Dt} = \frac{\partial \boldsymbol{\tau}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\,\boldsymbol{\tau} + \boldsymbol{\tau}\,\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T\boldsymbol{\tau}. \tag{5.7}$$

The corotational Maxwell model has $a = 0$. The integral model

$$\boldsymbol{\tau} = \frac{\eta}{\lambda^2} \int_{-\infty}^{t} \exp\,[-(t-\tau)/\lambda]\,[\boldsymbol{C}_t^{-1}(\tau) - \boldsymbol{1}]\,d\tau \tag{5.8}$$

is an alternative form of the upper convected Maxwell model ($a = 1$). The rate form (5.4, 6) may be obtained by differentiating (5.8) partially with respect to t, holding $\boldsymbol{x}$ fixed. The expression

$$\boldsymbol{\tau} = \frac{\eta}{\lambda^2} \int_{-\infty}^{t} \exp\,[-(t-\tau)/\lambda]\,[\boldsymbol{1} - \boldsymbol{C}_t(\tau)]\,d\tau \tag{5.9}$$

is equivalent to (5.4, 7) with $a = -1$, in the same way.

RUTKEVICH [40], [41] studies differential constitutive models (5.4) of Oldroyd type. He linearizes these equations and the equations of motion at a state of no motion ($\boldsymbol{u} = 0$) and constant stress. He finds that a change of type leading to imaginary wave speeds and a instability of Hadamard type occur if the principal values of $\boldsymbol{\tau}$ satisfy certain inequalities. If we denote these principal values by $\tau_1 \geqq \tau_2 \geqq \tau_3$, then RUTKEVICH's instability criterion reads as follows:

(5.10)
$$\tau_3 < -\frac{\eta}{\lambda}, \qquad a = 1,$$
$$\tau_1 > \frac{\eta}{\lambda}, \qquad a = -1,$$
$$\tau_1 - \tau_3 > \frac{2\eta}{\lambda}, \qquad a = 0.$$

In order to assess the significance of these results, it is necessary to look also at the integral models (5.8) and (5.9) corresponding to (5.4). Since $\boldsymbol{C}_t^{-1}$ and $\boldsymbol{C}_t$ are positive definite matrices, we find that $\tau_3 > -\frac{\eta}{\lambda}$ for $a = 1$ and $\tau_1 < \frac{\eta}{\lambda}$ for a -1. Thus Rutkevich's instability criterion cannot be achieved on solutions and must be considered unphysical. This was also noticed by CROCHET [10] who proved directly from the differential equation that $\tau_3 > -\frac{\eta}{\lambda}$ or, respectively, $\tau_1 < \frac{\eta}{\lambda}$ for all time, if this is the case initially.

M. RENARDY [38, 39] did two independent studies, related in part to our joint work. He has shown that the upper and lower convected Maxwell models are always evolutionary. These two models are special cases of the Kaye-BKZ model [22], [4], which has the form

(5.11)
$$\boldsymbol{\tau} = \int_{-\infty}^{t} a(t-s)\left(\frac{\partial W}{\partial I_1}\boldsymbol{C}_t^{-1}(s) - \frac{\partial W}{\partial I_2}\boldsymbol{C}_t(s)\right) ds.$$

Here a is a positive kernel and the "strain energy" W is a scalar function of $I_1 = \operatorname{tr} \boldsymbol{C}_t^{-1}(s)$ and $I_2 = \operatorname{tr} \boldsymbol{C}_t(s)$. The initial value problem is always well posed, *i.e.* instability of Hadamard type cannot occur, if W satisfies a strong ellipticity condition of the same form as in nonlinear elasticity. This condition is satisfied if W is monotone in both arguments and a convex function of $\sqrt{I_1}$ and $\sqrt{I_2}$. This obviously includes the cases $W = I_1$, $W = I_2$, corresponding to (5.8), (5.9). No such objections can be raised in the corotational case, $a = 0$. In fact, we need only restrict attention to motions where $\boldsymbol{\Omega} = 0$, and $\boldsymbol{\tau}$ and $\boldsymbol{D}$ are spatially homogeneous, to see that (5.4) allows $\tau_1 - \tau_3$ to reach any value. The instability in this case is therefore genuine.

It is instructive to look at special flow geometries. For time-dependent simple shear-flow, we have [21]

$$\tau = \int_{-\infty}^{t} \frac{\eta}{\lambda} e^{-(t-s)/\lambda} \varkappa(s) \cos\left(\sqrt{1-a^2}\int_{s}^{t} \varkappa(t')\,dt'\right) ds.$$

Differentiating this with respect to t, we find

$$\tau = \varkappa(t)\left[\frac{\eta}{\lambda} - \int_{-\infty}^{t} \frac{\eta}{\lambda} e^{-(t-s)/\lambda} \varkappa(s) \sqrt{1-a^2} \sin\left(\sqrt{1-a^2} \int_{s}^{t} \varkappa(t')\, dt'\right) ds\right]$$

$$+ \text{ terms of lower order.}$$

A change of type occurs when the expression in brackets changes sign. For $|a| < 1$, it is possible to construct histories for the shear rate $\varkappa(s)$ such that this is the case. For steady shear flows, $\varkappa \equiv \varkappa_0$, the expression in brackets is positive and there is no change of type, even though the shear stress-shear rate law for $a \neq \pm 1$ is not monotone. The case of simple elongation was considered by RENARDY [38]. He finds that a change of type can occur for $-1 < a < \frac{1}{2}$. Although RUTKEVICH's instability for $a = \pm 1$ is unphysical it is nevertheless relevant for numerical calculations (see CROCHET [10]). The reason is that at high Weissenberg numbers the eigenvalues of $\boldsymbol{\tau}$ can be arbitrarily close to the stability boundary $\pm\eta/\lambda$. Numerical errors can push them beyond this limit, with disastrous results.

Most of the constitutive equations which have been proposed, all the ones considered here, are simple fluids in which the stress is determined by the history of the relative gradient of the deformation

$$\boldsymbol{F}_t(\boldsymbol{x}, t) = \nabla \boldsymbol{\chi}_t(\boldsymbol{x}, t),$$

where

$$\boldsymbol{u}(\boldsymbol{\chi}_t(\boldsymbol{x}, \tau), \tau) = \partial \boldsymbol{\chi}_t / \partial \tau$$

is the velocity of the particle $\boldsymbol{x} = \boldsymbol{\chi}_t(\boldsymbol{x}, t)$ at times $\tau \leqq t$. The constitutive equation satisfying material frame-indifference may be expressed by a functional [48, p. 80]

$$\boldsymbol{\tau}(\boldsymbol{x}, t) = \underset{\tau=-\infty}{\overset{t}{F}} [\boldsymbol{C}_t(\boldsymbol{x}, \tau)] \tag{5.12}$$

on the history of the relative Cauchy strain

$$\boldsymbol{C}_t(\boldsymbol{x}, \tau) = \boldsymbol{F}_t^T \boldsymbol{F}_t.$$

In order to give a precise meaning to an expression such as (5.12), it is necessary to specify the set of arguments on which F is defined, called the domain of F or dom F. SAUT & JOSEPH [42], extending ideas of COLEMAN & NOLL [9], have proposed a classification of constitutive laws according to the choice of this domain. Roughly, this associates the nature of the stress-strain relation with the deformations which are allowed. In the linearized case the stresses which are allowed are functionals in the topological dual of dom F. The smaller dom F is, the larger is the dual, *i.e.* the more constitutive laws are allowed. To be more precise, let $\boldsymbol{C}_t^0(\tau)$ be some given history and $\boldsymbol{c}(\tau) = \boldsymbol{c}(\boldsymbol{x}, t-\tau)$ be a small perturbation. The linearization of $\boldsymbol{F}$ relative to $\boldsymbol{C}_t^0(\tau)$ can formally be written as an integral

$$\boldsymbol{F}_1\left[\underset{\tau=-\infty}{\overset{t}{\boldsymbol{C}_t^0(\tau)}} \,|\, \boldsymbol{c}(\tau)\right] = \int_{-\infty}^{t} \mathbf{K}\left(\underset{\tau'=-\infty}{\overset{t}{\boldsymbol{C}_t^0(\tau')}}, t-\tau\right) \boldsymbol{c}(\tau)\, d\tau, \tag{5.13}$$

where **K** is a function of $t-\tau$ and functional of the history of C_t^0 whose values are tensors of fourth order. The class of admissible kernels **K** depends on which deformations $c(\tau)$ are allowed. If, following COLEMAN & NOLL, c is restricted to a weighted L^2-space, then **K** must also be in a weighted L^2-space. If c is restricted to a Sobolev space, then Dirac measures and derivatives of Dirac measures can be included in **K**, thus admitting Jeffreys type models.

To see the effects of hyperbolicity clearly we exclude these cases and adopt (5.13) with a smooth kernel as the basis for our study of change of type.

It will be useful in what follows to specify the quantities in (5.13) more precisely. Let

$$\chi_t(\boldsymbol{x},\tau)=\xi^0(\boldsymbol{x},\tau)+\xi(\boldsymbol{x},\tau) \tag{5.14}$$

be the particle path for

$$\boldsymbol{x}=\chi(\boldsymbol{x},t)=\xi^0(\boldsymbol{x},t),$$

where ξ^0 is the position of $\boldsymbol{x}$ in some given motion and $\xi(\boldsymbol{x},\tau)$ is a perturbation with $\xi(\boldsymbol{x},t)=0$. Quadratic and terms of higher-order are neglected. To this order of approximation

$$C_t(\boldsymbol{x},\tau)=C_t^0(\boldsymbol{x},\tau)+c(\boldsymbol{x},\tau), \tag{5.15}$$

where

$$\begin{aligned} C_t^0(x,\tau)&=(\nabla\xi^0)^T\nabla\xi^0, \qquad C_t^0(x,t)=\mathbf{1},\\ c(x,\tau)&=(\nabla\xi^0)^T\nabla\xi+(\nabla\xi)^T\nabla\xi^0, \qquad c(x,t)=0. \end{aligned} \tag{5.16}$$

Let the perturbed extra stress be denoted by $\boldsymbol{\tau}$ and the unperturbed extra stress by $\boldsymbol{\tau}_0$. The total extra stress is $\boldsymbol{\tau}_0+\boldsymbol{\tau}$, where

$$\begin{aligned} \boldsymbol{\tau}_0&=\mathop{F}_{\tau=-\infty}^{t}\,[C_t^0(x,\tau)],\\ \boldsymbol{\tau}&=\int_{-\infty}^{t}\mathbf{K}\,[\mathop{C_t^0(x,\tau')}_{\tau'=-\infty}^{t},\,t-\tau]\,c(x,\tau)\,d\tau. \end{aligned} \tag{5.17}$$

We assume that **K** is a smooth function of τ and find that

$$\left(\frac{\partial}{\partial t}+\boldsymbol{u}^0\cdot\nabla\right)\boldsymbol{\tau}=\int_{-\infty}^{t}\mathbf{K}\left(\frac{\partial}{\partial t}+\boldsymbol{u}^0\cdot\nabla\right)c(x,\tau)\,d\tau \tag{5.18}$$

$$+\ \text{terms of lower differential order},$$

in which $\boldsymbol{u}^0$ pertains to the unperturbed motion. The position at time τ of any given particle is independent of the reference time t.

$$\frac{d\chi_t}{dt}=\frac{\partial\chi_t(x,\tau)}{\partial t}+\boldsymbol{u}(x,t)\cdot\nabla_x\chi_t(x,\tau)=0. \tag{5.19}$$

Hence

$$\frac{\partial \xi^0}{\partial t} + u^0(x, t) \cdot \nabla_x \xi^0 = 0, \tag{5.20}$$

$$\frac{\partial \xi}{\partial t} + u^0(x, t) \cdot \nabla_x \xi + v(x, t) \cdot \nabla_x \xi^0 = 0.$$

This yields

$$\begin{aligned}\left(\frac{\partial}{\partial t} + (u^0 \cdot \nabla)\right) c &= \left(\frac{\partial}{\partial t} + (u^0 \cdot \nabla)\right) (\nabla \xi^{0T} \nabla \xi + \nabla \xi^T \nabla \xi^0) \\ &= \nabla \xi^{0T} \nabla \left\{\left(\frac{\partial}{\partial t} + (u^0 \cdot \nabla)\right) \xi\right\} + \left(\nabla \left\{\left(\frac{\partial}{\partial t} + u^0 \cdot \nabla\right) \xi\right\}\right)^T \nabla \xi^0 \\ &\quad + \text{terms of lower order} \\ &= -\nabla \xi^{0T} \cdot \nabla(v \cdot \nabla \xi^0) - (\nabla(v \cdot \nabla \xi^0))^T \nabla \xi^0 \\ &\quad + \text{terms of lower order} \\ &= -(\nabla \xi^0)^T \nabla v \, \nabla \xi^0 + \text{transpose} + \text{terms of lower order}.\end{aligned}$$

By inserting this in (5.18), we find that the principal part of the constitutive equation is given by

$$\left[\frac{\partial}{\partial t} + u_0 \cdot \nabla\right] \tau = \mathbf{M}(x, t) \nabla v(x, t), \tag{5.21}$$

where

$$M_{ijmn} = - \int_{-\infty}^{t} K_{ijkl} \frac{\partial C_m^0}{\partial x_k} \frac{\partial C_n^0}{\partial x_l} d\tau.$$

Equations like (5.21) can also be derived in general, without linearization (§ 12).

6. Slip Surface Propagation in Problems Perturbing Rest

In problems which perturb rest we have $\xi^0 = x$, $\tau^0 = 0$ and

$$\tau = \int_{-\infty}^{t} G(t - \tau) A_1[u(x, \tau)] \, d\tau. \tag{6.1}$$

A Newtonian part of the stress arises when

$$G(s) = \mu \, \delta(s) + g(s),$$

where $\delta(s)$ is a Dirac measure and $g(s)$ is a smooth kernel. Thus

$$\tau = \mu A_1([u(x, t)] + \int_{-\infty}^{t} g(t - \tau) A_1[u(x, \tau)] \, d\tau. \tag{6.2}$$

In the special case when $g(s)$ is in exponential form (6.2) reduces to the model (5.3) of Jeffreys with $\mu = \eta\lambda_2/\lambda_1$. We introduce μ here and elsewhere only to notice that if μ is small the underlying dynamics is close to $\mu = 0$ dynamics. When $\mu = 0$, the dynamics is governed by

$$\frac{\partial \boldsymbol{\tau}}{\partial t} = G(0)\, A_1[\boldsymbol{u}(\boldsymbol{x}, t)] + \int_{-\infty}^{t} G'(t - \tau)\, A_1[\boldsymbol{u}(\boldsymbol{x}, \tau)]\, d\tau, \tag{6.3}$$

and

$$\varrho \frac{\partial \boldsymbol{u}}{\partial t} + \nabla p - \operatorname{div} \boldsymbol{\tau} = 0, \tag{$(6.3)_2$}$$

where

$$\operatorname{div} \boldsymbol{u} = 0. \tag{$(6.3)_3$}$$

Equations (6.3) are a first-order system, linear in the derivatives of p and the components of $\boldsymbol{u}$ and $\boldsymbol{\tau}$. If $G(t - \tau) = \text{const} \exp\{-(t - \tau)/\lambda\}$, then the last term in $(6.3)_1$ reduces to $-\boldsymbol{\tau}/\lambda$. In the analysis of characteristics using the Maxwell model the term $\boldsymbol{\tau}/\lambda$ is of lower order and it does not enter into the analysis of characteristics. In general, for smooth G', in (6.3) the integral is of order -1 in t and $+1$ in x, hence of order 0 as an operator in x and t, and is thus also of lower order. This shows that we can analyze first-order systems for general kernels; *we do not need special models.*

The case of one-dimensional shearing motion can be used to discuss the effects of viscosity, wave speed, wave amplitude and classify the kernels. In this case (6.3) reduces to

$$\begin{gathered} \frac{\partial \tau}{\partial t} = G(0) \frac{\partial u}{\partial x} + \int_{-\infty}^{t} G'(t - \tau) \frac{\partial u(x, \tau)}{\partial x}\, d\tau, \\ \varrho \frac{\partial u}{\partial t} = \frac{\partial \tau}{\partial x}. \end{gathered} \tag{6.4}$$

We may write (6.4) as

$$\frac{\partial \boldsymbol{q}}{\partial t} + \boldsymbol{B} \frac{\partial \boldsymbol{q}}{\partial x} + \boldsymbol{f} = 0, \tag{6.5}$$

where

$$\boldsymbol{q} = \begin{bmatrix} u \\ \tau \end{bmatrix},$$

$$\boldsymbol{B} = -\begin{bmatrix} 0 & 1/\varrho \\ G(0) & 0 \end{bmatrix},$$

$$\boldsymbol{f} = \begin{pmatrix} 0 \\ \int_{-\infty}^{t} G'(t - \tau) \frac{\partial u(x, \tau)}{\partial x}\, d\tau \end{pmatrix}.$$

The eigenvalues

$$C = \pm \sqrt{G(0)/\varrho} \tag{6.6}$$

of $\boldsymbol{B}$ are the wave speeds along the characteristics

$$x \pm ct = \text{const.}$$

It is instructive to review the results of analysis of the following problem, known as Stokes' or Rayleigh's problem, for a viscoelastic fluid. A fluid occupying a half space is at rest for $t \leqq 0$. At times $t \geqq 0$, the boundary of the fluid at $x = 0$ is made to move forward with a constant speed. The problem may be described by the following equations

$$\varrho \frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial x^2} + \int_{-\infty}^{t} g(t-\tau) \frac{\partial^2 u(x, \tau)}{\partial x^2} d\tau,$$

$$u(x, \tau) = 0, \qquad \tau \leqq 0, \tag{6.7}$$

$u(x, t)$ is bounded for positive values of x and t,

$$u(0, t) = H(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}.$$

This problem has been studied by NARAIN & JOSEPH [33] when $\mu \neq 0$ and $\mu = 0$ and by RENARDY [37] for $\mu = 0$ and singular kernels. Other authors, TANNER [46], STRAUB [45], BÖHME [5], KAZAKIA & RIVLIN [24] have studied the problem for special constitutive models. TANNER used an Oldroyd model B, which contains a contribution due to Newtonian viscosity. This model does not require linearization; it automatically leads to a linear problem. All authors solve the problem using a Laplace transform with respect to the time.

In the context of the present paper, the most interesting issue is the qualitative behavior of solutions in relation to properties of the kernel g. In particular, wave speeds and the presence or absence of discontinuities are of interest. These questions were addressed in [33], [37]. Not surprisingly, the crucial factor is the asymptotic behavior of the Laplace transform of g at infinity or in other words, the symbol of the operator in (6.7). The qualitative nature of solutions is thus determined by the type of the equation. With $\hat{u}(x, \omega)$ denoting the Laplace transform of $u(x, t)$, equation (6.7) becomes

$$\omega \varrho^2 \hat{u} = \omega \hat{G}(\omega)\, \hat{u}_{xx}, \qquad \hat{u}(x = 0) = \frac{1}{\omega}, \tag{6.8}$$

where $\hat{G}(\omega)$ is the Laplace transform of G:

$$\hat{G}(\omega) = \int_0^\infty G(t)\, e^{-\omega t}\, dt.$$

One finds that

$$\hat{G}(\omega) = \frac{G(0)}{\omega} + \frac{G'(0)}{\omega^2} + o\left(\frac{1}{\omega^2}\right),$$

if G'' is integrable. Since the character of solutions is determined by the symbol, we should look at the problem

$$\varrho \omega^2 \hat{u} = G(0)\, \hat{u}_{xx} + \frac{1}{\omega} G'(0)\, \hat{u}_{xx}.$$

Since

$$\frac{1}{G(0)+\frac{1}{\omega}G'(0)}=\frac{1}{G(0)}-\frac{1}{\omega}\frac{G'(0)}{G(0)^2},$$

an equivalent statement is

$$\frac{\varrho\omega^2}{G(0)}\hat{u}-\frac{\varrho\omega G'(0)}{G(0)^2}\hat{u}=\hat{u}_{xx}.$$

This corresponds to the equation

$$\frac{\varrho}{G(0)}u_{tt}-\frac{\varrho G'(0)}{G(0)^2}u_t=u_{xx}.$$

Hence the singularity may be expected to propagate at the wave speed $c=\sqrt{G(0)/\varrho}$ and to decay with an amplitude factor $\exp(tG'(0)/2G(0))$.

COLEMAN & GURTIN [6], [7] proved that if the acceleration was discontinuous, then its wave speed and amplitude had to be given by these expressions. In [33] and [37] it is shown that these expressions for the speed and amplitude could be derived for the problem (5.7) without assuming discontinuities of acceleration. The demonstration of propagation in [33] applies to propagating steps in displacement (propagating delta functions) as well as steps in velocity.

If a Newtonian term is included in the constitutive law, G contains a contribution $\mu\,\delta(0)$, and $\hat{G}(\omega)$ at ∞ behaves like $O(1)$. The equation becomes parabolic, and smooth (analytic) solutions are obtained. As $\mu\to 0$, a boundary layer forms around the shock front. This was shown numerically in [46]. An analysis of this boundary layer is given by NARAIN & JOSEPH [33].

RENARDY [37] studied the case where G does not contain a δ-function, but some weaker singularity. Specifically, he considers the kernels

$$-G'(t)=\sum_{n=1}^{\infty}e^{-n^{\alpha}t},\quad \alpha>\tfrac{1}{2}. \tag{6.9}$$

For $\alpha>1$, $G(0)$ is finite, but $G'(0)$ is not. The asymptotic behavior of $\hat{G}(\omega)$ is given by

$$\hat{G}(\omega)=\frac{G(0)}{\omega}+O(\omega^{-2+1/\alpha}). \tag{6.10}$$

As one expects, there is a finite wave speed $c=\sqrt{G(0)/\varrho}$. Solutions are zero in front of the wave and not zero behind it. Across the wave, however, they are of class C^∞. If $\alpha<1$, $G(0)$ is infinite, and the asymptotic behaviour of $\hat{G}(\omega)$ is dominated by a term $O(\omega^{1/\alpha-2})$ or $O(\ln\omega/\omega)$ in the limiting case $\alpha=1$. The wave speed is infinite and the solution becomes analytic everywhere except at $t=0$.

Similar studies can be done for small perturbations of rigid motions; see the work of KAZAKIA [23].

7. Slip Surface Propagation in Nonlinear Shearing Problems

COLEMAN & GURTIN [6], [7] studied simple shear flows of a viscoelastic liquid, which involve a surface across which the acceleration is discontinuous. They showed that in a material with a smooth memory function (no Newtonian viscosity), acceleration discontinuities propagate at the speed $c = \sqrt{G(0)/\varrho}$. If the amplitude is smaller than a critical amplitude, it decays with a factor $\exp(tG'(0)/2G(0))$, if the amplitude is larger than the critical amplitude, it will reach infinity in a finite time ("blow-up"). This blow-up of acceleration waves might be interpreted as development of a slip surface for the velocity. No such result has been proven.

SLEMROD [44], MALEK-MADANI & NOHEL [28], GRIPENBERG [51] and HATTORI [16] have studied simple nonlinear models. They show by contradiction using Riemann invariants and the method of characteristics that, for suitably chosen initial data, a global C^1-solution for the equation of motion cannot exist. This may mean the formation of a slip surface for the velocity [28]. Numerical evidence for the development of such discontinuities was found by MARKOVICH & RENARDY [29].

SLEMROD [43], following a suggestion of COLEMAN & GURTIN [7], uses his result to explain some of the various instabilities of shear flows collectively called melt fracture. Melt fracture is an instability of flow of molten polymers or polymeric solutions down capillaries. In the experiments (TORDELLA [47]), the polymer is forced down the capillary by high pressure. Extrudates leaving the capillary which at lower shear rates are smooth and continuous, become rough (shark skin effect), irregular and ultimately disintegrate. There are different explanations of the different types of instability which can occur. These are reviewed in the paper of PETRIE & DENN [35]. None of the explanations can be regarded as established. The mechanism proposed by SLEMROD has some possibilities when fracture is associated with a stick-slip phenomenon. There is some controversy about the presence or absence of slip in experiments. If it does occur periodically, as it might in SLEMROD'S theory, it would be a candidate for the explanation of the wavy surfaces shown in the pictures of TORDELLA [47]. It should be noted that the theory of HUNTER & SLEMROD [17] requires an entirely different type of shear stress to explain the type of hysteretic melt fracture which TORDELLA calls ripple.

It would be interesting to have the conditions under which slip surfaces for the velocity might develop from weaker slip surfaces or from smooth data.

8. Classification of Equations for Flows Perturbing Uniform Motion

ULTMAN & DENN [49] consider the equations for two-dimensional steady flow of an upper convected Maxwell fluid. They linearize at a motion with uniform velocity and zero stress, and they show that these linearized equations change type when a viscoelastic "Mach" number

$$M = \frac{U}{c} \tag{8.1}$$

exceeds one. Here U is the velocity of the unperturbed uniform flow, and c is the wave speed for propagation of singularities as considered in § 6: $c = \sqrt{G(0)/\varrho} = \sqrt{\eta/(\varrho\lambda)}$.

ULTMAN & DENN attempted to correlate some experimental observations of D. F. JAMES [18] with this change of type. JAMES observes a sudden change in the slope of the heat transfer curve as a function of velocity. This happens at a critical velocity, which, for the polyox solution used by JAMES, was about 1 cm/sec. It is not clear from the graphs how abrupt this change in slope is, but there is a change of slope. ULTMAN & DENN also suggest that the transition from subcritical to supercritical flow might explain abrupt changes in the drag coefficient observed by A. FABULA [13]. Again, the idea is that the critical velocity at transition is the wave speed c. They make an estimate of c from a molecular theory and correlate this prediction with the data of JAMES. Of course any such estimate can be expected to give at best an order of magnitude, since the fluids used in experiments are not really Maxwell fluids.

M. LUSKIN [27] studies the equations of ULTMAN & DENN as a first order system. He reduces them to canonical form and investigates characteristics. The streamline is a double characteristic, two characteristics are always complex, and the remaining two are complex for $M < 1$ and real for $M > 1$. Each characteristic has associated with it a canonical variable, which is a linear combination of the two velocity components, the three components of the extra stress and the pressure. This means that two of these variables can be discontinuous across streamlines, and two others can be discontinuous only for $M > 1$.

We shall now show how results similar to those of ULTMAN & DENN apply to general constitutive models without "viscosity". In linearization at uniform motion, the extra stress is given by

$$\boldsymbol{\tau} = \int_{-\infty}^{t} G(t-\tau)\, A[\boldsymbol{u}(\zeta, \tau)]\, d\tau, \tag{8.2}$$

where

$$\zeta = \begin{pmatrix} x_1 - U(t-\tau) \\ x_2 \end{pmatrix}.$$

We assume G is smooth, positive and monotone decreasing. By differentiating (8.2) with respect to t, holding $\boldsymbol{x}$ fixed, we find

$$\frac{\partial \boldsymbol{\tau}}{\partial t} + U \frac{\partial \boldsymbol{\tau}}{\partial x_1} = G(0)\, A[\boldsymbol{u}(\boldsymbol{x}, t)] + \int_{-\infty}^{t} G'(t-\tau)\, A[\boldsymbol{u}(\zeta, \tau)]\, d\tau, \tag{8.3}$$

where the last term is of lower differential order. The leading term is the same as for the Maxwell model, if $G(0)$ is replaced by η/λ.

It can be shown that the change of type is primarily associated with the behavior of the vorticity. Since we study motions in the plane, the vorticity curl $\boldsymbol{u}$ has only one component, which we denote by α. We take the curl of the equation of motion, apply the operation curl div to (8.3) and combine the two. In steady flow this leads to

$$(M^2 - 1)\frac{\partial^2 \alpha}{\partial x_1^2} - \frac{\partial^2 \alpha}{\partial x_2^2} = \int_{-\infty}^{t} G'(t-\tau)\, \Delta\alpha(\zeta)\, d\tau. \tag{8.4}$$

The right side is again of lower order. The left hand side is elliptic when $M < 1$ and hyperbolic when $M > 1$. The elliptic roots found by LUSKIN [27] correspond to the elliptic equatlon $\alpha = -\Delta\psi$ expressing the vorticity in terms of the stream function.

9. Classification of the Quasilinear System of Equations Governing the Steady Flow of an Upper Convected Maxwell Fluid

The flow of an upper convected Maxwell fluid is governed by the following system of equations

$$\varrho \frac{d\boldsymbol{u}}{dt} + \nabla p - \operatorname{div} \boldsymbol{\tau} = 0, \qquad \operatorname{div} \boldsymbol{u} = 0,$$
$$\boldsymbol{\tau} + \lambda \left[\frac{d\boldsymbol{\tau}}{dt} - \nabla \boldsymbol{u}\boldsymbol{\tau} - \boldsymbol{\tau} \nabla \boldsymbol{u}^T\right] = \eta A \overset{\text{def}}{=} \mu\lambda A. \tag{9.1}$$

We consider two-dimensional flows. In view of the applications in Chapter 10, we want to write the equations in both Cartesian and polar coordinates. In cartesian coordinates, we set

$$\boldsymbol{u} = (u, v), \qquad \boldsymbol{\tau} = \begin{bmatrix} \sigma & \tau \\ \tau & \gamma \end{bmatrix}. \tag{9.2}$$

We then obtain the equations

$$\begin{aligned}
\sigma_t + u\sigma_x + v\sigma_y - 2\sigma u_x - 2\tau u_y - 2\mu u_x &= -\sigma/\lambda, \\
\tau_t + u\tau_x + v\tau_y - \gamma u_y - \sigma v_x - \mu(u_y + v_x) &= -\tau/\lambda, \\
\gamma_t + u\gamma_x + v\gamma_y - 2\tau v_x - 2\gamma v_y - 2\mu v_y &= -\gamma/\lambda, \\
\varrho(u_t + uu_x + vu_y) + p_x - \sigma_x - \tau_y &= 0, \\
\varrho(v_t + uv_x + vv_y) + p_y - \tau_x - \gamma_y &= 0, \\
u_x + v_y &= 0.
\end{aligned} \tag{9.3}$$

The subscripts denote differentiation. In polar coordinates, we denote by u and v the radial and azimuthal velocities and by σ, τ, γ the components of the stress in polar coordinates. We then obtain the following system

$$\sigma_t + u\sigma_r + \frac{v\sigma_\theta}{r} - 2\sigma u_r - 2\tau \frac{u_\theta}{r} - 2\mu u_r = -\frac{\sigma}{\lambda}, \tag{9.4}$$

$$\tau_t + u\tau_r + \frac{v\tau_\theta}{r} - \frac{\gamma u_\theta}{r} - \sigma v_r - \mu\left(v_r + \frac{u_\theta}{r}\right) = -\frac{\tau}{\lambda} - \frac{\sigma v}{r} - \frac{\mu v}{r},$$

$$\gamma_t + u\gamma_r + \frac{v\gamma_\theta}{r} - 2\tau v_r - 2\gamma \frac{v_\theta}{r} - 2\mu \frac{v_\theta}{r} = -\frac{\gamma}{\lambda} - \frac{2v\tau}{r} + \frac{2u}{r}(\gamma + \mu),$$

$$\varrho\left(u_t + uu_r + v\frac{u_\theta}{r}\right) + p_r - \sigma_r - \frac{\tau_\theta}{r} = \varrho\frac{v^2}{r} + \frac{\sigma - \gamma}{r},$$

$$\varrho\left(v_t + uv_r + \frac{vv_\theta}{r}\right) + \frac{p_\theta}{r} - \tau_r - \frac{\gamma_\theta}{r} = -\frac{\varrho uv}{r} + \frac{2\tau}{r},$$

$$u_r + \frac{v_\theta}{r} = -\frac{u}{r}.$$

The terms on the right of (9.3) and (9.4) are of lower order and do not enter into the analysis of characteristics. The systems on the left are identical if we make the identifications $\frac{\partial}{\partial x} \sim \frac{\partial}{\partial r}$ and $\frac{\partial}{\partial y} \sim \frac{1}{r}\frac{\partial}{\partial \theta}$.

We consider steady solutions of (9.3) and (9.4). Thus we put the time derivatives equal to zero and introduce $q = (u, v, \sigma, \tau, \gamma, p)$. Equations (9.3) and (9.4) are of the respective forms

(9.5) $$\boldsymbol{A}q_x + \boldsymbol{B}q_y = f,$$

and

(9.5) $$\hat{\boldsymbol{A}}q_r + \hat{\boldsymbol{B}}\frac{q_\theta}{r} = \hat{f}.$$

Characteristics of (9.5) are determined by the equations

(9.7) $$\det\left(\frac{dy}{dx}\boldsymbol{A}(q) - \boldsymbol{B}(q)\right) = 0.$$

With $\alpha = -\frac{dy}{dx}$, this leads to

(9.8) $$(\alpha u + v)^2 (1 + \alpha^2) [2\alpha\tau - \varrho(\alpha u + v)^2 + \alpha^2\sigma + \gamma + (1 + \alpha^2)\mu] = 0.$$

Hence the streamlines

(9.9) $$\frac{dy}{dx}u - v = 0$$

are double characteristics. There are two imaginary roots $\alpha = \pm i$. Finally, the last bracket yields

(9.10)

$$-\frac{dy}{dx} = \frac{\varrho uv - \tau}{\mu + \sigma - \varrho u^2} \pm \left\{\frac{\tau^2 - 2\varrho\tau uv - (\mu + \gamma)(\mu + \sigma) + \varrho v^2(\mu + \sigma) + \varrho u^2(\mu + \gamma)}{(\mu + \sigma - \varrho u^2)^2}\right\}^{\frac{1}{2}}.$$

These characteristics are real or complex depending on whether the argument of the square root is positive or negative. This argument of course depends on the solution. It should be kept in mind here that the integral form of the Maxwell model imposes certain restrictions on the stresses (see Chapter 5). These roots are the ones which exhibit a change of type.

The imaginary roots $\frac{dy}{dx} = \pm i$ arise from the identity

(9.11) $$\zeta = -\nabla^2 \psi,$$

where ζ is the vorticity and ψ is the stream function. It is easy to show that

(9.12)
$$(\mu + \sigma - \varrho u^2)\frac{\partial^2 \zeta}{\partial x^2} + (\mu + \gamma - \varrho v^2)\frac{\partial^2 \zeta}{\partial y^2} + 2(\tau - \varrho uv)\frac{\partial^2 \zeta}{\partial x\, \partial y} = \text{lower order}.$$

This shows that the interesting characteristic roots (9.10) which can change type are associated with the vorticity. Using (9.11), we find that the quantity on left of (9.12) with ψ replacing ζ, is harmonic to within terms of lower order. In polar coordinates, the equation for characteristics becomes

(9.13) $$-r\frac{d\theta}{dr} = \text{the right-hand side of (9.10)},$$

with u, v, τ, γ, σ interpreted as the physical components of velocity and stress in polar coordinates.

10. Change of Type in Shear Flow, Extensional Flow, Sink Flow and Circular Couette Flow of an Upper Convected Maxwell Fluid

In this section we consider the problem of hyperbolicity of the linear equations perturbing some special solutions of (9.1). The characteristics equations (9.10) and (9.13) are useful for this. To compute the characteristics of the perturbation we need only to replace the u, v, τ, σ, γ in these formulas with the special values that (9.1) requires for special flows.

(i) First, we again conider the uniform flows discussed in § 8; $u = U$, $v = \tau = \gamma = \sigma = 0$. For these we find from (9.10) that

(10.1) $$\frac{dy}{dx} = \pm \sqrt{\frac{1}{M^2 - 1}},$$

where $M^2 = U^2/C^2$, $C^2 = \eta/\lambda\varrho \left(= \frac{G(0)}{\varrho}\right)$ is the wave speed. The characteristics are straight lines which start as lines perpendicular to the flow at $M = 1$ and tilt more toward the free stream as $M > 1$ is increased.

(ii) Our second application is to shear flow. We find, using (9.3), that

(10.2) $$u = \varkappa y, \quad \tau = \eta\varkappa; \quad \sigma = 2\eta\lambda\varkappa^2, \quad \gamma = v = 0,$$

where the shear rate $\varkappa$ is a constant. Inserting the fields (10.2) into (9.10), we find that

$$\frac{dy}{dx}=\frac{\lambda\varkappa}{1+2\lambda^2\varkappa^2-\frac{\varkappa^2y^2}{c^2}}\pm\sqrt{\frac{\frac{\varkappa^2y^2}{c^2}-\lambda^2\varkappa^2-1}{\left[1+2\lambda^2\varkappa^2-\frac{\varkappa^2y^2}{c^2}\right]^2}} \tag{10.3}$$

$$=\frac{1}{\lambda\varkappa\pm\sqrt{\frac{\varkappa^2y^2}{c^2}-\lambda^2\varkappa^2-1}}.$$

The vorticity is hyperbolic outside a strip defined by

$$\frac{\varkappa^2y^2}{c^2}>\lambda^2\varkappa^2+1. \tag{10.4}$$

(iii) In extensional flow, we find, using (9.3), that

$$u=sx,\ v=-sy,\quad \tau=0,\quad \sigma=2\eta s/(1-2s\lambda),\quad \gamma=-2\eta s/(1+2\lambda s), \tag{10.5}$$

where $0<s<1/2\lambda$. The stretch rate s is a positive constant, small enough to keep σ bounded and positive. The unbounded σ at $s=1/(2\lambda)$ is one of the many undesirable properties the upper-convected Maxwell model. Inserting the field (10.5) into (9.10), we find that

$$\frac{dy}{dx}=\frac{\varrho s^2xy}{\frac{\mu}{1-2\lambda s}-\varrho s^2x^2}\pm\frac{\mu}{\sqrt{1-4\lambda^2s^2}}\left\{\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right\}^{\frac{1}{2}}, \tag{10.6}$$

where

$$a^2=\frac{\mu}{\varrho s^2(1-2\lambda s)},\qquad b^2=\frac{\mu}{\varrho s^2(1+2\lambda s)}.$$

We get real characteristics, obeying (10.6), outside the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}>1.$$

(iv) Our fourth application is to sink flow. Here we must work with the polar equations (9.4) and (9.13). For sink flow

$$u=-Q/r,\quad Q>0 \text{ is the sink strength}, \tag{10.7}$$

$$v=\tau=0.$$

From (9.4), we find that

$$\frac{d\sigma}{dr}+\frac{2\sigma}{r}-\frac{r\sigma}{\lambda Q}+\frac{2\mu}{r}=0.$$

We find that

$$\sigma=2\eta Q/r^2. \tag{10.8}$$

From $(9.4)_3$, we find that

$$\frac{d\gamma}{dr} - \frac{2\gamma}{r} - \frac{r\gamma}{\lambda Q} - \frac{2\mu}{r} = 0.$$

We next introduce ϕ through the change of variables $\gamma = -\mu + \mu\phi/\lambda Q$, where

$$\frac{d\phi}{dr} - \phi\left[\frac{2}{r} + \frac{r}{\lambda Q}\right] + r = 0.$$

We require that ϕ be bounded as $r^2 \to 0$ and find

$$\phi = -r^2 e^{r^2/2\lambda Q} \int \frac{e^{-r^2/2\lambda Q}}{r}\, dr, \tag{10.9}$$

where the last factor is an indefinite integral. For small r, we find that

$$\phi \sim -r^2 \log r > 0.$$

Obviously ϕ is positive at every stationary point. The only stationary point is at $r = \infty$ and ϕ increases monotonically from zero to λQ.

Turning next to (9.13), using (10.7–9), we find that real characteristics exist and are given by (9.13) when

$$(\mu + \gamma)\left[\frac{\varrho Q^2}{r^2} - (\mu + \sigma)\right] = \frac{\mu\phi}{\lambda Q}\left[\frac{\varrho Q^2 - 2\eta Q}{r^2} - \mu\right] > 0. \tag{10.10}$$

The condition for hyperbolicity is satisfied if

$$\varrho Q > 2\eta;$$

that is, when the flow rate is large. In this case the flow is hyperbolic in the circle

$$r < \sqrt{\frac{\varrho Q^2 - 2\eta Q}{\mu}}. \tag{10.11}$$

The differential equations for the net of characteristics covering the hyperbolic circle at the origin is

$$r\frac{d\theta}{dr} = \pm\sqrt{\frac{\mu + \gamma}{\varrho u^2 - (\mu + \sigma)}} = \pm\sqrt{\frac{\mu\phi/\lambda Q\varrho}{\dfrac{Q^2 - 2\eta Q}{r^2} - \mu}}.$$

(v) Our fifth application is to Couette flow outside a rotating circular cylinder of radius a. We suppose that the fluid sticks to the rod, and the rod rotates with an angular frequency Ω. We assume that $v = v(r)$, $u = \sigma = 0$. Then using $(9.4)_5$ and $(9.4)_2$ we find that

$$v = \Omega a^2/r,$$

$$\tau = -2a^2\eta\Omega/r^2,$$

and from $(9.4)_3$

$$\gamma = 2\tau^2/\mu.$$

From (9.13), we find that

$$r\frac{d\theta}{dr} = \pm\sqrt{\frac{v^2}{c^2}\left(1-\frac{4\lambda^2c^2}{r^2}\right)-1}.$$

To have real characteristics it is necessary and sufficient

$$\Sigma(r^2) \overset{\text{def}}{=} \frac{a^4\Omega^2}{c^2r^2}\left(1-\frac{4\lambda^2c^2}{r^2}\right)-1>0.$$

Moreover, $\Sigma(\infty) = -1$, and

$$\frac{d\Sigma}{dr^2} = \frac{a^4\Omega^2}{c^2r^4}\left[\frac{8\lambda^2c^2}{r^2}-1\right].$$

Σ first increases, then decreases, with a single maximum at $r^2 = 8\lambda^2c^2$ given by

$$\Sigma_m = \Sigma(8\lambda^2c^2) = \frac{a^4\Omega^2}{16\lambda^2c^4}-1,$$

which is positive when

$$a^4 > \frac{16\lambda^2c^4}{\Omega^2}. \tag{10.12}$$

If (10.12) holds then there are two values

$$r_c^2 = \frac{8\lambda^2c^2}{1\pm\sqrt{1-\dfrac{16\lambda^2c^4}{\Omega^2a^4}}}$$

at which $\Sigma(r_c^2) = 0$. If

$$a^2 > 4\lambda^2c^2+\frac{c^2}{\Omega^2},$$

then $\Sigma(a^2) > 0$ and the vorticity is hyperbolic with real characteristics given by (9.13) in the annulus

$$a^2 \leqq r^2 \leqq \frac{8\lambda^2c^2}{1-\sqrt{1-\dfrac{16\lambda^2c^4}{\Omega^2a^4}}}.$$

If (10.12) holds and

$$a^2 < 4\lambda^2c^2+\frac{c^2}{\Omega^2},$$

then $\Sigma(a^2) < 0$ and the vorticity is hyperbolic with real characteristics given by (9.13) in the annulus

$$\frac{8\lambda^2c^2}{1+\sqrt{1-\dfrac{16\lambda^2c^4}{\Omega^2a^4}}} \leqq r^2 \leqq \frac{8\lambda^2c^2}{1-\sqrt{1-\dfrac{16\lambda^2c^4}{\Omega^2a^4}}}.$$

(vi) M. W. Johnson (private communication) has studied the flow of an upper convected Maxwell fluid between eccentric rotating cylinders. He assumes a small gap and uses the lubrication approximation. His analysis shows that change of type occurs.

11. Classification of the Quasilinear System of Equations Governing Steady Flows of a Class of Oldroyd Models

The Oldroyd models under discussion are given by (5.4) and (5.5). When $a = 1$, this model reduces to the upper convected Maxwell model which we studied in § 9 and §10.

We have

$$D = \begin{pmatrix} u_x & \frac{1}{2}(u_y + v_x) \\ \frac{1}{2}(u_y + v_x) & v_y \end{pmatrix},$$

$$\boldsymbol{\Omega} = \begin{pmatrix} 0 & \frac{1}{2}(u_y - v_x) \\ \frac{1}{2}(v_x - u_y) & 0 \end{pmatrix}.$$

This gives rise to the following quasilinear system

$$u\sigma_x + v\sigma_y + \tau(v_x - u_y) - a[2\sigma u_x + \tau(u_y + v_x)] - 2\mu u_x = -\sigma/\lambda,$$

$$u\tau_x + v\tau_y + \tfrac{1}{2}(\sigma - \gamma)(u_y - v_x) - \frac{a}{2}(\sigma + \gamma)(u_y + v_x) - \mu(u_y + v_x) = -\tau/\lambda,$$

$$(11.1) \quad u\gamma_x + v\gamma_y + \tau(u_y - v_x) - a[2\gamma v_y + \tau(u_y + v_x)] - 2\mu v_y = -\frac{\gamma}{\lambda},$$

$$\varrho(uu_x + vu_y) + p_x - \sigma_x - \tau_y = 0,$$

$$\varrho(uv_x + vv_y) + p_y - \tau_x - \tau_y = 0,$$

$$u_x + v_y = 0.$$

The analysis of characteristics follows the one used in § 9 exactly. We find characteristic directions, $\alpha = -dy/dx$ from the equation

(11.2)

$$(1 + \alpha^2)(\alpha u + v)^2 \left\{\varrho(\alpha u + v)^2 + \left(\frac{\gamma - \sigma}{2}\right)(\alpha^2 - 1) - 2\tau\alpha - (\alpha^2 + 1) \times \left(\mu + a\left(\frac{\gamma + \sigma}{2}\right)\right)\right\} = 0,$$

corresponding to (9.8). We can prove that the vorticity changes type from elliptic to hyperbolic when the sign of

(11.3)

$$\left[\varrho u^2 + \frac{\gamma}{2}(1 - a) - \frac{\sigma}{2}(1 + a) - \mu\right]\left[(1 + a)\frac{\gamma}{2} + (a - 1)\frac{\sigma}{2} + \mu - \varrho v^2\right] + (\varrho uv - \tau)^2$$

changes from negative to positive.

For shear flows, we find using (11.1) that

$$u = \varkappa y, \qquad v = 0, \qquad \tau = \frac{\eta\varkappa}{1 + \varkappa^2\lambda^2(1 - a^2)} \stackrel{\text{def}}{=} \frac{\eta\varkappa}{D},$$

$$\sigma = \lambda\varkappa(a + 1)\,\tau,$$

$$\gamma = \lambda\varkappa(a - 1)\,\tau,$$

where $\varkappa > 0$ is the shear rate.

Condition (11.3) now reads

(11.4)

$$\frac{\eta^2\varkappa^2}{D^2} + \left[\varrho\varkappa^2 y^2 - \frac{\varkappa^2\eta\lambda}{2D}(a - 1)^2 - \frac{\varkappa^2\eta\lambda}{2D}(1 + a)^2 - \mu\right]\left[\mu + \frac{a^2 - 1}{D}\varkappa^2\eta\lambda\right] > 0,$$

i.e.

$$\varrho\varkappa^2 y^2 B > \frac{-\eta^2\varkappa^2}{D^2} + B\left[\frac{\varkappa^2\eta\lambda}{2D}(a - 1)^2 + \frac{\varkappa^2\eta\lambda}{2D}(1 + a)^2 + \mu\right],$$

where $B \stackrel{\text{def}}{=} \frac{(a^2 - 1)}{D}\varkappa^2\eta\lambda + \mu = \frac{\mu}{D}$.

If a is in the range $-1 \leqq a \leqq 1$, B is always positive, and (11.4) describes the exterior of a strip in the x,y-plane (it is easy to see that the quantity on the right of the last inequality is always positive).

In extensional flow, we find using (11.1) that

$$u = sx, \qquad v = -sy, \qquad \tau = 0, \qquad \sigma = \frac{2\eta s}{1 - 2a\lambda s}, \qquad \gamma = \frac{-2\eta s}{1 + 2a\lambda s},$$

where $0 < s < 1/(2a\lambda)$ for $1 > a > 0$ or $0 < s < 1/(-2a\lambda)$ for $-1 \leqq a < 0$ (no restriction if $a = 0$).

Condition (11.3) is now evaluated as

$$\begin{aligned} &\varrho s^2 x^2\left[\mu + \frac{2\eta s(2a^2\lambda s - 1)}{1 - 4a^2\lambda^2 s^2}\right] + \varrho s^2 y^2\left[\mu + \frac{2\eta s(1 + 2a^2\lambda s)}{1 - 4a^2\lambda^2 s^2}\right] \\ &\qquad > \left[\mu + \frac{2\eta s(1 + 2a^2\lambda s)}{1 - 4a^2\lambda^2 s^2}\right]\left[\mu + \frac{2\eta s(2a^2\lambda s - 1)}{1 - 4a^2\lambda^2 s^2}\right]. \end{aligned} \tag{11.5}$$

Let us distinguish two cases

(i) $\mu + \frac{2\eta s(2a^2\lambda s - 1)}{1 - 4a^2\lambda^2 s^2} > 0$ (this is the case for $a = \pm 1$).

Then the region (11.5) is the exterior of the ellipse

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1,$$

where

$$A^2 \overset{\text{def}}{=} \frac{\mu + \dfrac{2\eta s(2a^2\lambda s + 1)}{1 - 4a^2\lambda^2 s^2}}{\varrho s^2},$$

$$B^2 \overset{\text{def}}{=} \frac{\mu + \dfrac{2\eta s(2a^2\lambda s - 1)}{1 - 4a^2\lambda^2 s^2}}{\varrho s^2}.$$

(ii) $\mu + \dfrac{2\eta s(2a^2\lambda s - 1)}{1 - 4a^2\lambda^2 s^2} < 0$. Then (11.5) yields

$$\frac{x^2}{A^2} - \frac{y^2}{B^2} < 1,$$

where

$$A^2 = \frac{\mu + \dfrac{2\eta s(1 + 2a^2\lambda s)}{1 - 4a^2\lambda^2 s^2}}{\varrho s^2},$$

$$B^2 = -\frac{\mu + \dfrac{2\eta s(2a^2\lambda s - 1)}{1 - 4a^2\lambda^2 s^2}}{\varrho s^2}.$$

Equation of the vorticity, steady case

One easily finds the equation satisfied by the vorticity $\zeta = -\Delta\psi$, where ψ is the stream function given by $\psi_y = u$, $v = -\psi_x$:

$$\text{(11.6)} \quad [-\varrho u^2 + \mu + \tfrac{1}{2}\sigma(1 + a) + \tfrac{1}{2}\gamma(a - 1)]\,\zeta_{xx} + 2(-\varrho uv + \tau)\,\zeta_{xy} + [-\varrho v^2 + \mu + \tfrac{1}{2}\sigma(a - 1) + \tfrac{1}{2}\gamma(a + 1)]\,\zeta_{yy} + \text{terms of lower order} \overset{\text{def}}{=} A\zeta_{xx} + 2B\zeta_{xy} + C\zeta_{yy}.$$

This is a mixed elliptic-hyperbolic second-order equation. It is hyperbolic for $\Delta = B^2 - AC > 0$, *i.e.*, when (11.3) holds (see the previous discussion).

Equation for vorticity in the unsteady case

In unsteady planar flow the equation for vorticity is

$$\text{(11.7)} \quad \varrho\left(\frac{\partial\zeta}{\partial t} + (\boldsymbol{u}\cdot\nabla)\,\zeta\right) + \lambda\left[\varrho\frac{\partial^2\zeta}{\partial t^2} + 2\varrho(\boldsymbol{u}\cdot\nabla)\frac{\partial\zeta}{\partial t} + \left(\varrho u^2 - \mu - \frac{\sigma}{2}(1 + a) + \frac{\gamma}{2}(1 - a)\right)\frac{\partial^2\zeta}{\partial x^2} + 2(\varrho uv - \tau)\frac{\partial^2\zeta}{\partial x\,\partial y} + \left(\varrho v^2 - \mu + \frac{\sigma}{2}(1 - a) - \frac{\gamma}{2}(a + 1)\right)\frac{\partial^2\zeta}{\partial y^2}\right] = \lambda[l.o.].$$

The term in brackets on the right-hand side is of lower order and does not involve any time derivative. The first term on the left-hand side is irrelevant for the analysis of the type of equation (11.7), but it shows how (11.7) reduces to the unsteady Newtonian case when $\lambda = 0$ (recall that $\mu = \eta/\lambda$).

Equation (11.7) is of evolution type (hyperbolic) when

(11.8)
$$\lambda^2\tau^2 - [\eta - \lambda(\tfrac{1}{2}\gamma(1-a) - \tfrac{1}{2}\sigma(1+a))] \cdot [\eta - \lambda(\tfrac{1}{2}\sigma(1-a) - \tfrac{1}{2}\gamma(1+a))] < 0,$$

(11.9)
$$\lambda[\tfrac{1}{2}\gamma(1-a) - \tfrac{1}{2}\sigma(1+a)] - \eta < 0.$$

12. Quasilinear Systems for Simple Fluids with Fading Memory of the Coleman-Noll Type

The determinate stress $\boldsymbol{\tau}$ in a simple fluid is given by an isotropic functional of the history of the relative Cauchy strain $G(s) = \boldsymbol{F}_t^T(t-s)\,\boldsymbol{F}_t(t-s) - \mathbf{1}$, *i.e.*

(12.1)
$$Q\boldsymbol{\tau}Q^T = \boldsymbol{F}\left[\overset{\infty}{\underset{s=0}{Q}}\, G(s)\, Q^T\right]$$

for all constant orthogonal tensors Q. By taking the material derivative of (12.1), we obtain

(12.2)
$$Q\frac{d\boldsymbol{\tau}}{dt}Q^T = \boldsymbol{F}_1\left[QGQ^T \middle| Q\frac{dG}{dt}Q^T\right],$$

where $\boldsymbol{F}_1$ denotes the first functional derivative of F. We have already noted in § 5 that different choices for the domain of the linear functional $\boldsymbol{F}_1[QGQ^T \mid \cdot]$ lead to different representations of $\boldsymbol{F}_1$. If, following COLEMAN & NOLL, we assume that the functional is defined on a weighted L^2-space, we obtain an integral

(12.3)
$$\boldsymbol{F}_1\left[G\middle|\frac{dG}{dt}\right] = \int_0^\infty \mathbf{K}(s, G)\frac{dG(s)}{dt}\,ds.$$

Here $\mathbf{K}(s, G)$ is a fourth-order tensor depending on s and on the values $\{G(\sigma), 0 \leqq \sigma < \infty\}$. For the following, we assume that $\mathbf{K}$ and its first derivative (as function of s) are integrable. In particular, it follows that $\mathbf{K}$ is uniformly bounded in s.

The isotropy condition can be written as

(12.4)
$$K_{ijkl}(s, G) = Q_{ai}Q_{bj}Q_{ck}Q_{dl}K_{abcd}(s, QGQ^T).$$

This consequence of isotropy does not appear to lead easily to very explicit representations for $\mathbf{K}$ in situations where G is not confined to special motions. Of course $\mathbf{K}$ is symmetric in the first two indices, and only the symmetric part in the last two indices enters into (12.3).

We next note that the material derivative of G is given by

$$(12.5)\qquad \frac{d}{dt}G(s) = \frac{d}{dt}(F^{-T}(t)\,F^T(t-s)\,F(t-s)\,F^{-1}(t))$$

$$= -F^{-T}(t)\,\dot{F}^T(t)\,G(s) - G(s)\dot{F}(t)\,F^{-1}(t) - \frac{d}{ds}G(s)$$

$$= -L^TG(s) - G(s)\,L - \frac{dG(s)}{ds},$$

where $L(x, t) = \nabla u(x, t)$ is the present value of the velocity gradient. Hence, we find

$$(12.6)\qquad \int_0^\infty K_{ijkl}(s, G)\frac{dG_{kl}(s)}{dt}\,ds = -\int_0^\infty (K_{ijkl} + K_{ijlk})\,G_{pl}(s)\,ds \cdot L_{pk}(t)$$

$$- \int_0^\infty K_{ijkl}(s, G)\frac{dG(s)}{ds}\,ds.$$

It follows that

$$(12.7)\qquad \frac{d\tau_{ij}}{dt} = M_{ijkp}(x, t)\,L_{pk}(x, t) + N_{ij}(x, t),$$

where

$$(12.8)\qquad M_{ijkp}(x, t) = -\int_0^\infty (K_{ijkl} + K_{ijlk})\,G_{pl}(s)\,ds,$$

and

(12.9)

$$N_{ij}(x, t) = \int_0^\infty K_{ijkl}\left(s, \overset{\infty}{\underset{s'=0}{G}}(s')\right)\frac{dG_{kl}(s)}{ds}\,ds = -\int_0^\infty G_{kl}(s)\frac{d}{ds}K_{ijkl}\left(s, \overset{\infty}{\underset{s'=0}{G}}(s')\right)ds.$$

The coefficients M_{ijkp} of L_{pk} and the terms $N_{ij}(x)$ are of lower order and $N_{ij}(x, t)$ does not enter into analysis of problems of change of type.

We may write (12.9) as

$$(12.10)\qquad \frac{d\tau_{ij}}{dt} = S_{ijkp}\,D_{kp} + A_{ijkp}\Omega_{kp} + N_{ij},$$

where

$$M_{ijkp} = S_{ijkp} + A_{ijkp},$$

and **S** is symmetric and **A** is skew symmetric in k and p. D and Ω denote the symmetric and skew symmetric part of L.

In its general form, (12.10) and the equations of motion are a quasilinear system of first order in the derivatives of p, u, τ. We could write out conditions for evolutionary character or for change of type in steady flow as before. However, we cannot in general isolate an equation for the vorticity, as we did for the Oldroyd models.

We can identify a class of models which is more special than the completely general equation (12.10), but much more general than the three-constant Oldroyd model. Let us assume that

$$S_{ijkp} = \tfrac{1}{2}(\delta_{ik}P_{jp} + \delta_{jk}P_{ip} + \delta_{ip}P_{jk} + \delta_{jp}P_{ik}), \tag{12.11}$$

where $\boldsymbol{P}$ is any second order tensor, expressible by integrals of the type (12.8). We need no assumptions at all on the anti-symmetric part A_{ijkp}. Using (12.11) we may reduce (12.10) to

$$\begin{aligned} \frac{d\tau_{ij}}{dt} &= P_{ik}\, D_{kj} + P_{jk}\, D_{ki} + A_{ijkp}\, \Omega_{kp} + N_{ij}, \\ \frac{d\boldsymbol{\tau}}{dt} &= \boldsymbol{PD} + \boldsymbol{DP}^T + \tfrac{1}{2}(\mathbf{A}\boldsymbol{\Omega} + (\mathbf{A}\boldsymbol{\Omega})^T) + \boldsymbol{N}, \end{aligned} \tag{12.12}$$

where A_{ijkp} is symmetric in ij and skew symmetric in kp.

The Oldroyd model with three constants arises from (12.12) with special choices for the tensors $\mathbf{A}$, $\boldsymbol{P}$ and $\boldsymbol{N}$:

$$\begin{aligned} -2A_{ijkl} &= \delta_{ik}\tau_{jl} + \delta_{jk}\tau_{il} - \delta_{il}\tau_{kj} - \delta_{jl}\tau_{ki}, \\ P_{ik} &= a\tau_{ik} + \frac{\eta}{\lambda}\,\delta_{ik}, \\ N_{ij} &= -\frac{1}{\lambda}\tau_{ij}. \end{aligned} \tag{12.13}$$

We now demonstrate that the quasilinear system associated with (12.12) is expressible in terms of the vorticity $\zeta = \operatorname{curl} \boldsymbol{u}$. The equations of motion are

$$\varrho \frac{d\boldsymbol{u}}{dt} = -\nabla p + \operatorname{div} \boldsymbol{\tau}, \qquad \operatorname{div} \boldsymbol{u} = 0. \tag{12.14}$$

We apply the operations curl and $\dfrac{d}{dt}$ to (12.14), and find that to leading order

$$\varrho \frac{d^2}{dt^2}\zeta = \operatorname{curl} \operatorname{div} \frac{d\boldsymbol{\tau}}{dt} + \dots. \tag{12.15}$$

The question is now whether the right side of (12.15) is expressible in terms of ζ, to leading order. Clearly $\boldsymbol{\Omega}$ is expressible in terms of ζ, and so we need no restrictions on $\mathbf{A}$. Then, working the part with tensor $\boldsymbol{P}$, we calculate

$$\begin{aligned} \varepsilon_{abi}\, \partial_b \partial_j (P_{ik}\, D_{kj} + P_{jk}\, D_{ki}) &= \varepsilon_{abi}[P_{ik}\, \partial_b\partial_j D_{kj} + P_{jk}\, \partial_b\partial_j D_{ki}] + \dots \\ &= \tfrac{1}{2}\varepsilon_{abi} P_{ik}\, \nabla^2 \frac{\partial u_k}{\partial x_b} + \tfrac{1}{2}\varepsilon_{abi} P_{jk} \frac{\partial^3 u_i}{\partial x_b \partial x_j \partial x_k} + \dots. \end{aligned}$$

Noting now that

$$\nabla^2 \boldsymbol{u} = -\text{curl}\ \zeta,$$

we get

$$= -\tfrac{1}{2}\varepsilon_{abi}P_{ik}\ \partial_b\ (\text{curl}\ \zeta)_k + \tfrac{1}{2}\ P_{jk}\ \partial_j\partial_k\zeta_a + \dots$$

It follows now that to leading order (12.15) is a second order quasilinear system of equations for the components of the vorticity.

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Note added in proof: BERNARD COLEMAN, in an unpublished work, has shown that our equation (12.10) can be consistent with his thermodynamics only if $S_{ijkp} = S_{kpij}$. This implies that the second order tensor $\boldsymbol{P}$ in (12.11) is symmetric.

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