

Hyperbolicity and change of type in sink flow

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We consider the problem of steady fast flow of a family of Oldroyd fluids into a hole, and show that the field of flow is partitioned into elliptic (subcritical) and hyperbolic (supercritical) regions. We analyse the characteristics and show that the vorticity changes type as in the experiments of Metzner, Uebler & Fong (1969).

1. Introduction

This paper is one of a series that addresses the analogue of the ‘transonic’ problem of aerodynamics that arises in the problem of steady flow of viscoelastic liquids. This problem is associated with the fact that the equations governing the distribution of vorticity in steady flow can change type (Joseph, Renardy & Saut 1984, hereinafter referred to as JRS); the flow is then partitioned into subcritical (elliptic) and supercritical (hyperbolic) regions for vorticity in a manner which has some analogies with the partitioning of high-speed flow of gases into sub- and supersonic regions. The viscoelastic problem is a very new one; there are only four papers: Rutkevich (1970, 1972), Ultman & Denn (1970), Luskin (1985) and JRS. Very little is known about the physical consequences of changing type; e.g. there is no theory of simple waves or shock waves. Many curious unexplained phenomena of flow of viscoelastic liquids may have their origin in the transition from sub- to supercritical flow. In this series of papers we try to move the theory closer to experiments by emphasizing issues that bear on the explanation of observations. We have also made some modest effort at identifying phenomena that could involve a change of type. Some of the entries on our list appear to fall in the frame of axisymmetric flow, and we have provided some elements of a theory for these. It cannot be said that we have achieved outstanding successes in our effort to identify clearly the type of observations that are really explained by mathematics of change of type. The obstacles that we encounter are partly theoretical, having to do with the fact that the nonlinear theory is not yet well developed, particularly with respect to the development of shocks. The other theoretical problem has to do with constitutive equations. It can be shown, and Rutkevich (1972) and JRS have shown, that the conditions for change of type or the equivalent conditions for loss of evolutionarity in unsteady problems are very sensitive to the choice of constitutive equations. This problem, which has always been troublesome, even in realms of fairly slow flows, is much more so for high-speed flows. JRS showed that all the Oldroyd models in the family that contains the upper and lower convected Maxwell models have a vorticity of changing type. In fact they showed that models that are very much more general than Oldroyd ones have vorticity changing type, and that the vorticity of all motions of *all* models with instantaneous elasticity that perturbs rigid motions changes type. But it appears that

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not all models that change type are such that it is precisely the vorticity that changes type. We do not know if a given fluid fits some one constitutive model in all of its possible motions. Probably it never does. But if we are to progress we must certainly try to verify that some real fluids do exhibit the more striking properties implied by the model. A change of type to hyperbolicity of the vorticity is one such striking property.

The reader may regard this paper as the second one of a series starting with JRS. In this paper we are going to give a theory that leads to vorticity of changing type in the perturbation of sink flow. We apply this theory to the problem of flow into a hole, which could conceivably be regarded as perturbing sink flow in some sense. Many experiments on flow into a hole have been reported, but those by Metzner, Uebler & Fong (1969) are notable because they exhibit a vorticity of changing type.

The plan of this paper is as follows. In §2 we specify our constitutive model. We consider Oldroyd models with instantaneous elasticity. These models contain the upper and lower convected and the co-rotational Maxwell models, and they exhibit vorticity of changing type. The equations are analysed for type in §3 and the characteristics are identified. The analysis of the quasilinear problem in §3 is without approximation. In §4 we linearize the quasilinear problem and show how the characteristics for any axisymmetric motion perturbing a basic one may be computed. In §5 we give the formulae for the characteristics of the axisymmetric motion that perturbs sink flow. In §6 we stretch our imagination and imagine that flow into a hole is a small perturbation of sink flow, and we apply the results of such imaginative thinking to explain the curious experimental results of Metzner *et al.* (1969).

2. Oldroyd models with instantaneous elasticity

The family of Oldroyd models given by

$$\lambda \frac{\mathcal{D}\boldsymbol{\tau}}{\mathcal{D}t} + \boldsymbol{\tau} = 2\eta\mathbf{D}, \quad (2.1)$$

where \mathbf{D} is the symmetric part of the velocity gradient, $\boldsymbol{\tau}$ is determinate stress, λ is a relaxation time and $\mathcal{D}/\mathcal{D}t$ is an invariant derivative given by (2.2), can be said to have instantaneous elasticity. The fluids (2.1) have instantaneous elasticity when $\lambda \neq 0$, and the elastic response is more persistent when λ is large. Viscous fluids (Newtonian) have no elasticity, $\lambda = 0$. The invariant derivative is given by

$$\frac{\mathcal{D}\boldsymbol{\tau}}{\mathcal{D}t} = \frac{\partial\boldsymbol{\tau}}{\partial t} + (\mathbf{u} \cdot \nabla)\boldsymbol{\tau} + \boldsymbol{\tau}\boldsymbol{\Omega} - \boldsymbol{\Omega}\boldsymbol{\tau} - a(\mathbf{D}\boldsymbol{\tau} + \boldsymbol{\tau}\mathbf{D}), \quad (2.2)$$

where $\boldsymbol{\Omega} = \frac{1}{2}(\nabla\mathbf{u} - \nabla\mathbf{u}^T)$ and $-1 \leq a \leq 1$. All these derivatives are frame-indifferent. The Jaumann derivative has $a = 0$. An upper convected Maxwell model arises from (2.2) when $a = +1$, while $a = -1$ corresponds to the lower convected Maxwell model and $a = 0$ to the co-rotational model. These models may be evaluated in viscometric flows. Comparison of this evaluation with observations suggests that, of the three distinguished values $a = -1, 0, 1$, $a = 1$ is best. More generally it is believed that the good a 's are positive and slightly less than one (Astarita & Marrucci 1974).

We are interested in the quasilinear dynamical system associated with (2.1), (2.2) and

$$\operatorname{div} \mathbf{u} = 0,$$

$$\rho \left[\frac{\partial\mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla\mathbf{u} \right] = -\nabla p + \operatorname{div} \boldsymbol{\tau}. \quad (2.3)$$

Fluids satisfying (2.1)–(2.3) are said to have instantaneous elasticity. They do not smooth discontinuities as is in the case of fluids of Jeffrey’s type, which have viscous terms which typically appear as time derivatives of \mathbf{D} multiplied by a retardation time. This quasilinear system (2.1)–(2.3) has been studied by Rutkevich (1970, 1972) and JRS. The latter authors showed that the vorticity, $\text{curl } \mathbf{u} = \boldsymbol{\omega}$, associated with (2.1) and (2.2) satisfies a second-order quasilinear equation

$$\rho \left[\frac{\partial^2 \omega_k}{\partial t^2} + 2\mathbf{u} \cdot \nabla \frac{\partial \omega_k}{\partial t} + u_i u_j \frac{\partial^2 \omega_k}{\partial x_i \partial x_j} \right] + \left[\frac{a-1}{2} \tau_{ii} + \mu \delta_{ii} \right] \epsilon_{kmi} \frac{\partial (\text{curl } \boldsymbol{\omega})_l}{\partial x_m} - \frac{a+1}{2} \tau_{ij} \frac{\partial^2 \omega_k}{\partial x_i \partial x_j} = \text{lower-order terms.} \quad (2.4)$$

This second-order equation shows that the vorticity is an important hyperbolic variable. There can be waves, possibly shock waves, of vorticity. In plane flow and in axisymmetric flow there is only one component of vorticity, so that (2.4) reduces to a scalar-valued second-order quasilinear equation. This second-order equation shows precisely that it is the vorticity that changes type. The analysis of the type of such equations is well developed, and JRS showed that in the plane case of steady flows the vorticity can change their type with subcritical and supercritical flow in different regions of the plane. In fact JRS defined a class of fluids with instantaneous elasticity more general than (2.1) and (2.2) that have vorticity of changing type.

3. Analysis of quasilinear systems in spherical coordinates

The analysis of quasilinear systems in orthogonal curvilinear systems is facilitated by the fact that only the terms with the highest derivatives enter into the analysis. This means that all of the extra terms, over and above Cartesian ones, which enter from geometry from the differentiation of base vectors, may be neglected. The systems of quasilinear equations that arise in this way have the same form, at highest order, as in Cartesian coordinates, except that the components are relative to the curvilinear system.

In the present analysis we consider (2.1)–(2.3) in spherical polar coordinates (r, θ, ϕ) where θ is the polar angle. In the axisymmetric problem we denote the velocity by

$$\mathbf{u} = (u, v, w) = (u, v, 0) \quad (3.1)$$

and the determinate stress by

$$\boldsymbol{\tau} = \begin{bmatrix} \sigma & \tau & 0 \\ \tau & \gamma & 0 \\ 0 & 0 & \beta \end{bmatrix}. \quad (3.2)$$

In addition, $\partial/\partial\phi = 0$. Equations (2.1)–(2.3) may then be written as

$$u_r + \frac{1}{r} v_\theta = -\frac{1}{r} (2u + v \cot \theta), \quad (3.3a)$$

$$\rho \left(uu_r + \frac{v}{r} u_\theta \right) + p_r - \sigma_r - \frac{1}{r} \tau_\theta = \frac{1}{r} [\rho v^2 + 2\sigma - \gamma - \beta + \tau \cot \theta], \quad (3.3b)$$

$$\rho \left(uv_r + \frac{v}{r} v_\theta \right) + \frac{1}{r} p_\theta - \tau_r - \frac{1}{r} \gamma_\theta = \frac{1}{r} [-\rho uv + 3\tau + (\gamma - \beta) \cot \theta], \quad (3.3c)$$

$$u\sigma_r + \frac{v}{r} \sigma_\theta - \left[2(a\sigma + \mu) u_r + (a-1) \tau v_r + (a+1) \frac{\tau}{r} u_\theta \right] = -\frac{\sigma}{\lambda} - (a-1) \frac{v\tau}{r}, \quad (3.3d)$$

$$u\gamma_r + \frac{v}{r}\gamma_\theta - \left[2(a\gamma + \mu)\frac{1}{r}u_\theta + (a-1)\frac{\tau}{r}u_\theta + (a+1)\tau v_r \right] = -\frac{\gamma}{\lambda} - \frac{1}{r}[(a-1)\tau v - 2(a\gamma + \mu)u], \quad (3.3e)$$

$$u\beta_r + \frac{v}{r}\beta_\theta = -\frac{\beta}{\lambda} + \frac{2}{r}(a\beta + \mu)(u + v \cot \theta), \quad (3.3f)$$

$$\begin{aligned} u\tau_r + \frac{v}{r}\tau_\theta - \frac{1}{2} \left[[(a-1)\sigma + (a+1)\gamma + 2\mu]\frac{1}{r}u_\theta + [(a-1)\gamma + (a+1)\sigma + 2\mu]v_r \right] \\ = -\frac{\tau}{\lambda} - \frac{1}{2r} \{ [(a+1)\sigma + (a-1)\gamma + 2\mu]v + 2aru + 2arv \cot \theta \}, \quad (3.3g) \end{aligned}$$

where the subscripts denote differentiation.

In axisymmetric flow there is only one component of vorticity, $\omega = \omega e_\phi$, and, using (2.4), we find that ω satisfies the following second-order equation:

$$A \frac{\partial^2 \omega}{\partial r^2} + 2B \frac{1}{r} \frac{\partial^2 \omega}{\partial \theta \partial r} + C \frac{1}{r^2} \frac{\partial^2 \omega}{\partial \theta^2} = \text{lower-order terms}, \quad (3.4)$$

where

$$\left. \begin{aligned} A &= \rho u^2 - \left[\frac{1}{2}(a+1)\sigma + \frac{1}{2}(a-1)\gamma + \mu \right], \\ B &= \rho uv - \tau, \\ C &= \rho v^2 - \left[\frac{1}{2}(a-1)\sigma + \frac{1}{2}(a+1)\gamma + \mu \right]. \end{aligned} \right\} \quad (3.5)$$

The differential equation for the characteristics of the vorticity is in the form

$$r \frac{d\theta}{dr} = \frac{-B \pm (B^2 - AC)^{1/2}}{A}. \quad (3.6)$$

There are real characteristics if and only if

$$0 < B^2 - AC. \quad (3.7)$$

If $B^2 - AC < 0$ the vorticity satisfies an elliptic equation. Since A, B, C depend on the solution, we might expect to see regions of the flow where the problem is hyperbolic and regions where it is elliptic.

It is also of interest to compute characteristics from the first-order system (3.3). We write this as

$$\mathbf{A} \frac{\partial \mathbf{q}}{\partial r} + \frac{\mathbf{B}}{r} \frac{\partial \mathbf{q}}{\partial \theta} = \mathbf{f}, \quad (3.8)$$

where

$$\mathbf{q} = (u, v, p, \sigma, \gamma, \beta, \tau),$$

and $\mathbf{A}, \mathbf{B}, \mathbf{f}$ depend on \mathbf{q} , but not on its derivatives. We suppose that \mathbf{q} is given on a curve $\phi(r, \theta) = 0$. Suppose s is the arclength on the curve. Then

$$\frac{d\mathbf{q}}{ds} = \frac{\partial \mathbf{q}}{\partial r} \frac{\partial r}{\partial s} + \frac{\partial \mathbf{q}}{\partial \theta} \frac{\partial \theta}{\partial s} \quad (3.9)$$

is known on the curve. The curve $\phi(r, \theta) = 0$ is characteristic if (3.8) and (3.9) cannot be uniquely solved for the derivatives of the components of \mathbf{q} .

This leads to the condition that

$$\det \left[\mathbf{A} \frac{d\theta}{ds} - \frac{\mathbf{B}}{r} \frac{dr}{ds} \right] = 0. \quad (3.10)$$

So $r \, d\theta/dr$ is an eigenvalue of \mathbf{B} relative to \mathbf{A} . Applying (3.10) to (3.3), we find that

$$(\alpha^2 + 1) (\alpha u + v)^3 \{ [\rho u^2 - [\frac{1}{2}(a + 1) \sigma + \frac{1}{2}(a - 1) \gamma + \mu]] \alpha^2 + 2(\rho uv - \tau) \alpha + [\rho v^2 - [\frac{1}{2}(a - 1) \sigma + \frac{1}{2}(a + 1) \gamma + \mu]] \} = 0, \tag{3.11}$$

where

$$\alpha = -r \frac{d\theta}{dr}.$$

It follows that streamlines are triply characteristic. Streamlines always have real characteristics and do not exhibit a change of type. On the other hand, the remaining real root is the one we have already associated with the vorticity, and is given by (3.5) and (3.6). It is a real root only if (3.7) is satisfied.

There is a large literature on slip at walls. Slip is a discontinuity of velocity across a streamline. Slip surfaces must lie along streamlines. Slip mechanisms are said to be important in some of the phenomena collectively called melt fracture, but the appearance of slip surfaces in such phenomena seems to be controversial (for an interesting discussion and extensive list of references see Petrie & Denn 1976). It is not known if the phenomenon called slip is a manifestation of some weak solution of the governing hyperbolic system that allows shocks or is something different, like cohesive fracture under shear.

The streamlines are characteristic but do not change type. In the models considered here the vorticity changes type and the characteristic directions for the vorticity do not lie along streamlines.

4. Quasilinear theory and linear theory

In §3 we classified the quasilinear system (2.1)–(2.3) without approximation. Certain definite results can be obtained from the quasilinear analysis under linearization. We first suppose that we have some exact solution of the governing equations. We are going to consider sink flow. We then linearize the equation around the exact solution and carry out an analysis of hyperbolicity and change of type of the linearized equations. The characteristics depend, of course, on the unperturbed flow. The analysis of the linearized equations is greatly simplified by the formulas we have already derived for the exact quasilinear system. To compute characteristic directions for the linearized systems we have only to put the values that the field variables take on in the unperturbed motion in the equations for the characteristics.

In the work to follow we treat the flow into a hole as a perturbation of sink flow. Sink flow is the axisymmetric irrotational solution in \mathbb{R}^3 given by

$$y = -Q/r^2, \quad v = 0. \tag{4.1}$$

The stresses in this flow are given by (3.3d, e, g) as

$$\left. \begin{aligned} \tau &= 0, \\ \frac{\sigma}{\mu} &= 4 e^{r^2/3\lambda Q} r^{-4a} \int_r^\infty s^{4a-1} e^{-s^2/3\lambda Q} ds, \\ \frac{\gamma}{\mu} &= -2 e^{r^2/3\lambda Q} r^{2a} \int_r^\infty s^{-2a-1} e^{-s^2/3\lambda Q} ds. \end{aligned} \right\} \tag{4.2}$$

We shall also find β from (3.3f) and then p from (3.3b, c). Formulas for these quantities are not given here, because they are not needed in the formula for the characteristics of the perturbed flow.

Now we consider axisymmetric perturbations of sink flow. We linearize (3.3) around (4.1) and (4.2). The characteristics for the vorticity of the linearized equations are given by (3.6), using the expressions (4.1) and (4.2):

$$r \frac{d\theta}{dr} = \pm \left\{ \frac{\frac{a-1}{2} \frac{\sigma}{\mu} + \frac{a+1}{2} \frac{\gamma}{\mu} + 1}{\frac{\rho Q^2}{\mu r^2} - \left[\frac{a+1}{2} \frac{\sigma}{\mu} + \frac{a-1}{2} \frac{\gamma}{\mu} + 1 \right]} \right\}^{\frac{1}{2}}. \tag{4.3}$$

It is generally true that the characteristics of the linearized problem are determined by the unperturbed problem. However, the streamlines, vorticity and stresses of the linearized problem cannot be determined by the unperturbed problem. To determine these quantities we must solve linearized boundary-value problems. In the present case we are considering the flow into a hole as representable in some sense by an axisymmetric perturbation of sink flow. Though we do not believe that such a large perturbation could be accurately represented by a linearized problem, we may hope to find results in some qualitative agreements with experiments.

The linearized problems should be solved relative to decay conditions on the solution as $r \rightarrow \infty$ and subject to the condition that $v = w = 0$ on the plane containing the hole and $u = -Q/r^2$ on this plane. In fact, we shall not solve this linearized problem, and seek now only to compare the characteristics given by (4.3) with observations.

5. Characteristics for the vorticity of axisymmetric flow perturbing sink flow

Our aim now is to evaluate σ and γ for sink flow to obtain explicit forms of the characteristics (4.3) of the vorticity of axisymmetric flow which perturb sink flow. We first introduce dimensionless parameters

$$[R, \Sigma, \Gamma, S] = \left[\frac{r}{(\lambda Q)^{\frac{1}{3}}}, \frac{\sigma}{\mu}, \frac{\gamma}{\mu}, \frac{\rho Q^{\frac{2}{3}}}{\eta \lambda^{\frac{1}{3}}} \right], \tag{5.1}$$

where $\Sigma(R)$ and $\Gamma(R)$ are given by

$$\frac{d\Sigma}{dR} + 4a \frac{\Sigma}{R} - R^2 \Sigma = -\frac{4}{R}, \tag{5.2a}$$

$$\frac{d\Gamma}{dR} - 2a \frac{\Gamma}{R} - R^2 \Gamma = \frac{2}{R}, \tag{5.2b}$$

$$\Sigma = 4R^{-4a} e^{\frac{1}{3}R^3} \int_R^{\infty} t^{4a-1} e^{-\frac{1}{3}t^3} dt, \tag{5.2c}$$

$$\Gamma = -2R^{2a} e^{\frac{1}{3}R^3} \int_R^{\infty} t^{-2a-1} e^{-\frac{1}{3}t^3} dt. \tag{5.2d}$$

The characteristic equation (4.3) may be written in the dimensionless variables as

$$R \frac{d\theta}{dR} = \pm \left[\frac{\frac{1}{2}(a-1)\Sigma + \frac{1}{2}(a+1)\Gamma + 1}{s/R^4 - [\frac{1}{2}(a+1)\Sigma + \frac{1}{2}(a-1)\Gamma + 1]} \right]^{\frac{1}{2}}. \tag{5.3}$$

These are real characteristics whenever

$$B^2 - AC = \{S/R^4 - [\frac{1}{2}(a+1)\Sigma + \frac{1}{2}(a-1)\Gamma + 1]\} \{\frac{1}{2}(a-1)\Sigma + \frac{1}{2}(a+1)\Gamma + 1\} > 0. \tag{5.4}$$

We shall consider three cases: $a = 1, 0$ and -1 . Then it can be readily shown by integration by parts from (5.2c, d) that

$$(\Sigma, \Gamma) = \left\{ \begin{array}{l} \left[\frac{4}{R^3} + \frac{4\Psi}{R^4}, -1 + R^2\Psi \right] \quad (a = 1), \\ (-2\Gamma, \Gamma) \quad (a = 0), \\ \left[1 - R^3 - \frac{1}{2}R^4\Phi, \frac{1}{R^2}\Phi \right] \quad (a = -1), \end{array} \right\} \quad (5.5)$$

where

$$\left. \begin{array}{l} \Psi = e^{\frac{1}{3}R^3} \int_R^\infty e^{-\frac{1}{3}t^3} dt = \frac{1}{R^2} - \frac{2}{R^5} + \frac{5 \cdot 2}{R^8} - \frac{8 \cdot 5 \cdot 2}{R^{11}} + \dots, \\ \Gamma = -2 e^{\frac{1}{3}R^3} \int_R^\infty t^{-1} e^{-\frac{1}{3}t^3} dt = -2 \left[\frac{1}{R^3} - \frac{3}{R^6} + \frac{6 \cdot 3}{R^9} - \frac{9 \cdot 6 \cdot 3}{R^{12}} + \dots \right], \\ \Phi = -2 e^{\frac{1}{3}R^3} \int_R^\infty t e^{-\frac{1}{3}t^3} dt = -\frac{2}{R} \left[1 - \frac{1}{R^3} + \frac{4}{R^6} - \frac{7 \cdot 4}{R^9} + \dots \right]. \end{array} \right\} \quad (5.6)$$

It is also useful to note by applying L'Hospital's rule to (5.2c, d) that as $R \rightarrow 0$

$$(\Sigma, \Gamma) \rightarrow \left\{ \begin{array}{l} \left[\frac{4}{3} 3^{\frac{1}{3}a} R^{-4a} \tilde{\Gamma}(\frac{1}{3}a), -\frac{1}{a} \right] \quad (0 < a \leq 1), \\ (-4 \ln R, 2 \ln R) \quad (a = 0), \\ \left[-\frac{1}{a}, -\frac{2}{3} 3^{-\frac{2}{3}a} R^{2a} \tilde{\Gamma}(-\frac{2}{3}a) \right] \quad (-1 \leq a < 0), \end{array} \right\} \quad (5.7)$$

where $\tilde{\Gamma}$ is the gamma function.

The regions of hyperbolicity $B^2 - AC > 0$ and the characteristic curves are now given by

$$\left[B^2 - AC, \frac{d\theta}{dR} \right] = \left\{ \begin{array}{l} \left[R^2\Psi \left\{ \frac{S}{R^4} - \left(\frac{4\Psi}{R^4} + \frac{4}{R^3} + 1 \right) \right\}, \pm \left\{ \frac{R^4\Psi}{S - (4\Psi + 4R + R^4)} \right\}^{\frac{1}{2}} \right] \quad (a = 1), \\ \left[\left(\frac{S}{R^4} + \frac{3}{2}\Gamma - 1 \right) \left(\frac{3}{2}\Gamma + 1 \right), \pm \left\{ \frac{R^2(\frac{3}{2}\Gamma + 1)}{S + R^4(\frac{3}{2}\Gamma - 1)} \right\}^{\frac{1}{2}} \right] \quad (a = 0), \\ \left[\left(\frac{S}{R^4} + \frac{\Phi}{R^2} - 1 \right) \left(R^3 + \frac{1}{2}R^4\Phi \right), \pm \left\{ \frac{R^2(R^3 + \frac{1}{2}R^4\Phi)}{S + R^2\Phi - R^4} \right\}^{\frac{1}{2}} \right] \quad (a = -1). \end{array} \right.$$

In the regions of hyperbolicity the characteristics are of the form $\theta = \theta_0 \pm g(R)$, where $\theta_0 \in [0, \pi)$ is a constant of integration. The functions $g(R)$ are more or less spirals in planes containing the axis of symmetry, the exact form of these spirals are shown in figure 1 and in the figures of §6. When $a = 1$ and $a = -1$ the vorticity is hyperbolic in a sphere around the origin. The characteristic surfaces are cones near the origin, and they are tangent to the spherical border between hyperbolic and elliptic regions. This border is an inflow boundary, so that the characteristics are tangent to spherical inflow boundary defining change of type. A typical characteristic net for $a = 1$ is shown in figure 1. The characteristic net for the corotational model $a = 0$ is in a spherical shell, hyperbolic in the shell, elliptic in a neighbourhood of the origin and outside the shell (see figure 4).

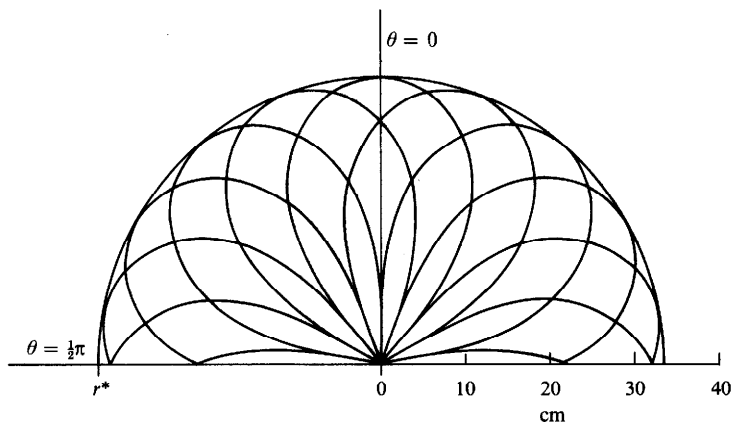


FIGURE 1. Typical characteristics net for the vorticity of an axisymmetric flow perturbing sink flow of an upper convected Maxwell model ($\alpha = 1$, $Q = 3354 \text{ cm}^3/\text{s}$, $r^* = 33.4 \text{ cm}$).

6. Discontinuities of vorticity in steady flow into a hole

In the experiments of Metzner *et al.* (1969) a fluid is sucked from a pipe of large diameter through a sudden contraction (figure 2). If the hole into which the flow goes is small the problem may be thought to be a form of sink flow. Because there are boundary walls, the flow through a sudden contraction is not a sink flow in a strict sense. We shall imagine first that the flow into the hole is not strongly influenced by the walls of the large pipe. We then have a hole in the semi-infinite regions above a plane. This flow is then regarded as an axisymmetric perturbation of sink flow without boundaries.

There are many reports of experiments on flows through a sudden contraction but the only one which has information about vorticity near the sink is the paper of Metzner *et al.* (1969). They consider high-speed flow of viscoelastic fluids into a sudden contraction (see figure 2). They say that

A tentative analysis of the observed velocity field suggests the flow upstream of the small duct to be radially directed toward the origin of the spherical coordinate system. If this is so the continuity equation gives

$$ur^2 = f(\theta). \quad (1)$$

They actually measure velocities in the cone, and they report that their measurements were accurate and that $f(\theta)$ may be taken as constant when $0 \leq \theta \leq 10^\circ$. They give experimental results and say that:

In the central conical region of the velocity field in which the fluid shearing deformations are negligibly small we may write equation (1) as

$$u = \frac{Q}{r^2}, \quad (1a)$$

where $f(\theta)$ has been replaced by the constant Q , which is proportional to the volumetric flow rate q . Under the conditions studied, equation (1a) applies to about 70% of all the fluid entering the tube from upstream. For purposes of further analysis we will restrict our attention to this central region ($0 \leq \theta \leq 10^\circ$); in this region the kinematics of the flow process, in spherical coordinates, are especially simple:

$$u_r = u = \frac{Q}{r^2}, \quad (1a)$$

$$u_\theta = u_\phi = 0. \quad (1b)$$

These equations may be shown to give rise to a diagonal deformation rate tensor.

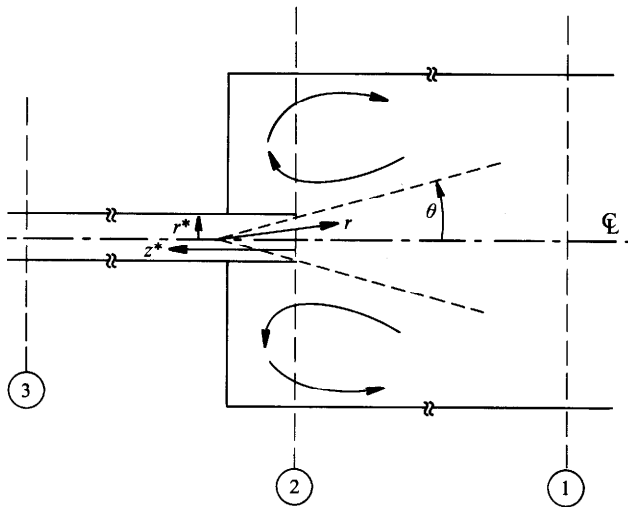


FIGURE 2. Schematic diagram of flow into a sudden contraction (Metzner *et al.* 1969).

Newtonian fluid: 70% Globe brand white corn syrup in water: $\mu = 1.96$ poise
 Viscoelastic fluid: 0.5% Separan AP 30, a partially hydrolysed polyacrylamide, in water
 Flow behaviour index $n = 0.424$, measured rheogonimetrically†

Flow rates:

Figure 3: 5.45 gal./min.

Figure 4: 7.25 gal./min.

Figure 5: 21.75 gal./min.

Geometry of equipment:

Downstream tube:

I.D. = 1.48 in.

O.D. = 2.00 in.

Upstream reservoir: square cross section 18 in. on a side. A distributor plate, to develop equal axial velocities at all radial positions, was mounted 10 in. upstream from the entrance to the small tube.

Illumination: through the vertical central plane of the apparatus. Width (depth) of illuminated plane: $\frac{1}{16}$ in.

Tracer particles: small air bubbles.

Temperature: all data reported were taken at 22.0 ± 2 °C.

† Curves of both the shearing stress and the first normal stress differences are available for the non-Newtonian fluid used (34).

TABLE 1. Experimental conditions corresponding to figures 3 to 5

The observed flow is irrotational and has a potential $\phi = Q/r$ in the central region. There is apparently a non-zero vorticity outside this central region. Derivatives of the vorticity must be discontinuous on the border of the central region.

The reader should now examine the characteristic nets shown for sink flow in figure 1. We take two members of this characteristic family separated by a cone angle of 10° . The hyperbolic character of the system allows potential flow in the cone and flow with vorticity outside of it. We wish to compare these theoretical cones of hyperbolicity with the three experiments reported by Metzner *et al.* The three experiments correspond to sinks of increasing strength. Their figure 8 is for $q = 7.25$ gal/min = 460 cm³/s, figure 9 is for $q = 14.5$ gal/min = 920 cm³/s, figure 10 is for $q = 21.75$ gal/min = 1380 cm³/s.

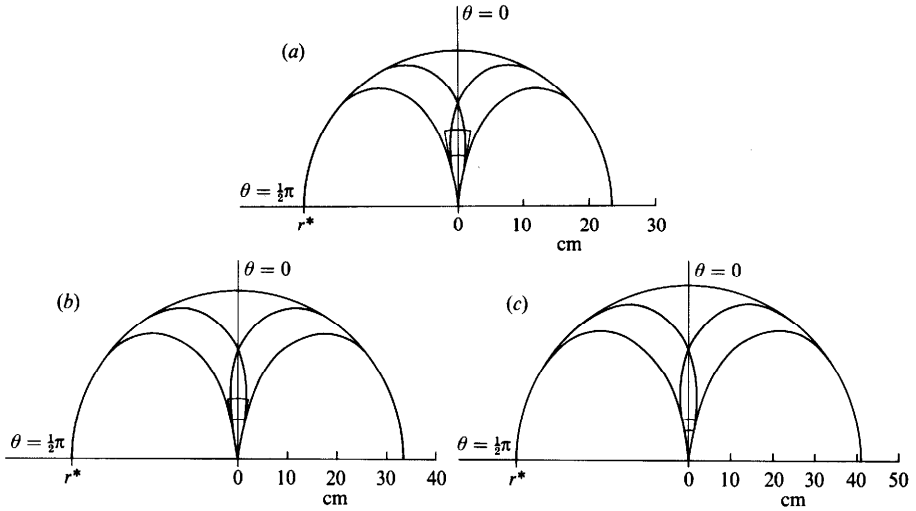


FIGURE 3. Comparison of the characteristics for the vorticity of an upper convected Maxwell model in an axisymmetric flow perturbing sink flow with experiments on flow into a hole: (a) $q = 460 \text{ cm}^3/\text{s}$, $S = 143.21$, $r^* = 23.430 \text{ cm}$, $R^* = 3.3723$ ($S \approx 4R^* + R^{*4}$); (b) $q = 920 \text{ cm}^3/\text{s}$, $S = 227.33$, $r^* = 33.396 \text{ cm}$, $R^* = 3.8150$ ($S \approx 4R^* + R^{*4}$); (c) $q = 1380 \text{ cm}^3/\text{s}$, $S = 297.89$, $r^* = 41.036 \text{ cm}$, $R^* = 4.0952$ ($S \approx 4R^* + R^{*4}$).

To compute the regions of hyperbolicity and the form of the characteristic surfaces we need values for the dimensionless flow-rate parameter S and the scale length $(\lambda Q)^{\frac{1}{2}}$, which is used to form the dimensionless radius. For these parameters we need values for the viscosity η , a relaxation time λ , the sink strength Q and the density ρ . The viscoelastic fluid used in the experiments was 0.5% Separan AP30, a partially hydrolysed polymer. We take the density of this solution as $\rho = 0.89 \text{ g/cm}^3$. We used the values $\eta = 3$ poise, $\lambda = 0.1 \text{ s}$ given by Metzner (1967) for a 0.45% Separan AP30 solution. The shear-wave speed is $C = (\eta/\lambda\rho)^{\frac{1}{2}} = 5.63 \text{ cm/s}$. To compute Q we use the observation that the sink-flow formula (1a) of Metzner *et al.* applies to 'about 70% . . .'. Then 0.7 q gal/min is equal to the mass into the conical region (of hyperbolicity?) with apex angle $\theta = 10^\circ$; that is

$$0.7q = u_r A = \frac{Q}{r^2} A = Q 2\pi(1 - \cos \theta),$$

where $2\pi r^2(1 - \cos \theta)$ is the area A of a spherical cap on the cone. Therefore

$$Q = 0.7q/2\pi(1 - \cos 10^\circ). \quad (5.1)$$

With these values given we compute the values given in table 1.

We shall now compare the results of our analysis with the experiment. This is done in three groups (3a-c; 4a-c; 5a-c) of figures corresponding to three entries of table 1. For each of the three sink strengths q we show the following.

(i) The region of hyperbolicity. These are spheres of outer radius r^* when $a = 1$ or $a = -1$ and annular regions between spheres of inner radius r_1^* and outer radius r_0^* when $a = 0$. The regions inside the circles of radius r^* in figures 3 and 5 are hyperbolic. The annular region in figure 4 between r_0^* and r_1^* is hyperbolic. The vorticity is elliptic where it is not hyperbolic.

(ii) On each of figures 3-5 we draw a cone with semivertex angle 10° centred at

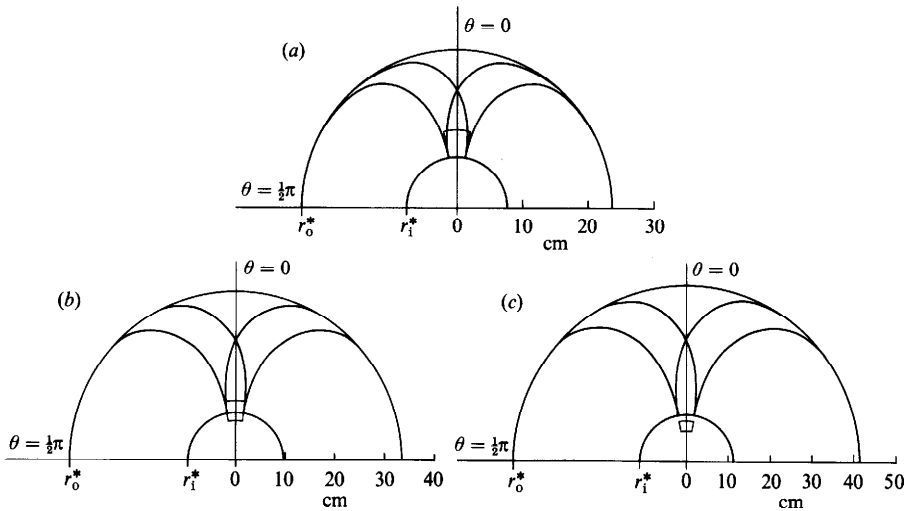


FIGURE 4. Comparison of the characteristics for the vorticity of a corotational Maxwell model in an axisymmetric flow perturbing sink flow with experiments on flow into a hole: (a) $q = 460 \text{ cm}^3/\text{s}$, $S = 143.21$, $r_i^* = 7.6009 \text{ cm}$, $R_i^* = 1.094$, $r_o^* = 23.625 \text{ cm}$, $R_o^* = 3.4003$ ($S \approx R_o^{*4}$); (b) $920 \text{ cm}^3/\text{s}$, $S = 227.33$, $r_i^* = 9.5765 \text{ cm}$, $R_i^* = 1.094$, $r_o^* = 33.572 \text{ cm}$, $R_o^* = 3.8351$ ($S \approx R_o^{*4}$); (c) $q = 1380 \text{ cm}^3/\text{s}$, $S = 297.89$, $r_i^* = 10.962 \text{ cm}$, $R_i^* = 1.094$, $r_o^* = 41.20 \text{ cm}$, $R_o^* = 4.1123$. ($S \approx R_o^{*4}$).

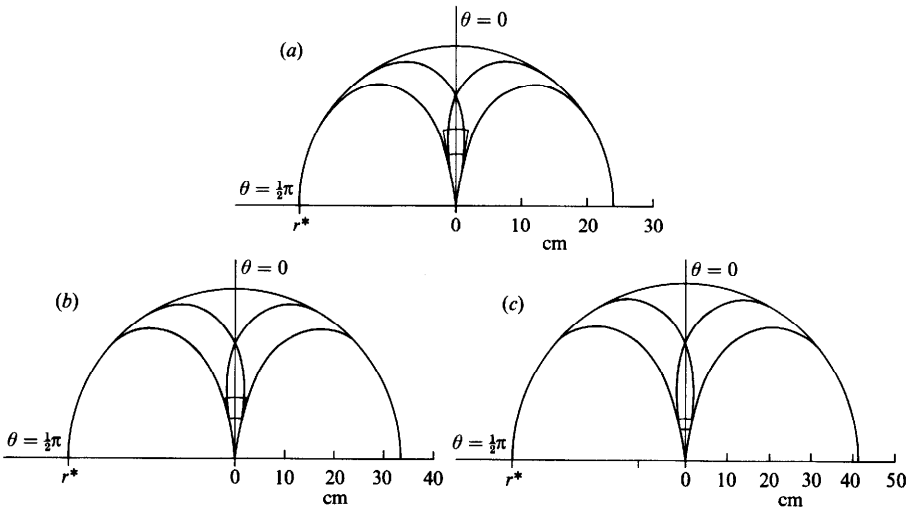


FIGURE 5. Comparison of the characteristics for the vorticity of a lower convected Maxwell model in an axisymmetric flow perturbing sink flow with experiments on flow into a hole: (a) $q = 460 \text{ cm}^3/\text{s}$, $S = 143.21$, $r^* = 23.89 \text{ cm}$, $R^* = 3.4378$, ($S \approx R^{*4}$); (b) $q = 920 \text{ cm}^3/\text{s}$, $S = 227.33$, $r^* = 33.70 \text{ cm}$, $R^* = 3.8542$ ($S \approx R^{*4}$); (c) $q = 1380 \text{ cm}^3/\text{s}$, $S = 297.89$, $r^* = 41.340 \text{ cm}$, $R^* = 4.1255$, ($S \approx R^{*4}$).

the origin. Metzner *et al.* found that the flow in this cone was the potential flow (1a, b). Actually they verified the relationships (1a, b) for different values of the radius r , say $A \leq r \leq B$ and different θ , $0 \leq \theta \leq 10^\circ$. The conical region between the spheres of radius A and B where (1a, b) was verified in the experiments is shown in figures 3–5.

(iii) We next draw on each of the figures 3–5 all of the cross-sections of characteristic

surfaces of revolution that are tangent to the central cone at the origin. There are two characteristic surfaces of revolution, which come in at an angle of 10° at the origin (see figure 1), and the four dark lines on the figure represent the cross-section of the two surfaces.

We regard the comparisons of theory and experiment of this paper as exploratory and not definitive. It is of course striking that the experiments of Metzner *et al.* (1969) do appear to involve a vorticity of changing type. It would be interesting to see if this striking type of experimental result could be repeated by other investigators using different fluids and experimental arrangements. We hasten to add that the Separan solution used in the experiment is not an Oldroyd model and surely cannot be characterized by a viscosity and relaxation time. In fact only special models give the vorticity precisely as the quantity which changes type. We have already remarked that models with true viscosity, e.g. retardation times, will smooth discontinuities, with only a little smoothing if the retardation 'viscosity' parameter is small. Probably all the polymer solutions used in experiments have some small smoothing. In view of all these uncertainties in theory and experiments, it would be premature to make strong claims.

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