Rheologica Acta Rheol Acta 23:345–354 (1984)

Remarks on the stability of viscometric flow

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Abstract: We study the stability of viscometric flow using the type of short memory introduced by Akbay, Becker, Krozer and Sponagel. The instability found by these researchers is recognized as a "change of type" leading to non-evolutionary character of the governing equations. We also address the question of justification for the short memory assumption and find that it cannot be justified for some of the more popular rheological models.

Key words: Viscometric flow, instability, short memory assumption, change of type

1. Introduction

In viscometric flow of polymeric liquids, instabilities are frequently observed at high shear rates. In many cases these instabilities occur even though the Reynolds number is low and have distinctly different characteristics from the instabilities that arise from inertial effects. Although various mathematical theories have been developed, the proper explanation of these instabilities is still an open problem (for review, see Petrie and Denn [18]). Among the mechanisms proposed are: Changes of type in the governing equations which may indicate fracture or phase changes in the material, development of shocks, free surface instabilities, slip on the wall, or bifurcations of the classical type. In a general context, equations describing linear perturbations of viscometric flow were derived by Pipkin and Owen [19] and Dunwoody and Joseph [11]. These equations provide a general framework; however, in order to arrive at definite conclusions, more specific assumptions are necessary. Such assumptions can be of two kinds: First, a particular constitutive model may be assumed, and second, one may seek approximate equations valid in certain limiting situations. Either approach seems to have its problems. In the former case, the analysis of one constitutive model tells nothing about others, and it is not clear at this time which models are adequate for which materials – in particular at high shear rates. In addition, the determination of the stability boundaries for simple shear flow – either analytically or numerically – does not appear to be easy even for the simplest constitutive models [13].

On the other hand, "approximate" constitutive laws must be used with great caution. There may be problems of consistency (e.g. does instability occur in a range of parameters consistent with the assumptions needed to justify the approximation?) or of stability. That is, the stability properties of approximate equations need not be the same as those of the "exact" equations, as is well known for "fluids of grade n", if $n > 1$ [10, 14, 21].

An approximate theory of perturbed viscometric flow based on a hypothesis of short memory (small relaxation time) has been developed by U. Akbay, E. Becker, S. Krozer and S. Sponagel in Darmstadt. In a series of papers, they study various flow situations such as plane Couette and Poiseuille flow, flow between eccentric rotating cylinders and boundary layer flow [1–7, 20]. The purpose of this paper is to work out some consequences of this theory and to give a critical assessment of it [1]). The short memory hypothesis can be formulated in terms of a perturbed extra stress in the form

$$\boldsymbol{S}(t) = \int_0^\infty \mathsf{K}(\varkappa, s)\, \boldsymbol{D}(t-s)\, ds \tag{1.1}$$

where K is assumed to decay rapidly to zero fast and to have appreciable values only for $s \ll \varkappa^{-1}$. E.g. we could take

$$\mathsf{K}(\varkappa, s) = \mathsf{A}(\varkappa)\, e^{-s/\lambda} \tag{1.2}$$

[1]) The theory of Akbay et al. is explained in Chapters 2 and 3.

where $\varkappa\lambda \ll 1$. We have denoted by $\boldsymbol{D}$ the symmetric part of the perturbed velocity gradient, and K is a fourth-order tensor so that $(\mathsf{K}\boldsymbol{D})_{ij} = K_{ijlm} D_{lm}$. $\varkappa$ is the shear rate of the basic flow. The approximation applied to (1.1), assuming (1.2), is that only terms linear in $\varkappa\lambda$ are kept, while higher powers are neglected.

The short memory hypothesis is applied only to the perturbation equations, not to the "basic" flow that is being perturbed. The viscometric functions, i.e. the viscosity $\eta(\varkappa)$ and the normal stress differences $N_1(\varkappa)$ and $N_2(\varkappa)$ are assumed to be given independently of any short memory hypothesis. As we shall see later, this leads to serious consistency problems at least for some of the constitutive laws presently used in rheology. The theory of Akbay et al. would work best for a constitutive law that has a different relaxation time for perturbations of simple shear flow than it has for simple shear flow itself. It is not clear to us how such constitutive laws should be constructed.

Akbay et al. make the additional approximation that the Reynolds number is negligible. With these assumptions they find that the following condition is necessary for the instability of plane Couette flow to two-dimensional disturbances.

$$\left[\frac{d}{d\varkappa}\left(\frac{N_1}{\varkappa}\right)\right]^2 \geqq 16\eta(\varkappa)\frac{d\tau}{d\varkappa}\cdot\frac{1}{\varkappa^2}. \tag{1.3}$$

A similar criterion is obtained for plane Poiseuille flow. They also give numerical evidence suggesting that (1.3) is sufficient for instability to occur. More recently they have also considered three-dimensional disturbances [20].

In this paper, we reconsider the stability of plane Couette flow and we show in Chapter 5 that, in the limit of zero Reynolds number, (1.3) is indeed a necessary and sufficient condition for instability. The instability is associated with a change of type, a notion which we explain in detail in Chapter 4. The character of this instability is quite different from those familiar from Newtonian fluid mechanics. At the critical "Weissenberg number" not only one mode becomes unstable, but a whole family of modes. At any supercritical Weissenberg number, there is an infinite number of unstable modes. These unstable modes have arbitrarily short wavelengths, and their growth rates actually tend to infinity as the wavelength tends to zero. As a consequence, this kind of instability must lead to a much more severe change in the flow field than the development of secondary flows. "Fracture" may be one possible consequence of such an instability.

For finite Reynolds numbers, the criterion (1.3) is still valid for a change of type. We do not know whether particular modes may become unstable at lower Weissenberg numbers, but numerical results indicate that this is not so at low Reynolds numbers. In fact, for disturbances of a fixed streamwise wavelength the critical Weissenberg number seems to increase with the Reynolds number.

In a final chapter (Chapter 6), we attempt to give a critical assessment of the theory of Akbay et al. First, we find that there is a problem with the short memory assumption. It turns out that if this assumption is applied to the basic flow, then the normal stresses are small compared to the shear stress, and (1.3) cannot be satisfied. We give a general argument for this, and also illustrate it for a particular model. It thus appears that (1.3) and the short-memory hypothesis are inconsistent, unless the fluid somehow has a different memory for perturbations of shear than it has for shear flow itself. We do not know if and how this could be made precise. Another question that might be asked is whether the result of Akbay et al. would hold without the short-memory hypothesis. The upper and lower convected Maxwell model satisfy (1.3) at high enough shear rates, in fact the quotient

$$\left[\frac{d}{d\varkappa}\frac{N_1}{\varkappa}\right]^2 \varkappa^2 \Big/ 16\eta(\varkappa)\frac{d\tau}{d\varkappa}$$

becomes arbitrarily large. However, it was recently shown [16, 22] that a change of type does not occur in those models. Thus any existing instability would not be of the kind considered by Akbay et al. Whether there are instabilities at all is a difficult problem. Numerical attempts to find them have so far led to negative results [13].

2. Perturbation of the stress in nearly viscometric flow

In this section we derive general equations for the perturbed part of the extra stress, with no restriction on the memory of the fluid. The essential features follow the method of Dunwoody and Joseph [11]. We refer the reader to that paper for a detailed derivation.

Consider a viscoelastic fluid in which the extra stress is given by a functional of the deformation gradients

$$\mathfrak{S} = \mathfrak{S}^{\mathsf{T}} = \boldsymbol{T} + P\boldsymbol{I} = \mathfrak{F}[\boldsymbol{G}(s)]_{s=0}^{\infty}. \tag{2.1}$$

Here $\boldsymbol{T}$ is the Cauchy stress, P the pressure and $\boldsymbol{G}$ the relative Cauchy strain: $\boldsymbol{G}(s) = \nabla\boldsymbol{X}^{\mathsf{T}}(s)\,\nabla\boldsymbol{X}(s) - \boldsymbol{I}$, where $\boldsymbol{X}(s) = \boldsymbol{X}(\boldsymbol{x}, t-s)$ is the position at time $t-s$ of the particle presently (i.e. at time t) at $\boldsymbol{x}$. We represent the basic flow (as yet unspecified) by a "o" subscript, and write the perturbation as follows:

$$
\begin{aligned}
&\text{(a)}\quad \boldsymbol{X}(s)=\boldsymbol{X}_o(s)+\boldsymbol{\xi}(s)\,,\\
&\text{(b)}\quad \boldsymbol{U}(s)=\boldsymbol{U}_o(s)+\boldsymbol{u}(s)\,,\\
&\text{(c)}\quad P=P_o+p\,,\\
&\text{(d)}\quad \mathfrak{S}=\boldsymbol{S}_o+\boldsymbol{S}\,.
\end{aligned}
\tag{2.2}
$$

Here the solenoidal velocity fields $\boldsymbol{U}_o$ and $\boldsymbol{u}$ satisfy

$$
\begin{aligned}
&\text{(a)}\quad \boldsymbol{U}_o(\boldsymbol{X}_o,t-s)=-\frac{\partial \boldsymbol{X}_o}{\partial s}\,,\quad \boldsymbol{X}_o(0)=\boldsymbol{x}\,,\\
&\text{(b)}\quad \boldsymbol{u}(\boldsymbol{X}_o,t-s)=-\frac{\partial \boldsymbol{\xi}}{\partial s}-(\boldsymbol{\xi}\cdot\nabla)\,\boldsymbol{U}_o\,,\quad \boldsymbol{\xi}(0)=\boldsymbol{0}\,.
\end{aligned}
\tag{2.3}
$$

The linearized perturbation of the extra stress is given by

$$\boldsymbol{S}_o=\mathfrak{F}[\boldsymbol{G}_o(s)]_{s=0}^{\infty}\,,\quad \boldsymbol{G}_o=\nabla\boldsymbol{X}_o{}^{\mathrm{T}}\nabla\boldsymbol{X}_o-\boldsymbol{I}\,,\tag{2.4}$$

$$\boldsymbol{S}=\mathfrak{F}_1[\boldsymbol{G}_o(s)\,|\,\delta\boldsymbol{G}]_{s=0}^{\infty}\,,\quad \delta\boldsymbol{G}=\nabla\boldsymbol{\xi}^{\mathrm{T}}\nabla\boldsymbol{X}_o+\nabla\boldsymbol{X}_o{}^{\mathrm{T}}\nabla\boldsymbol{\xi}\tag{2.5}$$

where $\mathfrak{F}_1$ is the first functional derivative of $\mathfrak{F}$, evaluated on the basic flow. That is, $\mathfrak{F}_1$ is a linear operator acting on δG. We assume an integral representation for $\mathfrak{F}_1$:

$$\mathfrak{F}_1[G_o\,|\,\delta G]_{s=0}^{\infty}=\int_0^{\infty}\mathsf{K}(s,\boldsymbol{G}_o)\,\delta\boldsymbol{G}(s)\,ds\,.\tag{2.6}$$

The isotropy of $\mathfrak{F}$ implies that

$$\boldsymbol{Q}\boldsymbol{S}_o\boldsymbol{Q}^{\mathrm{T}}=\mathfrak{F}[\boldsymbol{Q}\boldsymbol{G}_o\boldsymbol{Q}^{\mathrm{T}}]_{s=0}^{\infty}\,,\tag{2.7}$$

$$\boldsymbol{Q}\mathfrak{F}_1[\boldsymbol{G}_o\,|\,\delta\boldsymbol{G}]_{s=0}^{\infty}\boldsymbol{Q}^{\mathrm{T}}=\mathfrak{F}_1[\boldsymbol{Q}\boldsymbol{G}_o\boldsymbol{Q}^{\mathrm{T}}\,|\,\boldsymbol{Q}\delta\boldsymbol{G}\boldsymbol{Q}^{\mathrm{T}}]_{s=0}^{\infty}\tag{2.8}$$

which holds for all orthogonal $\boldsymbol{Q}$ and symmetric $\delta\boldsymbol{G}$.

For simple shear, with shear rate $\varkappa=U/h$, we have

$$\boldsymbol{X}_o=(\boldsymbol{I}-s\boldsymbol{N})\,\boldsymbol{x}\,,\tag{2.9}$$

$$\boldsymbol{G}_o=-s\,(\boldsymbol{N}+\boldsymbol{N}^{\mathrm{T}})+s^2\boldsymbol{N}^{\mathrm{T}}\boldsymbol{N}\,,\tag{2.10}$$

$$\delta\boldsymbol{G}=\nabla\boldsymbol{\xi}+\nabla\boldsymbol{\xi}^{\mathrm{T}}-s\,(\nabla\boldsymbol{\xi}^{\mathrm{T}}\boldsymbol{N}+\boldsymbol{N}^{\mathrm{T}}\nabla\boldsymbol{\xi})\tag{2.11}$$

where

$$\boldsymbol{N}=\varkappa\begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix}.\tag{2.12}$$

The extra stress of the basic shear flow is given by the viscometric functions

$$\boldsymbol{S}_o=\boldsymbol{T}_o-\left(\frac{1}{3}\operatorname{tr}\boldsymbol{T}_o\right)\boldsymbol{I}=\frac{1}{3}\cdot\begin{pmatrix}2N_1+N_2&3\tau&0\\3\tau&-N_1+N_2&0\\0&0&-N_1-2N_2\end{pmatrix}\tag{2.13}$$

where τ is the shear stress and $N_1=T_{11}-T_{22}$, $N_2=T_{22}-T_{33}$ are the first and second normal-stress differences.

Consistency relations between the kernel functions K and the viscometric functions can be obtained as follows. First, we consider families $\boldsymbol{Q}(\lambda)$ of orthogonal matrices such that $\boldsymbol{Q}(0)=\boldsymbol{I}$ and then differentiate (2.7) with respect to λ, at $\lambda=0$. Since $\dot{\boldsymbol{Q}}(0)$ is an arbitrary skew-symmetric matrix, this will yield three matrix equations. A fourth matrix equation is obtained by differentiating (2.7) with respect to $\varkappa$, i.e. by perturbing the shear rate. Using the notation $(\mathsf{K})_{ijlm}=\boldsymbol{K}_{lm}$, we find the following four equations [11]

$$\begin{pmatrix}\dfrac{2N_1'+N_2'}{3}&\tau'&0\\ \tau'&\dfrac{-N_1'+N_2'}{3}&0\\ 0&0&\dfrac{-N_1'-2N_2'}{3}\end{pmatrix}=\int_0^{\infty}-2s\,[\boldsymbol{K}_{12}-\varkappa s\boldsymbol{K}_{22}]\,ds\,,\tag{2.14}$$

$$\begin{pmatrix}-2\tau&N_1&0\\N_1&2\tau&0\\0&0&0\end{pmatrix}=\int_0^{\infty}2\varkappa s\,[\boldsymbol{K}_{11}-\boldsymbol{K}_{22}-\varkappa s\boldsymbol{K}_{12}]\,ds\,,\tag{2.15}$$

$$\begin{pmatrix}0&0&N_1+N_2\\0&0&\tau\\N_1+N_2&\tau&0\end{pmatrix}=\int_0^{\infty}-2\varkappa s\boldsymbol{K}_{23}\,ds\,,\tag{2.16}$$

$$\begin{pmatrix}0&0&\tau\\0&0&N_2\\ \tau&N_2&0\end{pmatrix}=\int_0^{\infty}2\varkappa s\,(\varkappa s\boldsymbol{K}_{23}-\boldsymbol{K}_{13})\,ds\,.\tag{2.17}$$

Clearly, only (2.14) and (2.15) are relevant for two-dimensional flows.

Eq. (2.8) for the isotropy of $\mathfrak{F}_1$ can be used to show [11], by choosing special forms for $\boldsymbol{Q}$, that

$$K_{ijlm}=0\tag{2.18}$$

whenever there are an odd number of 3's in the set $\{i,j,l,m\}$. Finally the symmetry of $\boldsymbol{S}$ implies that

$$K_{ijlm}=K_{jilm}\,,\tag{2.19}$$

and the incompressibility condition implies [19]

$$\sum_{l=1}^{3}\boldsymbol{K}_{ll}+2\varkappa s\boldsymbol{K}_{12}+\varkappa^2s^2\boldsymbol{K}_{11}=0\,.\tag{2.20}$$

From (2.6) and (2.11), we find the following expression for the perturbed extra stress:

$$S = \int_0^\infty \sum_{\alpha,\beta=1}^{3} K_{\alpha\beta}\, \delta G_{\alpha\beta}\, ds \tag{2.21}$$

$$= 2\int_0^\infty \{K_2 \xi_{1,1} + K_1 (\xi_{1,2} + \xi_{2,1}) + K_{13} (\xi_{1,3} + \xi_{3,1}) + K_{23} (\xi_{2,3} + \xi_{3,2} - \varkappa s \xi_{1,3}) + (K_{33} - K_{22})\, \xi_{3,3}\}\, ds$$

where

$$K_1 = -\varkappa s K_{22} + K_{12}, \tag{2.22}$$

$$K_2 = K_{11} - K_{22} - \varkappa s K_{12}. \tag{2.23}$$

Note that K_1, K_2, K_{13}, and K_{23} are all related to viscometric functions by the consistency relations (2.14 to 2.17), but that $K_{22} - K_{33}$ does not occur in any of the consistency relations.

3. Short memory

We consider in this section the short memory assumption as applied by Akbay, Becker, Krozer and Sponagel. In general, smooth deformation histories may be approximated asymptotically in terms of the Rivlin-Ericksen tensors.

$$G(s) = \sum_{n=1}^{N} \frac{(-s)^n}{n!} A_n[U(t)] + O(s^{N+1}) \tag{3.1}$$

where

$$A_n[U(t)] = (-1)^n \frac{d^n}{ds^n} G(s)\bigg|_{s=0}$$

is the n-th Rivlin-Ericksen tensor. For the perturbation term δG, we obtain

$$\delta G = -s A_1[u(t)] + O(s^2)$$
$$= -2s\, D[u(t)] + O(s^2) \tag{3.2}$$

where $D[u] = \frac{1}{2}(\nabla u + \nabla u^{\mathrm{T}})$. Under the assumption that this leading term is sufficient for a short memory fluid, (2.21) may be written as

$$S = -2\int_0^\infty s \left\{\sum_{\alpha,\beta=1}^{3} K_{\alpha\beta} D_{\alpha\beta}\right\} ds$$

$$= -2\int_0^\infty s (K_{11} - K_{22})\, ds \frac{\partial u_1}{\partial x_1}$$

$$-2\int_0^\infty s K_{12}\, ds \left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)$$

$$-2\int_0^\infty s K_{13}\, ds \left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right)$$

$$-2\int_0^\infty s K_{23}\, ds \left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right)$$

$$-2\int_0^\infty s (K_{33} - K_{22})\, ds \frac{\partial u_3}{\partial x_3}. \tag{3.3}$$

The first four integrals are precisely those occurring in the consistency relations, and if terms of order $\varkappa s$ can be neglected compared to others, they are approximately equal to expressions (2.14–2.17) involving viscometric functions. With this approximation, we arrive at the following expression for the perturbed extra stress:

$$S = -\frac{1}{\varkappa}\begin{pmatrix} -2\tau & N_1 & 0 \\ N_1 & 2\tau & 0 \\ 0 & 0 & 0 \end{pmatrix} \frac{\partial u_1}{\partial x_1}$$

$$+ \begin{bmatrix} \dfrac{2N_1' + N_2'}{3} & \tau' & 0 \\ \tau' & \dfrac{-N_1' + N_2'}{3} & 0 \\ 0 & 0 & \dfrac{-2N_2' - N_1'}{3} \end{bmatrix} \left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)$$

$$+ \frac{1}{\varkappa}\begin{pmatrix} 0 & 0 & \tau \\ 0 & 0 & N_2 \\ \tau & N_2 & 0 \end{pmatrix} \left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right)$$

$$+ \frac{1}{\varkappa}\begin{pmatrix} 0 & 0 & N_1 + N_2 \\ 0 & 0 & \tau \\ N_1 + N_2 & \tau & 0 \end{pmatrix} \cdot \left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right)$$

$$-2\int_0^\infty s (K_{33} - K_{22})\, ds \frac{\partial u_3}{\partial x_3}. \tag{3.4}$$

This is the form used by Akbay et al. [2]. We see that even in this approximation there remains a term not related to viscometric functions (see [20]). More such terms would occur, if higher order terms were included.

In two-dimensional flows, eq. (3.4) reduces to

$$S = -\frac{1}{\varkappa}\begin{pmatrix} -2\tau & N_1 \\ N_1 & 2\tau \end{pmatrix} \frac{\partial u_1}{\partial x_1} + \begin{bmatrix} \dfrac{2N_1' + N_2'}{3} & \tau' \\ \tau' & \dfrac{-N_1' + N_2'}{3} \end{bmatrix} \left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right). \tag{3.5}$$

The equation of motion, after linearizing at plane Couette flow and eliminating the pressure, becomes

$$\varrho\left(\frac{\partial}{\partial t} + \varkappa x_2 \frac{\partial}{\partial x_1}\right)\left(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}\right)$$

$$= \left(\frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_1^2}\right) S_{12} + \frac{\partial^2 (S_{11} - S_{22})}{\partial x_1\, \partial x_2}. \tag{3.6}$$

We introduce a stream function Ψ such that $(u_1, u_2) = \left(-\frac{\partial \Psi}{\partial x_2}, \frac{\partial \Psi}{\partial x_1}\right)$.

Then we obtain

$$\varrho\left(\frac{\partial}{\partial t} + \varkappa x_2 \frac{\partial}{\partial x_1}\right)\left(\frac{\partial^2 \Psi}{\partial x_2^2} + \frac{\partial^2 \Psi}{\partial x_1^2}\right) = \left(N_1' - \frac{N_1}{\varkappa}\right)\frac{\partial^2}{\partial x_1 \partial x_2} L\Psi + \tau' L^2 \Psi + \frac{4\tau}{\varkappa}\frac{\partial^4 \Psi}{\partial x_1^2 \partial x_2^2} \tag{3.7}$$

where

$$L = \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_1^2}.$$

The boundary conditions are

$$\Psi = \frac{\partial \Psi}{\partial x_2} = 0 \text{ at } x_2 = 0 \text{ and } x_2 = h. \tag{3.8}$$

4. What is a change of type and why should we care about it?

We shall see below that eq. (3.7) will have an instability if a certain "Weissenberg number" exceeds a critical value. This instability will be associated with what is called a change of type. The reason why this is important is that change-of-type instabilities have quite different mathematical properties and will lead to quite different physical effects than other instabilities.

In order to explain what we mean by a change of type, we must first explain what we mean by the symbol of a differential operator. To obtain the symbol, one simply replaces the derivatives $\partial/\partial x_i$ by numbers D_i. Thus we can assign to any differential operator a polynomial. For example, the differential operator on the right-hand side of (3.7) would have the symbol

$$\left(N_1' - \frac{N_1}{\varkappa}\right) D_1 D_2 (D_2^2 - D_1^2) + \tau' (D_2^2 - D_1^2)^2 + \frac{4\tau}{\varkappa} D_1^2 D_2^2. \tag{4.1}$$

Two differential operators are said to be of the same type, if – up to a transformation of the independent variables – their symbols have the same asymptotic behavior at infinity. If the asymptotic behavior of the symbol changes, we say that a change of type occurs. For example, the operator $\frac{\partial^2}{\partial x^2} + y\frac{\partial^2}{\partial y^2}$ changes type as y goes through zero: For $y > 0$ the symbol $D_1^2 + yD_2^2$ is a positive definite quadratic form (the operator is then called "elliptic"), for $y < 0$ the symbol is an indefinite quadratic form (the operator is then called "hyperbolic").

Mathematical problems such as existence, uniqueness and regularity of solutions or continuous dependence on the data depend essentially on the type of the differential operator. For second-order operators it is well known that in the elliptic case the "good" problem to look at is a boundary value problem, while in the hyperbolic case it is an initial value problem.

Heuristically, the reason for this is that the type describes how the operator acts on rapidly varying functions. For such cases, one expects that locally the variation of the coefficients can be neglected, and that, in the interior of the domain where the problem is posed, the influence of boundary solutions can also be neglected. For constant coefficient problems without boundaries, the symbol is just the Fourier transform of the differential operator.

How does the foregoing apply to equation (3.7)? If we formally set $\partial/\partial t = \sigma$, $\partial/\partial x_1 = D_1 = i\alpha$, $\partial/\partial x_2 = D_2 = i\beta$, then the left side of the equation becomes

$$\varrho(\sigma + \varkappa x_2 i\alpha)(-\beta^2 - \alpha^2), \tag{4.2}$$

and the right side becomes

$$-\left(N_1' - \frac{N_1}{\varkappa}\right)(\alpha^2 - \beta^2)\alpha\beta + \tau'(\alpha^2 - \beta^2)^2 + \frac{4\tau}{\varkappa}\alpha^2\beta^2. \tag{4.3}$$

The expression (4.3) is a homogeneous fourth-degree polynomial. It turns out that this polynomial is positive definite if and only if

$$We^2 = \left[\frac{d}{d\varkappa}\frac{N_1}{\varkappa}\right]^2 \varkappa^2/(\eta(\varkappa)\,\tau'(\varkappa)) < 16. \tag{4.4}$$

If $We^2 > 16$, the polynomial is indefinite. If we now formally put (4.2) equal to (4.3), then we find that σ is negative if α, β are large and $We^2 < 16$. If $We^2 > 16$, then there is a sector in the α, β-plane for which σ becomes positive for α, β large, moreover σ becomes arbitrarily large in this case as $\alpha, \beta \to \infty$.

This heuristic argument suggests the following: If $We^2 < 16$, then at least highly oscillatory modes will be stable. If instabilities exist, then they will only affect a finite number of modes. If, on the other hand, $We^2 > 16$, then we expect to find an infinite number of unstable modes. In addition, the growth constants of these modes can be arbitrarily large. Thus the change of type leads to an instability which is in a sense much stronger than those studied in bifurcation theory. One cannot expect a secondary flow whose dynamics would be governed by the evolution of one or several modes. Rather, the initial value problem becomes ill-posed, i.e. there are flow fields for $We^2 > 16$, which would not occur even as transient states, since "random" disturbances containing all modes would blow up instantly. In situations where this "forbidden" unstable region lies between two stable ones, a phase transition can occur. That is, there can be discontinuous solutions whose values lie in both of the stable regions. The two regions are then referred to as different "phases". If the forbidden region stretches all the way to infinity and a second phase does not exist, the result of a change of type may be something like "fracture".

5. Instability of plane Couette flow with short memory

In this chapter, we give a more detailed analysis of eq. (3.7), and we provide a rigorous basis for some of the heuristic conclusions obtained in the last chapter. The stream function Ψ is represented by normal modes

$$\Psi = e^{\sigma t} e^{i\alpha x_1} \varphi(x_2). \tag{5.1}$$

We wish to find eigenvalues σ and eigenfunctions $\varphi(\cdot)$. For simplicity, we set

$$E = \frac{4\eta(\varkappa)}{\tau'(\varkappa)}, \quad \Gamma = \frac{\varkappa}{\tau'(\varkappa)}\left(\frac{N_1(\varkappa)}{\varkappa}\right)',$$
$$R = \frac{\varrho\varkappa}{\tau'(\varkappa)}, \quad \sigma = -i\alpha\frac{R}{\varrho}c. \tag{5.2}$$

Moreover, we assume that the width h of the channel is 1. Eqs. (3.7, 3.8) then become

$$i\alpha R(x_2 - c)(\varphi'' - \alpha^2\varphi) \tag{5.3}$$
$$= \varphi^{iv} + \alpha^2(2-E)\varphi'' + \alpha^4\varphi + i\alpha\Gamma(\varphi''' + \alpha^2\varphi'),$$

$$\varphi = \varphi' = 0 \quad \text{at} \quad x_2 = 0, \ x_2 = 1. \tag{5.4}$$

Let us assume that $\tau' > 0$ and hence $R > 0$. In this case we want to show that when $\Gamma^2 < 4E$ the imaginary part of c must be negative provided that α is large enough. For this, we form an energy integral

$$\int_0^1 [\varphi^{iv} + \alpha^2(2-E)\varphi'' + \alpha^4\varphi + i\alpha\Gamma(\varphi''' + \alpha^2\varphi')]\bar{\varphi}\,dx_2$$
$$= i\alpha R\int_0^1 (x_2 - c)(\varphi'' - \alpha^2\varphi)\bar{\varphi}\,dx_2 \tag{5.5}$$

and integrate by parts, to find

$$\int_0^1 |\varphi'' + \alpha^2\varphi|^2 dx_2 + E\alpha^2\int_0^1 |\varphi'|^2 dx_2$$
$$- i\alpha\Gamma\int_0^1 (\varphi'' + \alpha^2\varphi)\bar{\varphi}'\,dx_2 \tag{5.6}$$
$$= -i\alpha R\int_0^1 (x_2 - c)[|\varphi'|^2 + \alpha^2|\varphi|^2]dx_2 - i\alpha R\int_0^1 \varphi'\bar{\varphi}\,dx_2.$$

There is a constant γ_0 such that

$$\left|\int_0^1 \varphi'\bar{\varphi}\,dx_2\right| \leqq \gamma_0\int_0^1 |\varphi'|^2 dx_2$$

for any φ satisfying $\varphi(0) = \varphi(1) = 0$. Taking the real part of (5.6) and using this inequality, we find

$$-\alpha R\,\mathrm{Im}\,c\int_0^1 (|\varphi'|^2 + \alpha^2|\varphi|^2)\,dx_2$$
$$\geqq \int_0^1 |\varphi'' + \alpha^2\varphi|^2 dx_2 + \left(E - \frac{R\gamma_0}{\alpha}\right)\alpha^2\int_0^1 |\varphi'|^2 dx_2$$
$$+ \alpha\Gamma\,\mathrm{Im}\int_0^1 (\varphi'' + \alpha^2\varphi)\bar{\varphi}'\,dx_2. \tag{5.7}$$

By applying the Cauchy-Schwarz theorem to the last term, we find that the right-hand side of this is positive as long as $\Gamma^2 < 4\left(E - \frac{R\gamma_0}{\alpha}\right)$. Thus, if $\Gamma^2 < 4E$ and α is large enough, we must have $\mathrm{Im}\,c < 0$. If $\alpha R x_2$ is put to zero in (5.3), the term proportional to R in (5.7) disappears, and stability for any α is guaranteed if $\Gamma^2 < 4E$.

In the following, we neglect the Rx_2 term. Then with $\lambda = -i\alpha Rc$, we have (5.3) in the form

$$\lambda(\varphi'' - \alpha^2\varphi) = \varphi^{iv} + (2-E)\alpha^2\varphi''$$
$$+ \alpha^4\varphi + i\alpha\Gamma(\varphi''' + \alpha^2\varphi'). \tag{5.8}$$

We shall prove that there is an eigenvalue $\lambda = 0$ at a value of $\Gamma(\alpha)$ which converges to $\sqrt{4E}$ as $\alpha \to \infty$ (this proof is due to M. Ahrens). Since (5.8) is a linear fourth order ODE, it can be solved exactly. We look for exponential solutions of the form $\varphi(x_2) = e^{i\alpha r x_2}$ and obtain the following algebraic equation for r:

$$r^4 + \Gamma r^3 - (2-E)r^2 - \Gamma r + 1 + \frac{\lambda}{\alpha^2}(r^2+1) = 0. \tag{5.9}$$

We are interested in neutral stability, $\lambda = 0$, so that

$$r^4 + \Gamma r^3 - (2-E)r^2 - \Gamma r + 1 = 0. \tag{5.10}$$

Eq. (5.10) is a fourth degree polynomial in r and hence solvable by standard methods. Letting $\gamma = \Gamma^2/4 - E$, so that $\gamma \leqq 0$ corresponds to linear stability, we can write the discriminant for (5.10) in the form

$$\Delta = 16\gamma^2\{16\gamma + (E+4)^2\}. \tag{5.11}$$

Thus $\Delta > 0$ for $\gamma > 0$, assuring us that (5.10) has distinct roots in the case of interest. In addition, the four roots of (5.10) are real. For completeness, we list the roots here. Define $R_k = r_k + \Gamma/4$, $k = 1, 2, 3, 4$ where r_k are the roots of (5.10). Thus

$$R_1 = \tfrac{1}{2}\{-\sqrt{-y_1} + \sqrt{-y_2} + \sqrt{-y_3}\},$$
$$R_2 = \tfrac{1}{2}\{-\sqrt{-y_1} - \sqrt{-y_2} - \sqrt{-y_3}\},$$

$$R_3 = \tfrac{1}{2}\{\sqrt{-y_1} + \sqrt{-y_2} - \sqrt{-y_3}\},$$
$$R_4 = \tfrac{1}{2}\{\sqrt{-y_1} - \sqrt{-y_2} + \sqrt{-y_3}\}, \tag{5.12}$$

where y_1, y_2, y_3 are real and negative and are solutions of the cubic

$$Y^3 + b_1 Y^2 + b_2 Y + b_3 = 0, \tag{5.13}$$

where

$$b_1 = 2\left(\frac{3}{8}\Gamma^2 + 2 - E\right),$$

$$b_2 = \frac{3}{16}\Gamma^4 + (1-E)\Gamma^2 - 4E + E^2,$$

$$b_3 = \frac{1}{64}\Gamma^2(\Gamma^2 - 4E)^2. \tag{5.14}$$

It is easy to check that b_1, b_2, b_3 are all positive for the case of interest, $\Gamma^2 > 4E$, if we assume $E > -4$. The rule of signs applied to (5.13) thus shows y_1, y_2, y_3 are negative, so that r_k, $k = 1, 2, 3, 4$ are all real, as asserted previously. In fact, the y_1, y_2, y_3 may be found from

$$y_1 = \frac{1}{3}\left\{-b_1 + \sqrt[3]{I} + \sqrt[3]{II}\right\},$$

$$y_2 = \frac{1}{3}\left\{-b_1 + \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\sqrt[3]{I} + \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\sqrt[3]{II}\right\},$$

$$y_3 = \frac{1}{3}\left\{-b_1 + \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\sqrt[3]{I} + \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\sqrt[3]{II}\right\} \tag{5.15}$$

where $\overline{II}$ is the complex conjugate of II and

$$I = \overline{II} = -\ (E+4)\,[18\gamma + (E+4)^2] + \tfrac{3}{2}\sqrt{3}\sqrt{\Delta}\, i. \tag{5.16}$$

We are now able to write down a solution to the Orr-Sommerfeld type problem, in the form

$$\varphi = \sum_{k=1}^{4} a_k\, e^{i\alpha r_k x_2} \tag{5.17}$$

where r_k are determined by (5.12–5.13) above, and the a_k are constants. The boundary conditions allow a non-trivial $\varphi(x_2)$ if and only if

$$\det\begin{bmatrix} 1 & 1 & 1 & 1 \\ r_1 & r_2 & r_3 & r_4 \\ e^{i\alpha r_1} & e^{i\alpha r_2} & e^{i\alpha r_3} & e^{i\alpha r_4} \\ r_1 e^{i\alpha r_1} & r_2 e^{i\alpha r_2} & r_3 e^{i\alpha r_3} & r_4 e^{i\alpha r_4} \end{bmatrix} = 0. \tag{5.18}$$

This implicitly defines a relationship $\Gamma(\alpha)$ which must hold for neutral stability. Reduction to the y_k, $k = 1, 2, 3$ given in (5.15) gives (5.18) in the form

$$f(\alpha, \Gamma) = (y_3 - y_2)\cos(\alpha\sqrt{-y_1}) + (y_1 - y_3)\cos(\alpha\sqrt{-y_2}) + (y_2 - y_1)\cos(\alpha\sqrt{-y_3}) = 0. \tag{5.19}$$

Recall that y_1, y_2, y_3 are independent of α, and are smooth functions of Γ in our case of interest, $\Gamma^2 > 4E$.

Eq. (5.19) was numerically solved to display the neutral stability curve $\Gamma(\alpha)$, for several choices of the parameter E. A typical case, $E = 2$, is given in figure 1. Note the behavior of Γ as $\alpha \to \infty$, indicating that $\Gamma^2 = 4E$ is a horizontal asymptote.

We shall show that $\Gamma^2 = 4E$ is indeed a horizontal asymptote. Thus, loss of linear stability occurs at this value of Γ^2 for disturbances with short wavelength (α large). The numerical results of Akbay and Sponagel [4] indicate the same result.

Asymptotic behavior

We will take the limit of (5.19) as $\Gamma^2 \downarrow 4E$, i.e. as $\gamma \downarrow 0$. First, from the form of y_1, y_2, y_3 given in (5.15) we find

$$\left.\begin{aligned} y_1 &\to -(E+4) \\ y_2, y_3 &\to 0 \end{aligned}\right\} \quad \text{as } \gamma \downarrow 0. \tag{5.20}$$

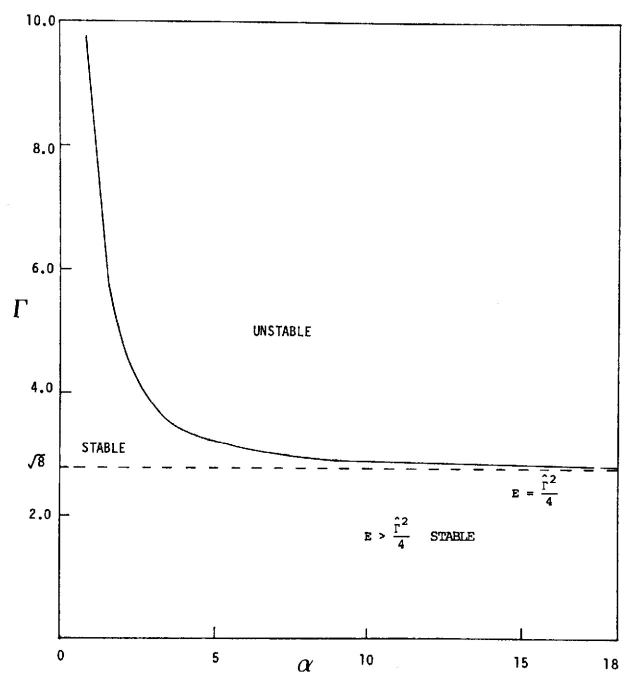

Fig. 1. Stability boundary for $E = 2$

Then computing f, $\partial f/\partial\alpha$ from (5.18), and using this last result, (5.20), gives

$$f, \frac{\partial f}{\partial\alpha} \to 0 \text{ as } \gamma \downarrow 0 . \tag{5.21}$$

We now consider $\partial f/\partial\Gamma$. From the definition of y_1, y_2, y_3, we can compute

$$\left.\begin{aligned} \frac{\partial y_1}{\partial\Gamma} &\to -\frac{E+8}{E+4}\sqrt{E} \\ \frac{\partial y_2}{\partial\Gamma} &\to -\frac{E}{E+4}\sqrt{E} \\ \frac{\partial y_3}{\partial\Gamma} &\to -\sqrt{E} \end{aligned}\right\} \text{ as } \gamma\downarrow 0 . \tag{5.22}$$

Hence, (5.19) implies

$$\frac{\partial f}{\partial\Gamma} \to -\frac{4}{E+4}\sqrt{E}$$

$$\cdot\left\{\cos\left(\alpha\sqrt{E+4}\right) - 1 + \frac{E+4}{2}\alpha^2\right\} \text{ as } \gamma\downarrow 0 . \tag{5.23}$$

Let

$$g(\alpha) = \left\{\cos\left(\alpha\sqrt{E+4}\right) - 1 + \frac{E+4}{2}\alpha^2\right\} .$$

Then

$g(0) = 0$ and

$g'(\alpha) = \sqrt{E+4}\,\{\sqrt{E+4}\,\alpha - \sin(\alpha\sqrt{E+4})\}$.

Clearly $g'(\alpha) > 0$ for $\alpha > 0$. Thus, $g(\alpha) > 0$ for $\alpha > 0$. Hence, for $\alpha > 0$, $\partial f/\partial\Gamma$ does not reduce to zero as $\gamma\downarrow 0$. The behavior of f near $\gamma = 0$ is approximately described by

$$f(\alpha,\Gamma) \approx y_1\left(\cos\left(\alpha\sqrt{-y_2}\right) - \cos\left(\alpha\sqrt{-y_3}\right)\right) . \tag{5.24}$$

Since $\partial y_2/\partial\Gamma$ and $\partial y_3/\partial\Gamma$ have different limits as $\gamma\downarrow 0$, the term in brackets will have an increasing number of zeros for small values of γ as α grows large. It is clear that this property persists for the exact f. Hence we find an infinite number of curves $\Gamma = \Gamma(\alpha)$, which must approach the line $\Gamma^2 = 4E$ as $\alpha \to \infty$. Since

$$\frac{d\Gamma}{d\alpha} = -\left(\frac{\partial f}{\partial\Gamma}\right)^{-1}\frac{\partial f}{\partial\alpha}, \tag{5.25}$$

the slope of each of these curves must approach 0 as $\alpha \to \infty$. Hence $\Gamma^2 = 4E$ is a horizontal asymptote for these curves.

We turn next to a description of results obtained by numerical calculation at finite Reynolds numbers. Figure 2 shows the neutral stability curve for the case $R = 1$, $E = 4$, showing an asymptotic behavior $\Gamma \to 4$ as $\alpha \to \infty$. The method of calculation follows that of Orszag [17]. Eq. (5.3) is discretised by representing φ as a sum of Chebyshev polynomials. The resulting matrix equation is solved for c, given α and Γ. The computations were checked against Gallagher and Mercer's [12] results for the case $\Gamma = 0$, $R = 1, 10, 20, 30$ using 20 Chebyshev polynomials. The number of

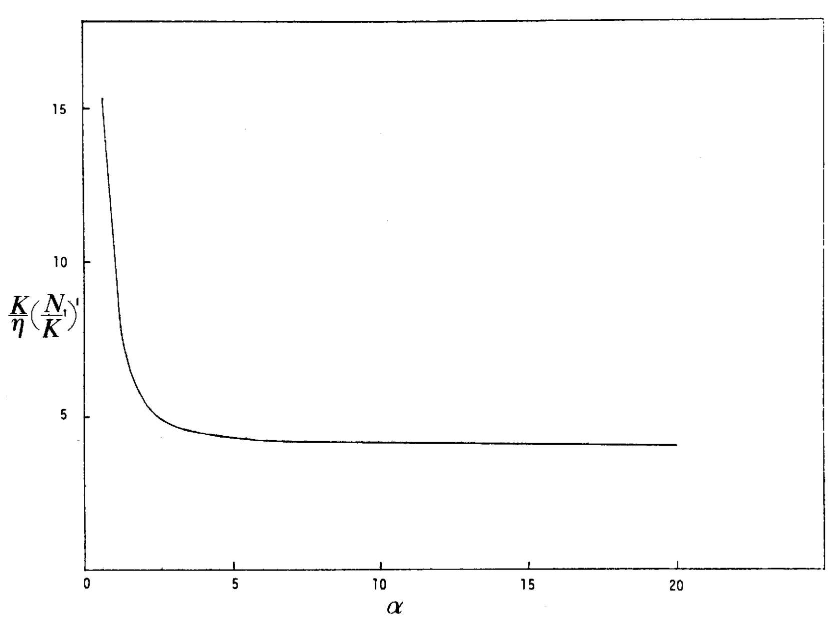

Fig. 2. Neutral stability curve for $R = 1$

Chebyshev polynomials used for figure 2 varies from 40 at $\alpha = 20$ up to 100 at $\alpha = 50$. For negative Γ, the neutral stability curve is a reflection across the α-axis. When R is increased this symmetry is lost and the neutral stability curve for Γ positive is raised. These trends are shown for $\alpha = 4$ in table 1 for which 40 Chebyshev polynomials are sufficient for 4-digit accuracy in c. A similar calculation at $\alpha = 50$ requires over 80 Chebyshev polynomials for the same accuracy. At $R = 1$ the neutral stability curve lies near $\Gamma = 4.00$ while at $R = 25$, it lies near $\Gamma = 4.04$. Hence it appears that the curve asymptotes to $\Gamma = 4$ as $\alpha \to \infty$ but more slowly as R increases. Curiously, the influence of finite R seems to be stabilizing rather than destabilizing, at least for low values of R.

Table 1. Weissenberg numbers for which $Re\,\sigma \approx 0$, $\alpha = 4$ for various Reynolds numbers

R	Positive $\frac{K}{\eta}\left(\frac{N_1}{K}\right)'$	Negative $\frac{K}{\eta}\left(\frac{N_1}{K}\right)'$
1	4.4	− 4.4
5	4.5	− 4.4
15	4.7	− 4.4
25	5.0	− 4.5

6. A critical assessment of the short memory theory

A difficulty in the theory of Akbay et al. is that, if the short memory approximation is applied to the basic flow one finds that the first normal stress is negligible and hence $\Gamma \approx 0$. That is, if short memory is assumed, then all terms in (3.1) except the first can be neglected, and we get the Reiner-Rivlin fluid, which has $N_1 = 0$. The instability criterion will not be achieved. We illustrate this for the following model constitutive law of Oldroyd type

$$\begin{aligned} S + \lambda\,[\dot{S} + (v \cdot \nabla)\,S + (\omega \cdot S - S \cdot \omega) \\ + a\,(D \cdot S + S \cdot D)] = 2\mu D \end{aligned} \qquad (6.1)$$

where S is the stress and D and ω are the symmetric and antisymmetric parts of the velocity gradients. μ and λ are arbitrary positive constants, and a is supposed to be between $+1$ and -1. In simple shear flow, we get

$$\tau = \frac{\mu\varkappa}{1 + \varkappa^2\lambda^2(1-a^2)},$$

$$N_1 = \frac{2\mu\lambda\varkappa^2}{1 + \varkappa^2\lambda^2(1-a^2)}.$$

If $|a| < 1$, τ' becomes negative for large $\varkappa$. This will lead to an instability, but not of the kind considered by Akbay et al. With $n = (1-a^2)\,\lambda^2$, it can be shown that

$$\frac{\varkappa^3}{16}\left(\frac{N_1}{\varkappa}\right)' \cdot \frac{1}{\tau\tau'} = \frac{\lambda^2\varkappa^2(1-\varkappa^2 n)}{4(1+\varkappa^2 n)} < \frac{\lambda^2\varkappa^2}{4}.$$

Hence the instability criterion does not apply if $\lambda\varkappa$ is small. The maximum value of the left hand side is

$$\frac{1}{4(1-a^2)} \cdot (\sqrt{2}-1)^2,$$

which never reaches 1 unless a^2 is sufficiently close to 1. The problem of consistency in the theory of Akbay et al. raises the question whether changes of type (or their absence) can be discussed without resorting to a short memory hypothesis. For a $K-BKZ$ fluid [8, 15], this has in fact been done [22]. The $K-BKZ$ model includes the special cases $a = \pm 1$ of (6.1). In these cases, the condition (1.3) is satisfied for large enough $\varkappa$, however, it was shown in [22] that a change of type does not occur! Thus the criterion (1.3) is invalid without the short memory hypothesis and the question whether large values of N_1 would produce instabilities in shear flow remains open.

Acknowledgements

The work of Ahrens and Joseph was supported by United States Army under Contract No. DAAG-29-82-K0051 by the Fluid Mechanics Branch of the National Science Foundation. The work of M. and Y. Renardy was supported by United States Army under Contract DAAG-29-80-C0051 and by the National Science Foundation MCS-8210950 and MCS-8215064 under Grant No. MCS-7927062, Mod. 2.

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(Received August 11, 1983)

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