Systematic Linearization for Stability of Shear Flows of Viscoelastic Fluids

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This paper is dedicated to J. L. Ericksen on the occasion of his 60th birthday

§ 1. Introduction

The phenomenon of melt fracture occuring in the process of polymer extrusion (see TORDELLA, 1963) has attracted the attention of research workers in the past two decades. In order to understand the mechanisms which might give rise to this phenomenon, the behaviour of perturbations of plane Couette flow of viscoelastic fluids with fading memory has been studied by various authors. Among them are COLEMAN & GURTIN (1968) who proposed and studied the formation of shear shocks and DUNWOODY (1970), who took a similar view, but incorporated heating effects. SLEMROD (1978, 1979) also considered the physical conditions necessary to produce the existence of non-smooth solutions to the perturbation problem, and proposed a further mechanism for instability based on the proposition that the shear stress in the basic flow is not a convex function of the shear rate (see J. L. ERICKSEN (1975)).

The most recent studies of this stability problem have been by AKBAY, BECKER, KROZER & SPONAGEL (1980) and AKBAY & SPONAGEL (1982), both based on an approximate constitutive theory for slow flows of fluids with short memory proposed by AKBAY & BECKER (1979) and BECKER (1980). In both cases it is assumed that stability can be studied by linear theory using standard spectral analysis of an eigenvalue problem and other approximations. Here we adopt the same approach, except that we have derived linear stability equations for infinitesimal perturbations of the history of a simple shear flow of a viscoelastic fluid with fading memory without further approximations. Simplifying assumptions with regard to material response are only introduced in order to draw conclusions from our exact analysis of stability in the final section 7. We show that periodic disturbances of long wave length in the flow direction may lead to instability when the modified Weissenberg number function (of $\varkappa$)

$$\frac{\eta}{\left(\dfrac{d(\eta\varkappa)}{d\varkappa}\right)}\left[\frac{\varkappa}{\eta}\,\frac{d}{d\varkappa}\left(\frac{N_1}{\varkappa}\right)\right]^2,$$

where $\varkappa$ is the shear rate, $\eta(\varkappa)$ is the shear viscosity function and $N_1(\varkappa)$ is the first normal stress difference, is sufficiently large and the memory of the fluid is short in the sense of AKBAY & BECKER (1979).

§ 2. Kinematics

The path lines in past time τ of particles $\boldsymbol{x}$ of a fluid filling a region Ω of $\mathbb{E}^3$ at present time t are given by

$$X = X(\boldsymbol{x}, \tau), \quad X(\boldsymbol{x}, t) = \boldsymbol{x}, \quad -\infty < \tau < t, \tag{2.1}$$

where X and the (solenoidal) velocity $V(., \tau)$ are related by

$$\frac{dX}{d\tau} = V(X, \tau), \quad X(\boldsymbol{x}, t) = \boldsymbol{x}. \tag{2.2}$$

Here $d/d\tau$ is the material time derivative at past time τ, following the particle $\boldsymbol{x}$.

The path lines and velocity of steady simple shear flows are related by

$$V(X, \tau) = N(\varkappa)\, X, \quad N^2 = \mathbf{0}, \tag{2.3}$$

$$[N] = \begin{bmatrix} 0 & \varkappa & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

where $\varkappa = U/h$ is the constant shear rate, h is the depth of the channel and Ux_2/h, $0 \leqq x_2 \leqq h$, is the velocity of the simple shear flow. Let $V(X)$ satisfy (2.3) and let $\boldsymbol{v}(X, \tau)$ be a (solenoidal) perturbation of $V(X)$. Then

$$\frac{dX}{d\tau} = V(X) + \boldsymbol{v}(X, \tau), \quad X(\boldsymbol{x}, t) = \boldsymbol{x}, \tag{2.4}$$

where $\boldsymbol{v}(., \tau) = \varepsilon \boldsymbol{u}(., \tau)$, $0 \leqq \varepsilon < 1$, is $O(\varepsilon)$ and $V(.)$, $\boldsymbol{u}(., \tau)$ are solenoidal over all of $\mathbb{E}^3$.

Solutions of (2.4) continuous in ε exist and are conveniently expressed in terms of the lapse time

$$s = t - \tau, \quad 0 \leqq s < \infty;$$

$$X = \chi_0 + \chi; \tag{2.5}$$

$$\chi_0 \stackrel{\text{def}}{=} [\mathbf{1} - sN(\varkappa)]\, \boldsymbol{x};$$

$$\chi = \chi(\boldsymbol{x}, \tau, \varepsilon), \quad \chi(\boldsymbol{x}, t, \varepsilon) = \chi(\boldsymbol{x}, \tau, 0) = \mathbf{0}.$$

Hence

$$\frac{d}{d\tau}(\chi_0 + \chi) = V(\chi_0) + N(\varkappa)\, \chi + \boldsymbol{v}(\chi_0, \tau) + O(\varepsilon^2), \tag{2.6}$$

where

$$N(\varkappa) = \partial V(\chi_0)/\partial \chi_0.$$

Putting $X = \chi_0$ in (2.2), we find that (2.6) may be reduced to

$$\frac{d\chi}{d\tau} = N(\varkappa)\,\chi + v(\chi_0, \tau) + O(\varepsilon^2) \tag{2.7}$$

and

$$\frac{dN\chi}{d\tau} = Nv(\chi_0, \tau) + O(\varepsilon^2). \tag{2.8}$$

The following identities are used in § 3 to express the perturbed extra stress in terms of the velocity v alone:

$$\begin{aligned}
&\frac{d}{ds}\nabla\chi(x, t-s) + \nabla(N\chi + v) = O(\varepsilon^2),\\
&\frac{d}{ds}(N^T\nabla\chi) + N^T\nabla(N\chi + v) = O(\varepsilon^2),\\
&\frac{d}{ds}\nabla N\chi + \nabla(Nv) = O(\varepsilon^2),
\end{aligned} \tag{2.9}$$

where $\nabla \overset{\text{def}}{=} \partial/\partial x$ and $v = v(\chi_0(x, s), t-s)$ is evaluated on the path of the shear flow.

The history of the relative deformation gradient of a perturbed simple shear flow can be computed from (2.5).

Thus

$$F(x, t-s, \varepsilon) = \nabla X(x, t-s, \varepsilon) = \nabla\chi_0(x, s) + \nabla\chi(x, t-s) + O(\varepsilon^2),$$

where F is the relative deformation tensor usually denoted by F_t. The relative Cauchy strain history is given by

$$\begin{aligned}
G(x, t-s, \varepsilon) &\overset{\text{def}}{=} F^T(x, t-s, \varepsilon)\,F(x, t-s, \varepsilon) - 1\\
&= G_0(\varkappa s) + g(x, t-s) + O(\varepsilon^2),\\
G_0(\varkappa s) &= F_0^T F_0 - 1,\ F_0(\varkappa s) = \nabla\chi_0(x, s),\\
g(x, t-s) &= f(x, t-s) + f^T(x, t-s),
\end{aligned} \tag{2.10}$$

where from (2.5)

$$[G_0(\varkappa s)] = \begin{bmatrix} 0 & -s\varkappa & 0\\ -s\varkappa & -s^2\varkappa^2 & 0\\ 0 & 0 & 0 \end{bmatrix}, \tag{2.11}$$

$$f(x, t-s) = \nabla\chi(x, t-s) - sN^T\nabla\chi(x, t-s). \tag{2.12}$$

In the linearized theory we neglect all terms of $O(\varepsilon^2)$.

§ 3. Constitutive Equation for Stress

We follow GREEN & RIVLIN (1957) and COLEMAN & NOLL (1961) in expressing the perturbation stresses by integrals, and confine our attention to the linear case. The integral representations may be broadly interpreted as representing all the possible forms of the linearized stresses associated with many fading memory topologies. The linearized stresses are tensor-valued functionals on the space of Cauchy strain histories and hence they lie in the topological dual of this space. This dual space consists of functionals expressible by integrals against fading memory kernels. The theory of COLEMAN & NOLL (1961) is set in the largest domain space consistent with kinematics and it has the smallest topological dual. In particular the kernels appearing in the theory of COLEMAN & NOLL (1961) must be quadratically integrable. By restricting the domain space the topological dual is enlarged and a different material response results (see SAUT & JOSEPH (1982)). In the sequel we adopt kernels consistent with theories such as that proposed by COLEMAN & NOLL.

Suppose now that $G_0(\varkappa s)$ is a viscometric strain history at a particle x. Let the extra stress be expressed generally as

(3.1)
$$\begin{aligned} S &= \mathop{\mathfrak{T}}_{s=0}^{\infty}[G(x, t-s)] = \mathop{\mathfrak{T}}_{s=0}^{\infty}[G_0(\varkappa s) + g(x, t-s)] \\ &= \mathop{\mathfrak{T}}_{s=0}^{\infty}[G_0] + \mathop{\mathfrak{T}_1}_{s=0}^{\infty}[G_0 \mid g] + \mathop{\mathfrak{T}_2}_{s=0}^{\infty}[G_0 \mid g, g] + \dots, \end{aligned}$$

where $\mathfrak{T}_l[G_0 \mid g, \dots, g]$ is the l^{th} Fréchet derivative at G_0, and

(3.2)
$$S^0 = \mathop{\mathfrak{T}}_{s=0}^{\infty}[G_0(\varkappa, s)] = T^0 - \tfrac{1}{3}\operatorname{tr} T^0\, \mathbf{1}$$

is the extra stress in viscometric flow if T^0 is the stress.

(3.3)
$$[S^0] = \tfrac{1}{3}\begin{bmatrix} 2N_1 + N_2 & 3\tau & 0 \\ 3\tau & -N_1 + N_2 & 0 \\ 0 & 0 & -2N_2 - N_1 \end{bmatrix}$$

and $[N_1, N_2] = [T_{11} - T_{22}, T_{22} - T_{33}]$ are first and second normal stress differences and $\tau = T_{12}$ is the shear stress. We are assuming one of those theories of fading memory in which $\mathfrak{T}_1$ may be expressed as an integral. Thus

(3.4)
$$\begin{aligned} \mathop{\mathfrak{T}_1}_{s=0}^{\infty}[G_0(\varkappa s) \mid g(x, t-s)] &= \int_0^\infty \tilde{K}(G_0(\varkappa s), s)\, g(x, t-s)\, ds \\ &= \int_0^\infty K(\varkappa, s)\,[\nabla\chi(x, t-s) - sN^T(\varkappa)\,\nabla\chi(x, t-s) + \text{transpose}]\, ds, \end{aligned}$$

where $K(\varkappa, s)$ is a fourth order tensor. From isotropy of material it follows that

$$(3.5)\quad Q \sum_{s=0}^{\infty} [Q^T \chi_0(\varkappa s)\, Q \mid Q^T g(x, t-s)\, Q]\, Q^T = \sum_{s=0}^{\infty} [G_0(\varkappa s) \mid g(x, t-s)],$$

or equivalently that

$$(3.6)\qquad Q_{im} \tilde{K}_{mnlk}(Q^T G_0 Q, s)\, Q_{jn} Q_{pl} g_{pq} Q_{qk} = \tilde{K}_{ijkl}(G_0, s)\, g_{kl}$$

for all orthogonal Q and all symmetric g. Hence, from consideration of "viscometric" symmetries the symmetries of $\tilde{K}$ and therefore K may be found as has been indicated by PIPKIN & OWEN (1967).

We can express the extra stress in terms of $v(\chi_0(x, s), t-s)$ alone. To eliminate $\chi(x, t-s)$ from (3.4) we use the identities (2.9) and integrate by parts, using $\chi(x, t) = 0$ and $K(\varkappa, \infty) = 0$, to obtain (3.4) in the form

$$(3.7)\qquad \begin{aligned} &= \int_0^\infty \{M(\varkappa, s)\, [\nabla v + \nabla v^T] + P(\varkappa, s)\, [N^T \nabla v + (\nabla v)^T N] \\ &\quad + \overline{\overline{M}}(\varkappa, s)\, [\nabla(Nv) + (\nabla(Nv))^T] \\ &\quad + \overline{\overline{P}}(\varkappa, s)\, [N^T \nabla(Nv) + (\nabla(Nv))^T N]\}\, ds, \end{aligned}$$

where

$$\nabla v = \partial v(\chi_0(x, s), t-s)/\partial x$$

and

$$(3.8)\qquad \begin{aligned} M(\varkappa, s) &= \int_\infty^s K(\varkappa, \eta)\, d\eta, \\ \overline{\overline{M}}(\varkappa, s) &= \int_\infty^s M(\varkappa, \eta)\, d\eta, \\ P(\varkappa, s) &= -\int_\infty^s \eta \cdot K(\varkappa, \eta)\, d\eta = -sM(\varkappa, s) + \overline{\overline{M}}(\varkappa, s), \\ \overline{\overline{P}}(\varkappa, s) &= \int_\infty^s P(\varkappa, \eta)\, d\eta. \end{aligned}$$

These four kernels have the same symmetries as K. For example since K_{ijkl} is symmetric in (i, j) and (k, l), the components of M, $\overline{\overline{M}}$, P and $\overline{\overline{P}}$ have the same symmetry. We shall assume that the kernels vanish for large times.

We shall say that $K(\varkappa, \eta)$ is a kernel of the Maxwell type with relaxation time λ if all integrals and moments of $K(\varkappa, \eta)$ satisfy estimates of the same order as if

$$K(\varkappa, \eta) = \lambda^{-1} e^{-\frac{\eta}{\lambda}} k(\varkappa);$$

that is

$$
\begin{aligned}
M(\varkappa, s) &= O(1),\\
\bar{\bar{M}}(\varkappa, s) &= O(\lambda),\\
P(\varkappa, s) &= O(\lambda),\\
\bar{\bar{P}}(\varkappa, s) &= O(\lambda^2).
\end{aligned} \tag{3.9}
$$

A fluid has a short memory for disturbances if λ is small. For some fluids a short memory is possible only if the Weissenberg number is also small. The short memory assumption of AKBAY, *et al.*, which we use in § 1, is not useful in such cases.

§ 4. Consistency Relations

PIPKIN & OWEN (1967) have shown that symmetry and isotropy reduce the number of independent components of the kernels $K(\varkappa, s)$ to thirteen. Moreover these independent components of $K_{ijkl}(\varkappa, s)$, hence M_{ijkl} and P_{ijkl} as well, can be related to the three viscometric functions $N_1(\varkappa)$, $N_2(\varkappa)$ and $\tau(\varkappa)$. They note that

(i) The symmetry of the stress implies that

$$K_{ijkl} = K_{jikl}, \tag{4.1}$$

(ii) Since $\operatorname{tr} S = 0$,

$$K_{iikl} = 0. \tag{4.2}$$

(iii) Since $g = g^T$ in (3.4), there is no loss of generality in putting

$$K_{ijkl} = K_{ijlk}. \tag{4.3}$$

(iv) The symmetry of the basic simple shear is such that all the components in which the subscript 3 appears once or thrice vanish. For by (3.6), on setting

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

we find $Q G_0(\varkappa s)\, Q^T = G_0(\varkappa s)$ and then, for example

$$Q_{im} K_{mnlk} Q_{jn} Q_{qk} g_{pq} = K_{ijkl} g_{kl}. \tag{4.4}$$

It follows from (4.4) that all K_{ijkl} with subscript 3 appearing once or thrice, vanish.

Since we consider only two dimensional perturbations in the plane, all other components of K containing the index 3 may be neglected here. Hence a knowledge of the nine quantities $K_{\alpha\beta\gamma\delta}$, where the indices range over 1 and 2, is all that is required.

Now we derive the consistency relations by our own version of the method of PIPKIN & OWEN (1967). First, in an obvious notation, we write

$$\mathfrak{T}_1 = \int_0^\infty K_{\gamma\delta}(\varkappa, s)\, g_{\gamma\delta}(\boldsymbol{x}, t-s)\, ds, \tag{4.5}$$

where $\boldsymbol{K}_{\gamma\delta}$ is a two-dimensional second order tensor for each pair of indices (γ, δ). We then write (i), (ii), and (iii) in the forms

$$\boldsymbol{K}_{\gamma\delta} = \boldsymbol{K}_{\gamma\delta}^T, \quad \boldsymbol{K}_{\gamma\delta} = \boldsymbol{K}_{\delta\gamma}, \quad \operatorname{tr} \boldsymbol{K}_{\gamma\delta} = \boldsymbol{0}. \tag{4.6}$$

Less obvious restrictions follow from the fact that $\mathfrak{T}_1[G_0(\varkappa, s) \mid \cdot]$ is evaluated on a viscometric history given by (2.11). Since $\mathfrak{T}$ is an isotropic functional

$$QSQ^T = \underset{s=0}{\overset{\infty}{\mathfrak{T}}} [QG(\boldsymbol{x}, s)\, Q^T] \tag{4.7}$$

for all orthogonal Q and all histories G. Hence, no matter what the form of $\mathfrak{T}$

$$S^0(\varkappa) = \underset{s=0}{\overset{\infty}{\mathfrak{T}}} [G_0(\varkappa, s)] \tag{4.8}$$

is given in terms of the functions $N_1(\varkappa)$, $N_2(\varkappa)$ and $\tau(\varkappa)$ as in (3.3) ($\mathfrak{T}$ determines the form of these functions) and

$$Q(\lambda)\, S^0(\varkappa)\, Q^T(\lambda) = \underset{s=0}{\overset{\infty}{\mathfrak{T}}} [Q(\lambda)\, G_0(\varkappa, s)\, Q^T(\lambda)] \tag{4.9}$$

is also the constitutive equation for a viscometric flow identically for all values of λ and $\varkappa$. Indeed we may interpret the relation of (4.9) to (4.8) in the following way: the stress in a simple fluid at a particle $\boldsymbol{x}$, due to a motion obtained from a given motion by a rotation at the present instant centered at $\boldsymbol{x}$, is exactly that given by transforming the stress by the rotation tensor relating the particle paths in the second motion to the first.

Since (4.9) holds identically in λ and $\varkappa$, we have for rotations in the plane of flow

(4.10)
$$S^0\dot{Q}^T + \dot{Q}S^0 = \mathfrak{T}_1[G_0 \mid G_0\dot{Q}^T + \dot{Q}G_0] = \int_0^\infty K(\varkappa, s)\{G_0\dot{Q}^T + \dot{Q}G_0\}\, ds$$

and for changing shear rate

$$\frac{dS^0}{d\varkappa} = \mathfrak{T}_1\left[G_0 \mid \frac{\partial G_0}{\partial \varkappa}\right] = \int_0^\infty K(\varkappa, s)\, \frac{\partial G_0}{\partial \varkappa}\, ds, \tag{4.11}$$

where G_0 is given by (2.8), $Q(0) = \boldsymbol{1}$ and $\dot{Q} = dQ(\lambda)/d\lambda$ at $\lambda = 0$. The relations (4.10) and (4.11) contain all the consistency relations.

From (4.11) we obtain

$$\frac{d}{d\varkappa}\begin{pmatrix} 2N_1 + N_2 & 3\tau \\ 3\tau & N_2 - N_1 \end{pmatrix} \tag{4.12}$$
$$= 6\int_0^\infty [\varkappa s^2 \boldsymbol{K}_{22} - s\boldsymbol{K}_{12}]\, ds = 6\int_0^\infty [\boldsymbol{M}_{12} + \varkappa \boldsymbol{P}_{22}]\, ds$$

after integration by parts using (3.8). Since $\dot{Q}$ is a skew tensor, we find that

$$[\dot{Q}]|_{\gamma=0} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and this leads us to

$$\begin{pmatrix} -2\tau & N_1 \\ N_1 & 2\tau \end{pmatrix} = 2\varkappa \int_0^\infty [s(\boldsymbol{K}_{11} - \boldsymbol{K}_{22}) - \varkappa s^2 \boldsymbol{K}_{12}]\, ds \tag{4.13}$$
$$= 2\varkappa \int_0^\infty [\boldsymbol{M}_{22} - \boldsymbol{M}_{11} - \varkappa \boldsymbol{P}_{12}]\, ds.$$

The stress (4.5) in two dimensions may be expanded as

$$\mathfrak{T}_1 = \int_0^\infty \{\boldsymbol{K}_{11} g_{11} + \boldsymbol{K}_{22} g_{22} + 2\boldsymbol{K}_{12} g_{12}\}\, ds, \tag{4.14}$$

where $\boldsymbol{K}_{11}$, $\boldsymbol{K}_{22}$, $\boldsymbol{K}_{12}$ have components which may be represented by 2×2 matrices. The consistency conditions we require are from (4.12)

$$2N_1' + N_2' = 6\int_0^\infty [M_{1112} + \varkappa P_{1122}]\, ds,$$
$$N_2' - N_1' = 6\int_0^\infty [M_{2212} + \varkappa P_{2222}]\, ds, \tag{4.15}$$
$$\tau' = 2\int_0^\infty [M_{1212} + \varkappa P_{1222}]\, ds,$$

and (4.13)

$$\tau/\varkappa \stackrel{\text{def}}{=} \eta(\varkappa) = \int_0^\infty [M_{1111} - M_{1122} + \varkappa P_{1112}]\, ds$$
$$= \int_0^\infty [M_{2222} - M_{2211} - \varkappa P_{2212}]\, ds, \tag{4.16}$$
$$N_1/\varkappa = 2\int_0^\infty [M_{1222} - M_{1211} - \varkappa P_{1212}]\, ds.$$

To simplify our equations it is useful to introduce kernels which are suggested by the consistency conditions. Thus

$$G \overset{\text{def}}{=} 2(M_{22} - M_{11} - \varkappa P_{12}),$$
$$F \overset{\text{def}}{=} 2(M_{12} + \varkappa P_{22}), \tag{4.17}$$

$$\begin{bmatrix} 2N_1' + N_2' & 3\tau' \\ 3\tau' & N_2' - N_1' \end{bmatrix} = 3 \int_0^\infty F(\varkappa, s)\, ds \tag{4.18}$$

and

$$\begin{bmatrix} -2\tau & N_1 \\ N_1 & 2\tau \end{bmatrix} = \varkappa \int_0^\infty H(\varkappa, s)\, ds$$

$$N_1'(\varkappa) = \int_0^\infty (F_{11} - F_{22})\, ds,$$
$$\tau'(\varkappa) = \int_0^\infty F_{12}\, ds, \tag{4.19}$$

$$N_1(\varkappa) = \varkappa \int_0^\infty H_{12}\, ds,$$
$$\tau(\varkappa) = \tfrac{1}{2}\varkappa \int_0^\infty H_{22}\, ds = -\tfrac{1}{2}\varkappa \int_0^\infty H_{11}\, ds. \tag{4.20}$$

§ 5. Equations of Perturbed Motion

We linearize the equations of motion for perturbations $v = \varepsilon u$ of plane Couette flow $V = (\varkappa x_2, 0)$, $0 \leqq x_2 \leqq h$, $\varkappa h = U$. Hence neglecting terms $O(\varepsilon^2)$

$$\varrho \left[\frac{\partial u_1}{\partial t} + \varkappa x_2 \frac{\partial u_1}{\partial x_1} + \varkappa u_2\right] = -\frac{\partial p}{\partial x_1} + \frac{\partial S_{11}}{\partial x_1} + \frac{\partial S_{12}}{\partial x_2},$$
$$\varrho \left[\frac{\partial u_2}{\partial t} + \varkappa x_2 \frac{\partial u_2}{\partial x_1}\right] = -\frac{\partial p}{\partial x_2} + \frac{\partial S_{12}}{\partial x_1} + \frac{\partial S_{22}}{\partial x_2}. \tag{5.1}$$

The perturbation is solenoidal with respect to X, *i.e.* $\partial u_i/\partial X_i = 0$, and hence is derivable from a stream function $\Psi(X, t - s) = \Psi(\chi_0, t - s) + O(\varepsilon)$. Therefore, we may write

$$u(\chi_0, \tau) = \left(-\frac{\partial \Psi}{\partial \chi_{02}}, \frac{\partial \Psi}{\partial \chi_{01}}\right) = (-\Psi_{,2} - \varkappa s\, \Psi_{,1}, \Psi_{,1}) \tag{5.2}$$

where $\Psi_{,l} = \partial\Psi/\partial x_l$, $l = 1, 2$. In terms of this stream function, the equations of motion (5.1) may be reduced to

$$\varrho \left[\frac{\partial}{\partial t} + \varkappa x_2 \frac{\partial}{\partial x_1}\right] \nabla^2 \Psi = \left[\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2}\right] S_{12} + \frac{\partial^2 (S_{22} - S_{11})}{\partial x_1\, \partial x_2}. \tag{5.3}$$

Expressions for the extra stress may be found from (4.14) with the kernels expressed as in (3.7). To facilitate the computation we write

(5.4)

$$S=\int_0^\infty \{M_{11}A_{11}+2M_{12}A_{12}+M_{22}A_{22}+P_{11}B_{11}+2P_{12}B_{12}+P_{22}B_{22}$$
$$+\bar{\bar{M}}_{11}C_{11}+2\bar{\bar{M}}_{12}C_{12}+\bar{\bar{M}}_{22}C_{22}+\bar{\bar{P}}_{11}D_{11}+2\bar{\bar{P}}_{12}D_{12}+\bar{\bar{P}}_{22}D_{22}\}\,ds$$

where

$$[A]=[\nabla u+\nabla u^T]=\begin{bmatrix}2u_{1,1} & u_{1,2}+u_{2,1}\\ u_{1,2}+u_{2,1} & 2u_{2,2}\end{bmatrix}$$
$$=\begin{bmatrix}-2\Psi_{,12} & \Psi_{,11}-\Psi_{,22}\\ \Psi_{,11}-\Psi_{,22} & 2\Psi_{,12}\end{bmatrix}-\varkappa s\begin{bmatrix}2\Psi_{,11} & \Psi_{,12}\\ \Psi_{,12} & 0\end{bmatrix},$$

$$[B]=[N^T\nabla u+\nabla u^T N]=\varkappa\begin{bmatrix}0 & u_{1,1}\\ u_{1,1} & 2u_{1,2}\end{bmatrix}$$
$$=-\varkappa\begin{bmatrix}0 & \Psi_{,21}\\ \Psi_{,21} & 2\Psi_{,22}\end{bmatrix}-\varkappa^2 s\begin{bmatrix}0 & \Psi_{,11}\\ \Psi_{,11} & 2\Psi_{,12}\end{bmatrix},$$

$$[C]=[\nabla(Nu)+\nabla(Nu)^T]=\varkappa\begin{bmatrix}2u_{2,1} & u_{2,2}\\ u_{2,2} & 0\end{bmatrix}=\varkappa\begin{bmatrix}2\Psi_{,11} & \Psi_{,12}\\ \Psi_{,12} & 0\end{bmatrix},$$

$$[D]=[N^T\nabla(Nu)+\nabla(Nu)^T N]=\varkappa^2\begin{bmatrix}0 & u_{2,1}\\ u_{2,1} & 2u_{2,2}\end{bmatrix}=\varkappa^2\begin{bmatrix}0 & \Psi_{,11}\\ \Psi_{,11} & 2\Psi_{,12}\end{bmatrix}.$$

After collecting the coefficients of different derivatives of $\Psi(x_1, x_2)$ in (5.4), we find that

$$S=\int_0^\infty [\mathbb{P}(\varkappa,s)\,\Psi_{,12}+\mathbb{Q}(\varkappa,s)\,\Psi_{,11}+\mathbb{R}(\varkappa,s)\,\Psi_{,22}]\,ds, \tag{5.5}$$

where

$$\mathbb{P}=2(-M_{11}+M_{22}-\varkappa^2 sP_{22}+\varkappa^2\bar{\bar{P}}_{22}),$$
$$\mathbb{Q}=2(M_{12}+\varkappa P_{11}+\varkappa^2\bar{\bar{P}}_{12}-\varkappa^2 sP_{12}), \tag{5.6}$$
$$\mathbb{R}=-2(M_{12}+\varkappa P_{22}).$$

We now use (4.12) and (4.13) and definitions of the type (4.17)–(4.20) to write (5.6) as

$$\mathbb{P} = \boldsymbol{H} + \varkappa\tilde{\boldsymbol{F}}, \tag{5.7}$$

$$\mathbb{R} = -\boldsymbol{F} \tag{5.8}$$

where, for any $\mathbb{A}(s)$,

$$\frac{d}{ds}\,\overline{\overline{\mathbb{A}}}(s) = \mathbb{A}(s),$$

$$\overline{\overline{\mathbb{A}}}(s) = \tilde{A}(s) + sA\,(s).$$

Using (3.8), we eliminate $\boldsymbol{P}_{22}$ from $(5.6)_3$ to get

$$2\boldsymbol{M}_{12} = -\mathbb{R} - 2\varkappa\boldsymbol{P}_{22} = -\mathbb{R} - 2\varkappa\tilde{\boldsymbol{M}}_{22},$$

and we eliminate $\boldsymbol{M}_{12}$ from $(5.6)_2$ and use $\tilde{\boldsymbol{M}} = \boldsymbol{P}$ to get

$$\mathbb{Q} = -\mathbb{R} + 2\varkappa(-\tilde{\boldsymbol{M}}_{22} + \tilde{\boldsymbol{M}}_{11} + \varkappa\tilde{\boldsymbol{P}}_{12})$$

or

$$\mathbb{Q} = \boldsymbol{F} - \varkappa\tilde{\boldsymbol{H}}. \tag{5.9}$$

Expressions (5.7), (5.8) and (5.9) show that the kernels in the expression (5.5) for the extra stress are expressible in terms of kernels involved in the definitions of viscometric functions.

For the study of stability in two dimensions we form the components S_{12} and $S_{22} - S_{11}$ of the extra stress:

$$S_{12} = \int_0^\infty [\mathbb{P}_{12}\Psi_{,12} + \mathbb{Q}_{12}\Psi_{,11} + \mathbb{R}_{12}\Psi_{,22}]\,ds,$$

where

$$\mathbb{P}_{12} = H_{12} + \varkappa\tilde{F}_{12},$$

$$\mathbb{R}_{12} = -F_{12},$$

$$\mathbb{Q}_{12} = F_{12} - \varkappa\tilde{H}_{12},$$

$$S_{22} - S_{11} = \int_0^\infty [(\mathbb{P}_{22} - \mathbb{P}_{11})\,\Psi_{,12} + (\mathbb{Q}_{22} - \mathbb{Q}_{11})\,\Psi_{,11} + (\mathbb{R}_{22} - \mathbb{R}_{11})\,\Psi_{,22}]\,ds,$$

where

$$\mathbb{P}_{22} - \mathbb{P}_{11} = H_{22} - H_{11} - \varkappa(\tilde{F}_{22} - \tilde{F}_{11}),$$

$$\mathbb{R}_{22} - \mathbb{R}_{11} = F_{11} - F_{22},$$

$$\mathbb{Q}_{22} - \mathbb{Q}_{11} = F_{22} - F_{11} - \varkappa(\tilde{H}_{22} - \tilde{H}_{11}).$$

§ 6. The Spectral Problem of Linearized Theory

We are going to assume that stability can be determined from the linearized theory associated with a spectral problem derived from (5.3). We note that the stream function Ψ under the integrals in (5.10) and (5.11) are of the form

$$\Psi = \Psi(\chi_{01}, x_2, t - s), \tag{6.1}$$

where

$$\chi_0 = (\chi_{01}, \chi_{02}) = (x_1 - \varkappa s x_2, x_2).$$

We look for Ψ in the class of functions which are periodic in χ_{01} with period $2\pi/\alpha$. Then

$$\Psi = e^{i\alpha\chi_{01}} \tilde{\Psi}(x_2, t - s). \tag{6.2}$$

The spectral problem governing stability may then be obtained formally by using the well-known method of the exponential time factor, following procedures used by CRAIK (1968) and JOSEPH (1976) to study the stability of the rest state. Thus

$$\tilde{\Psi}(x_2, t - s) = e^{\sigma t} e^{-\sigma s} \psi(x_2). \tag{6.3}$$

The stability of plane Couette flow is judged by eigenvalues σ associated with the spectral problem (6.11). This problem arises from (5.3) for disturbances of the form (6.2) and (6.3) or, alternatively, by the method of Laplace transforms (see DIXIT, NARAIN & JOSEPH (1982)). We may write (6.2) and (6.3) as

$$\Psi = e^{\phi_1} f(x_2) = e^{\phi_1} e^{-s\phi_2} \psi(x_2),$$

where

$$\phi_1 = \sigma t + i\alpha x_1, \quad \phi_2 = \sigma + i\alpha\varkappa x_2. \tag{6.4}$$

Using (6.4), we find that

$$\begin{gathered} S = e^{\phi_1} \mathscr{S}, \\ \mathscr{S} = \alpha^2 \psi \mathbb{J}(a) + i\alpha\psi' \mathbb{J}(b) + \psi'' \mathbb{J}(c), \end{gathered} \tag{6.5}$$

where, for any $f(s)$,

$$\mathbb{J}(f) \stackrel{\text{def}}{=} \int_0^\infty e^{-\phi_2 s} f(s)\, ds \tag{6.6}$$

and

$$\begin{gathered} a(s) = \varkappa s \mathbb{P}(\varkappa, s) - \mathbb{Q}(\varkappa, s) - \varkappa^2 s^2 \mathbb{R}(\varkappa, s), \\ b(s) = \mathbb{P}(\varkappa, s) - 2\varkappa s \mathbb{R}(\varkappa, s), \\ c(s) = \mathbb{R}(\varkappa, s). \end{gathered} \tag{6.7}$$

We next note that if $f(s)$ is independent of x_2, then

$$\frac{\partial}{\partial x_2}\mathbb{J}(f) = -i\alpha\varkappa\mathbb{J}(sf). \tag{6.8}$$

Hence

$$\mathscr{S}' = -i\alpha^3\varkappa\psi\mathbb{J}(sa) + \alpha^2\psi'\mathbb{J}(a + \varkappa sb') + i\alpha\psi''\mathbb{J}(b - cs\varkappa) + \psi'''\mathbb{J}(c), \tag{6.9}$$

$$\begin{aligned}\mathscr{S}'' + \alpha^2\mathscr{S} &= \alpha^4\psi\mathbb{J}[(1 - \varkappa^2 s^2)\, a] + i\alpha^3\psi'\mathbb{J}[(1 - \varkappa^2 s^2)\, b - 2\varkappa sa] \\ &+ \alpha^2\psi''\mathbb{J}[(1 - \varkappa s^2)\, c + 2\varkappa sb + a] + i\alpha\psi'''\mathbb{J}[b - 2\varkappa sc] + \psi^{iv}\mathbb{J}(c).\end{aligned} \tag{6.10}$$

Finally, combining (6.5), (6.3), (6.9) and (6.10) with (5.3) we get

$$\begin{aligned}\varrho\phi_2(\psi'' - \alpha^2\psi) &= -(\mathscr{S}_{12}'' + \alpha^2\mathscr{S}_{12}) + i\alpha(\mathscr{S}_{22}' - \mathscr{S}_{11}') \\ &= \alpha^4\psi\mathbb{C}_0 + i\alpha^3\psi'\mathbb{C}_1 + \alpha^2\psi''\mathbb{C}_2 + i\alpha\psi'''\mathbb{C}_3 + \psi^{iv}\,\mathbb{C}_4,\end{aligned} \tag{6.11}$$

$$\psi(0) = \psi'(0) = \psi(h) = \psi'(h) = 0, \tag{6.12}$$

where

$$\mathbb{C}_n = \mathbb{J}(l_n),$$

$$l_0 = -[(1 - \varkappa^2 s^2)\, a_{12} + s\varkappa(a_{11} - a_{22})],$$

$$l_1 = -[(1 - \varkappa^2 s^2)\, b_{12} - 2\varkappa s a_{12} + (a_{11} - a_{22}) + \varkappa s(b_{11} - b_{22})],$$

$$l_2 = -[(1 - \varkappa s^2)\, c_{12} + 2\varkappa s b_{12} + a_{12} - (b_{11} - b_{22}) + \varkappa s(c_{11} - c_{22})],$$

$$l_3 = -[b_{12} - 2\varkappa s c_{12} + (c_{11} - c_{22})],$$

$$l_4 = -[c_{12}],$$

and

$$a_{12} = -F_{12} + \varkappa^2 s\overline{\overline{F}}_{12} + \varkappa\overline{\overline{H}}_{12},$$

$$b_{12} = H_{12} + \varkappa s F_{12} + \varkappa\overline{\overline{F}}_{12},$$

$$c_{12} = -F_{12}$$

$$a_{11} - a_{22} = F_{22} - F_{11} - \varkappa^2 s(\overline{\overline{F}}_{11} - \overline{\overline{F}}_{22}) + \varkappa(\overline{\overline{H}}_{11} - \overline{\overline{H}}_{22}),$$

$$b_{11} - b_{22} = H_{11} - H_{22} + \varkappa s(F_{11} - F_{22}) + \varkappa(\overline{\overline{F}}_{22} - \overline{\overline{F}}_{11}),$$

$$c_{11} - c_{22} = -F_{11} + F_{22}.$$

The velocity U of the moving wall, the gap width h and viscosity $\eta(\varkappa) = \tau(\varkappa)/\varkappa$ are now introduced as dimensional parameters for defining the dimensionless parameters

$$x_2 = h\bar{x}, \quad t = \frac{h}{U}\bar{t}, \quad \varkappa = \frac{U}{h}, \quad R = \frac{\varrho U h}{\eta(\varkappa)},$$

$$\psi = Uh\bar{\psi}, \quad \mathbb{C}_n(\sigma, x) = \eta(\varkappa)\,\bar{\mathbb{C}}_n(\bar{\sigma}, \bar{x}),$$

$$\sigma = \frac{U}{h}\bar{\sigma}, \quad \alpha = \frac{\bar{\alpha}}{h}.$$

The dimensionless parameters are introduced into (6.11) and, after dropping the overbars, we find the spectral problem in dimensionless form

$$R(\sigma + i\alpha x)(\psi'' - \alpha^2\psi) = \alpha^4\mathbb{C}_0(\sigma, x)\,\psi + i\alpha^3\mathbb{C}_1(\sigma, x)\,\psi' + \alpha^2\mathbb{C}_2(\sigma, x)\,\psi'' + i\alpha\mathbb{C}_3(\sigma, x)\,\psi''' + \mathbb{C}_4(\sigma, x)\,\psi'''', \tag{6.13}$$

$$\psi(0) = \psi'(0) = \psi(1) = \psi'(1) = 0,$$

Equation (6.13) is nearly the same as (6.11) except that R appears in the right hand side and $\varkappa$ has been set equal to 1 everywhere except in the dimensionless numbers R and $\mathbb{C}_n$. In particular, we retain $\varkappa$ in the non-dimensional ratios $l_n(\varkappa, s)/\eta(\varkappa)$ and therefore in the evaluation of $\boldsymbol{F}$, $\boldsymbol{H}$ etc.

§ 7. Long Wave Solutions

The wave length of disturbances is $2\pi/\alpha$. Long waves are those for which α is small. The analyticity of the coefficients in (6.13) makes it natural to seek solutions as power series in α. Thus

$$\begin{aligned} \sigma &= \sigma_0 + \alpha\sigma_1 + \alpha^2\sigma_2 + O(\alpha^3), \\ \psi(x) &= \psi_0(x) + \alpha\psi_1(x) + \alpha^2\psi_2(x) + O(\alpha^3). \end{aligned} \tag{7.1}$$

Hence

$$\begin{aligned} \phi_2 &= \sigma_0 + \alpha(\sigma_1 + ix) + \alpha^2\sigma_2 + O(\alpha^3), \\ e^{-\phi_2 s} &= e^{-\sigma_0 s}\left[1 - \alpha\,(\sigma_1 + ix)\,s + \alpha^2\left\{-\sigma_2 s + \frac{s^2}{2}(\sigma_1 + ix)^2\right\} + O(\alpha^3)\right] \end{aligned} \tag{7.2}$$

and then from (6.11)

$$\begin{aligned} \mathbb{C}_n &= H_{n0} - \alpha(\sigma_1 + ix)\,H_{n1} - \alpha^2\left\{\sigma_2 H_{n1} - \frac{(\sigma_1 + ix)^2}{2}H_{n2}\right\} + O(\alpha^3), \\ H_{np} &= \int_0^\infty e^{-\sigma_0 s}s^p\{l_n(s)/\eta\}\,ds. \end{aligned} \tag{7.3}$$

In the analysis to follow we shall need the functions

$$l_4 = F_{12},$$

(7.4) $$l_3 = -\{H_{12} + 3\varkappa s F_{12} + \varkappa \overline{\overline{F}}_{12} + F_{22} - F_{12}\},$$

$$l_2 = -\{-2F_{12} + 3\varkappa^2 s^2 F_{12} + 3\varkappa^2 s \overline{\overline{F}}_{12} + 2\varkappa s(F_{22} - F_{11}) - \varkappa(\overline{\overline{F}}_{22} - \overline{\overline{F}}_{11}) + 2\varkappa s H_{12} + \varkappa \overline{\overline{H}}_{12} - H_{11} + H_{22}\}.$$

Inserting the representations (7.1) and (7.3) into (6.13), we identify independent powers of α to obtain the system of equations for $(\psi_0, \psi_1, \psi_2, \ldots)$ and $(\sigma_0, \sigma_1, \sigma_2, \ldots)$

$$H_{40}\psi_0^{iv} = R\sigma_0\psi_0'',$$

$$H_{40}\psi_1^{iv} - (\sigma_1 + ix)\, H_{41}\psi_1^{iv} + iH_{30}\psi_1''' = R\{\sigma_0\psi_1'' + (\sigma_1 + ix)\,\psi_0'',$$

(7.5) $$H_{40}\psi_2^{iv} - (\sigma_1 + ix)\, H_{41}\psi_1^{iv} + iH_{30}\psi_1''' - \left\{\sigma_2 H_{41} - \frac{(\sigma_1 + ix)^2}{2} H_{42}\right\}\psi_0^{iv} - (\sigma_1 + ix)\, iH_{31}\psi_0''' + H_{20}\psi_0'' = R\{\sigma_0(\psi_2'' - \psi_0) + (\sigma_1 + ix)\,\psi_1'' + \sigma_2\psi_0''\},$$

and for each $m = 0, 1, 2, \ldots$

(7.6) $$\psi_m(0) = \psi_m'(0) = \psi_m(1) = \psi_m'(1) = 0.$$

All solutions of $(7.5)_1$ and (7.6) are of the form

(7.7) $$\psi_0 = \cos \Lambda x - 1, \qquad \Lambda = \sqrt{-R\sigma_0/H_{40}} = 2m\pi.$$

If we assume that

(7.8) $$e^{-\mathrm{Re}\,\sigma_0 s}\, l_4 \left(\tau' = \int_0^\infty l_4(\varkappa, s)\, ds\right)$$

is positive, where $\mathrm{Re}\,\sigma_0$ represents the real part of σ_0, and monotone decreasing, which is essential if H_{40} is to exist, then $l_4 = O(e^{\alpha s})$, $s \to \infty$, $\alpha < 0$ and $\mathrm{Re}\,\sigma_0 > \alpha$, *i.e.* the spectrum of (7.5), has a lower bound. Also, if (7.8) is differentiable with respect to s then (7.7) in the form

$$R\sigma_0 = -\frac{4m^2\pi^2}{\eta} \int_0^\infty l_4 e^{-\sigma_0 s}\, ds, \qquad m = 0, 1, 2, \ldots$$

implies

(7.9)
(i) $\sigma_0 < 0$ if σ_0 is real,
(ii) $\mathrm{Re}\,\sigma_0 < 0$ if σ_0 is complex.

The first follows automatically, while the second follows from a contradiction on assuming the opposite (see CRAIK (1968)), since

$$\int_0^\infty l_4 e^{-\mathrm{Re}\,\sigma_0 s} \cos[(\mathrm{Im}\,\sigma_0)\, s]\, ds > 0.$$

A necessary condition for the existence of a solution to $(7.5)_2$ and (7.6) is

$$\langle R(\sigma_1 + ix)\, \psi_0'', \psi_0\rangle - iH_{30}\langle \psi_0''', \psi_0\rangle + H_{41}\langle(\sigma_1 + ix)\, \psi_0^{iv}, \psi_0\rangle = 0, \tag{7.10}$$

where

$$\langle f, g\rangle = \int_0^1 fg\, dx.$$

The second term vanishes, while the third is simplified using $(7.5)_1$ and (7.6). Then, (7.10) gives

$$\sigma_1 = -i\langle x\psi_0', \psi_0'\rangle / \langle \psi_0'\rangle = -i/2, \tag{7.11}$$

and so, *to the first order in α, $e^{\sigma t}$ has a time-periodic factor with period $2\pi/\omega$, where the frequency $\omega \cong \mathrm{Im}\,\sigma_0 - i\alpha/2 + O(\alpha^2)$.*

Returning to $(7.5)_2$ with (7.11) we obtain

$$\psi_1^{iv} + \Lambda^2 \psi_1'' = (x - \tfrac{1}{2})\, A\psi_0'' + B\psi_0''', \tag{7.12}$$

where

$$A = \frac{iR}{H_{40}} \left\{1 - \frac{H_{41}\Lambda^2}{R}\right\},$$

$$B = -iH_{30}H_{40}.$$

The relevant particular integral of (7.12) and (7.6) is

$$\begin{aligned} \psi_1 &= ax[\cos \Lambda x - 1] + \Lambda b(x^2 - x) \sin \Lambda x, \\ a &= \frac{iR}{4H_{40}\Lambda^2} \left\{5\left(1 - \frac{H_{41}\Lambda^2}{R}\right) - 2\frac{H_{30}\Lambda^2}{R}\right\}, \\ b &= \frac{iR}{4H_{40}\Lambda^2} \left\{1 - \frac{H_{41}\Lambda^2}{R}\right\}. \end{aligned} \tag{7.13}$$

With ψ_1 known we apply the Fredholm alternative to (7.5) and (7.6) to find the following expression for σ_2:

$$\begin{aligned} &\sigma_2\{-H_{41}\langle \psi_0'', \psi_0''\rangle + R\langle \psi_0', \psi_0'\rangle\} \\ &\quad = -R\sigma_0\langle \psi_0, \psi_0\rangle + iR\langle(x - \tfrac{1}{2})\, \psi_1'', \psi_0\rangle - iH_{30}\langle \psi_1''', \psi_0\rangle \\ &\qquad + H_{20}\langle \psi_0', \psi_0'\rangle + iH_{41}\langle(x - \tfrac{1}{2})\, \psi_1^{iv}, \psi_0\rangle \\ &\qquad - H_{31}\langle(x - \tfrac{1}{2})\, \psi_0''', \psi_0\rangle + \tfrac{1}{2} H_{42}\langle(x - \tfrac{1}{2})^2\, \psi_0^{iv}, \psi_0\rangle. \end{aligned} \tag{7.14}$$

The values of the integrals in (7.14)

$$\frac{\langle\psi_0,\psi_0\rangle}{\langle\psi_0',\psi_0'\rangle}=\frac{3}{\Lambda^2},\quad \frac{\langle\psi_0'',\psi_0''\rangle}{\langle\psi_0',\psi_0'\rangle}=\Lambda^2,$$

$$\frac{\langle(x-\frac{1}{2})\psi_0''',\psi_0\rangle}{\langle\psi_0',\psi_0'\rangle}=\frac{3}{2},\quad \frac{\langle(x-\frac{1}{2})^2\psi_0^{iv},\psi_0\rangle}{\langle\psi_0',\psi_0'\rangle}=\tfrac{1}{12}(\Lambda^2-42),$$

(7.15) $$\frac{\langle\psi_1''',\psi_0\rangle}{\langle\psi_0',\psi_0'\rangle}=-\tfrac{3}{2}a+\tfrac{1}{6}b(\Lambda^2+3),$$

$$\frac{\langle(x-\frac{1}{2})\psi_1'',\psi_0\rangle}{\langle\psi_0',\psi_0'\rangle}=-\frac{a}{12\Lambda^2}(\Lambda^2-6)+\frac{b}{12\Lambda^2}(4\Lambda^2+3),$$

$$\frac{\langle(x-\frac{1}{2})\psi_1^{iv},\psi_0\rangle}{\langle\psi_0',\psi_0'\rangle}=\frac{a}{12}(\Lambda^2+30)-\frac{b}{12}(8\Lambda^2+15),$$

where a and b are as in (7.13). On substituting the above values into (7.14) we find that

$$\sigma_2\{-H_{41}\Lambda^2+R\}=3H_{40}-R\left[\frac{-R}{48H_{40}\Lambda^2}\left\{5\left(1-\frac{H_{41}\Lambda^2}{R}\right)-\frac{2H_{30}\Lambda^2}{R}\right\}(\Lambda^2-6)\right.$$

$$\left.+\frac{R}{48H_{40}\Lambda^2}\left\{1-\frac{H_{41}\Lambda^2}{R}\right\}(4\Lambda^2+3)\right]$$

$$+H_{30}\left[-\frac{3R}{8H_{40}\Lambda^2}\left\{5\left(1-\frac{H_{41}\Lambda^2}{R}\right)-\frac{2H_{30}\Lambda^2}{R}\right\}\right.$$

(7.16) $$\left.+\frac{R}{24H_{40}\Lambda^2}\left\{1-\frac{H_{41}\Lambda^2}{R}\right\}(\Lambda^2+3)\right]$$

$$+H_{20}-H_{41}\left[\frac{R}{48H_{40}\Lambda^2}\left\{5\left(1-\frac{H_{41}\Lambda^2}{R}\right)-\frac{2H_{30}\Lambda^2}{R}\right\}(\Lambda^2+30)\right.$$

$$\left.-\frac{R}{48H_{40}\Lambda^2}\left\{1-\frac{H_{41}\Lambda^2}{R}\right\}(8\Lambda^2+33)\right]$$

$$-\tfrac{3}{2}H_{31}+\frac{1}{24}H_{42}(\Lambda^2-42).$$

The expression

(7.17)

$$\sigma = \sigma_0 + \alpha\sigma_1 + \alpha^2\sigma_2 + O(\alpha^3) = \sigma_0 - i\alpha/2 + \alpha^2\sigma_2 + O(\alpha^3), \quad \mathrm{Re}\,\sigma_0 < 0,$$

together with (7.14) gives the explicit formula for σ through terms of $O(\alpha^3)$.

To compute σ_2 from (7.16) we need values for the quantities defining the H_{nl}. This rheological information is not available even for one single non-Newtonian fluid. To obtain some more definite, if approximate, results, we follow BECKER (1980) in assuming that the kernels (3.8) have the short memory form (3.9) and that integrals of $e^{-\sigma_0 s}$ times kernels of the Maxwell type with small relaxation time λ are nearly the same as when $e^{-\sigma_0 s} = 1$. Naturally such an assumption requires that $|\sigma_0|\,\lambda$ be small when λ is small. Thus our analysis applies for all eigenvalues σ_0 such that $|\sigma_0| = O(1)$. Using the short memory idea, we find that

$$H_{40} \cong \frac{\tau'(\varkappa)}{\eta(\varkappa)},$$

$$H_{30} \cong \frac{N_1' - N_1/\varkappa}{\eta} = \frac{\varkappa\left(\dfrac{N_1}{\varkappa}\right)'}{\eta(\varkappa)}, \tag{7.18}$$

$$H_{20} \cong \frac{2\tau'(\varkappa)}{\eta(\varkappa)} - 4$$

and all the other H_{nl} are of lower order. Retaining only H_{40}, H_{30} and H_{20}, given by (7.18), we reduce (7.16) to

$$\sigma_2 R = 5\frac{\tau'}{\eta} - 4 + \tfrac{3}{4}\frac{\eta}{\tau'}\left(\frac{\varkappa\left[\dfrac{N_1}{\varkappa}\right]'}{\eta}\right)^2 + O(R) + O(\lambda) \tag{7.19}$$

for small Reynolds numbers. We note that the shear rate may be large if the gap width is small enough for any Reynolds number. The same short memory assumption also implies that

$$R\sigma_0 = -\Lambda^2 H_{40} \cong -\Lambda^2\frac{\tau'}{\eta}. \tag{7.20}$$

After combining these expressions with (7.17), we get

$$\sigma = \frac{1}{R}\left\{\frac{\tau'}{\eta}[-\Lambda^2 + 5\alpha^2] - 4\alpha^2 + \tfrac{3}{4}\frac{\eta}{\tau'}\left(\frac{\alpha\varkappa}{\eta}\left[\frac{N_1}{\varkappa}\right]'\right)^2\right. \tag{7.21}$$

$$+ [O(\lambda) + O(R)]\,\alpha^2 - \frac{i\alpha}{2} + O(\alpha^3).$$

We now consider special fluids for which $\tau' = \eta$ and $N_1/\varkappa^2$ are constants. Then (7.21) becomes

$$\sigma = \frac{1}{R}\left\{(\alpha^2 - \Lambda^2) + \tfrac{3}{4}\,\alpha^2\,\frac{N_1^2}{\tau^2} + [O(\lambda) + O(R)]\,\alpha^2\right\} - \tfrac{1}{2}\,i\alpha + O(\alpha^3), \qquad \tau = \varkappa\eta. \tag{7.22}$$

Since $(N_1/\tau)^2 = K\varkappa^2$ for $K > 0$, (7.22) implies that the instability criterion $\operatorname{Re}\sigma > 0$ may be realized at low Reynolds number for sufficiently high rates of shear $\varkappa$, if the first two terms in (7.22) dominate the sign of $\operatorname{Re}\sigma$. This condition can be satisfied in situations in which the magnitude of the ratio of the first normal to shear stress is sufficiently large.

Acknowledgements. J. Dunwoody acknowledges the support of a Fulbright Senior Fellowship while enjoying the hospitality of the Department of Aerospace Engineering and Mechanics, University of Minnesota, Minnapolis, MN 55455. D. D. Joseph acknowledges support from the Mathematics Division, U. S. Army Research Station, Durham.

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(Received July 15, 1981; revised June 21, 1983)

Note added in proof. AKBAY *et al.* (1980, 1981) use the assumption of short memory and neglect another term to derive a critical value of four for the modified Weissenberg number defined in § 1. They show that when this number is less than four the flow is stable and when it is greater than four then numerical calculations show that it is unstable to short waves, $\alpha \to \infty$.