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Classification of linear viscoelastic solids based on a failure criterion

AMITABH NARAIN * and DANIEL D. JOSEPH **

* *Department of Mechanical Engineering and Engineering Mechanics, Michigan Technological University, Houghton, MI 49931, USA*
** *Department of Aerospace Engineering and Mechanics, University of Minnesota, Minneapolis, MN 55455, USA*

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Abstract

An isotropic, incompressible linear viscoelastic solid subjected to a step shear displacement fails if the relaxation function $G(s)$ is such that $0 < G(0) < \infty$ and $-\infty < G'(0) \leqslant 0$. In this case, the discontinuity in displacement propagates into the interior of the body. The discontinuity will not propagate however if $G(0) = \infty$ or $G'(0) = -\infty$. In the former case there is a diffusion-like smoothening of discontinuous data characteristic of parabolic equations. The case $G(0) = \infty$ may be achieved by composing the kernel as a sum of a smooth kernel and a delta function at the origin times a viscosity coefficient. If the viscosity is small, the smoothing will take place in a propagating layer which scales with the small viscosity. The case of $G'(0) = -\infty$ is interesting in the sense that the solution is C^∞ smooth but the boundary of the support of the solution propagates at a constant wave speed. If $0 < G(0) < \infty$ and $-\infty < G'(0) < 0$, then the material accomodates stress waves under step traction leading to an elastic steady state.

§1. Introduction

Let the position of a particle P of a solid in its natural configuration be $\boldsymbol{X}$ at $t = 0$ and $\boldsymbol{x}$ at a later time t. Let the motion be given by χ: $E^3 \times \mathbb{R} \to E^3$ such that

$$\boldsymbol{x} = \chi(\boldsymbol{X}, t).$$

Let $\boldsymbol{u} \overset{\text{def}}{=} \boldsymbol{x} - \boldsymbol{X}$,

$$2\boldsymbol{E} \overset{\text{def}}{=} \operatorname{grad}_X \boldsymbol{u}(\boldsymbol{X}, t) + (\operatorname{grad}_X \boldsymbol{u}(\boldsymbol{X}, t))^T.$$

Then the Cauchy stress $\boldsymbol{T}$ for an isotropic, incompressible linear viscoelastic solid [1,4] is given by:

$$\begin{aligned} \boldsymbol{T} &= -p\mathbf{1} + 2\mu \boldsymbol{E}(t) + 2\int_0^\infty \frac{\mathrm{d}G(s)}{\mathrm{d}s}\{\boldsymbol{E}(t-s) - \boldsymbol{E}(t)\}\mathrm{d}s \\ &= -p\mathbf{1} + 2(\mu + G(0^+))\boldsymbol{E}(t) + 2\int_0^\infty \frac{\mathrm{d}G(s)}{\mathrm{d}s}\boldsymbol{E}(t-s)\mathrm{d}s. \end{aligned} \tag{1.1}$$

Equation (1.1) also follows as the first order linearized Cauchy-stress [1,2] for an arbitrary isotropic incompressible simple solid in the weighted $L_2(0, \infty)$ fading memory space of Coleman and Noll [1].

The parameter μ is the shear modulus or the second Lamé constant in the theory of linear elasticity. The relaxation function G: $[0, \infty) \to \mathbb{R}^+$ is assumed to have the following reasonable properties:

(i) $G \in C[0, \infty) \cap PC^1(0, \infty)$; that is, G is continuous and piecewise continuously differentiable.

(ii) G is monotonically decreasing and $\lim_{s\to\infty} G(s) = 0$, $G(0^+) = G(0)$.

(iii) $G(s)$ and $G'(s)$ are $O(e^{-\bar{\lambda}s})$ as $s \to \infty$ for some $\bar{\lambda} > 0$.

§2. Propagation of discontinuous shearing displacements

Let a viscoelastic solid at rest occupy the region $0 \leqslant x < \infty$, $-\infty < y < \infty$, $-\infty < z < \infty$. The bottom plane is given a step displacement in the y-direction. The resulting displacement field is in a form

$$x = X, \quad y = Y + v(x, t), \quad z = Z \tag{2.1}$$

which automatically satisfies the incompressibility condition. Using (1.1) and the momentum equation we find that

$$\left.\begin{aligned}
&\rho\frac{\partial^2 v}{\partial t^2} = (\mu + G(0))\frac{\partial^2 v}{\partial x^2} + \int_0^t \frac{dG}{ds}(s)\frac{\partial^2 v}{\partial x^2}(x, t-s)ds,\\
&v(0, t) = \begin{cases} 1 & \text{for } t > 0,\\ 0 & \text{for } t \leqslant 0,\end{cases}\\
&\lim_{x\to\infty} v(x, t) = 0,\\
&v(x, 0) = \frac{\partial v}{\partial t}(x, 0) = 0,\\
&v(x, t) \text{ is bounded for } x \geqslant 0 \text{ and } t \geqslant 0.
\end{aligned}\right\} \tag{2.2}$$

In deriving (2.2) we assumed that the material was initially at rest; hence $\partial v/\partial t(x, 0^+) = \partial v/\partial t(x, 0^-) = 0$. We solve (2.2) using Laplace transforms following Narain and Joseph [5].

Let

$$\left.\begin{aligned}
\bar{v}(x, u) &\overset{\text{def}}{=} \int_0^\infty e^{-us} v(x, s)ds\\
\bar{G}(u) &\overset{\text{def}}{=} \int_0^\infty e^{-us} G(s)ds
\end{aligned}\right\} \quad \forall u \in \mathbb{C} \ni \operatorname{Re} u > 0.$$

Then the transform of (2.2) yields

$$\bar{v}(x, u) = \frac{1}{u}\, e^{-xu\sqrt{\frac{\rho}{(\mu + u\bar{G}(u))}}}. \tag{2.3}$$

The properties of $\bar{G}(u)$ are specified in §4 of [5]. Here we note that

$$\lim_{\substack{|u|\to\infty\\ |\arg u| \leqslant \frac{1}{2}\pi}} u\bar{G}(u) = G(0), \tag{2.4}$$

$$\left.\begin{aligned}
&\operatorname{Re} u > 0 \text{ and } \operatorname{Im} u = 0 \Rightarrow \operatorname{Re}\bar{G}(u) > 0 \text{ and } \operatorname{Im}\bar{G}(u) = 0,\\
&\operatorname{Re} u \geqslant 0 \text{ and } \operatorname{Im} u > 0 \Rightarrow \operatorname{Re}\bar{G}(u) > 0 \text{ and } \operatorname{Im}\bar{G}(u) < 0,\\
&\operatorname{Re} u > 0 \text{ and } \operatorname{Im} u < 0 \Rightarrow \operatorname{Re}\bar{G}(u) > 0 \text{ and } \operatorname{Im}\bar{G}(u) > 0.
\end{aligned}\right\} \tag{2.5}$$

It follows from (2.5) that

$$\operatorname{Re} u > 0 \Rightarrow \operatorname{Re}(u\bar{G}(u)) > 0. \tag{2.6}$$

From (2.6) it follows that $\bar{v}(x, u)$ given in (2.3) is analytic in the half-plane $\operatorname{Re} u > 0$. $v(x, t)$ is then given by the inverse Laplace transform

$$v(x, t) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} \frac{e^{ut - ux\eta(u)}}{u} du, \qquad \gamma > 0$$

where

$$\eta(u) \overset{\text{def}}{=} \sqrt{\frac{\rho}{(\mu + u\bar{G}(u))}}. \tag{2.7}$$

Let,

$$\alpha \overset{\text{def}}{=} \frac{1}{C} \overset{\text{def}}{=} \sqrt{\frac{\rho}{\mu + G(0)}}. \tag{2.8}$$

Using (2.4) and the analyticity of $\bar{v}(x, u)$ for $\operatorname{Re} u > 0$, we can invert (2.7) arguing along lines given in §5 of [5] to show that

$$\begin{aligned}
&t < \alpha x \Rightarrow v(x, t) = 0,\\
&t > \alpha x \Rightarrow v(x, t) \overset{\text{def}}{=} f(x, t)\\
&= \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \frac{1}{y}\exp\left(-xy\sqrt{\frac{\rho}{q(y)}}\cos\tfrac{1}{2}(\pi - r(y))\right)\sin(yt - \theta(y))dy,
\end{aligned} \tag{2.9}$$

where

$$\theta(y) \overset{\text{def}}{=} xy\sqrt{\frac{\rho}{q(y)}}\sin\tfrac{1}{2}(\pi - r(y)), \tag{2.10}$$

$$\mu + iy\bar{G}(iy) \overset{\text{def}}{=} q(y)\, e^{ir(y)}. \tag{2.11}$$

Now $(2.5)_2$ implies that for $\operatorname{Re} u = 0$, $\operatorname{Im} u > 0$

$$\arg(u\bar{G}(u)) = \arg(iy\bar{G}(iy)) \in (0, +\tfrac{1}{2}\pi),$$

and since

$$\mu > 0, \quad r(y) \in (0, \tfrac{1}{2}\pi), \tag{2.12}$$

it follows from (2.12) that (2.10) converges uniformly for $t > \alpha x$.

Equations (2.9) and (2.10) together imply that

$$\begin{aligned}
&v(x, t) = f(x, t)H(t - \alpha x),\\
&H(\xi) \overset{\text{def}}{=} \begin{cases} 1 & \text{for } \xi > 0,\\ 0 & \text{for } \xi < 0.\end{cases}
\end{aligned} \tag{2.13}$$

Now we show that a discontinuity in the displacement field (failure) propagates into the interior. For this, it is enough to show that $f(x, \alpha x^+) = e^{\alpha x G'(0)/2(\mu + G(0))} > 0$ for $0 < G(0) < \infty$ and $-\infty < G'(0) \leqslant 0$. Assuming that $G(s)$ is regular to justify the following calculation, we note that

$$\bar{G}(u) = \frac{G(0)}{u} + \frac{G'(0)}{u^2} + O\left(\frac{1}{u^3}\right). \tag{2.14}$$

(For weaker assumptions on G see Renardy [6].) Then, using (2.14) we rewrite (2.7) as follows:

$$v(x, t) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} \frac{e^{ut}}{u} \exp\left[-xu\sqrt{\frac{\rho}{\mu + G(0) + \frac{G'(0)}{u} + O\left(\frac{1}{u^2}\right)}}\right] du$$

$$= \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} \frac{\exp\left\{ut - \alpha x u\left[1 - \left(\frac{G'(0)}{\mu + G(0)}\right)\left(\frac{1}{2u}\right) + O\left(\frac{1}{u^2}\right)\right]\right\}}{u} du$$

$$= \exp\left[\frac{\alpha x}{2}\left(\frac{G'(0)}{\mu + G(0)}\right)\right]\frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty}\frac{e^{ut - \alpha x u}}{u}\left[1 + O\left(\frac{1}{u}\right)\right] du$$

$$= \exp\left[\frac{\alpha x G'(0)}{2(\mu + G(0))}\right] H(t - \alpha x)$$

$$+ \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} \exp\left[\frac{\alpha x G'(0)}{2(\mu + G(0))}\right]\frac{e^{ut - \alpha x u}}{u} O\left(\frac{1}{u}\right) du. \tag{2.15}$$

The second term in (2.15) is continuous for $x, t \in \mathbb{R}^+$ because it is uniformly convergent for all positive x, t. Comparing (2.15) with (2.13) we get:

$$f(x, \alpha x^+) = \exp(\alpha x G'(0)/2\mu + 2G(0)). \tag{2.16}$$

Equation (2.16) establishes that viscoelastic solids with kernels satisfying $0 < G(0) < \infty$ and $-\infty < G'(0) \leqslant 0$ fail under step displacement data on the boundary. The case $G'(0) = 0$ implies an undamped discontinuity.

§3. Kernel functions for which the material does not fail under step displacement

[(i)] Let, $0 < G(0) < \infty$ and $G'(0) = -\infty$.
Then $f(x, \alpha x^+) = 0$. But the boundary of the support of $v(x, t)$ disturbing the initial state of rest propages with a constant wave speed $C = \sqrt{[\mu + G(0)]/\rho}$. The solution $v(x, t)$ is smooth.
[(ii)(a)] $G(0) = \infty$. In this case we again have smooth solutions and the discontinuity at the boundary is instantly smoothed as in parabolic problems.
[(ii)(b)] Let T be allowed to have distributions in its kernel as in the Theory of Saut and Joseph [7]. For example,

$$G(s) = a\delta(s) + g(s), \qquad a > 0 \tag{3.1}$$

where $g(s)$ has the properties (i)–(iii) of §1. In this case we have to solve the problem:

$$\left.\begin{aligned}
&\rho\frac{\partial^2 v}{\partial t^2} = (\mu + g(0))\frac{\partial^2 v}{\partial x^2} + a\frac{\partial^3 v}{\partial x^2 \partial t} + \int_0^t g'(s)\frac{\partial^2 v}{\partial x^2}(x, t - s)ds,\\
&v(0, t) = \begin{cases}1 & \text{for } t > 0,\\ 0 & \text{for } t \leqslant 0.\end{cases}\\
&\lim_{x \to \infty} v(x, t) = 0,\\
&v(x, 0) = \frac{\partial v}{\partial t}(x, 0) = 0,\\
&v(x, t) \text{ is bounded for } x \geqslant 0 \text{ and } t \geqslant 0.
\end{aligned}\right\} \tag{3.2}$$

There is a viscous term "$a(\partial^3 v)/(\partial x^2 \partial t)$" in Eqn. (3.2). The solution of problem (3.2) is given by:

$$v(x, t, a) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty}\frac{1}{u}\exp\left[ut - ux\sqrt{\frac{\rho}{\mu + au + u\bar{g}(u)}}\right] du. \tag{3.3}$$

Following the argument given in §18 [5], it can be shown that (3.3) yields smooth C^∞ solutions for $a \geqslant a_0 > 0$ for any a_0. Also, following §18 [5], we can establish that for small values of viscosity a, $v(x, t, a)$ has a transition layer around the discontinuous solution given by (2.13) and (2.16). This smooth transition layer propagates with the discontinuous solution for bounded x and t and scales with the small viscosity a.

§4. On stress-waves and their reflections due to step-traction data on the boundary for $0 < G(0) < \infty$ and $-\infty < G'(0) < 0$

In this section we consider a problem in which the solid occupies the region $0 \leqslant x \leqslant \ell$, $-\infty < y < \infty$, $-\infty < z < \infty$. A step shear traction is applied at $x = 0$ and the boundary at $x = \ell$ is held fixed. The resulting shearing motion is of the type (2.1) and the displacement field $v(x, t)$ is governed by:

$$\left.\begin{aligned}
&\rho\frac{\partial^2 v}{\partial t^2} = (\mu + G(0))\frac{\partial^2 v}{\partial x^2} + \int_0^t \frac{dG}{ds}(s)\frac{\partial^2 v}{\partial x^2}(x, t - s)ds,\\
&v(\ell, t) = 0 \qquad \forall t \geqslant 0,\\
&[T(-\hat{i})\cdot\hat{j}] = \left[(\mu + G(0))\frac{\partial v}{\partial x}(0, t) + \int_0^t \frac{dG}{ds}(s)\frac{\partial v}{\partial x}(0, t - s)ds\right]\\
&\qquad = \begin{cases}1 & \text{for } t > 0,\\ 0 & \text{for } t \leqslant 0,\end{cases}\\
&v(x, 0) = \frac{\partial v}{\partial t}(x, 0) = 0 \qquad \forall x > 0,\\
&v(x, t) \text{ is bounded for } x \geqslant 0 \text{ and } t \geqslant 0.
\end{aligned}\right\} \tag{4.1}$$

Now solving (4.1) by the Laplace transform, using the notation introduced in §2, we find that

$$\bar{v}(x, u) = \frac{\eta(u)}{u^2\rho} \times \frac{\sinh u\eta(u)(\ell - x)}{\cosh u\eta(u)\ell},$$

where

$$\eta(u) \overset{\text{def}}{=} \sqrt{\frac{\rho}{(\mu + uG(u))}}, \qquad u \in \mathbb{C} \ni \operatorname{Re} u > 0. \tag{4.2}$$

From (4.2) we see that

$$-\overline{T^{\langle xy\rangle}}(x, u) = \frac{\rho}{\eta^2}\frac{d\bar{v}}{dx}(x, u) = \frac{1}{u}\frac{\cosh u\eta(u)(\ell - x)}{\cosh u\eta(u)\ell}$$
$$= \frac{1}{u}\, e^{-u\eta(u)x}\left[1 + e^{-2u\eta(u)(\ell - x)}\right]\frac{1}{\left[1 + e^{-2u\eta(u)\ell}\right]}. \tag{4.3}$$

Using (2.14), we write

$$u\eta(u) = u\sqrt{\frac{\rho}{(\mu + G(0)) + \left(\frac{G'(0)}{u} + O\left(\frac{1}{u^2}\right)\right)}}$$
$$= u\sqrt{\frac{\rho}{\mu + G(0)}} - \frac{1}{2}\frac{G'(0)}{\mu + G(0)} + O\left(\frac{1}{u}\right). \tag{4.4}$$

If we choose $\operatorname{Re} u = \gamma$ sufficiently large (say larger than γ^*) then, since $G'(0) < 0$, we have

$$\operatorname{Re}(u\eta(u)) > 0. \tag{4.5}$$

From (4.5) it follows that

$$\operatorname{Re} u = \gamma > \gamma^* \Rightarrow |e^{-2u\eta(u)\ell}| = e^{-2\ell \operatorname{Re}(u\eta(u))} < 1. \tag{4.6}$$

Then we can write

$$\frac{1}{1 + e^{-2u\eta(u)\ell}} = 1 - e^{-2u\eta(u)\ell} + e^{-4u\eta(u)\ell} - e^{-6u\eta(u)\ell} \pm \dots \tag{4.7}$$

The right hand side of (4.7) is absolutely uniformly convergent, and (4.7) and (4.3) give

$$-\overline{T^{\langle xy\rangle}}(x, u) = \frac{1}{u}\Big[e^{-u\eta(u)x} + \left\{e^{-u\eta(u)(2\ell - x)} - e^{-u\eta(u)(2\ell + x)}\right\}$$
$$- \left\{e^{-u\eta(u)(4\ell - x)} - e^{-u\eta(u)(4\ell + x)}\right\} \pm \dots\Big]. \tag{4.8}$$

The right hand side of (4.8) is absolutely uniformly convergent for $\operatorname{Re} u > \gamma^*$. Thus using (2.7) with $\gamma > \gamma^*$, we can invert (4.8) term by term. Using (2.13), we find that

$$L^{-1}\left[\frac{1}{u}\, e^{-\theta u\eta(u)}\right] = f(\theta, t)H(t - \alpha\theta). \tag{4.9}$$

Using (4.9) in the inversion of (4.8), we get

$$-T^{\langle xy\rangle}(x, t) = \big[f(x, t)H(t - \alpha x) + \{f(2\ell - x, t)H(t - \alpha(2\ell - x))$$
$$- f(2\ell + x, t)H(t - \alpha(2\ell + x))\} \pm \dots\big]. \tag{4.10}$$

Equations (4.10) and (2.16) imply the existence of damped stress-waves which reflect repeatedly from the bounding walls. This is a generalization of Hunter's [3] result for exponential kernels. The characteristic lines along which the stress waves are reflected are shown in Fig. 4.1.

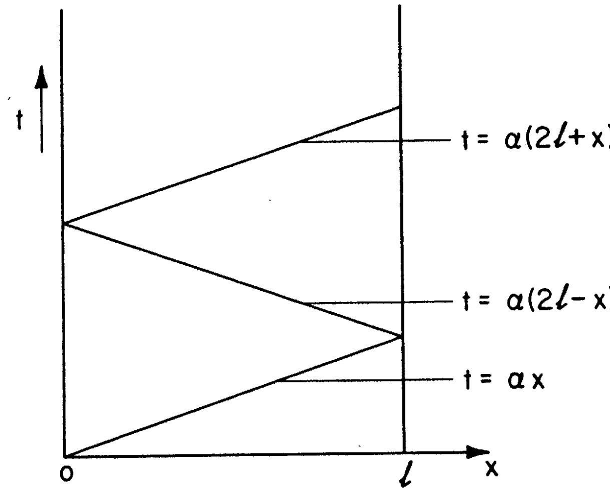

Figure 1.

Using (4.2) we observe that the asymptotic steady displacement is given by:

$$\lim_{t\to\infty} v(x, t) = \lim_{u\to 0^+} u\bar{v}(x, u)$$
$$= \lim_{u\to 0^+} \frac{\eta(u)}{u\rho}\frac{\sinh u\eta(u)(\ell - x)}{\cosh u\eta(u)\ell}$$
$$= \lim_{u\to 0} \eta^2(u) \times \frac{(\ell - x)}{\rho}$$
$$= (\ell - x)/\mu. \tag{4.11}$$

The displacement (4.11) is same as the displacement of an elastic material due to unit shear stress at the boundary $x = 0$.

Conclusion

Kernels which are commonly used to describe viscoelastic solids lead to failure under step shear displacement. (The same remark holds for elastic solids.) This failure can be avoided by adding a small "Newtonian" viscous term to the constitutive equation or by some other choices of kernels $G(s)$ which have $G'(0) = -\infty$.

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Note added in proof

(a) The amplitude derived in (2.16) is the same as the one derived by Coleman and Gurtin (*Arch. Rational Mech. Anal.*, Vol. 19, pp. 239–265, 1965) as a necessary condition for the amplitude of acceleration waves.

(b) The precise arguments leading to the assertion (i) in §3 of this paper can be found in the Ph.D. thesis of A. Narain (Univ. of Minnesota, 1983).