EXAMPLES AND SIGNIFICANCE OF CHANGE OF TYPE IN VISCOELASTICITY

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ABSTRACT. The equations governing the flow of viscoelastic fluids are classified according to the symbol of their differential operators. Conditions for a change of type in steady two-dimensional flows are derived for a three-constant Oldroyd model. We find a change of type in the vorticity equation when a critical condition involving speeds and stresses is satisfied. We also sketch how change of type can be discussed for more general models.

I. INTRODUCTION. An important dimensionless quantity characterizing the flow of viscoelastic fluids is the Weissenberg or Deborah number. The exact definition of this quantity varies with the constitutive model and the flow under consideration, but, roughly speaking, it measures the ratio of elastic to viscous forces, or, alternatively, of a time characteristic of the fluid to a time characteristic of the flow.

Numerical calculations of steady flows in viscoelastic fluids typically fail if this Weissenberg number is high or even moderate. It is not well understood why and the reason is probably not always the same. Experimentally, qualitative changes in the flow behaviour are often observed at high Weissenberg numbers.

In a recent paper [6], we advance the idea that some of these effects are related to a change of type in the governing equations. We discuss change of type in detail for a three-constant Oldroyd model, but also sketch an analysis for more general models. This study extends earlier work of Rutkevich [10], Ultman and Denn [11], and Luskin [7]. When discussing change of type we have to distinguish between two cases:

1. There is a change of type for the equations governing steady flow as well as for the time-dependent equations. This leads to Hadamard instability and ill-posedness of the initial value problem. This kind of situation is familiar from the theory of phase transitions.
2. There is a change of type in the steady equations, but not in the unsteady equations. This happens when the speed of the fluid exceeds a wave propagation speed as in a sonic transition in gas dynamics. There is no Hadamard instability associated with this.

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Several papers in the literature have attempted to link experimental observations to change of type. Hunter and Slemrod [4], and, on the basis of a different model, Becker and his coworkers [2] have tried to explain melt fracture by a change of type leading to Hadamard instability (see [1] for a detailed and critical discussion of Becker's theory). Ultman and Denn [11] refer to an observation of James [5] on heat transfer in flows past a cylinder. It appears that there is a discontinuity in slope when heat transfer coefficient is plotted against the speed of the fluid. Ultman and Denn suggest that a sonic transition occurs at the speed where the slope is discontinuous. Recently, Yoo, Ahrens and Joseph [12] have discussed experiments by Metzner, Uebler and Fong [8] on tube entry flows from a conical region. At high Weissenberg number, the flow partitions into an interior cone, where the streamlines are approximately straight towards the sink, and an outer region of recirculation. The boundary between these regions seems to be rather sharp, and there is an apparent discontinuity in the vorticity (see Fig. 11 in [8]). Yoo, Ahrens and Joseph relate this observation to our analysis of Oldroyd models. All these studies are rather tentative, and at present not enough is known either experimentally or theoretically to make strong claims.

In section 2, we give basic definitions relating to change of type in first order systems of partial differential equations. These are applied in section 3 to the study of two-dimensional steady flows for a class of three-constant Oldroyd models [9]. A criterion for criticality is given, and the vorticity is identified as the variable associated with the change of type. In section 4 we demonstrate how similar ideas can be extended to general fluids with fading memory. However, it is in general not possible to decouple the characteristic equation and isolate a vorticity equation as in the case of the three-constant Oldroyd model.

2. BASIC DEFINITIONS. The equations for viscoelastic flow discussed below have the form of quasilinear first order systems. In this section, we give some definitions relating to characteristics and change of type in such systems (see e.g. [3]). We are concerned with equations of the form

$$\sum_{\ell=0}^{n} \underline{\underline{A}}_\ell(\underline{x},\underline{u}) \frac{\partial \underline{u}}{\partial x_\ell} = \underline{f}(\underline{x},\underline{u}) \tag{2.1}$$

where $\underline{u}$ is a k-vector and the $\underline{\underline{A}}_\ell$ are $k \times k$-matrices. The term "quasilinear" means that $\underline{\underline{A}}_\ell$ and $\underline{f}$ may depend on $\underline{x}$ and $\underline{u}$, but not on derivatives of $\underline{u}$, i.e. the highest order derivatives occur in the equations in a linear way. For every choice of $\underline{x}$ and $\underline{u}$, we define characteristic surfaces as follows: A surface given by an equation $\phi(t,x_1,\dots,x_n) = 0$ is characteristic if

$$\det\Big(\sum_{\ell=0}^{n} \underline{\underline{A}}_\ell \frac{\partial \phi}{\partial x_\ell}\Big) = 0 \quad . \tag{2.2}$$

The system is called elliptic if there are no real characteristic surfaces. Hyperbolic systems are characterized as the opposite extreme, namely, there is a maximal number of real characteristics. More precisely, a system is called hyperbolic, if one of the matrices $\underline{\underline{A}} = \underline{\underline{A}}_\mu$ is non-singular and, for every choice of real parameters $(\lambda_\ell,\ \ell = 0,1,\dots,n;\ \ell \neq \mu)$, the roots α of the eigenvalue problem

(2.3)
$$\det\Big(\alpha \underline{\underline{A}} - \sum_{\substack{\ell=0\\ \ell\neq\mu}}^{n} \lambda_\ell \underline{\underline{A}}_\ell\Big) = 0$$

are real and semisimple. The equations of viscoelastiticity are neither elliptic nor hyperbolic. However, we will encounter situations where the number of real characteristic surfaces changes. In this case, we say there is a change of type.

The phenomenon of Hadamard instability is closely related to this. It is evident that, if (2.3) has complex roots, then $\operatorname{Im}(\alpha)$ can be made arbitrarily large by making the λ_ℓ large. If we choose $\mu = 0$ and interpret the first coordinate $x_0 = t$ as time, then this means that the linearization of (2.1) will have rapidly growing solutions when the initial data are very oscillatory. This kind of catastrophic instability is referred to as "Hadamard instability".

3. CHANGE OF TYPE IN TWO-DIMENSIONAL STEADY FLOWS OF THREE-CONSTANT OLDROYD FLUID. We consider differential models with a constitutive law of the form

(3.1)
$$\lambda \frac{D\underline{\underline{\tau}}}{Dt} + \underline{\underline{\tau}} = 2\eta\, \underline{\underline{D}}$$

where D/Dt denotes a frame invariant time derivative expressed as

(3.2)
$$\frac{D\underline{\underline{\tau}}}{Dt} = \frac{\partial\underline{\underline{\tau}}}{\partial t} + (\underline{u}\cdot\nabla)\underline{\underline{\tau}} + \underline{\underline{\tau}}\,\underline{\underline{\Omega}} - \underline{\underline{\Omega}}\,\underline{\underline{\tau}} - a(\underline{\underline{\tau}}\,\underline{\underline{D}} + \underline{\underline{D}}\,\underline{\underline{\tau}}) \quad .$$

Here we have split the velocity gradient $\nabla\underline{u}$ with components $(\nabla\underline{u})_{ij} = \partial u_i/\partial x_j$ into its symmetric part $\underline{\underline{D}} = 1/2\,(\nabla\underline{u} + (\nabla\underline{u})^T)$ and its anti-symmetric part $\underline{\underline{\Omega}} = 1/2\,(\nabla\underline{u} - (\nabla\underline{u})^T)$. The special cases $a = 1$, $a = -1$ and $a = 0$ are known as the upper convected, lower convected and corotational Maxwell model, respectively.

In steady two-dimensional flows, we denote velocity components by u and v, and the extra stress tensor is written in the form

(3.3)
$$\underline{\underline{\tau}} = \begin{pmatrix} \sigma & \tau \\ \tau & \gamma \end{pmatrix} .$$

The constitutive law (3.1), together with the equation of motion and the incompressibility condition leads to the following quasilinear first order system

(3.4)
$$\begin{aligned}
&u\sigma_x + v\sigma_y + \tau(v_x - u_y) - a[2\sigma u_x + \tau(u_y+v_x)] - 2\,\frac{\eta}{\lambda}\,u_x = -\frac{\sigma}{\lambda}\\
&u\tau_x + v\tau_y + \frac{1}{2}\,(\sigma-\gamma)(u_y - v_x) - \frac{a}{2}\,(\sigma+\gamma)(u_y+v_x) - \frac{\eta}{\lambda}\,(u_y+v_x) = -\frac{\tau}{\lambda}\\
&u\gamma_x + v\gamma_y + \tau(u_y - v_x) - a[2\gamma v_y + \tau(u_y+v_x)] - 2\,\frac{\eta}{\lambda}\,v_y = -\frac{\gamma}{\lambda}\\
&\rho(uv_x + vu_y) + p_x - \sigma_x - \tau_y = 0\\
&\rho(uv_x + vv_y) + p_y - \tau_x - \gamma_y = 0
\end{aligned}$$

$$u_x + v_y = 0 \quad .$$

We can apply the definitions of section 2 to this system. This leads to the following equation for the slope $\alpha = dy/dx$ of characteristic lines.

$$(3.5)\quad (1+\alpha^2)(-\alpha u+v)^2\{\rho(-\alpha u+v)^2 + (\frac{\gamma-\sigma}{2})(\alpha^2-1) + 2\tau\alpha - (\alpha^2+1)(\frac{\eta}{\lambda} + a(\frac{\gamma+\sigma}{2}))\} = 0 \quad .$$

We see that the stream lines are double characteristics, and that two characteristic values are always complex. The interesting factor is the last one. The roots of this factor change from complex to real when the sign of

$$(3.6)[\rho u^2 + \frac{\gamma}{2}(1-a) - \frac{\sigma}{2}(1+a) - \frac{\eta}{\gamma}][(1+a)\frac{\gamma}{2} + (a-1)\frac{\sigma}{2} + \frac{\eta}{\lambda} - \rho v^2] + (\rho uv-\tau)^2$$

changes from negative to positive.

The reason why (3.5) decouples into quadratic factors becomes evident in a streamfunction-vorticity formulation. When the equations are rewritten in this way, one can see that the roots $\alpha = \pm i$ are associated with the equation expressing the vorticity as the Laplacian of the stream function. The third factor is associated with an equation which involves a linear combination of second derivatives of the vorticity and only contains lower order terms otherwise. It is therefore the vorticity which is associated with the change of type. It is interesting in this context that the experiments of Metzner, Uebler and Fong [8] can be interpreted as suggesting a discontinuity in the vorticity.

One can also derive a time dependent vorticity equation, which leads to a criterion for Hadamard instability. Hadamard instability occurs if one of the following conditions is violated

$$(3.7)\quad \lambda^2\tau^2 - [\eta - \lambda(\frac{\gamma}{2}(1-a) - \frac{\sigma}{2}(1+a))][\eta - \lambda(\frac{\sigma}{2}(1-a) - \frac{\gamma}{2}(1+a))] < 0$$

$$(3.8)\quad \lambda[\frac{\gamma}{2}(1-a) - \frac{\sigma}{2}(1+a)] - \eta < 0 \quad .$$

Note that (3.7) agrees exactly with (3.6) for zero speeds. Changes of type in steady flow which do not involve Hadamrd instability must therefore require a non-zero speed of the fluid. In fact, the criterion is that the speed of the fluid is faster than a viscoelastic wave speed. In particular, if the stresses vanish, a change of type occurs when the fluid speed exceeds the wave speed of linear viscoelasticity. Since this requires a finite (but not large) Reynolds number, such changes of type are more likely to be found in dilute polymer solutions rather than in melts.

In discussing the criteria (3.6) or (3.7), (3.8), it must be kept in mind that the values of the extra stresses are not arbitrary. The constitutive law (3.1) can be regarded as an evolution problem for the stress with given deformation. However, in the discussion of materials with fading memory, we are not interested in arbitrary solutions of this evolution problem, but only in those that behave reasonably as time tends to $-\infty$. This imposes restrictions on the values of the extra stresses, which can be shown to preclude Hadamard instability if $a = \pm 1$.

For a discussion of particular flow geometries we refer to [6] and [12].

4. CHANGE OF TYPE IN FLUIDS WITH FADING MEMORY. The extra stress $\underline{\underline{\tau}}$ in a simple fluid is given by an isotropic functional of the history of the relative Cauchy strain $\underline{\underline{G}}(s) = \underline{\underline{F}}_t^T(t-s)\underline{\underline{F}}_t(t-s) - \underline{\underline{1}}$, i.e.

$$\underline{\underline{\tau}} = \underline{\underline{F}}[\underline{\underline{G}}(s)]_{s=0}^{\infty} \quad . \tag{4.1}$$

By taking the material derivative of (4.1), we obtain

$$\frac{d\underline{\underline{\tau}}}{dt} = \underline{\underline{F}}_1[\underline{\underline{G}} \mid \frac{d\underline{\underline{G}}}{dt}] \quad . \tag{4.2}$$

Following Coleman and Noll, we assume that the Fréchet derivative $\underline{\underline{F}}_1$ of the functional $\underline{\underline{F}}$ can be represented in the form

$$\underline{\underline{F}}_1[\underline{\underline{G}} \mid \frac{d\underline{\underline{G}}}{dt}] = \int_0^\infty \underline{\underline{K}}(s,\underline{\underline{G}}) \frac{d\underline{\underline{G}}(s)}{dt} ds \quad . \tag{4.3}$$

Here $\underline{\underline{K}}(s,\underline{\underline{G}})$ is a fourth order tensor depending on s and the values $\{\underline{\underline{G}}(\sigma), 0 < \sigma < \infty\}$. For the following, we assume that K and its first derivative with respect to s are integrable.

The material derivative of $\underline{\underline{G}}$ is given by

$$\frac{d\underline{\underline{G}}}{dt} = -\underline{\underline{L}}^T\underline{\underline{G}} - \underline{\underline{G}}\,\underline{\underline{L}} - \frac{d\underline{\underline{G}}}{ds} \tag{4.4}$$

where $\underline{\underline{L}} = \nabla \underline{u}$ is the present value of the velocity gradient. Hence we find

$$\begin{aligned} \int_0^\infty K_{ijk\ell}(s,\underline{\underline{G}}) \frac{dG_{k\ell}}{dt}(s)ds = &- \int_0^\infty (K_{ijk\ell} + K_{ij\ell k})G_{p\ell}(s)ds \cdot L_{pk}(t) \\ &- \int_0^\infty K_{ijk\ell}(s,\underline{\underline{G}}) \frac{d\underline{\underline{G}}(s)}{ds} ds \quad . \end{aligned} \tag{4.5}$$

The last term can be integrated by parts and treated as a perturbation of lower differential order. With

$$M_{ijkp} = - \int_0^\infty (K_{ijk\ell} + K_{ij\ell k})G_{p\ell}(s)ds \tag{4.6}$$

we can therefore write the equations of viscoelastic fluid motion in the form

$$\begin{aligned} &\frac{d\tau_{ij}}{dt} = M_{ijkp} \frac{\partial u_p}{\partial x_k} + N_{ij} \\ &\rho \frac{du_i}{dt} = - \frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j} + f_i \\ &\frac{\partial u_i}{\partial x_i} = 0 \quad . \end{aligned} \tag{4.7}$$

This again has the form of a quasilinear first order system, and the definitions of characteristics and change of type apply. In general, however, it is not possible to decouple this system as in section 3 and isolate a vorticity equation. In two-dimensional steady flow, we would still find the stream lines as double characteristics, but the remaining characteristic values would be determined by a fourth order equation, which cannot easily be factored. In [6], we identify a class of constitutive models which has certain structural similarities with the Oldroyd models above and permits the derivation of a vorticity equation.

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