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# LINEARIZED DYNAMICS OF SHEARING DEFORMATION PERTURBING REST IN VISCOELASTIC MATERIALS

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This paper extends our earlier work [6, 7] on the propagation of jumps in velocity and displacement for shearing deformations imposed impulsively at the boundary of viscoelastic fluids and solids obeying constitutive equations in integral form with arbitrary kernels of fading memory type. The earlier work is briefly reviewed in §1 and we give new results. In §2 we relate old results to experiments. The limiting velocity distribution for start-up of Couette flow between parallel plates is a linear shear. It is common practice to assume that the real motion is close to linear shear long before the stress approaches its asymptotic steady state value. When the simplified kinematics are assumed, the evolution of the wall shear stress is determined by material functions, independent of deformation. These material functions are then determined by experimental measurements. We argue that in some cases only very special features of the material functions can be determined by this method because (in all cases) the early time behavior of the motion is incorrectly given by the kinematic assumption. The assumption that the early part of the stress response can be ignored is at best an approximation when the dynamics shows the presence of a delta function singularity in the wall shear stress at time t=0 and at subsequent discrete times of reflection off bounding walls. This delta function contribution cannot be ignored even if the steady state is achieved rapidly. In fact the early time behavior of the material functions can be obtained from experiments only by using a correct theory based on dynamics rather than kinematical assumptions. When this is done it is possible to interpret data showing stress jumps with linear theories based on commonly used constitutive equations and to interpret early oscillations in the observed values of material functions in terms of repeated reflections off bounding walls. The foregoing remarks apply equally to the interpretation of stress relaxation experiments and other experiments involving impulsive changes in velocity and displacement. In §3 we derive formulas for the amplitude of jumps and reflections for fluids sheared between concentric cylinders. In §4 we develop integral methods of solution analogous to Duhamel integrals for inverting start up problems with arbitrary data perturbing rest. In §5 we apply our analysis to start up for viscoelastic solids and show how creep depends on the kernel of the integral equation.

## §1. A Summary of Previous Work on Step Jumps of Velocity and Displacement.

In our earlier work [6], we treated the problems of step increase in velocity and displacement using a constitutive expression of the type:

$$(1.1) \qquad \underset{\sim}{T} = -p\underset{\sim}{1} + \mu \underset{\sim}{A}_1 + \int_0^\infty \tilde{\mu}(s) \underset{\sim}{G}(s)\,ds$$

where, $\tilde{\mu}(s) \equiv \frac{dG}{ds}$ and $G: [0,\infty) \to \mathbf{R}^+ = \{x \in \mathbf{R} \mid x>0\}$

is assumed to be (i) strictly monotonically decreasing, (ii) continuous and piecewise continuously differentiable, (iii) of $0(e^{-\bar{\lambda}s})$ as $s\to\infty$ for some $\bar{\lambda}>0$ and, whenever needed, we may assume (iv) $G'(s) < 0$ is strictly monotonically increasing to $\lim_{s\to\infty} G'(s) = 0$.

Constitutive equations such as (1.1) may be justified in various ways (see Saut and Joseph [11] and Renardy [9]). We considered two singular problems in which the velocity is assumed to be in the form $\underset{\sim}{V} = \hat{e}_y v(x,t)$ in the semi-infinite space above a flat plate and

$$\Omega = [x,y,z;\ 0<x<\infty,\ -\infty<y<\infty,\ -\infty<z<\infty].$$

At x=0 we imagine either a step-jump in velocity or displacement, satisfying

$$(1.2) \qquad \mu\frac{\partial^2 v}{\partial x^2}(x,t) + \int_0^t G(s)\frac{\partial^2 v}{\partial x^2}(x,\ t-s)\,ds = \rho\frac{\partial v}{\partial t}(x,t).$$

$$v(x,\ 0) = 0,$$

$v(x,\ t)$ is bounded as $x,\ t\to\infty$.

And for step-increase in velocity at x=0

$$(1.3) \qquad v(0,\ t) = H(t-0)\ .$$

For the step-increase in displacement of the bottom plate we have

$$(1.4) \qquad v(0,t) = \delta(t).$$

### §1.1 Linearized Simple Fluids of Maxwell Type ($\mu$=0).

The solution of problem (1.2) and (1.3) is given in §4-6 of [6] as:

$$v(x,\ t) = f(x,\ t)\ H(t-\alpha x)$$

where

$$c = \frac{1}{\alpha} = \sqrt{G(0)/\rho}$$

and f(x,t) is defined in (5.10) of [6]. Here it will suffice to note that (see [10], [6], and [2])

$$a(x) \overset{\text{def}}{=} f(x,\alpha x^{+}) = \exp(\alpha x G'(0)/2G(0)).$$

(1.5) $$\frac{\partial f}{\partial t}(x,\ \alpha x^{+}) = -\alpha x\ \exp\left(\frac{\alpha x G'(0)}{2G(0)}\right)\left[\frac{3}{8}\left(\frac{G'(0)}{G(0)}\right)^2 - \frac{1}{2}\frac{G''(0)}{G(0)}\right].$$

$$\frac{\partial f}{\partial x}(x,\ \alpha x^{+}) = \alpha f(x,\ \alpha x^{+})\left[\frac{G'(0)}{2G(0)} + \alpha x\left\{\frac{3}{8}\left(\frac{G'(0)}{G(0)}\right)^2 - \frac{1}{2}\frac{G''(0)}{G(0)}\right\}\right].$$

If $G(s) = ke^{-\mu s}$, then $\frac{3}{8}\left(\frac{G'(0)}{G(0)}\right)^2 - \frac{1}{2}\frac{G''(0)}{G(0)} = -\frac{1}{2}\mu^2 < 0$ .

The solution of step-displacement problem (1.2) and (1.4) is given as (see in (10.7) of [6]).

(1.6) $$v(x,\ t) = \frac{\partial f}{\partial t}(x,t)\ H(t-\alpha x) + f(x,\ \alpha x^{+})\ \delta(t-\alpha x)$$

where f(x, t) is the same as in (1.5).

## §1.2 Special Kernels for Fluids of the Maxwell Type ($\mu$=0).

There are two special cases ($G'(0) = -\infty$, $G'(0)=0$):

(i) $G'(0) = -\infty$ and $0<G(0)<\infty$ .

In this case the amplitude a(x) of the shock (given in (1.5)) is zero. Thus the discontinuity of the data is removed but the support of the solution propagates with the speed $c = \frac{1}{\alpha}$.

In fact Renardy [8] has shown that for a kernel (used in certain molecular models)

$$G'(s) = -\sum_{n=1}^{\infty} \exp(-n^{\alpha}s),\ \alpha>1,$$

$$G'(0) = -\infty,$$

$$G(0) = \sum_{n=1}^{\infty} \frac{1}{n^{\alpha}}$$

the solution is $C^{\infty}$ smooth at the support (see Fig. 1.1). It may be noted that the special kernel used by Renardy is such that all of its derivatives at s=0 are unbounded; that is the contact between the vertical axis and the curve G(s) at s=0 is $C^{\infty}$ smooth. Some form of continuity of solution on kernels possessing nearly identical features globally might be expected. For example we may construct kernels with $G'(0)=-\infty$, and even with $C^{\infty}$ contact at the vertical axis whose graphs are indistinguishable from kernels for which G'(0) is finite in all neighborhoods bounded away from s=0. This may lead to smooth, shock like solutions (see Fig. 1.1). Such problems are in some sense like the ones which are perturbed with a small viscosity $\mu$. We shall remark in §1.3, that the small viscosity leads to a transition layer of size $\mu$ which collapses onto a shock as $\mu\to 0$. For small $\mu$ the solution is smooth, but shock like (see Figs. 1.1, 1.3). The heuristic argument for the equivalence of problems for kernels of type (i) with those perturbed by a small viscosity is as follows. We are given G(s), s>0 such that G(0) is finite, $G'(s) < 0$, $s > 0$, and $G'(0) = -\infty$. Now we implement the construction of a comparison kernel of Maxwell type. First choose a small time $\varepsilon$. Then, at $G(\varepsilon)$ draw the tangent $G'(\varepsilon)$. This tangent pierces s=0 at the value $G_M(0)$. Define $G_M(s)$

$$G_M(s) = \begin{cases} G'(\varepsilon)s + G_M(0), & s \le \varepsilon \\ G(s), & s > \varepsilon . \end{cases}$$

We may write

$$\int_0^t G(s)\frac{\partial^2 v}{\partial x^2}(x,t-s)\,ds = \int_0^t G_M(s)\frac{\partial^2 v}{\partial x^2}(x,t-s)\,ds$$

$$+ \int_0^{\varepsilon}(G(s) - G_M(s))\frac{\partial^2 v}{\partial x^2}(x,t-s)\,ds .$$

Using the mean value theorem the last integral may be written as

$$\varepsilon[G(\bar{s}) - G_M(\bar{s})]\,\frac{\partial^2 v}{\partial x^2}(x,\ t-\bar{s}),\quad 0<\bar{s}<\varepsilon.$$

Then with $\varepsilon\to 0$ we get $\bar{s}(\varepsilon)\to 0$ and we approximate the perturbing term with

$$\varepsilon[G(0) - G_M(0)]\frac{\partial^2 v}{\partial x^2}(x,\ t).$$

The approximating problem is like one perturbed by a small viscosity $\mu=\varepsilon[G(0)-G_M(0)]$.

The reader may notice that the heuristic argument just given applies to <u>any</u> two kernels which coincide for $s > \varepsilon$. The implication is that an approximation to the solution corresponding to one kernel may be obtained by solving a problem with the other kernel, perturbed by a viscous term with a suitably selected viscosity coefficient.

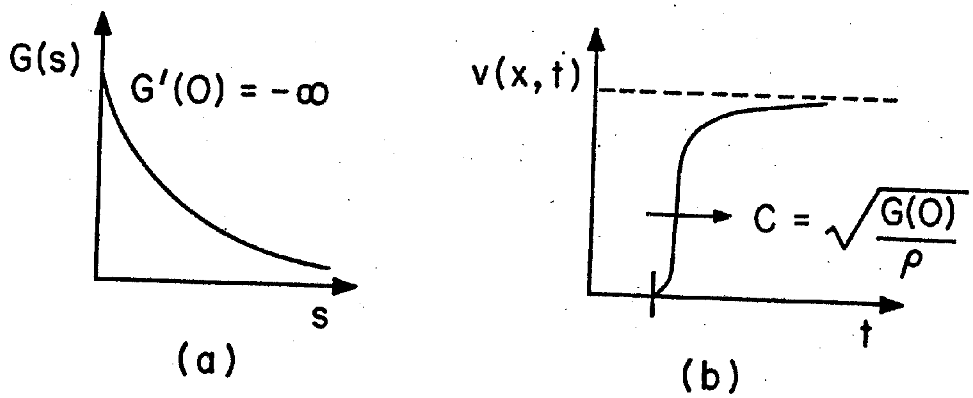



Fig. 1.1.: Propagating smooth solutions (b) occur when G(s), satisfying (i), is as sketched in (a).

To establish the above heuristic argument, we let $f(x,t) = f_1(x,t)$ in (5.10) of [6] be the solution for the kernel with $G'(0) = -\infty$ and let $f(x, t) = f_2(x, t)$ be the solution for the comparison kernel $\tilde{G}(s)$, $G(0)= \tilde{G}(0)$, $\tilde{G}'(s)$ is finite for $0 \leq s < \varepsilon$ and $\tilde{G}(s) = G(s)$ for $s \geq \varepsilon$. Then by choosing small $\varepsilon$ we reduce the value of $|\bar{G}(iy) - \bar{\tilde{G}}(iy)|$. Now invoking the continuity of (5.10) of [6] with respect to $r(y)$ and $p(y)$, we find that $|f_1(x,t) - f_2(x,t)|$ is small.

In the second special case we have

$$\text{(ii)} \quad G'(0) = 0.$$

In this case, $a(x) = 1$, and

$$\frac{\partial f}{\partial t}(x, \alpha x^+) = \tfrac{1}{2}\alpha x\left[\frac{G''(0)}{G(0)}\right].$$

It is necessary that $G''(0) \geq 0$ if $G$ is to be monotonically decreasing in $[0,\infty)$. For the case in which $G''(0) > 0$ there will be a velocity overshoot in the neighborhood of $t=\alpha x$ at all $x$.

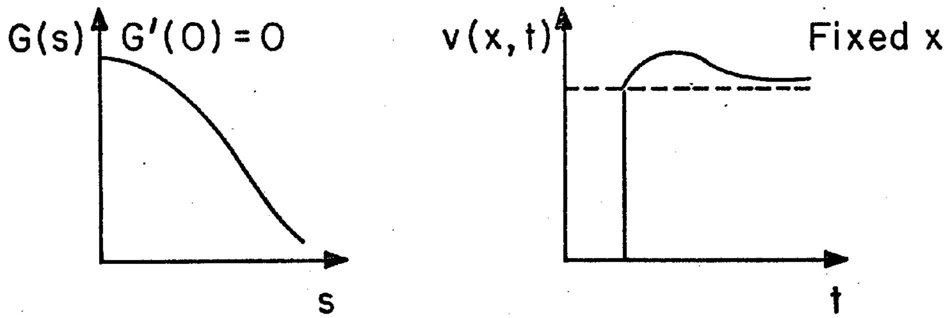



Fig. 1.2: Shock profile for the case G'(0) = 0.

## §1.3 Viscosity and Transition Layers

Consider the problem of a step increase of velocity for Newtonian fluids ($\mu>0$, $\tilde{\mu}(s) \equiv 0$ in (1.1)). The classical solution of this problem ((1.2), (1.3)) is given by:

(1.7) $$v(x, t) = \operatorname{erfc}(x/\sqrt{4\nu t})$$

where $\nu=\frac{\mu}{\rho}$, and erfc is the complementary error function.

If $\mu>0$ is small and G has the assumed properties, it can be shown (see §18 of [6]) that there is a transition layer around the shock solution with $\mu=0$. This smooth transition layer exists in a bounded domain of $\{(x,t)\ x\geq 0$ and $t\geq 0\}$ and its thickness scales with $\mu$. Thus:

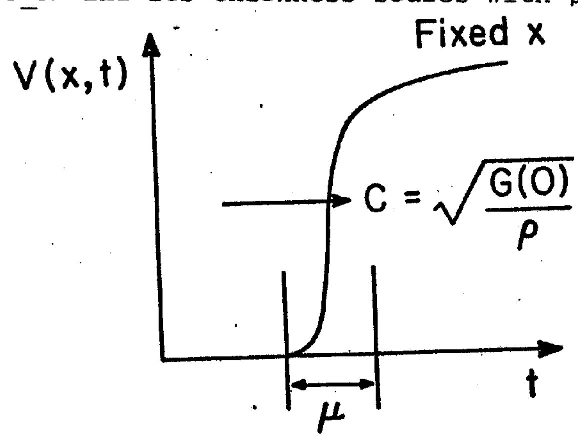



Fig. 1.3.: Transition layers when $\mu>0$ is small.

## §2 Remarks on the Experimental Determination of Relaxation Functions.

Many experimental measurements of relaxation functions are based on the incorrect assumption that a linear velocity profile (which is the $t \to \infty$ asymptotic state for the problem of step change in velocity) can be achieved impulsively (see Fig. 2.1-2.3)

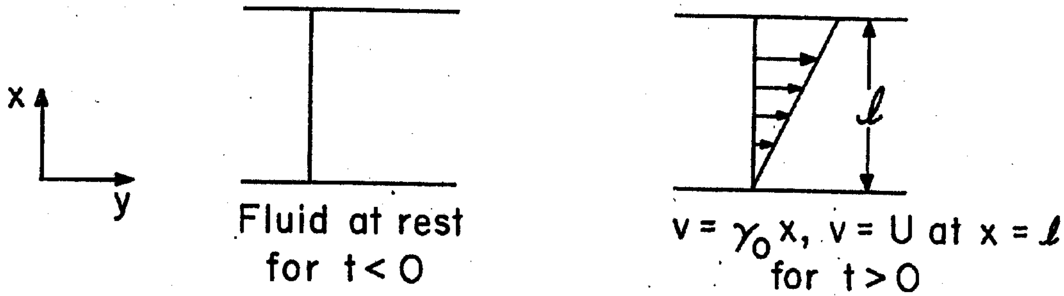



Fig. 2.1: Assumed "solution for the step increase in velocity. The stress is measured after times t>0. The relaxation function is determined from the constitutive equation on the assumed, dynamically inadmissible, velocity field.

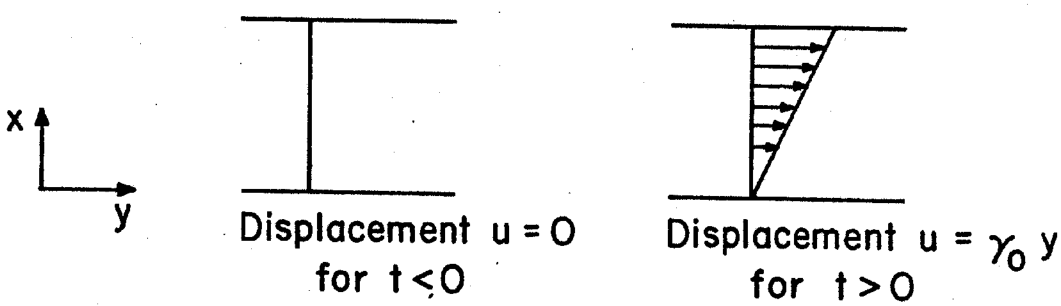



Fig. 2.2: Assumed "solution" for the step increase in displacement. The stress is measured at times t>0. The relaxation function is determined from evaluating the constitutive equation on the assumed dynamically inadmissible, deformation field.

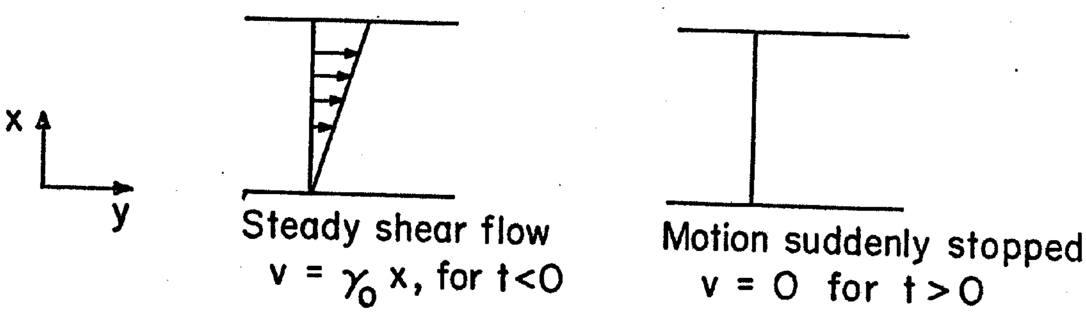



Fig. 2.3: Assumed "solution" for sudden cessation of motion. The stress is measured at times t>0. The relaxation function is determined

> from evaluating the constitutive equation on the assumed dynamically inadmissible deformation field.

However we have shown that the deformation assumed in Figs. 2.1 - 2.3 cannot be achieved at finite times on solutions of the initial-boundary value problem in the realm of linear viscoelasticity. The deformations assumed are in fact limiting cases for $t \to \infty$. It is therefore necessary to explain how and in what sense the customary methods of determining relaxation functions have validity. The following observations are important:

(1) The customary methods can always be used to measure "viscosity" $(\mu + \int_0^\infty G(s)ds)$ by measurement at large times. But the test is inadequate to determine separately $\mu$ and $\int_0^\infty G(s)ds$.

(2) Suppose $\mu=0$, then the customary methods measure the stress on the stationary plate as a funciton of time. We are here concerned with the question whether the experimental measurement is going to be close to the relaxation function as indicated by the assumed kinematics of Fig. 1.4. In the context of linear viscoelasticity, we will show that this experimental measurement will never give the integral $\int_0^t G(s)ds$ for small time t near zero. However, this integral can be close to measured values for large times provided that the half life time of discontinuities is small. For simple Maxwell models with non-zero values of $|G'(0)|$, this time can be estimated as $-G(0)/G'(0)$.

To obtain expressions for the shear stress at the wall we consider the dynamics solution given in §8 of [6] for the step increase in velocity (see Fig. 2.1). In that solution the moving plate is at x=0 and the stationary plate is at $x=\ell$. For the case in which the moving plate is at $x=\ell$ we ultimately have simple shear $U(1-\frac{x}{\ell})$ as $t \to \infty$ with shear rate $\frac{\partial v}{\partial x} \overset{def}{=} -\dot{\gamma}_0 = -\frac{U}{\ell}$. The solution of this problem is:

$$(2.1) \qquad v(x,t) = U[f(x,t)H(t-\alpha x) + \{f(x + 2\ell,t)H(t-\alpha(x + 2\ell)) - f(2\ell-x,t)H(t-\alpha(2\ell-x))\} + \{\cdots\} + \cdots].$$

The stress at the wall x=0 and $x=\ell$ is given by:

$$(2.2) \qquad T^{\langle xy \rangle}(0,t) = \int_0^t G(s) \frac{\partial v}{\partial x}(0,t-s)ds$$

and

$$(2.3) \qquad T^{\langle xy \rangle}(\ell,t) = \int_0^t G(s) \frac{\partial v}{\partial x}(\ell,t-s)ds.$$

If we assume an instantaneous deformation as in Fig. 2.1, then (1.5) implies that

(2.4) $$T^{<xy>}(x,t) = -\frac{U}{\ell}\int_0^t G(s)ds, \quad x\in[0,\ell].$$

However (2.1) implies that

(2.5) $$\frac{\partial v}{\partial x}(0,t) = U[\{\frac{\partial f}{\partial x}(0,t)H(t-0) - \alpha f(0,t)\delta(t-0)\}$$

$$+2\{\frac{\partial f}{\partial x}(2\ell,t)H(t-(2\alpha\ell)) - \alpha f(2\ell,t)\delta(t-(2\alpha\ell))\}$$

$$+2\{\cdots\} + \cdots],$$

and

(2.6) $$\frac{\partial v}{\partial x}(\ell,t) = 2U[\frac{\partial f}{\partial x}(\ell,t)H(t-\alpha\ell) - \alpha f(\ell,t)\delta(t-\alpha\ell)$$

$$+\{\frac{\partial f}{\partial x}(3\ell,t)H(t-(3\alpha\ell)) - \alpha f(3\ell,t)\delta(t-(3\alpha\ell))\}$$

$$+\{\cdots\} + \cdots].$$

Combining (2.5) and (2.2), we find that in the time interval $0<t<\alpha(2\ell)$, the stress at the driving plate is

(2.7) $$T^{<xy>}(0,t) = U\int_0^t G(t-s)\frac{\partial f}{\partial x}(0,s)ds - U\alpha G(t)f(0,0^+),$$

but equation (1.5) implies that

$$Uf(0,0^+) = v(0,0^+) = U.$$

Hence,

(2.8) $$-T^{<xy>}(0,0^+) = U\sqrt{\rho G(0)}.$$

Combining (1.13) and (1.10) we get

(2.9) $T^{<xy>}(\ell,t)=0$ for $0<t<\alpha\ell$ and

(2.10) $$-T^{<xy>}(\ell,\alpha\ell^+) = 2U\sqrt{\rho G(0)}\exp(\frac{\alpha\ell G'(0)}{2G(0)}).$$

In general, for $t>(2n\alpha\ell)$; $n=1, 2, \ldots$ we find by combining

(2.5) and (2.6) with (2.2) and (2.3) that

$$(2.11)\qquad -T^{<xy>}(0,t) = [-\int_0^t G(t-s)\frac{\partial f}{\partial x}(0,s)\,ds + \alpha G(t)]$$
$$+2[-\int_{(2\alpha\ell)}^{t} G(t-s)\frac{\partial f}{\partial x}(2\ell,s)\,ds + G(t-(2\alpha\ell))\exp\left(\frac{G'(0)}{2G(0)}2\alpha\ell\right)]$$
$$+2[\cdots] + \cdots$$

and

$$(2.12)\qquad -T^{<xy>}(\ell,t) = 2U[\alpha G(t-\alpha\ell)f(\ell,\alpha\ell^{+}) - \int_{\alpha\ell}^{t} G(t-s)\frac{\partial f}{\partial x}(\ell,s)\,ds]$$
$$+2U[\alpha G(t-(3\alpha\ell))f(3\ell,(3\alpha\ell)^{+}) - \int_{(3\alpha\ell)}^{t} G(t-s)\frac{\partial f}{\partial x}(3\ell,s)\,ds]$$
$$+2[\cdots] + \cdots$$

In order to understand (2.11) and (2.12), we need to know some features of the function $\frac{\partial f}{\partial x}(2n\ell,t)$ for n=0, 1, 2,.... For a Maxwell fluid $G(s) = Ke^{-\mu s}$ and (see (7.3) of [6]):

$$(2.13)\qquad -\frac{\partial f}{\partial x}(x,t) = -U\sqrt{\frac{\rho\mu^2}{K}}\,\frac{\partial\hat{f}}{\partial\hat{x}}(\hat{x},\hat{t}) \quad\text{where } x = \sqrt{\frac{K}{\rho\mu^2}}\,\hat{x}\ ,\ t = \frac{1}{\mu}\hat{t}$$

$$\frac{\partial\hat{f}}{\partial\hat{x}}(\hat{x},\hat{t}) = -\frac{1}{2}e^{-\frac{\hat{x}}{2}} - \frac{\hat{x}}{8}e^{-\frac{\hat{x}}{2}} + \frac{1}{2}\int_{\hat{x}}^{\hat{t}} \frac{e^{-\frac{\sigma}{2}}}{\sqrt{\sigma^2-\hat{x}^2}}\, I_1\left(\tfrac{1}{2}\sqrt{\sigma^2-\hat{x}^2}\right)d\sigma$$
$$+\frac{\hat{x}^2}{2}\int_{\hat{x}}^{\hat{t}} \frac{e^{-\frac{\sigma}{2}}}{(\sigma^2-\hat{x}^2)}\left\{\frac{I_1(\frac{1}{2}\sqrt{\sigma^2-\hat{x}^2})}{\sqrt{\sigma^2-\hat{x}^2}} - \frac{1}{2}I_1'\left(\tfrac{1}{2}\sqrt{\sigma^2-\hat{x}^2}\right)\right\}d\sigma$$

We also recall that when a steady state $v(x,\infty) = \frac{U(\ell-x)}{\ell}$ is approached we have

$$(2.14)\qquad \lim_{t\to\infty} T(x,t) = -\frac{U}{\ell}\int_0^\infty G(s)\,ds,\ x \in [0,\ell].$$

There are two cases to consider: (i) $\sqrt{G(0)\rho} > \ell^{-1}\int_0^\infty G(s)\,ds$ and (ii) $\sqrt{G(0)\rho} < \ell^{-1}\int_0^\infty G(s)\,ds$. In the first case the initial value of the stress is larger than the final value (overshoot). A typical graph is sketched in Fig. 2.4(i). In the second case there is a jump of stress less than the steady state value. This case is sketched in Fig. 2.4(ii).

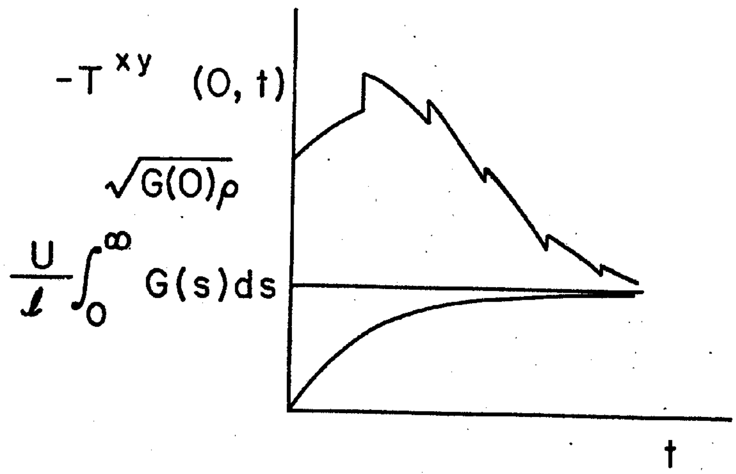



Fig. 2.4 (i)

Stress development at the lower wall of a channel filled with a visco-elastic fluid of Maxwell type under a step change of shear.

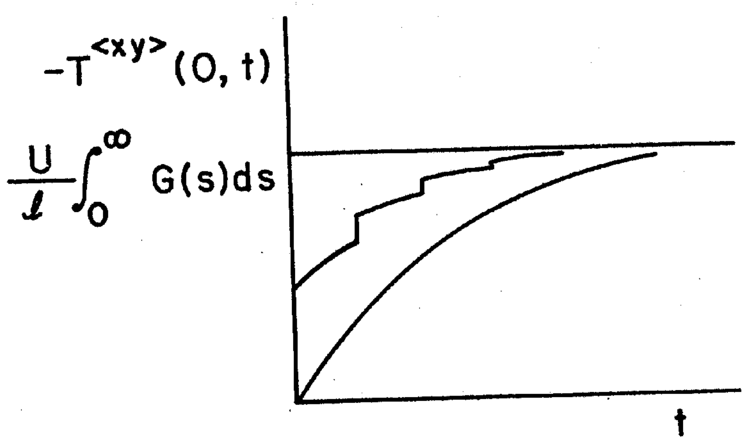



Fig. 2.4 (ii)

Stress development at the lower wall of a viscoelastic fluid of Maxwell type under a step change of shear.

Of course the amplitude of jumps in Fig. 2.4 (i), (ii) ultimately tend to steady state value. Moreover in the two special cases $G'(0) = -\infty$ or $\mu>0$ and small we will have essentially the same response as in Figs. 2.4 with smooth bumps replacing jumps. In any experiment the jumps (for $\mu=0$) would not be vertical because step changes at the boundary are discontinuous idealizations of smooth rapid changes and if $v(0,t)$ is a continuous function close to $UH(t-0)$, then $T^{<xy>}(0,0^+) = 0$ but $T^{<xy>}(0, \varepsilon_1) \cong U\sqrt{G(0)\rho}$ and $T^{<xy>}(\ell, \varepsilon_2) \cong 2U\sqrt{G(0)\rho}\, \exp(\frac{\alpha\ell G'(0)}{2G(0)})$ for some $\varepsilon_1, \varepsilon_2 > 0$ and small. This observation follows as a consequence of the continuous dependence of the solution on the data [6] and our solution for arbitrary initial data.

The aforementioned results may be applied to the interpretation of experiments by Meissner [5], Huppler et al [3], among others. They plot

$$\frac{T^{\langle xy\rangle}(0,t)}{T^{\langle xy\rangle}(0,\infty)} \overset{\text{def}}{=} \frac{\eta^{+}(t)}{\eta_0},$$

where

$$T^{\langle xy\rangle}(0,t) \overset{\text{def}}{=} -\dot{\gamma}_0\, \eta^{+}(t)$$

$$= -\frac{U}{\ell}\eta^{+}(t),$$

$$\eta_0 = \int_0^{\infty} G(s)\, ds$$

Our analysis shows that at the driving plate

$$\frac{\eta^{+}(0^{+})}{\eta_0} = \frac{\ell\sqrt{G(0)\rho}}{\int_0^{\infty} G(s)\, ds}$$

where $\frac{\eta^{+}(\infty)}{\eta_0} = 1.$

The stress response at the stationary wall is given by

$$\frac{T^{\langle xy\rangle}(\ell,\alpha\ell^{+})}{T^{\langle xy\rangle}(\ell,\infty)} = \frac{\eta^{+}(\alpha\ell^{+})}{\eta_0} = \frac{2\ell\sqrt{\rho G(0)}}{\int_0^{\infty} G(s)\, ds} \exp\left(\frac{\alpha\ell G'(0)}{2G(0)}\right)$$

where, $\frac{\eta^{+}(\infty)}{\eta_0} = 1$

Typical representations of experimental results of various authors are represented schematically in Fig. 2.5 (cf. Bird, Armstrong and Hassager, [1] Fig. A.4-9).

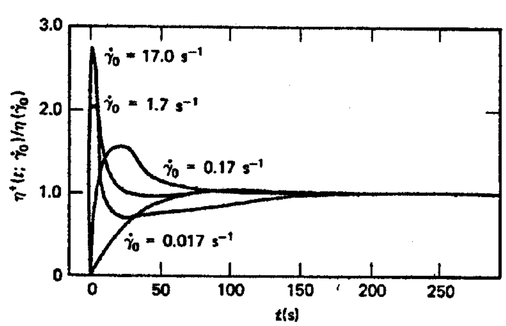



Fig. 2.5

Representations of stress development in a parallel plate channel under a step change of velocity. These representations are supposed to represent the results of experiments.

The experimental results represented in Fig. 2.5 do not exhibit the stress jumps, at small rates of shear, which are required by linearized dynamics. It is possible that the conditions of the experiments were such as to make the initial jumps in stress small relative to asymptotic ($t\to\infty$) levels of stress. However, stress overshoot could possibly occur even in the realm of linear theory. We cannot know whether or not overshoots do occur without reliable estimates of G(0). The methods which are presently used to determine G(0) are inadequate because they do not apply at small times. Some micro-molecular models like those of Kee and Carreau [4], have tried to explain this overshoot by allowing for such features in a "non-linear stress-strain history model" when evaluated at a kinematic assumption of Fig. 1.4. We believe it is now apparent that any such modeling on the above experimental data is meaningless if the dynamics are going to be neglected.

It is perhaps also possible to explain the oscillations at small times in the stress observed by Meissner [5] in terms of larger amplitudes of stress which are generated by reflections off bounding walls for fluids of the type which support shocks or near shocks (fluids with $G(0)<\infty$, $-G'(0)\leq\infty$ with or without a small viscosity.) Nonlinearity also participates in the results observed at high shears. For example, the narrowing of the width of peak region in the graphs shown in Fig. 2.5 may not be entirely explained by linear theory.

## §2.1 The stress response for the step displacement problem

This problem is associated with Fig. 2.2. The kinematic assumption mentioned in the caption of that figure leads to a direct formula

$$(2.15)\qquad \frac{U}{\ell}\, G(t) = -T^{\langle xy\rangle}(0,t)\ .$$

The dynamic solution for the linearized problem associated with that experiment is given in §12 of Eq. [6]. Following procedures used to obtain (2.8), (2.10) and (2.14) we find

$$(2.16)\qquad -T^{\langle xy\rangle}(0,0^+) = \frac{U}{2}\, G'(0)\sqrt{\frac{\rho}{G(0)}} < 0$$

and

$$(2.17)\qquad T^{\langle xy\rangle}(0,\infty) = 0\ .$$

At the stationary plate we have

$$(2.18) \qquad -T^{<xy>}(\ell,\alpha\ell^+) = 2U[-G(0)\frac{\partial f}{\partial x}(\ell,\alpha\ell^+) + \alpha\, G'(0)\ f(\ell,\alpha\ell^+)]$$

and

$$T^{<xy>}(\ell,\infty) = 0 \ .$$

Eq. (2.15) may be a correct representation of linearized dynamics for large t but it is a false representation of linearized dynamics for small t.

## §2.2 Summary

The asymptotic values of G(t) for large t can be obtained in the usual way using the kinematic assumptions exhibited in Figs. 1.5 and 1.6. The early time behavior of G(t) is not well represented by the asymptotic solution and at least should be correlated with the results of dynamic analysis. In the context of linearized dynamics which should be valid at least for small shears, we find that

$$-T^{<xy>}(0,0^+) = U\sqrt{G(0)\rho} \ ,$$

$$-T^{<xy>}(\ell,\alpha\ell^+) = 2U\sqrt{\rho G(0)}\ \exp\ (\alpha\ell\ G'(0)/2G(0))$$

for the step change in velocity. Here $\ell$ is the distance from the driving plate at x=0 to the stationary plate and $\alpha\ell$ is the time of first reflection. In the problem of the step change in displacement, we find that

$$-T^{<xy>}(0,0^+) = \frac{U}{2}\ G'(0)\sqrt{\frac{\rho}{G(0)}} < 0 \ ,$$

$$-T^{xy}\ (\ell,\alpha\ell^+) = 2U\ [\ -G(0)\ \frac{\partial f}{\partial x}(\ell,\alpha\ell^+) + \alpha G'(0)\ \ f(\ell,\alpha\ell^+)]$$

It may be useful to reinterpret existing experimental results in terms of the dynamic theory. For example, the constants $\kappa_i$ and $\mu_i$ appearing in the Maxwell model with finitely many relaxation times

$$G(s) = \sum_{i=1}^{N} \kappa_i e^{-\mu_i s}$$

could, in principle, be determined by comparing experimental results with formulas which could be obtained from the analysis of the type of Kazakia and Rivlin [10].

It may be true that conclusions similar to the ones which we have considered here for experiments with viscoelastic fluids apply in the theory of viscoelastic solids [7].

## §3 Cylindrical vortex sheets generated by sudden spin up of a cylinder in a fluid

The problem of spin up was considered in §14 of [6]. In this case the velocity of shearing motion is in circles

$$\underline{V}\ (\underline{x},t) = w(r,t)\underline{e}_\theta$$

and w(r,t) is defined in

$$\mathcal{D} = \{r \geq a,\ 0 \leq \theta \leq 2\pi,\ -\infty < z\}.$$

The boundary value problem for sudden spin up is given (see (14.8) of [6]) by

$$\rho \frac{\partial w}{\partial t}(r,t) = \int_0^t G(s)\left[\frac{\partial^2 w}{\partial r^2}(r,t-s) + \frac{1}{r}\frac{\partial w}{\partial r}(r,t-s) - \frac{w(r,t-s)}{r^2}\right] ds$$

(3.1)

$$w(a,t) = \begin{cases} a\ \Omega = 1 & \text{for } t > 0 \\ 0 & \text{for } t < 0, \end{cases}$$

$$w(r,0) = 0\ ,\qquad r \geq a > 0,$$

w(r,t) is bounded as r, $t \to \infty$ .

We showed in [6] that the solution of (3.1) is given by

(3.2) $$w(r,t) = g(r,t)\ H(t-(r-a)\alpha).$$

where g(r,t) is defined in (14.16) of [6]. Here we derive a simpler form for $g(r,\ \alpha(r-a)^+)$ than the one given by (14.19) of [6]. This derivation follows along lines leading to the formulas (5.21), (5.23) in [6].

We know from (14.11) of [6] that

(3.3) $$w(r,t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{ut}}{u}\ \frac{K_1\left(r\sqrt{\frac{\rho u}{G(u)}}\right)}{K_1\left(a\sqrt{\frac{\rho u}{G(u)}}\right)}\ du; \qquad \mathrm{Re}\ u > 0$$

where $K_1$ is a modified Bessel function whose asymptotic form is given by

$$(3.4) \qquad K_1(z) = \sqrt{\frac{\pi}{2z}}\ \exp(-z) + 0(\frac{1}{z}) .$$

The asymptotic expansion

$$(3.5) \qquad \bar{G}(u) = \frac{G(0)}{u} + \frac{G'(0)}{u^2} + 0(\frac{1}{u^3})$$

was established as (5.16) of [6].

It is easy to verify that:

$$(3.6) \qquad \sqrt{\frac{\rho u}{\bar{G}(u)}} = \sqrt{\frac{\rho}{G(0)}}\ u - \sqrt{\frac{\rho}{G(0)}}\ \frac{G'(0)}{2G(0)} + 0(\frac{1}{u})$$

$$= \alpha u - \frac{\alpha G'(0)}{2G(0)} + 0(\frac{1}{u}).$$

Equation (3.4) and (3.6) imply that

$$(3.7) \qquad \frac{K_1\left(r\sqrt{\frac{\rho u}{\bar{G}(u)}}\right)}{K_1\left(a\sqrt{\frac{\rho u}{\bar{G}(u)}}\right)} = \sqrt{\frac{a}{r}}\ \exp\left[(r-a)\frac{\alpha G'(0)}{2G(0)}\right]\ \exp\left[(-\alpha u(r-a)\ \right] + 0(\frac{1}{u})$$

$$= \sqrt{\frac{a}{r}}\ \exp\left[\ (r-a)\frac{\alpha G'(0)}{2G(0)}\right]\ \exp\left[\ -\alpha u(r-a)\right] + 0(\frac{1}{u}) .$$

Substituting (3.7) into (3.3), we get:

$$(3.8) \qquad w(r,t) = \frac{1}{2\pi i}\ \exp\left[\frac{(r-a)\alpha G'(0)}{2G(0)}\right] \int_{\gamma-i\infty}^{\gamma+i\infty} \sqrt{\frac{a}{r}}\ \frac{e^{u\{t-\alpha(r-a)\}}}{u}\, du$$

$$+ \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{ut}\ 0(\frac{1}{u^2})\ du$$

$$= \sqrt{\frac{a}{r}}\ \exp\left[\frac{(r-a)\ \alpha\ G'(0)}{2G(0)}\right] H(t-\alpha(r-a))$$

$$+ \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{ut}\ 0(\frac{1}{u^2})\ du\ .$$

The last term in (3.8) is continuous $r \geq a$ and $t \geq 0$ because the integral is uniformly convergent for any fixed r,t. Comparing (3.8) with (3.2) while using the continuity of the second term in (3.8) we get

$$(3.9) \qquad g(r,\ \alpha(r-a)^+) = \sqrt{\frac{a}{r}}\ \exp\left[\frac{(r-a)\ \alpha\ G'(0)}{2G(0)}\right].$$

The decay with r of cylindrical vortex sheets is more rapid than plane sheets which damp according to (1.5) without the factor $r^{-1/2}$.

We next consider the problem of reflections off the walls of concentric cylinders which bound a fluid occupying the region

$$\hat{\mathcal{D}} = \{a < r \leq b,\ 0 \leq \theta < 2\pi,\ -\infty < z < \infty\}.$$

The spin up problem may be stated as follows

$$\text{(3.10)}\qquad \begin{aligned} \rho \frac{\partial w}{\partial t} &= \int_0^t G(s)\left[\frac{\partial^2 w}{\partial r^2}(r,\ t-s) + \frac{1}{r}\frac{\partial w}{\partial r}(r,t-s) - \frac{w(r,t-s)}{r^2}\right] ds, \\ w(a,\ t) &= \begin{cases} a\Omega = 1 & \text{for } t > 0 \\ 0 & \text{for } t \leq 0, \end{cases} \\ w(b,\ t) &= 0 \qquad \forall\ t \in \mathbb{R}, \\ w(r,\ 0) &= 0 \qquad \forall\ r \in [a,\ b], \\ w(r,\ t) &\text{ is bounded as } r,\ t \to \infty. \end{aligned}$$

We now utilize the method of Laplace transforms, following arguments given in §6 of [6] and find that

$$\text{(3.11)}\qquad w(r,\ t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{ut}\,\bar{w}(r,\ u)\,du$$

where

$$\text{(3.12)}\qquad \begin{aligned} \bar{w}(r,\ u) &= \frac{1}{u}\ \frac{I_1(b\eta(u))K_1(r\eta(u)) - K_1(b\eta(u))I_1(r\eta(u))}{K_1(a\eta(u))I_1(b\,\eta(u)) - K_1(b\eta(u))I_1(a\eta(u))}, \\ \eta(u) &= \sqrt{\frac{\rho u}{\bar{G}(u)}}. \end{aligned}$$

An asymptotic form for (3.11) follows from combining the asymptotic expressions for $|z| \to \infty$

$$\text{(3.13)}\qquad \begin{aligned} I_1(z) &= \frac{e^z}{\sqrt{2\pi z}} + 0(\tfrac{1}{z}), \\ K_1(z) &= \sqrt{\frac{\pi}{2z}}\ e^{-z} + 0(\tfrac{1}{z}) \end{aligned}$$

with (3.12). Thus

$$\bar{w}(r, u) = \sqrt{\frac{a}{r}} \frac{e^{(b-r)\eta(u)} - e^{-(b-r)\eta(u)}}{e^{(b-a)\eta(u)} - e^{-(b-a)\eta(u)}} + 0(\frac{1}{u}) .$$

Hence

$$w(r, t) = \sqrt{\frac{a}{r}} \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{ut}}{u} \frac{e^{(b-r)\eta(u)} - e^{-(b-r)\eta(u)}}{e^{(b-a)\eta(u)} - e^{-(b-a)\eta(u)}} du \tag{3.14}$$

$$+ \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{ut} \; 0(\frac{1}{u^2}) du .$$

We next note that the first term in (3.14) is the same as in (8.3)-(8.7) of §8 of [6] if we set r-a = x, and b-a = ℓ. The second term in (3.14), being uniformly convergent for any r and t, is a continuous function of r and t. Thus

$$w(r, t) = \sqrt{\frac{a}{r}} \Big[ f(x, t) H(t-\alpha x) + \{ f(x+2l, t)$$

$$H(t-\alpha(x+2l)) - f(2l-x, t) H(t-\alpha(2l-x)) \} \tag{3.15}$$

$$+ \cdots\cdots \Big] + h(x, t) .$$

The function f in (3.15) is the same f appearing in (1.5) while h(x,t) is continuous for $x = r-a \in [0, \ell]$ and $t \geq 0$. It follows from (3.15) that discontinuities are reflected along the characteristic lines shown in Fig. 3.1.

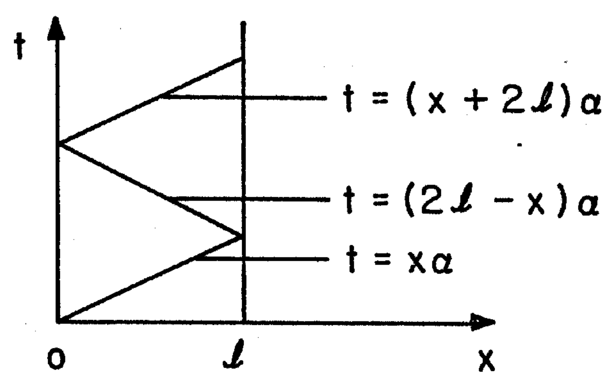



Fig. 3.1. Characteristic lines for reflection from the walls of concentric cylinders, x = r-a, ℓ = b-a

The asymptotic steady state in case of flow governed by (1.20) is given by:

$$\lim_{t\to\infty} w(r,t) = \lim_{u\to 0} u\,\bar{w}(r,u)\ .$$

Using (1.22) and

$$\left.\begin{aligned} I_1(z) &\sim \frac{z}{4} \\ K_1(z) &\sim \frac{1}{z} \end{aligned}\right\} \qquad \text{as } z \to 0 \text{ and } \mathrm{Re}\ z > 0$$

we get

$$\text{(3.16)} \qquad \lim_{t\to\infty} w(r,t) = \frac{\frac{b}{r} - \frac{r}{b}}{\frac{b}{a} - \frac{a}{b}}$$

$$= \frac{a}{r}\,\frac{b^2-r^2}{b^2-a^2}\ , \quad \text{for } a\Omega = 1.$$

## §4 Solutions of start up problems with arbitrary boundary data by integrals of Duhamel's type

A shearing motion is initiated at $x=0$ by data of the form

$$V(0,t) = \begin{cases} g(t) & , \ t > 0 \\ 0 & , \ t < 0 \end{cases}$$

where $g(t)$ is an arbitrary function (possessing a Laplace transform). The velocity $V(x,t)$ then satisfies

$$(4.1) \qquad \int_0^t G(s) \frac{\partial^2 v}{\partial x^2}(x,t-s)\,ds = \rho \frac{\partial v}{\partial t}(x,t),$$

$$v(0,t) = g(t), \text{ where } g(t) \equiv 0, \quad t < 0,$$

$$v(x,0) = 0, \quad x \geq 0,$$

$v(x,t)$ is bounded.

We shall solve (4.1) by superposition using the solution of the following singular problem:

$$\int_0^t G(s) \frac{\partial^2 u}{\partial x^2}(x,t-s)\,ds = \rho \frac{\partial u}{\partial t}(x,t),$$

$$u(0,t) = \delta(t-\tau), \quad \tau \in (0,t),$$

$$(4.2) \qquad u(x,0) = 0, \quad x \geq 0,$$

$u(x,t)$ is bounded for $x,\ t \to \infty$.

It is easy to see and not hard to prove that the solution of (4.2) is the time-derivative of the solution of (3.1) where

$$v(0,t) = \begin{cases} 1 \text{ for } t > \tau \\ 0 \text{ for } t \leq \tau. \end{cases}$$

It then follows that the solution of (4.2) is

$$(4.3) \qquad u(x,t) = \frac{\partial f}{\partial t}(x, t-T)\, H(t-\tau-\alpha x) + f(x, \alpha x^+) \quad (t-\tau-\alpha x).$$

Of course (4.3) can be obtained directly as the inverse of the Laplace transform of (4.2). (The details of this type of calculation are given in §10 of [6]). We note that t in the upper limit of integration in the integral on the left of (4.2), may be replaced with $t+\delta$, $\delta>0$ because $u(x,-\delta) = 0$ for $\delta>0$. The interpretation of the $\delta$ function which this implies may be expressed as follows: for any h(s) such that h(s) = 0, s<0 we have

$$\int_{-\infty}^{\infty} h(s)\ \delta(s)ds = \int_{0}^{\infty} h(s)\ \delta(s)ds = h(0)\ .$$

We now assert that the solution of problem (4.1) is a linear superposition (integration) of the function $g(\tau)\ u(x,t)$. This is true because

$$(4.4)\qquad v(0,t) = g(t) = \int_0^{\infty} g(t-\eta)\ \delta(\eta)d\eta = \int_0^{t} g(t-\eta)\ \delta(\eta)d\eta$$

$$= \int_0^t g(\tau)\ \delta(t-\tau)d\tau\ .$$

Using (4.3) and (4.4), we find that the solution of (4.1) is

$$(4.5)\qquad v(x,t) = \int_0^t g(\tau)\ u(x,t)d\tau$$

$$= \int_0^t g(\tau)\ \left[\frac{\partial f}{\partial t}\ (x,\ t-\tau)\ H(t-\tau-\alpha x) + f(x,\ \alpha x^+)\delta(t-\tau-\alpha x)\right] d\tau .$$

It follows from (2.5) that if

$$(4.6)\qquad t-\alpha x < 0 \text{ then } v(x,t) = 0\ .$$

This implies that the information of rest prior to start-up is always preserved. On the other hand, when $t-\alpha x > 0$, (4.5) gives

$$(4.7)\qquad v(x,t) = \int_0^{t-\alpha x} g(\tau)\ \frac{\partial f}{\partial t}\ (x,t-\tau)\ d\tau\ + f(x,\ \alpha x^+) \int_0^{t-\alpha x} g(t-\alpha x-\eta)\delta(\eta)d\eta$$

$$= \int_0^{t-\alpha x} g(\tau)\ \frac{\partial f}{\partial t}\ (x,t-\tau)\ d\tau\ +\ f(x,\ \alpha x^+)\ g(t-\alpha x)$$

$$= \int_0^{t-\alpha x} g(\tau)\ \frac{\partial f}{\partial t}\ (x,t-\tau)\ d\tau\ +\ \exp\left(\frac{\alpha x G'(0)}{2G(0)}\right)\ g(t-\alpha x)\ .$$

It is easy to verify that (4.7) reduces to (1.5) for $g(\tau) = H(\tau)$ and (1.6) for $g(\tau) = \delta(\tau)$. Eqs. (4.6) and (4.7) together constitute the solution of the problem posed in (4.1). Thus we conclude that discontinuities in the boundary values of g or its derivatives propagate into the interior with speed $C=1/\alpha$. Hence (4.7) also proves that any discontinuity in a start-up problem of linear viscoelasticity can come only through the boundary data. It is also clear from (4.7), that this propagating discontinuity is exponentially damped.

We turn next to the construction of the solution of start-up problems between parallel plates. The problem to be solved may be expressed as:

$$(4.8) \qquad \int_0^t G(s) \frac{\partial^2 v}{\partial x^2} (x, t-s)\, ds = \rho \frac{\partial v}{\partial t},$$

$$v(0,t) = g(t) \quad ; \quad \text{when } g(t) \equiv 0, \quad t<0,$$

$$v(\ell,t) = 0,$$

$$v(x,0) = 0,$$

$$v(x,t) \text{ is bounded as } t \to \infty.$$

Proceeding as in the previous problem we first consider the case in which $g(t) = \delta(t-\tau)$. The $\hat{v}(x,t)$ for this singular problem is given by

$$\hat{v}(x,t) = [\psi(x,t-\tau) + \{\psi(x+2l, t-\tau) - \psi(2l-x,t-\tau)\} + \cdots\cdots]$$

where

$$(4.9) \qquad \psi(x,t) = \frac{\partial f}{\partial t} (x,t)\, H(t-\alpha x) + f(x,\alpha x^+)\, \delta(t-\alpha x).$$

The function $f(x, t)$ in (4.9) is defined by (1.5). We now use the principle of superposition to compose the solution of (4.8) in Duhamel form

$$(4.10) \qquad v(x,t) = \int_0^t g(\tau)\, \hat{v}(x,t)\, d\tau$$

$$= \int_0^t g(\tau) \left[ \frac{\partial f}{\partial t}(x,t-\tau) + \left\{ \frac{\partial f}{\partial t}(x+2l,\ t-\tau) - \frac{\partial f}{\partial t}(2l-x,t-\tau) \right\} \right.$$

$$\left. + \cdots \right] H(t-\tau-\alpha x)$$

$$+ [f(x,\alpha x^+)\ g(t-\alpha x) + \{f(2l+x,\ \alpha(2l+x)^+)\ g(t-\alpha x)$$

$$- f(2l-x,\ \alpha(2l-x))\ g(t-\alpha x)\} + \cdots ]\ d\tau$$

The dots in (4.10) represent similar terms arising out of repeated reflection between the walls at $x=0$ and $x=\ell$ of the original characteristic $t-\alpha x$ = const. It also follows from (4.10) that $v(x,t)=0$ when $t-\alpha x < 0$ and, when $t-\alpha x > 0$ we find that:

$$(4.11) \qquad v(x,t) = \int_0^{t-\alpha x} g(\tau) \left[ \frac{\partial f}{\partial t}(x,t-\tau) + \left\{ \frac{\partial f}{\partial t}(x+2l,t-\tau) - \frac{\partial f}{\partial t}(2l-x,t-\tau) \right. \right.$$

$$\left. + \cdots \right]\ d\tau + \left[ f(x,\alpha x^+) g(t-\alpha x) + \{f(2l+x,\ \alpha(2l+x)^+) \right.$$

$$\left. g(t-\alpha x) - f(2l-x,\alpha(2l-x)^+)\ g(t-\alpha x)\} + \cdots \right].$$

We may use (4.11) to study the interactions of multiple shocks generated by multiple discontinuities in the boundary data $g(t)$. For example, consider

$$(4.12) \qquad g(t) = \begin{cases} 0 & \text{for } t < 0, \\ 1 & \text{for } 0 \le t \le 1, \\ 0 & \text{for } t > 1. \end{cases}$$

It follows from (4.11) that the discontinuities of $g(t)$ propagate along the characteristic lines $t-\alpha x=0$ and $t-\alpha x=1$ and their repeated reflections, as in Fig. 4.1

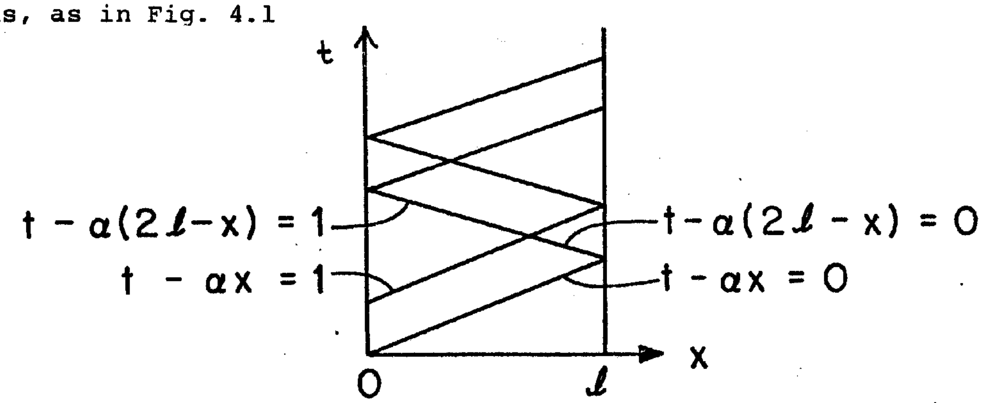



Fig. 4.1. Propagation of the singular data given by (4.12)

## §5. Linear viscoelastic solids

In [7] we used Laplace transforms to study problems of singular boundary data in viscoelastic solids. If $G(0) > 0$ and $G'(0) < 0$ are finite, steps in displacement initiated at the boundary will propagate into the interior of the solid. We interpreted this to mean that the material fails in shear. Stress relaxation experiments in solids are in some sense modeled by steps in displacement. As in fluids, it is necessary to understand the underlying dynamics of such problems. To pursue such an understanding we study the following initial-boundary value problem with smooth, but otherwise arbitrary boundary data $g(t)$, $t \geq 0$:

$$\rho \frac{\partial^2 v}{\partial t^2} = (\mu+G(0)) \frac{\partial^2 v}{\partial x^2} + \int_0^t \frac{dG}{ds}(s) \frac{\partial^2 v}{\partial x^2}(x,t-s)\,ds,$$

(5.1) $$v(0,t) = g(t) \; ; \quad g(t) = 0 \quad t < 0,$$

$$v(x,0) = \frac{\partial v}{\partial t}(x,0) = 0,$$

$$v(x,t) \text{ is bounded as } x,\ t \to \infty.$$

We can solve (5.1) using the methods which led to the Duhamel type of integrals displayed in equations (4.1-4.7). Thus

(5.2) $$v(x,t) = \int_0^t g(\tau) \frac{\partial \hat{f}}{\partial t}(x,t-\tau)\,d\tau + \hat{f}(x, \alpha x^+)\, g(t-\alpha x)$$

where

$$\alpha = \sqrt{\frac{\rho}{\mu+G(0)}}$$

and $\hat{f}(x,t) = f(x,t)$ where $f(x,t)$ is defined by equation (3.10,11) of [7] and $f(x,t)$ has the properties specified in §1 of this paper.

The implications of this type of solution for the rheometry of viscoelastic solids should resemble those discussed in §2 for fluids. We defer a detailed comparison of theory and experiment in solids to a later paper. For now it will suffice to note that in theory of solids the notion of homogeneous strain and stress is frequently used, especially in the study of the creep of viscoelastic solids. Such homogeneous strains and stresses are undoubtedly incompatible with exact analysis of the underlying dynamics. Following the usual path, assuming a homogeneous state of stress, we prove the following intuitive result: If the homogeneous stress in a linear viscoelastic solid relaxes monotonically in step-strain tests, then the longitudinal strain in the same solid increases monotonically in creep tests (see Fig. 5.1). To prove this we note that the stress T in a linear viscoelastic solid undergoing uni-axial strain $\varepsilon(x,t) = \frac{\partial u}{\partial x}(x,t)$ is given by $T = (\mu + G(0))\ \varepsilon(t) + \int_0^\infty \frac{dG}{ds}\ \varepsilon(t-s)\, ds$. A montonically decreasing stress relaxation for a homogeneous step-strain implies that G satisfies assumptions (i)-(ii) listed under (1.1). We have assumed either that $G'(0) \neq 0$ or $G''(0) \neq 0$. The strain $\varepsilon$ defining creep is governed by

$$(5.3) \qquad T = (\mu + G(0))\ \varepsilon(t) + \int_0^t \frac{dG}{ds}(s)\ \varepsilon(t-s)\, ds$$

$$= \begin{cases} 1 & \text{for } t > 0 \\ 0 & \text{for } t < 0 \,. \end{cases}$$

By taking various limits of (5.3) and its derivative we can show that

$$\left.\begin{aligned} \varepsilon(0^+) &= \frac{1}{\mu+G(0)}\,, \\ \varepsilon'(0^+) &= \frac{-G'(0)}{\{\mu+G(0)\}^2} > 0\,, \text{ if } G'(0) \neq 0 \\ \varepsilon''(0^+) &= -G''(0)/\{\mu+G(0)\}^2 \text{ if } G'(0) = 0 \text{ and } G''(0) \neq 0\,, \\ \lim_{t\to\infty} \varepsilon(t) = \varepsilon^* &= \frac{1}{\mu} \end{aligned}\right\}.$$

It is easy to verify, using (5.3) that $\varepsilon(t)$ is continuous and $\varepsilon'(t)$ exists for any $t > 0$. We want to prove that

$$(5.5) \qquad \varepsilon'(t) > 0, \quad \forall\ t > 0.$$

If (5.5) is not true, then (using (5.4)) there exists a $\bar{t} > 0$ such that

$$(5.6) \qquad \varepsilon'(\bar{t}) = 0 \quad \text{and} \quad \varepsilon'(t) > 0, \forall t \in [0,\bar{t}]\,.$$

By differentiating (5.3) once with respect to t, we find that

$$(5.7) \qquad (\mu + G(0))\ \varepsilon'(t) + G'(t)\ \varepsilon(0) + \int_0^t G'(s)\ \varepsilon'(t-s)\, ds = 0, \forall\ t > 0.$$

After evaluating (5.7) at $t = \bar{t}$, using (5.6), we get:

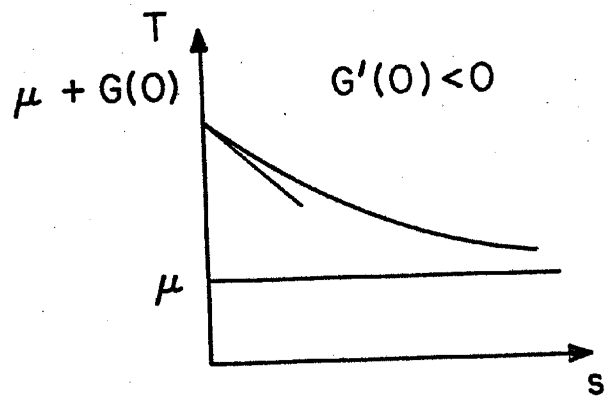
$$(5.8) \qquad G'(t)\,\varepsilon(0) + \int_0^{\bar{t}} G'(s)\,\varepsilon'(\bar{t}-s)\,ds = 0 .$$

But (5.8) then leads to a contradiction because the assumptions about G(s) make the left side of (5.8) strictly negative. It follows that $\varepsilon'(t) > 0$ and not $\leq 0$. It is not hard to demonstrate that $\varepsilon'(t) > 0$ when

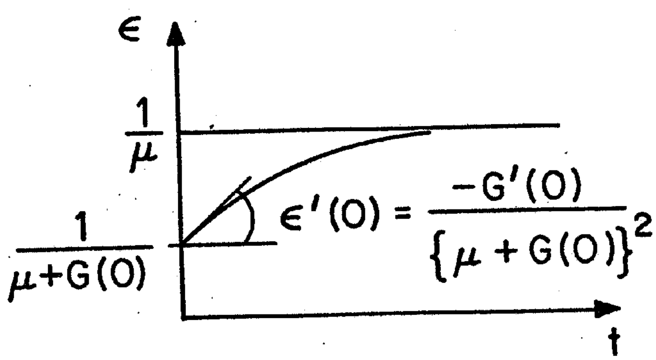
$$(5.9) \qquad G(s) = a\,\delta(s) + h(s)$$

where $a > 0$, h(s) satisfies the assumptions under (1.1) and $\delta(s)$ is a Dirac measure at the origin.

Graphical represntations of the monotonicity result are exhibited in Fig. 5.1 below:




(a): Homogeneous step-strain relaxation




(b) Creep response to a homogeneous step in stress

Fig. 5.1: Relation between stress relaxation and creep.

When $a > 0$ in (5.9), the response to a step increase in stress is monotonic as in Fig. 5.1 (b), but it passes through the origin as in Fig. 5.2.

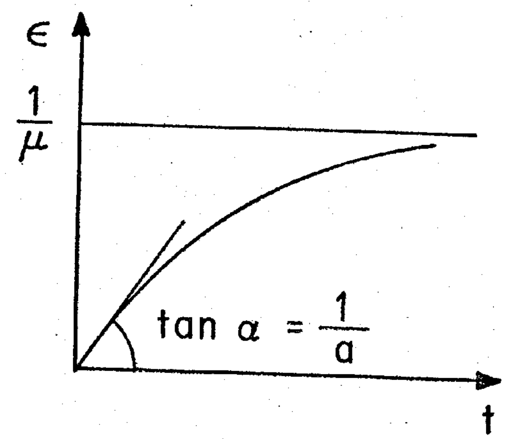



Fig. 5.2: Creep response for kernels of the type (5.9)

We close by reminding the reader that the type of response which we have described above depends tacitly on the unfounded and actually incorrect assumption that homogeneous step-strain (relaxation) and step-stress (creep) tests are admissible deformations compatible with dynamics.

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Note added in proof: The following works which are relevant to the analysis of this paper have been brought to our attention:
P. W. Bucher and F. Mainaridi, Asymptotic expansions for transient viscoelastic waves, Journal de Mécanique, Vol. 14, No. 4, 1975 and R. M. Christensen, Theory of Viscoelasticity: An introduction, Academic Press, 1971. These works give expansions of solutions near the shock and develop full solutions for special choices of the kernel G(s).

We can extend the results of our work in [7] and in §5 of this paper to one dimensional longitudinal motion in the following way. If the symmetric part of the displacement gradient is denoted by $\hat{\nabla}\underset{\sim}{u}$, then the Cauchy stress $\underset{\sim}{T}$ for an isotropic linear viscoelastic solid is given by:

$$\underset{\sim}{T} = 2\mu_0 \hat{\nabla}\underset{\sim}{u} + \lambda_0 (\mathrm{tr}(\hat{\nabla}\underset{\sim}{u}))\underset{\sim}{1} + 2\int_0^\infty \frac{d\mu}{ds}(s)\{\hat{\nabla}\underset{\sim}{u}(\underset{\sim}{x},t-s)-\hat{\nabla}\underset{\sim}{u}(\underset{\sim}{x},t)\}ds$$

$$+\left\{\int_0^\infty \frac{d\lambda}{ds}(s)\{\mathrm{tr}(\hat{\nabla}\underset{\sim}{u}(\underset{\sim}{x},t-s)) - \mathrm{tr}(\hat{\nabla}\underset{\sim}{u}(\underset{\sim}{x},t))\}ds\right\}\underset{\sim}{1}$$

On the one hand a longitudinal displacement, $\underset{\sim}{u} = u(x,t)\hat{i}$ is governed by (5.3) with

$$\mu \overset{\text{def}}{=} 2\mu_0 + \lambda_0 \quad ,$$

$$G(s) \overset{\text{def}}{=} 2\mu(s) + \lambda(s) \quad ,$$

$$\varepsilon(t) \overset{\text{def}}{=} \frac{\partial u}{\partial x}(x,t) \quad .$$

On the other hand shearing motion, $\underset{\sim}{u} = v(x,t)\hat{j}$ satisfy (5.3) with

$$\mu \overset{\text{def}}{=} \mu_0 \ ,$$

$$G(s) \overset{\text{def}}{=} \mu(s) \ ,$$

$$\varepsilon(t) \overset{\text{def}}{=} \frac{\partial v}{\partial x}(x,t) .$$

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# CORRIGENDUM I

Linearized Dynamics for Step Jumps of Velocity and Displacement of Shearing Flows of A Simple Fluid, by A. Narain and D. D. Joseph (Rheol. Acta 21, 228-250 (1982)).

1) The quantities G(s), $C_t(t)$ and $A_1(t)$ are tensors and should be in boldface.

2) The equations under (3.4) should read

$$\lambda^t(x,s) = 0 \quad \text{for} \quad t \leq 0$$

and

$$\frac{d\lambda^t}{ds}(x,s) = -\frac{\partial v}{\partial x}(x,t-s).$$

3) The first sentence under Fig. 5.1 should read "Now for $t-\alpha x<0$ ..."

4) Eqn (6.8) should be replaced by:

$$Mv_n \to Mv \equiv \int_{\alpha x}^{t} G(t-s) f_{xx}(x,s)ds - 2\alpha G(t-\alpha x) f_x(x,\alpha x^+)$$
$$+\alpha^2 G'(t-\alpha x) f(x,\alpha x^+) - \alpha^2 G(t-\alpha x) \frac{\partial f}{\partial t}(x,\alpha x^+)$$
$$- \rho\frac{\partial f}{\partial t}(x,t).$$

5) Eqn. (6.11) should be replaced by:

$$2\frac{\partial f}{\partial t}(x, \alpha x^+) + \frac{2}{\alpha}\frac{\partial f}{\partial x}(x,\alpha x^+) = \frac{G'(0)}{G(0)} f(x,\alpha x^+)$$

6) The left side of (14.11) should be replaced by

$$\frac{\omega(r,t)}{a\Omega}$$

7) Eqn. $(14.12)_{(ii)}$ is:

$$K_1(Z) \sim \sqrt{\frac{\pi}{2z}} \exp(-z) \quad \text{as} \quad |z| \to \infty.$$

8) Eqn. (16.5) can be ignored.

9) The sentence under Eqn. (4.5) should read "Eqs. (4.3, 4.5) imply ... ... half-plane Re $u>-\lambda$."

10) The equations between (14.3) and (14.6) should be numbered (14.4) and (14.5).

11) The left side of the equation above (14.6) should read $\mathcal{G}\langle r\theta\rangle(t)$ in place of $\mathcal{G}\langle\pi\theta\rangle(t)$.

12) The left side of Eqn (6.7) should read:

$$\frac{\partial^2 v_n}{\partial x^2}$$

13) The definition of $\eta(u)$ underneath (10.4) is

$$\eta(u) = \sqrt{\frac{\rho u}{\bar{G}(u)}}$$

14) The eqn. (5.17) should read

$$\left[1 + \frac{G'(0)}{G(0)u} + \frac{G''(0)}{G(0)u^2} + 0(\frac{1}{u^3})\right]^{-\frac{1}{2}}$$

$$= 1 - \frac{\bar{\gamma}'}{2u} + \frac{3}{8}\frac{\bar{\gamma}''}{u^2} + \frac{\bar{\gamma}''}{u^2} + 0(\frac{1}{u^3}).$$

where $\bar{\gamma}' \stackrel{\text{def}}{=} \frac{G'(0)}{G(0)}$

$$\bar{\gamma}'' \stackrel{\text{def}}{=} -\frac{1}{2}\,\frac{G''(0)}{G(0)}$$

15) The right side of (12.5) should read:

$$v(x,t) = \bar{U}[g(x,t) + \{g(x+21,\ t) - g(2\ell-x,t\} + \{\cdot\cdot\} + \cdots\cdots]$$

CORRIGENDUM II

For the paper: Linearized Dynamics of Shearing Deformation Perturbing Rest in Viscoelastic Materials" by A. Narain and D.D. Joseph.

1) In §1.2, case (ii) of $G'(0) = 0$ we have $G''(0) \leq 0$. For the case of $G''(0) < 0$, Fig. 1.2 should show an undershoot as opposed to overshoot shown in the Figure.

2) Equation $(2.13)_2$ should read:

$$\frac{\partial\hat{f}}{\partial\hat{x}}(\hat{x},\hat{t}) = -\frac{1}{2}\exp(-\frac{\hat{x}}{2}) - \frac{\hat{x}}{2}\exp(-\frac{\hat{x}}{2}) + \frac{1}{2}\int_{\hat{x}}^{\hat{t}} \frac{\exp(-\frac{\sigma}{2})}{\sqrt{\sigma^2-\hat{x}^2}}\, I_1(\tfrac{1}{2}\sqrt{\sigma^2-\hat{x}^2})\,d\sigma$$

$$+ \frac{\hat{x}^2}{2}\int_{\hat{x}}^{\hat{t}} \frac{e^{-\frac{\sigma}{2}}}{(\sigma^2-\hat{x}^2)}\left\{\frac{I_1(\frac{1}{2}\sqrt{\sigma^2-\hat{x}^2})}{(\sigma^2-\hat{x}^2)^{-\frac{3}{2}}} - \frac{1}{2}\, I_1^1(\tfrac{1}{2}\sqrt{\sigma^2-\hat{x}^2})\right\}d\sigma$$

3) Equation (2.16) should read

$$-T^{\langle xy\rangle}(0,0^+) = -\frac{U}{2}G'(0)\sqrt{\frac{\rho}{G(0)}} > 0.$$

4) Equation (2.18) should read

$$-T^{<xy>}(\ell,\alpha\ell^{+}) = -2U\alpha G(0)\exp\left(\frac{\alpha\ell G'(0)}{2G(0)}\right)\left[\alpha\ell\left\{\frac{3}{8}\left(\frac{G'(0)}{G(0)}\right)^{2} - \frac{G''(0)}{2G(0)}\right\} + \frac{G'(0)}{2G(0)}\right].$$

This correction should also be noted in the summary of §2.2

5) Equation (4.10) should read

$$v(x,t) = \int_0^t g(\tau)\left[\frac{\partial f}{\partial t}(x,t-\tau)H(t-\tau-\alpha x) + \left\{\frac{\partial f}{\partial t}(x+2\ell,t-\tau)H(t-\tau-\alpha(x+2\ell))\right.\right.$$

$$\left.\left.\frac{\partial f}{\partial t}(2\ell-x,t)H(t-\tau-\alpha(2\ell-x))\right\} + \cdots\right]d\tau$$

$$+[f(x,\alpha x^{+})\, g(t-\alpha x) + \{f(2\ell+x,\alpha(2\ell+x)^{+}g(t-\alpha(2\ell+x))$$

$$-f(2\ell-x,\alpha(2\ell-x)^{+})g(t-\alpha(2\ell-x))\} + \{\cdots\} + \cdots]$$

6) Equation (4.11) should read

$$v(x,t) = \left[\int_0^{t-\alpha x} g(\tau)\ \frac{\partial f}{\partial t}(x,t-\tau)d\tau + \left\{\int_0^{t-\alpha(x+2\ell)} g(\tau)\ \frac{\partial f}{\partial t}(2\ell+x,t-\tau)d\tau\right.\right.$$

$$\left.\left.-\int_0^{t-\alpha(2\ell-x)} g(\tau)\ \frac{\partial f}{\partial t}(2\ell-x,t-\tau)d\tau\right\} + \{\cdots\} + \cdots\right]$$

$$+[f(x,\alpha x^{+})g(t-\alpha x) + \{f(2\ell+x,\alpha(2\ell+x)^{+})$$

$$g(t-\alpha(2\ell+x) - f(2\ell-x,\alpha(2\ell-x)^{+})g(t-\alpha(2\ell-x))\}$$

$$+ \{\cdots\} + \cdots].$$