COURSE 5

STABILITY AND BIFURCATION THEORY

Daniel D. JOSEPH

Department of Aerospace Engineering and Mechanics, The University of Minnesota, Minneapolis, MN 55455, U.S.A.

Contents

G. Iooss, R.H.G. Helleman and R. Stora, eds.
Les Houches, Session XXXVI, 1981 – Comportement Chaotique des Systèmes Déterministes/
Chaotic Behaviour of Deterministic Systems

1. Bifurcation in $\mathbb{R}'$

In this lecture we consider the theory of singular points of plane curves. And to these considerations we add the study of stability. To make a point, not to be taken literally, I will say that sixty per cent of the ideas of bifurcation theory can be most readily understood from this elementary study in $\mathbb{R}'$.

We study the evolution equation

$$du/dt = F(\mu, u); \qquad \mu, u \in \mathbb{R}', \tag{1.1}$$

where $F(\cdot,\cdot)$ has two continuous derivatives with respect to μ and u. Equilibrium of eq. (1.1) satisfy $u=\varepsilon$, independent and

$$F(\mu, \varepsilon) = 0. \tag{1.2}$$

The study of bifurcation of equilibrium solutions of the automous problem (1.1) is equivalent to the study of singular points of the curve (1.2) in the (μ, ε) plane.

It is desirable to classify points of the curves (1.2):

(i) *A regular point* of $F(\mu, \varepsilon)=0$ is one for which the implicit function theorem works

$$F_\mu \neq 0 \text{ or } F_\varepsilon \neq 0. \tag{1.3}$$

If eq. (1.3) holds, then we can find a unique curve $\mu=\mu(\varepsilon)$ or $\varepsilon=\varepsilon(\mu)$ through the point.

(ii) *A regular turning point* is a point at which $\mu_\varepsilon(\varepsilon)$ changes sign and $F_\mu(\mu, \varepsilon) \neq 0$.

(iii) *A singular point* of the curve $F(\mu, \varepsilon)=0$ is a point at which

$$F_\mu = F_\varepsilon = 0. \tag{1.4}$$

(iv) *A double point* of the curve $F(\mu, \varepsilon)=0$ is a singular point through which pass two and only two branches of $F(\mu, \varepsilon)=0$ possessing distinct tangents. We shall assume that all second derivatives of F do not simultaneously vanish at a double point.

(v) *A singular turning (double) point* of the curve $F(\mu, \varepsilon)=0$ is a double point at which μ_ε changes sign on one branch.

(vi) *A cusp point* of the curve $F(\mu, \varepsilon) = 0$ is a point of second order contract between two branches of the curve. The two branches have the same tangent at a cusp point.

(vii) *A conjugate point* is an isolated singular point solution of $F(\mu, \varepsilon) = 0$.

(viii) *A higher-order singular point* of the curve $F(\mu, \varepsilon) = 0$ is a singular point at which all three second derivatives of $F(\mu, \varepsilon)$ are null.

Double points are most important for bifurcation. Suppose (μ_0, ε_0) is a singular point. Then equilibrium curves passing through the singular points satisfy

$$2F(\mu, \varepsilon) = F_{\mu\mu}\,\delta\mu^2 + 2F_{\varepsilon\mu}\,\delta\varepsilon\,\delta\mu + F_{\varepsilon\varepsilon}\,\delta\varepsilon^2 + O[(|\delta\mu| + |\delta\varepsilon|)^2] = 0 \qquad (1.5)$$

where $\delta\mu = \mu - \mu_0$, $\delta\varepsilon = \varepsilon - \varepsilon_0$ and $F_{\mu\mu} = F_{\mu\mu}\,(\mu_0, \varepsilon_0)$, etc. In the limit, as $(\mu, \varepsilon) \to (\mu_0, \varepsilon_0)$ the eq. (1.5) for the curves reduces to the quadratic equation

$$F_{\mu\mu}\,\mathrm{d}\mu^2 + 2F_{\varepsilon\mu}\,\mathrm{d}\varepsilon\,\mathrm{d}\mu + F_{\varepsilon\varepsilon}\,\mathrm{d}\varepsilon^2 = 0 \qquad (1.6)$$

for the tangents to the curve. We find that

$$\left[\frac{\mu_\varepsilon^{(1)}(\varepsilon_0)}{\mu_\varepsilon^{(2)}(\varepsilon_0)}\right] = -\frac{F_{\varepsilon\mu}}{F_{\mu\mu}}\begin{bmatrix}1\\1\end{bmatrix} + \left(\frac{D}{F_{\mu\mu}^2}\right)^{1/2}\begin{bmatrix}1\\-1\end{bmatrix}, \qquad (1.7)$$

$$\left[\frac{\mu_\varepsilon^{(1)}(\varepsilon_0)}{\varepsilon_\mu^{(2)}(\mu_0)}\right] = -\frac{F_{\varepsilon\mu}}{F_{\varepsilon\varepsilon}}\begin{bmatrix}1\\1\end{bmatrix} - \left(\frac{D}{F_{\varepsilon\varepsilon}^2}\right)^{1/2}\begin{bmatrix}1\\-1\end{bmatrix}, \qquad (1.8)$$

where

$$D = F_{\varepsilon\mu}^2 - F_{\mu\mu}F_{\varepsilon\varepsilon}.$$

If $D < 0$ there are not real tangents through (μ_0, ε_0) and the point (μ_0, ε_0) is an isolated (conjugate) point solution of $F(\mu, \varepsilon) = 0$.

We shall consider the case when (μ_0, ε_0) is not a higher-order singular point. Then (μ_0, ε_0) is a double point if and only if $D > 0$. If the two curves pass through the singular point and $D = 0$ then the slope at the singular point of higher contact is given by eqs. (1.7) or (1.9). If $D > 0$ and $F_{\mu\mu} \neq 0$ then there are two tangents with slopes $\mu_\varepsilon^{(1)}(\varepsilon_0)$ and $\mu_\varepsilon^{(2)}(\varepsilon_0)$ given by eq. (1.7). If $D > 0$ and $F_{\mu\mu} = 0$, then $F_{\varepsilon\mu} \neq 0$ and

$$\mathrm{d}\varepsilon\;(2\mathrm{d}\mu\,F_{\varepsilon\mu} + \mathrm{d}\varepsilon\,F_{\varepsilon\varepsilon}) = 0$$

and there are two tangents $\varepsilon_\mu(\mu_0) = 0$ and $\mu_\varepsilon(\varepsilon_0) = -F_{\varepsilon\varepsilon}/2F_{\varepsilon\mu}$. If $\varepsilon_\mu(\mu_0) = 0$ then $F_{\mu\mu}(\mu_0, \varepsilon_0) = 0$. So all the possibilities are covered in the following two cases:

(A) $D>0$, $F_{\mu\mu}\neq 0$ with tangents $\mu_\varepsilon^{(1)}(\varepsilon_0)$ and $\mu_\varepsilon^{(2)}(\varepsilon_0)$.

(B) $D>0$, $F_{\mu\mu}=0$ with $\varepsilon_\mu(\mu_0)=0$ and $\mu_\varepsilon(\varepsilon_0)=-F_{\varepsilon\varepsilon}/2F_{\varepsilon\mu}$.

The existence of two branches passing through the point (μ_0,ε_0) is guaranteed by the implicit function theorem when $D>0$ (see ESBT*, section II.4).

When $D=0$ and all second derivatives are not zero there is a cusp at the origin (ESBT, section II.5). There are two typical situations:

(i) Bifurcation with two curves having common tangents and different curvatures at $(\mu_0,\varepsilon_0)=(0,0)$, an example is given by fig. 1.

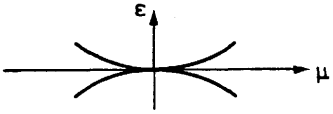

Fig. 1. Bifurcation at a cusp point.

(ii) A cusp point of a single curve. This is degenerate form of a turning point. An example is given at fig. 2.

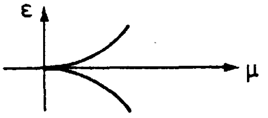

Fig. 2. A degenerate turning point.

When $D=0$ because all second derivatives are null it is necessary to consider cubic equation to determine the number real tangents. If there are three real, distinct roots then three bifurcating solutions pass through the singular point (μ_0,ε_0). If two roots are complex, then there is no bifurcation.

We next consider the stability of bifurcating solutions using the linearized theory of stability. The linearized equation is

$$Z_t=F_\varepsilon(\mu,\varepsilon)Z. \tag{1.10}$$

The general solution of eq. (1.10) is

$$Z=e^{\sigma t}Z_0, \qquad \sigma=F_\varepsilon(\mu,\varepsilon). \tag{1.11}$$

* We will use the abbreviation ESBT to denote ref. [4].

Since all solutions of eq. (1.10) are in the form eq. (1.11) we find that disturbances Z of ε grow when $\sigma > 0$ and decay when $\sigma < 0$. The linearized theory implies that $[\mu(\varepsilon), \varepsilon]$ satisfying $F(\mu, \varepsilon) = 0$ is stable when $\sigma < 0$ and is unstable when $\sigma > 0$. This criterion applies even to the nonlinear problem when the initial disturbance is sufficiently small (cf. section II.7 in ESBT).

A very general and important result is easy to deduce from the second part of eq. (1.11) under the hypothesis that eq. (1.2) may be solved for $\mu = \mu(\varepsilon)$. Then, differentiating $F[\mu(\varepsilon), \varepsilon] = 0$ with respect to ε we find that

$$\sigma(\varepsilon) = F_\varepsilon[\mu(\varepsilon), \varepsilon] = -\mu_\varepsilon F_\mu[\mu(\varepsilon), \varepsilon]. \tag{1.12}$$

It follows easily from eq. (1.12) that $\sigma(\varepsilon)$ must change sign as ε is varied across a regular turning point. This implies that the $u = \varepsilon$, $\mu = \mu(\varepsilon)$ is stable on one side of regular turning point and is unstable on the other side (see fig. 3).

The study of stability may be tied to the study of bifurcation by the hypothesis of strict loss of stability which was introduced by Hopf. This hypothesis is a non-degeneracy condition which guarantees double-point bifurcation. More precisely, we have the following theorem: *Suppose that* (μ_0, ε_0) *is a singular point* (A) $\sigma_\varepsilon(\varepsilon_0) \neq 0$ *or* (B) $\sigma_\mu(\mu_0) \neq 0$. *Then* (μ_0, ε_0) *is a double point.* For the proof under hypothesis (A) see ESBT, section II.9. For case B we must solve $F(\mu, \varepsilon)$ for $\varepsilon(\mu)$. At the singular point (μ_0, ε_0) we have strict loss of stability because $\sigma_\mu = F_{\varepsilon\mu} + F_{\varepsilon\varepsilon}\varepsilon_\mu = F_{\varepsilon\mu} = D^{1/2} \operatorname{sgn} F_{\varepsilon\mu}$.

It is easy to derive formulas which show that there is an exchange of stability at a double point (ESBT, section II.10). These formulas can be used to prove the following theorem. *Assume that all singular points of solutions of* $F(\mu, \varepsilon) = 0$ *are double points. The stability of such solutions must change at each regular turning point and at each singular point (which is not a turning point), and only at such points.*

We shall prove this theorem for the case in which $u = 0$ is a solution of the evolution problem

$$F(\mu, 0) = 0 \qquad \forall \mu \in \mathbb{R}. \tag{1.13}$$

Fig. 3. Exchange of stability at a regular turning point. The same type of exchange of stability can be demonstrated for degenerate case shown in fig. 2.

Then, differentiating the second part of eq. (1.11) with respect to μ on the solution $\varepsilon = 0$, we get

$$\sigma_\mu^{(1)}(0) = F_{\mu\varepsilon}(0,0) \neq 0, \qquad \text{say} > 0. \tag{1.14}$$

On the bifurcating branch $F[\mu(\varepsilon), \varepsilon] = 0$ and

$$\begin{aligned} \sigma^{(2)} &= F_\varepsilon[\mu(\varepsilon), \varepsilon] = -\mu_\varepsilon F_\mu[\mu(\varepsilon), \varepsilon], \\ &= -\mu_\varepsilon[F_{\mu\varepsilon}(0,0)\varepsilon + \mathrm{O}(\varepsilon)], \\ &= -\mu_\varepsilon \sigma_\mu^{(0)}(0)\{\varepsilon + \mathrm{O}(\varepsilon)\}. \end{aligned} \tag{1.15}$$

The following bifurcation diagrams are implied by eqs. (1.14) and (1.15) (fig. 4).

To bring the ideas developed so far we give a demonstration here of the stability and bifurcation of the bent wire arch described in fig. II.5 of ESBT. We replace u by σ, the angle of deflection, and μ by l, the length. We imagine that the equation of motion of the bent arch is

$$\mathrm{d}\theta/\mathrm{d}t = \theta[l - l(\theta)], \tag{1.16}$$

where

$$l(-\theta) = l(\theta)$$

is even. The upright position is $\theta = 0$ and the bifurcating solution is $l = l(\theta)$, shown in fig. 5.

The ideas developed so far have a much wider range of applicability than might at first be supposed. The local analysis near turning point and singular point applies even to partial differential equations under rather common conditions (called bifurcation at a simple eigenvalue) under which the important part of the problem is a part which can be projected into one dimension.

There is a very important global result which holds strictly in $\mathbb{R}'$ and not

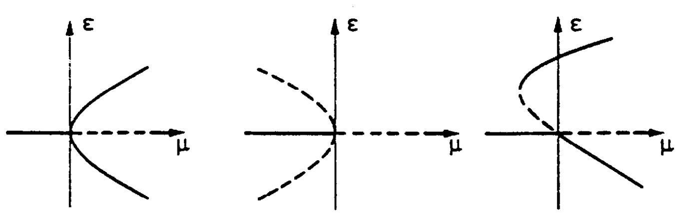

Fig. 4. Bifurcation and stability at a double point.

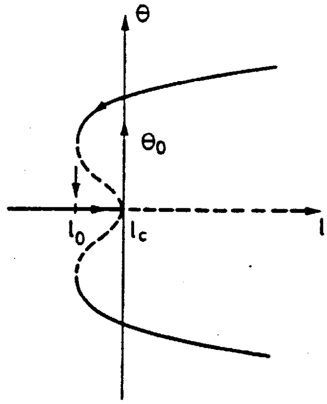

Fig. 5. Bifurcation diagram for the bent arch. The arch bifurcates subcritically and exhibit hysteresis. A demonstration of the actual bifurcation is given here and is described in ESBT.

necessarily in $\mathbb{R}'$ in projection. In the one-dimensional case it is possible to prove that the stability of solutions which pierce the line $\mu = \text{constant}$ is of alternating sign, as in fig. 6. In higher dimensions curves of solutions which appear to intersect when projected onto the plane of the bifurcation diagram actually do not intersect in the higher-dimensional space. We may write the evolution eq. (1.1) in following factored form

$$du/dt = F_1F_2F_3F_4F_5F_6F_7, \tag{1.17}$$

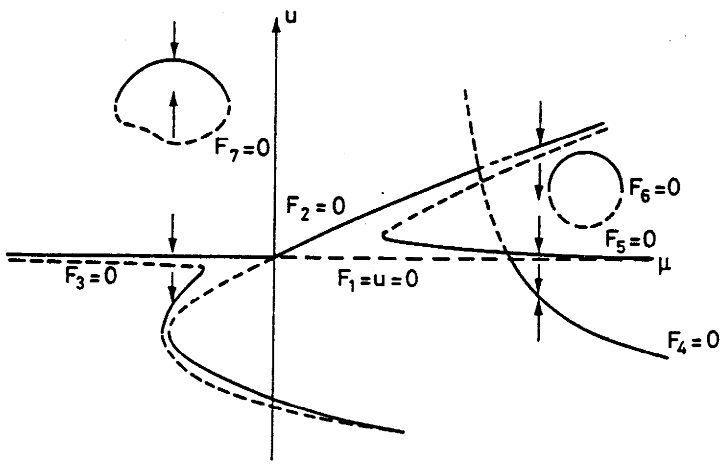

Fig. 6. Bifurcation, stability and domains of attraction of equilibrium solutions of eq. (1.17).

where each $F_i = F_i(\mu, u) = 0$ gives an equilibrium solution. An example of seven equilibrium solutions is shown in fig. 6.

In this simple example we see bifurcating solutions, solutions which perturb bifurcation and isolated solutions. I call the intersecting solutions F_1, F_2 and F_4 "bifurcating" (from one another). All the remaining solutions can be isolated F_3 and F_5 perturb bifurcation. F_6 is an "isola" which can be treated as a perturbation of a conjugate singular point. The stability of solutions on the line $\mu = \text{constant}$ alternates. You see that nonuniqueness is endemic, even in $\mathbb{R}'$.

2. Bifurcation in $\mathbb{R}^2$

For the moment we will use a general notation for our fundamental (autonomous) problem

$$du/dt = f(\mu, u) = f_u(\mu \mid u) + N(\mu, u). \tag{2.1}$$

Here $u \in \mathbb{R}^n$ or, say, u is an element in a normed space and $u = 0$ is a solution for all μ

$$f(\mu, 0) = 0,$$

$f_u(\mu \mid u)$ is the derivative of f with respect to u at $u = 0$, a linear operator $f_u(\mu \mid \cdot)$ and $N(\mu, u) = O(|u|^2)$. The linearized stability problem for the stability of the solution $u = 0$ is

$$dv/dt = f_u(\mu \mid v). \tag{2.2}$$

A spectral problem for stability can be obtained from solutions of eq. (2.2) in the form

$$v = e^{\sigma t} \zeta, \tag{2.3}$$

where ζ is independent of t and

$$\sigma = \zeta + i\eta$$

is an eigenvalue of

$$\sigma \zeta = f_u(\mu \mid \zeta). \tag{2.4}$$

We say $u = 0$ is stable (according to spectral theory) if $\zeta(\mu) < 0$ for all eigenvalues of eq. (2.4). The problem (2.1) arises when we have a problem governed by differential equations which is forced by steady data. Then there is a forced steady solution and u is the difference between the forced solution and any other solution of the same forced problem. Many very general problems may be represented by eq. (2.1).

Consider eq. (2.4) in $\mathbb{R}^2$. Then σ is an eigenvalue of

$$f_u(\mu\,|\,\cdot)=\begin{bmatrix} a(\mu) & b(\mu) \\ c(\mu) & d(\mu)\end{bmatrix}\overset{\text{def}}{=}A, \tag{2.5}$$

a root of

$$\sigma^2-\sigma(a+d)+ad-bc=0. \tag{2.6}$$

There are two roots

$$\sigma_1=\frac{a+d}{c}+\Delta^{1/2},$$

and

$$\sigma^2=\frac{a+d}{2}-\Delta^{1/2},$$

where

$$\Delta=\frac{(a-d)^2}{4}+bc=\frac{(a+d)^2}{4}-ad+bc.$$

The adjoint matrix, the transpose

$$A^T=\begin{bmatrix} a & b \\ c & d\end{bmatrix}$$

has the same two eigenvalues but, if $c\neq b$, different eigenvectors.

There are four cases in two categories to consider. Category one are the algebraically simple eigenvalues.

Case 1: $\Delta>0$. $\sigma_1\neq\sigma_2$ are real. There are two adjoint eigenvectors.

Case 2: $\Delta<0$. σ_1 and σ_2 are complex and $\sigma_2=\bar{\sigma}_1$. There are two eigenvectors and they are conjugate. Category two are the algebraically double eigenvalues $\sigma_1=\sigma_2$; i.e., $\Delta=0$.

Case 3: $\sigma_1=\sigma_2$ is a semi-simple double eigenvalue, $(a-d)^2=b=c=0$. Then $A=al=A^T$ and every vector ζ is an eigenvector belonging to $\sigma_1=\sigma_2=a$. We can select two orthonormal ones $\zeta_1\cdot\zeta_2=0$. The eigenvalue a is said to have a Riesz index 1.

Case 4: Riesz index two. There is only one eigenvector, one vector satisfying $(A-\sigma I)\zeta=0$ and one generalized eigenvector satisfying $(A-\sigma I)^2X=0$ (section IV.4 of ESBT).

We shall not consider bifurcation for cases 3 and 4 (double eigenvalues) in these lectures. Case 1 can be formulated as a problem in $\mathbb{R}'$ by the method of projection. I will show this in the next lecture. The remaining case 2 in which $\sigma=\zeta+i\eta$ is complex and

$$v=e^{\xi t}e^{i\eta t}\zeta \tag{2.7}$$

is time periodic, leads to a time-periodic bifurcating solution as I now shall show.

The evolution eq. (2.1) may be written in component form

$$\dot{u}_j=A_{ij}(\mu)u_j+B_{ijk}u_ju_k+\text{higher order terms.} \tag{2.8}$$

We suppose that near $\mu=0$ the discriminant Δ is negative, so that

$$\sigma(\mu)\zeta=A\zeta,$$

$$\sigma(\mu)\bar{\zeta}^*=A^T\bar{\zeta}^*, \tag{2.9}$$

where $\zeta^*(\mu)$ is the adjoint vector with eigenvalue $\bar{\sigma}(\mu)=\xi(\mu)-i\eta(\mu)$ in the scalar product

$$\langle X\cdot Y\rangle\overset{\text{def}}{=}X\cdot\bar{Y}. \tag{2.10}$$

We may normalize so that

$$\langle\zeta,\zeta^*\rangle=\zeta\cdot\bar{\zeta}^*=\zeta_k\bar{\zeta}_k^*=1. \tag{2.11}$$

It is easy to deduce

$$\langle\zeta,\bar{\zeta}^*\rangle=\zeta_k\zeta_k^*=0. \tag{2.12}$$

We suppose that the loss of stability of $u=0$ occurs at $\mu=0$ so that $\xi(0)=0$. We will get bifurcation into periodic solutions if

$$\eta(0)=\omega_0\neq 0 \qquad \text{and } d\xi(0)/d\mu=\xi_\mu(0)\neq 0 \tag{2.13}$$

[say $\xi_\mu(0)>0$].

Since ζ and $\bar{\zeta}$ are linearly independent any real-valued two-dimensional vector $u=(u_1,u_2)$ may be represented as

$$u_i=a(t)\zeta_i+\bar{a}(t)\bar{\zeta}_i. \tag{2.14}$$

Combining eqs. (2.14) and (2.8), using eq. (2.9) we get

$$\dot{a}\zeta_i + \dot{\bar{a}}\bar{\zeta}_i = \sigma(\mu)\zeta_i + \bar{\sigma}(\mu)\bar{\zeta}_i + a^2 B_{ijk}\zeta_j\zeta_k + 2|a|^2 B_{ijk}\zeta_i\bar{\zeta}_k + \bar{a}^2 B_{ijk}\bar{\zeta}_i\bar{\zeta}_k + \mathrm{O}(|a|^3).$$

The orthogonality properties, [eqs. (2.11) and (2.12)], are now employed to reduce the preceding into a single, complex-valued, amplitude equation

$$\dot{a} = f(\mu, a) = \sigma(\mu)a + \alpha(\mu)a^2 + 2\beta(\mu)|a|^2 + \gamma(\mu)\bar{a}^2 + \mathrm{O}(|a|^3), \tag{2.15}$$

where, for example, $\alpha(\mu) = B_{ijk}(\mu)\zeta_j\zeta_k\zeta_i^*$. (For simplicity we shall suppress cubic terms of $f(\mu, a)$ here.) These terms come into the bifurcating solution at second order but do not introduce new features. The linearized stability of the solution $a=0$ of eq. (2.15) is determined by $\dot{a} = \sigma(\mu)a$, $a =$ constant $\times\, \mathrm{e}^{\sigma(\mu)t}$. At criticallity ($\mu = 0$), $a =$ constant $\times\, \mathrm{e}^{\mathrm{i}\omega_0 t}$ is 2π-periodic in $s = \omega_0 t$.

We shall show that a bifurcating time-periodic solution may be constructed from the solution of the linearized problem at criticallity. This bifurcating solution is in the form

$$a(t) = b(s, \varepsilon),\ s = \omega(\varepsilon)t,\ \omega(0) = \omega_0,\ \mu = \mu(\varepsilon), \tag{2.16$_1$}$$

where ε is the amplitude of a defined by

$$\varepsilon = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{-\mathrm{i}s} b(s, \varepsilon)\, \mathrm{d}s = [b]. \tag{2.16$_2$}$$

The solution, eq. (2.16), of eq. (2.15) is unique to within an arbitrary translation of the time origin. This means that under translation $t \to t + c$ the solution $b(s + c\omega(\varepsilon), \varepsilon)$ shifts its phase. This unique solution is analytic in ε when $f(\mu, a)$ is analytic in the variables $(\mu, a, \bar{a})$ and it may be expressed as a series:

$$\begin{bmatrix} b(s, \varepsilon) \\ \omega(\varepsilon) - \omega_0 \\ \mu(\varepsilon) \end{bmatrix} = \sum_{n=1}^{\infty} \varepsilon^n \begin{bmatrix} b_n(s) \\ \omega_n \\ \mu_n \end{bmatrix}. \tag{2.17}$$

The perturbation problems which govern $b_n(s)$, ω_n and μ_n can be obtained by identifying the coefficient of ε^n which arise when eq. (2.17) is substituted into the two equations: $\omega \dot{b} = f(\mu, b)$ and $\varepsilon = [b]$. We find that at order one

$$\omega_0 \dot{b}_1 - \mathrm{i}\omega_0 b_1 = 0,\ [b_1] = 1,\ b_1(s) = \mathrm{e}^{\mathrm{i}s}.$$

At order two we find that $[b_2] = 0$ and

$$\omega_0[\dot{b}_2 - \mathrm{i}b_2] + \omega_1\dot{b}_1 = \mu_1\sigma_\mu b_1 + \alpha_0 b_2^2 + 2\beta_0|b_1|^2 + \gamma_0\bar{b}_1^2,$$

where $\sigma_\mu = \mathrm{d}\sigma(0)/\mathrm{d}\mu$ and, for example, $\alpha_0 = \alpha(0)$.

Equations of the form $\dot{b}(s) - \mathrm{i}b(s) = f(s) = f(s+2\pi)$ are solvable for $b(s) = b(s+2\pi)$ if and only if the Fourier expansion of $f(s)$ has no term proportional to $\mathrm{e}^{\mathrm{i}s}$. Hence, because $\xi_\mu \neq 0$ we obtain

$$\mu_1 = \omega_1 = 0$$

in eq. (2.17) and

$$\dot{b}_2 - ib_2 = (\alpha_0 \mathrm{e}^{2\mathrm{i}s} + 2\beta_0 + \gamma_0 \mathrm{e}^{-2\mathrm{i}s})/\omega_0.$$

We find that

$$b_2(s) = [\alpha_0 \mathrm{e}^{2\mathrm{i}s} - 2\beta_0 - (\gamma_0 \mathrm{e}^{-2\mathrm{i}s}/3)]/\mathrm{i}\omega_0.$$

The problem which governs at order three, with cubic terms in b neglected, is

$$\dot{b}_3 - \mathrm{i}b_3 = [-\omega_2\dot{b}_1 + \mu_2\sigma_\mu b_1 + 2\alpha_0 b_1 b_2 + 2\beta_0(b_1\bar{b}_2 + \bar{b}_1 b_2) + 2\gamma_0\bar{b}_1\bar{b}_2]/\omega_0. \tag{2.18}$$

To solve eq. (2.19), we must eliminate terms proportional to $\mathrm{e}^{\mathrm{i}s}$ from the right-hand side of eq. (2.19). This is done if $[b_3] = 0$; that is, if

$$\mathrm{i}\omega_2 - \mu_2\sigma_\mu = -[4\alpha_0\beta_0 - 4|\beta_0|^2 - 2\alpha_0\beta_0 - (2|\gamma_0|^2/3)]/\mathrm{i}\omega_0. \tag{2.19}$$

The real part of eq. (2.19) is solvable for μ_2 provided that $\zeta_\mu \neq 0$. The imaginary part of eq. (2.19) is always solvable for ω_2.

Proceeding to higher orders, it is easy to verify that all of the perturbation problems are solvable when eq. (2.13) holds and, in fact $\omega(\varepsilon) = \omega(-\varepsilon)$ are even functions. It follows that periodic solutions which bifurcate from steady solutions bifurcate to one or the other side of criticality and never to both sides; periodic bifurcating solutions cannot undergo two-sided or transcritical bifurcation.

We now search for the conditions under which the bifurcating periodic solutions are stable. We consider a small disturbance $z(t)$ of $b(s, \varepsilon)$. Setting $a(t) = b(s, \varepsilon) + z(t)$ in eq. (2.15), we find the linearized equation $\dot{z}(t) = f_a[\mu(\varepsilon), b(s, \varepsilon)]z(t)$ where $f_a = \partial f/\partial a$ and $s = \omega(\varepsilon)t$. Then, using Floquet theory, we set $z(t) = \mathrm{e}^{\gamma t}y(s)$ where $y(s) = y(s+2\pi)$ and find that

$$\gamma y(s) = -\omega\dot{y}(s) + f_a(\mu, b)y(s) \overset{\mathrm{def}}{=} [J(s, \varepsilon)y](s) \tag{2.20}$$

where $\dot{y}(s) = \mathrm{d}y(s)/\mathrm{d}s$.

The stability result we need may be stated as a factorization theorem. To prove this theorem we use the fact that $\gamma=0$ is always an eigenvalue of J with eigenfunction $\dot{b}(s,\varepsilon)$

$$J\dot{b}=0 \tag{2.21}$$

and the relation

$$\omega_\varepsilon(\varepsilon)\dot{b}(s,\varepsilon)=\mu_\varepsilon(\varepsilon)f_\mu(\mu(\varepsilon),b(s,\varepsilon))+Jb_\varepsilon \tag{2.22}$$

which arises from differentiating $\omega\dot{b}=f(\mu,b)$ with respect to ε at any ε.

Factorization theorem. The eigenfunction y of eq. (2.20) and the Floquet exponent γ are given by the following formulas:

$$y(s,\varepsilon)=c(\varepsilon)\left(\frac{\tau}{\gamma}\dot{b}(s,\varepsilon)+b_\varepsilon(s,\varepsilon)+\mu_\varepsilon(\varepsilon)\varepsilon q(s,\varepsilon)\right),$$

$$\tau(\varepsilon)=\omega_\varepsilon(\varepsilon)+\mu_\varepsilon(\varepsilon)\hat{\tau}(\varepsilon),$$

$$\gamma(\varepsilon)=\mu_\varepsilon(\varepsilon)\hat{\gamma}(\varepsilon), \tag{2.23}$$

where $c(\varepsilon)$ is an arbitrary constant and $q(s,\varepsilon)=q(s+2\pi,\varepsilon)$, $\hat{\tau}(\varepsilon)$ and $\hat{\gamma}(\varepsilon)$ satisfy the equation

$$\hat{\tau}\dot{b}+\hat{\gamma}b_\varepsilon+f_\mu(\mu,b)+\varepsilon(\gamma q-Jq)=0 \tag{2.24}$$

and are smooth functions in a neighborhood of $\varepsilon=0$. Moreover $\hat{\tau}(\varepsilon)$ and $\hat{\gamma}(\varepsilon)/\varepsilon$ are aven functions and such that

$$\hat{\gamma}_\varepsilon(0)=-\zeta_\mu(0), \qquad \hat{\tau}(0)=-\eta_\mu(0). \tag{2.25}$$

Remark. If $\omega_\varepsilon(0)\neq 0$, $c(\varepsilon)$ may be chosen so that $y(s,\varepsilon)\to b(s,\varepsilon)$ when $\varepsilon\to 0$.

Proof. Substitute the representations (2.23) into (2.20) utilizing eq. (2.21) to eliminate $J\dot{b}$ and eq. (2.22) to eliminate Jb_ε. This leads to eq. (2.23) which may be solved by series

$$\begin{bmatrix} q(s,\varepsilon) \\ \hat{\gamma}(\varepsilon)/\varepsilon \\ \hat{\tau}(\varepsilon) \end{bmatrix}=\sum_{l=0}^{\infty}\begin{bmatrix} q_l(s) \\ \hat{\gamma}_l \\ \hat{\tau}_l \end{bmatrix}\varepsilon^l, \tag{2.26}$$

where $\gamma_0=\hat{\gamma}_\varepsilon(0)$ and $\hat{\tau}_0=\hat{\tau}(0)$. Using the fact that to the lowest order $b=\varepsilon e^{is}$, $\gamma=0(\varepsilon^2)$ and $f_\mu(\mu,b)=\sigma_\mu(0)e^{is}\varepsilon$ we find that

$$e^{is}[i\hat{\tau}(0)+\hat{\gamma}_\varepsilon(0)+\sigma_\mu]-J_0q_0=0, \qquad J_0\overset{\text{def}}{=}J(\cdot,0). \tag{2.27}$$

Eq. (2.27) is solvable for $q_0(s)=q_0(s+2\pi)$ if and only if the term in the bracket vanishes; that is if eq. (2.25) holds. The remaining properties asserted in the theorem may be obtained by mathematical induction using the power series (2.26).

The linearized stability of the periodic solution for small values of ε may now be obtained from the spectral problem: $u(s,\varepsilon)=u(s+2\pi,\varepsilon)$ is stable when $\gamma(\varepsilon)<0$ [$\gamma(\varepsilon)$ is real] and is unstable when $\gamma(\varepsilon)>0$ where

$$\gamma(\varepsilon)=\mu_\varepsilon(\varepsilon)\hat{\gamma}(\varepsilon)=-\mu_\varepsilon(\varepsilon)[\xi_\mu(0)\varepsilon+O(\varepsilon^3)].$$

Two examples are given in fig. 7.

3. Projections into $\mathbb{R}^2$

In this section we shall show that the analysis of bifurcation of periodic solutions from steady ones in $\mathbb{R}^2$, also applies in $\mathbb{R}^n$ and in infinite dimensions; say, for partial differential equations and for functional differential equations, when the steady solution looses stability at a simple, complex-valued eigenvalue.

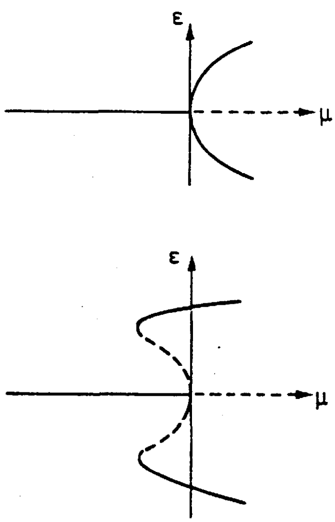

Fig. 7. (a) Supercritical (stable) Hopf bifurcation. (b) Subcritical (unstable) Hopf bifurcation with a turning point. In (b), if zero loses stability strictly as μ is increased past zero, then $\xi_\mu>0$ and zero is unstable for $\mu>0$ (as shown); the double eigenvalue of J_0 splits into two simple eigenvalues of $J(\cdot,\varepsilon)$: one eigenvalue is 0 and the other, $\gamma(\varepsilon)$, controls stability.

Our basic problem is again

$$\dot{u}=f(\mu,u)=f_\mu(\mu\,|\,u)+N(\mu,u), \tag{3.1}$$

where $N(\mu,u)=O(|u|^2)$. A small disturbance $v=e^{\sigma t}\zeta$ of $u=0$ satisfies

$$\sigma\zeta=f_u(\mu\,|\,\zeta). \tag{3.2}$$

The adjoint problem is

$$\sigma\bar{\zeta}^*=f_u^*(\mu\,|\,\bar{\zeta}^*) \tag{3.3}$$

and very often in applications, there are a countably infinite number of eigenvalues $\{\sigma_n\}$ which are arranged in a sequence corresponding to the size of their real parts

$$\xi_1\geq\xi_2\geq\cdots\geq\xi_n\geq\cdots,$$

clustering at $-\infty$. To each eigenvalue there corresponds, at most, a finite number of eigenvectors ζ_n and adjoint eigenvectors ζ_n^*. In the case of a semi-simple eigenvalue σ_n we may choose the eigenvectors of $f_u(\mu\,|\,\cdot)$ and $f_u^*(\mu\,|\,\cdot)$ such that they form biorthonormal families

$$\langle\zeta_{nk},\zeta_{nj}^*\rangle=\delta_{kj},\qquad k,j=1,\ldots,m_n, \tag{3.4}$$

m_n being the multiplicity of the eigenvalue φ_n (assumed to be semi-simple). Taking now the scalar product of eq. (3.1) with ζ_n^* we obtain

$$\begin{aligned}\frac{d}{dt}\langle u,\zeta_n^*\rangle&=\langle f_u(\mu\,|\,u),\zeta_n^*\rangle+\langle N(\mu,u),\zeta_n^*\rangle\\&=\langle u,f_u^*(\mu\,|\,\zeta_n^*)\rangle+\langle N(\mu,u),\zeta_n^*\rangle\\&=\sigma_n\langle u,\zeta_n^*\rangle+\langle N(\mu,u),\zeta_n^*\rangle.\end{aligned} \tag{3.5}$$

When u is small the linearized equations lead to

$$\langle u(t),\zeta_n^*\rangle\simeq\langle u(0),\zeta_n^*\rangle e^{\xi_n(\mu)t}e^{i\eta_n(\mu)t},$$

so that if $\xi_n(\mu)<0$, the projection $\langle u(t),\zeta_n^*\rangle$ decays to zero. In fact, for the full nonlinear problem there is a coupling between different projections, and if some of these do not decay, this last result is no longer true. Nevertheless, the important part of the evolution problem (3.1) is related to the part of the spectrum of $f_u(\mu\,|\,\cdot)$ for which $\zeta_n(\mu)\geq 0$.

In the problem of bifurcation studied in this section we shall assume that the real part of two complex-conjugate simple eigenvalues $\sigma(\mu)$, $\bar{\sigma}(\mu)$

changes sign when μ crosses 0 and the remainder of the spectrum stays on the left hand side of the complex plane. Suppose ζ and $\bar{\zeta}^*$ are the eigenvectors of $f_u(\mu\,|\,\cdot), f_u^*(\mu\,|\,\cdot)$ belonging to the eigenvalue $\sigma(\mu)$. Then, the equation governing the evolution of the projection

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle u, \zeta^*\rangle = \sigma(\mu)\langle u, \zeta^*\rangle + \langle N(\mu, u), \zeta^*\rangle, \tag{3.6}$$

is complex-value, that is, two-dimensional. So our problem is essentially two-dimensional whenever

$$u - \langle u, \zeta^*\rangle\zeta - \langle u, \bar{\zeta}^*\rangle\bar{\zeta}$$

is an "extra little part".

Now we shall delineate the sense in which the essentially two-dimensional problem is strictly two-dimensional. We first decompose the bifurcating solution u into a real-valued sum

$$u(t) = a(t)\zeta + \bar{a}(t)\bar{\zeta} + w(t), \tag{3.7}$$

where

$$\langle w, \zeta^*\rangle = \langle \bar{\zeta}, \zeta^*\rangle = \langle \zeta, \zeta^*\rangle - 1 = 0. \tag{3.8}$$

Substituting eq. (3.7) into eq. (3.1), we find, using eq. (3.2) that

$$[\dot{a} - \sigma(\mu)]\zeta + [\dot{\bar{a}} - \bar{\sigma}(\mu)\bar{a}]\bar{\zeta} + \frac{\mathrm{d}w}{\mathrm{d}t} = f_u(\mu\,|\,w) + N(\mu, u). \tag{3.9}$$

Projecting eq. (3.9) with ζ^* leads us to an evolution problem for the "little part" w on a supplementary space of the space spanned by ζ and $\bar{\zeta}$:

$$\frac{\mathrm{d}w}{\mathrm{d}t} = f_u(\mu\,|\,w) + [N(\mu, u) - \langle N(\mu, u), \zeta^*\rangle\zeta - \langle N(\mu, u), \bar{\zeta}^*\rangle\bar{\zeta}]. \tag{3.10}$$

and to an evolution equation for the projected part

$$\dot{a} - \sigma(\mu)a = \langle N(\mu, u), \zeta^*\rangle. \tag{3.11}$$

In deriving eq. (3.11) we made use of the relations

$$\left\langle \frac{\mathrm{d}w}{\mathrm{d}t}, \zeta^* \right\rangle = \frac{\mathrm{d}}{\mathrm{d}t}\langle w, \zeta^*\rangle = 0$$

and

$$\langle f_u(\mu\,|\,w), \zeta^*\rangle = \langle w, f_u^*(\mu\,|\,\zeta^*)\rangle = \sigma\langle w, \zeta^*\rangle = 0.$$

Eq. (3.10) now follows easily from eqs. (3.9) and (3.11).

In sum, eq. (3.11) governs the evolution of the projection of the solution u into the eigensubspace belonging to the eigenvalue $\sigma_1(\mu)=\sigma(\mu)$ and eq. (3.10) governs the evolution of the part of the solution which is orthogonal to the subspace spanned by ζ^* and $\bar{\zeta}^*$.

In bifurcation problems the complementary projection w plays a minor role; it arises only as a response generated by nonlinear coupling to the component of the solution spanned by ζ and ζ. To see this we note that

$$\begin{aligned}
&\langle N(\mu,u),\zeta^*\rangle=\tfrac{1}{2}\langle (f_{uu}(\mu|u|u)+\mathrm{O}(\|u\|^3)),\zeta^*\rangle\\
&\tfrac{1}{2}\langle f_{uu}(\mu|u|u),\zeta^*\rangle=\alpha(\mu)a^2+2\beta(\mu)|a|^2+\gamma(\mu)\bar{a}^2+2a\langle f_{uu}(\mu|\zeta|w),\zeta^*\rangle\\
&\qquad\qquad+2\bar{a}\langle f_{uu}(\mu|\bar{\zeta}|w),\zeta^*\rangle+\langle f_{uu}(\mu|w|w),\zeta^*\rangle,\\
&\alpha(\mu)=\tfrac{1}{2}\langle f_{uu}(\mu|\zeta|\zeta),\zeta^*\rangle,\\
&\beta(\mu)=\tfrac{1}{2}\langle f_{uu}(\mu|\zeta|\bar{\zeta}^*\rangle,\\
&\gamma(\mu)=\tfrac{1}{2}\langle f_{uu}(\mu|\bar{\zeta}|\bar{\zeta}^*\rangle.
\end{aligned}\tag{3.12}$$

It follows that amplitude equation (3.11) may be written as

$$\dot{a}-\sigma(\mu)a=\alpha(\mu)a^2+2\beta(\mu)|a|^2+\gamma(\mu)\bar{a}^2+\mathrm{O}(|a|^3+|a|\,\|w\|+\|w\|^2).\tag{3.13}$$

Returning now to eq. (3.10) with eq. (3.12) we find that after a long time $w=\mathrm{O}(|a|^2)$ and dramatize the two-dimensional structure of Hopf bifurcation in the general case by comparing eq. (3.13) with eq. (2.5) which governs the stability of the strictly two-dimensional problem.

4. Bifurcation from periodic orbits. Normal forms

We consider the equation

$$\mathrm{d}V/\mathrm{d}t=F(t,\mu,V).\tag{4.1}$$

Here $V(t,\mu)$ lies in a real Hilbert space $(H,\langle\ \rangle)$, μ is a real bifurcation parameter, and F is T-periodic i.e., $F(T,\mu,V)=F(t+T,\mu,V)$. Assume that there is a T-periodic solution

$$V=U(t,\mu)=U(t+T,\mu).\tag{4.2}$$

We rewrite eq. (4.1) in local form about U. If $u=V-U$, then

$$\mathrm{d}u/\mathrm{d}t=f(t,\mu,u)$$

where

$$f(t, \mu, u) = F(t, \mu, U+u) - F(t, \mu, U). \tag{4.3}$$

We shall study eq. (4.3) with

$$f(t, \mu, \cdot) = f_u(t, \mu \,|\, \cdot) + N(t, \mu, \cdot),$$

where $f_u(t, \mu \,|\, \cdot)$ is linear and $N(t, \mu, v) = O(\|v\|^2)$.

We assume that the periodic orbit U, that is the orbit $u=0$ of eq. (4.3) is stable if $\mu<0$, and loses stability for $\mu>0$. To express this precisely consider the linearisation of eq. (4.3)

$$dv/dt = f_u(t, \mu \,|\, v). \tag{4.4}$$

This is to be thought of as a complex linear equation (with real coefficients) on $H^{\mathbb{C}}$, the complexification of H. Associated with eq. (4.4) is a linear operator on the space $\mathbb{P}_T^{\mathbb{C}}$ of T-periodic vector fields on $H^{\mathbb{C}}$,

$$J_\mu = -d/dt + f_u(t, \mu \,|\, \cdot) \tag{4.5}$$

Eigenvalues of J_μ are called *Floquet exponents*. The orbit $u=0$ is stable if all Floquet exponents have negative real part, and unstable if any has positive real part. The loss of stability at $\mu=0$ is assumed to occur in the simplest way.

Bifurcation assumptions:

There is a Floquet exponent $\sigma(\mu) = \xi(\mu) + i\eta(\mu)$ such that

(i) $\sigma(0) = i\omega_0 = 2\pi r/T$, $0 \le r < 1$.

(ii) $\sigma(\mu)$ and $\bar{\sigma}(\mu)$ are isolated algebraically simple eigenvalues of J_μ.

(iii) $d\xi/d\mu(0) > 0$.

(iv) all eigenvalues of J_0 other than $\sigma(0)$ and $\bar{\sigma}(0)$ have negative real part.

The type of bifurcation that occurs depends on the value of r.

(i) Strong resonance: if $r = m/n$ and $n = 1, 2, 3$, or $n=4$ and a certain inequality holds then nT-periodic solutions bifurcate.

(ii) Wan [6] has shown that there is an invariant torus when $n=4$ and the inequality does not hold.

(iii) Weak resonance: If $r = m/n$, $n \ge 5$, and certain exceptional conditions hold then nT-periodic solutions bifurcate.

(iv) If $r \ne m/n$, $n = 1, 2, 3, 4$ there is a Hopf bifurcation to an invariant torus.

The next section describes how to approximate the original problem (4.3) with an autonomous equation in $\mathbb{R}^2$. It should be mentioned that the asymptotic representations can be constructed directly, without normal forms, by methods of applied analysis (see appendices to chapter X in ref. [4]).

4.1. Derivation of the autonomous equation

We assume that $r \neq 0, \frac{1}{2}$ (see refs. [3-5] for a study of these cases). This means that the periodic orbit $u=0$ loses stability in two real dimensions instead of just one. The first step is to decompose eq. (4.3) into a part in this plane and a complementary part.

There is an inner product on $\mathbb{P}_T^{\mathbb{C}}$,

$$[\xi_1, \xi_2] = \frac{1}{T}\int_0^T \langle \xi_1(t), \xi_2(t) \rangle \, \mathrm{d}t.$$

Let J_μ^* be the adjoint of J_μ with respect to $[\cdot, \cdot]$. It can be verified that

$$J_\mu^* = \mathrm{d}/\mathrm{d}t + f_u^*(t, \mu \,|\, \cdot), \tag{4.6}$$

where $f^*(t, \mu \,|\, \cdot)$ is the adjoint of $f(t, \mu \,|\, \cdot)$ with respect to $\langle\, , \rangle$. Now $\sigma(\mu), \bar{\sigma}(\mu)$ are eigenvalues of J_μ, J_μ^* respectively; let ξ_μ, ξ_μ^* be corresponding eigenfunctions. Using eq. (4.6) and the assumption that $r \neq 0, \frac{1}{2}$, one can show that

$$\langle \xi_\mu(t), \xi_\mu^*(t) \rangle \equiv \langle \xi_\mu(0), \xi_\mu^*(0) \rangle$$

$$\langle \bar{\xi}_\mu(t), \xi_\mu^*(t) \rangle \equiv 0.$$

Normalise ξ_μ, ξ_μ^* so $\langle \xi_\mu, \xi_\mu^* \rangle = 1$. Now we can write

$$u = z\xi_\mu + \bar{z}\bar{\xi}_\mu + W,$$

where $z = \langle u, \xi_\mu^* \rangle$ and W is real. Eq. (4.3) becomes

$$\mathrm{d}z/\mathrm{d}t = \sigma(\mu)z + b, \tag{4.7a}$$

$$\mathrm{d}W/\mathrm{d}t = f_u(t, \mu \,|\, W) + B, \tag{4.7b}$$

where

$$b(t, \mu, z, \bar{z}, W) = \langle N(t, \mu, u), \xi_\mu^*(t) \rangle,$$

$$B(t, \mu, z, \bar{z}, W) = N(t, \mu, u) - \langle N(t, \mu, u), \xi_\mu^* \rangle \xi_\mu - \langle N(t, \mu, u), \bar{\xi}_\mu^* \rangle \bar{\xi}_\mu.$$

We have $b=b_0+b_1$, $B=B_0+B_1$, where $b_0=b(t,\mu,z,\bar{z},0)$, $b_1=O(|z|\,\|W\|+\|W\|^2)$ $B_0=B(t,\mu,z,\bar{z},0)B_1=O(|z|\,\|W\|+\|W\|^2)$.

Roughly speaking eq. (4.7b) will be eliminated and eq. (4.7a) made autonomous up to $0(|z|^{N+1})$. To do this we change variables

$$y=z+\gamma(t,\mu,z,\bar{z})=z+\sum_{p+q\geq 2}^{N} z^p\bar{z}^q\gamma_{pq}(t,\mu),$$

$$Y=W+\Gamma(t,\mu,z,\bar{z})=W+\sum_{p+q\geq 2}^{N} z^p\bar{z}^q\Gamma_{pq}(t,\mu), \tag{4.8}$$

where N is arbitrary, γ_{pq} and Γ_{pq} are T-periodic, and $\Gamma_{pq}\perp\xi^*_\mu,\bar{\xi}^*_\mu$. We chose γ_{pq}, Γ_{pq} later, after eq. (4.7) has been rewritten in terms of y, Y. Now

$$\frac{dy}{dt}=\sigma z+b+\frac{\partial\gamma}{\partial t}+\frac{\partial\gamma}{\partial z}(\sigma z+b)+\frac{\partial\gamma}{\partial\bar{z}}(\bar{\sigma}\bar{z}+\bar{b})$$

$$=\sigma y+\left(\frac{\partial\gamma}{\partial t}+\sigma z\frac{\partial\gamma}{\partial z}+\bar{\sigma}\bar{z}\frac{\partial\gamma}{\partial\bar{z}}-\sigma\gamma+\tilde{b}\right)+b_1\left(1+\frac{\partial\gamma}{\partial z}\right)+\bar{b}_1\frac{\partial\gamma}{\partial\bar{z}},$$

where

$$\tilde{b}(t,\mu,z,z)=b_0\left(1+\frac{\partial\gamma}{\partial z}\right)+b_0\frac{\partial\gamma}{\partial\bar{z}};$$

$$\frac{dY}{dt}=f_u(t,\mu\,|\,Y)+\left(\frac{\partial\Gamma}{\partial t}+\sigma z\frac{\partial\Gamma}{\partial z}+\bar{\sigma}\bar{z}\frac{\partial\Gamma}{\partial z}-f_u(t,\mu\,|\,\Gamma)+\tilde{B}\right)$$

$$+B_1\left(1+\frac{\partial\Gamma}{\partial z}\right)+\bar{B}_1\frac{\partial\Gamma}{\partial\bar{z}},$$

where

$$\tilde{B}(t,\mu,z,\bar{z})=B_0\left(1+\frac{\partial\Gamma}{\partial z}\right)+\bar{B}_0\frac{\partial\Gamma}{\partial\bar{z}}.$$

Expand

$$\begin{matrix}\tilde{b}\\ \tilde{B}\end{matrix}=\sum_{p+q\geq 2}^{N}\begin{matrix}\tilde{b}_{pq}\\ \tilde{B}_{pq}\end{matrix}(t,\mu)z^p\bar{z}^q+O(|z|^{N+1}),$$

where $\tilde{b}_{pq},\tilde{B}_{pq}$ are T periodic and $\tilde{B}_{pq}\perp\xi^*,\bar{\xi}^*$. Then

$$\frac{dy}{dt}=\sigma y+\sum_{p+q\geq 2}^{N}\left(\frac{\partial\gamma_{pq}}{\partial t}+[\sigma(p-1)+\bar{\sigma}q]\gamma_{pq}+\tilde{b}_{pq}\right)z^p\bar{z}^q$$

$$+O(|z|\,\|w\|+\|w\|^2+|z|^{N+1}),$$

$$\frac{dY}{dt}=f_u(t,\mu\mid Y)+\sum_{p+q\geq 2}\left(\frac{\partial\Gamma_{pq}}{\partial t}-f_u(t,\mu\mid\Gamma_{pq})+[\sigma p+\bar{\sigma}q]\Gamma_{pq}+\tilde{B}_{pq}\right)z^p\bar{z}^q$$
$$+\mathrm{O}(|z|\,\|w\|+\|w\|^2+|z|^{N+1}).$$

Finally use eq. (4.8) on the right hand side to get

$$\frac{dy}{dt}=\sigma y+\sum_{p+q\geq 2}\left(\frac{\partial\gamma_{pq}}{\partial t}+[\sigma(p-1)+\bar{\sigma}q]\gamma+\bar{\bar{b}}_{pq}\right)y^p\bar{y}^p$$
$$+\mathrm{O}(|y|\,\|Y\|+\|Y\|^2+|y|^{N+1}), \tag{4.9a}$$

$$\frac{dY}{dt}=f_u(t,\mu\mid Y)+\sum_{p+q\geq 2}^{N}\{-J_\mu(\Gamma_{pq})+[\sigma p+\bar{\sigma}q]\Gamma_{pq}+\bar{\bar{B}}_{pq}\}$$
$$+\mathrm{O}(|y|\,\|Y\|+\|Y\|^2+|y|^{N+1}), \tag{4.9b}$$

where $\bar{\bar{b}}_{pq}$ and $\bar{\bar{B}}_{pq}$ are functions of γ_{ij}, Γ_{ij} with $i+j<p+q$ with T-periodic coefficients and such that all terms in eq. (4.9b) are orthogonal to $\xi^*_\mu, \bar{\xi}^*_\mu$.

Now γ_{pq}, Γ_{pq} are chosen successively for $p+q=2,3,\ldots,N$ so as to simplify eq. (4.9). This is the key step. We choose Γ_{pq} to make $-J_\mu(\Gamma_{pq})+[\sigma p+\bar{\sigma}q]+\bar{\bar{B}}_{pq}\equiv 0$ for small μ. This is always possible since $\Gamma_{pq}, \bar{\bar{B}}_{pq}\in\{\xi\in\mathbb{P}_T:\xi\perp\xi^*_\mu,\bar{\xi}^*_\mu\}$ and the bifurcation assumptions mean that for small μ none of the eigenvalues of J_μ on this space has real part as small as $\mathrm{Re}(\sigma p+\bar{\sigma}q)$. This reduces eq. (4.9b) to

$$\frac{dY}{dt}=f_u(t,\mu\mid Y)+\mathrm{O}(\|Y\|\,|y|+\|Y\|^2+|y|^{N+1}). \tag{4.10a}$$

In order to choose γ_{pq}, write

$$\bar{\bar{b}}_{pq}(t,\mu)=\sum_{l\in\mathbb{R}} b_{pql}(\mu)\exp(2\pi i l t)/T,$$

$$\gamma_{pq}(t,\mu)=\sum_{l\in\mathbb{R}}\gamma_{pql}(\mu)\exp(2\pi i l t)/T.$$

Then

$$\frac{\partial\gamma_{pq}}{\partial t}+[\sigma(p-1)+\sigma q]\gamma_{pq}+\bar{\bar{b}}_{pq}=\sum_{l\in\mathbb{Z}}\alpha_{pql}(\mu)\exp(2\pi i l t)/T,$$

where

$$\alpha_{pql}(\mu)=\left(\frac{2\pi i l}{\mathrm{T}}+[\sigma(p-1)+\bar{\sigma}q]\right)\gamma_{pql}(\mu)+b_{pql}(\mu),$$

$$\alpha_{pql}(0)=\frac{2\pi i}{T}\,\{l+r[p-1-q]\}\gamma_{pql}(0)+b_{pql}(0).$$

$$\frac{\mathrm{d}Y}{\mathrm{d}t} = f_u(t,\mu \mid Y) + \sum_{p+q\geq 2}\left(\frac{\partial \Gamma_{pq}}{\partial t} - f_u(t,\mu \mid \Gamma_{pq}) + [\sigma p + \bar{\sigma} q]\Gamma_{pq} + \tilde{B}_{pq}\right) z^p \bar{z}^q$$
$$+ \mathrm{O}(|z|\,\|w\| + \|w\|^2 + |z|^{N+1}).$$

Finally use eq. (4.8) on the right hand side to get

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \sigma y + \sum_{p+q\geq 2}\left(\frac{\partial \gamma_{pq}}{\partial t} + [\sigma(p-1) + \bar{\sigma} q]\gamma + \bar{\bar{b}}_{pq}\right) y^p \bar{y}^p$$
$$+ \mathrm{O}(|y|\,\|Y\| + \|Y\|^2 + |y|^{N+1}), \tag{4.9a}$$

$$\frac{\mathrm{d}Y}{\mathrm{d}t} = f_u(t,\mu \mid Y) + \sum_{p+q\geq 2}^{N} \{-J_\mu(\Gamma_{pq}) + [\sigma p + \bar{\sigma} q]\Gamma_{pq} + \bar{\bar{B}}_{pq}\}$$
$$+ \mathrm{O}(|y|\,\|Y\| + \|Y\|^2 + |y|^{N+1}), \tag{4.9b}$$

where $\bar{\bar{b}}_{pq}$ and $\bar{\bar{B}}_{pq}$ are functions of γ_{ij}, Γ_{ij} with $i+j<p+q$ with T-periodic coefficients and such that all terms in eq. (4.9b) are orthogonal to $\xi^*_\mu, \bar{\xi}^*_\mu$.

Now γ_{pq}, Γ_{pq} are chosen successively for $p+q=2,3,\dots,N$ so as to simplify eq. (4.9). This is the key step. We choose Γ_{pq} to make $-J_\mu(\Gamma_{pq}) + [\sigma p + \bar{\sigma} q] + \bar{\bar{B}}_{pq} \equiv 0$ for small μ. This is always possible since $\Gamma_{pq}, \bar{\bar{B}}_{pq} \in \{\xi \in \mathbb{P}_T : \xi \perp \xi^*_\mu, \bar{\xi}^*_\mu\}$ and the bifurcation assumptions mean that for small μ none of the eigenvalues of J_μ on this space has real part as small as $\mathrm{Re}(\sigma p + \bar{\sigma} q)$. This reduces eq. (4.9b) to

$$\frac{\mathrm{d}Y}{\mathrm{d}t} = f_u(t,\mu \mid Y) + \mathrm{O}(\|Y\|\,|y| + \|Y\|^2 + |y|^{N+1}). \tag{4.10a}$$

In order to choose γ_{pq}, write

$$\bar{\bar{b}}_{pq}(t,\mu) = \sum_{l\in\mathbb{R}} b_{pql}(\mu)\exp(2\pi\mathrm{i}lt)/T,$$

$$\gamma_{pq}(t,\mu) = \sum_{l\in\mathbb{R}} \gamma_{pql}(\mu)\exp(2\pi\mathrm{i}lt)/T.$$

Then

$$\frac{\partial \gamma_{pq}}{\partial t} + [\sigma(p-1) + \sigma q]\gamma_{pq} + \bar{\bar{b}}_{pq} = \sum_{l\in\mathbb{Z}} \alpha_{pql}(\mu)\exp(2\pi\mathrm{i}lt)/T,$$

where

$$\alpha_{pql}(\mu) = \left(\frac{2\pi\mathrm{i}l}{\mathrm{T}} + [\sigma(p-1) + \bar{\sigma} q]\right)\gamma_{pql}(\mu) + b_{pql}(\mu),$$

$$\alpha_{pql}(0) = \frac{2\pi\mathrm{i}}{T}\,\{l + r[p-1-q]\}\gamma_{pql}(0) + b_{pql}(0).$$

We have $b=b_0+b_1$, $B=B_0+B_1$, where $b_0=b(t,\mu,z,\bar{z},0)$, $b_1=O(|z|\,\|W\|+\|W\|^2)$ $B_0=B(t,\mu,z,\bar{z},0)B_1=O(|z|\,\|W\|+\|W\|^2)$.

Roughly speaking eq. (4.7b) will be eliminated and eq. (4.7a) made autonomous up to $0(|z|^{N+1})$. To do this we change variables

$$y=z+\gamma(t,\mu,z,\bar{z})=z+\sum_{p+q\geq 2}^{N} z^p\bar{z}^q\gamma_{pq}(t,\mu),$$

$$Y=W+\Gamma(t,\mu,z,\bar{z})=W+\sum_{p+q\geq 2}^{N} z^p\bar{z}^q\Gamma_{pq}(t,\mu), \tag{4.8}$$

where N is arbitrary, γ_{pq} and Γ_{pq} are T-periodic, and $\Gamma_{pq}\perp\xi_\mu^*,\bar{\xi}_\mu^*$. We chose γ_{pq}, Γ_{pq} later, after eq. (4.7) has been rewritten in terms of y, Y. Now

$$\frac{dy}{dt}=\sigma z+b+\frac{\partial\gamma}{\partial t}+\frac{\partial\gamma}{\partial z}(\sigma z+b)+\frac{\partial\gamma}{\partial\bar{z}}(\bar{\sigma}\bar{z}+\bar{b})$$

$$=\sigma y+\left(\frac{\partial\gamma}{\partial t}+\sigma z\frac{\partial\gamma}{\partial z}+\bar{\sigma}\bar{z}\frac{\partial\gamma}{\partial\bar{z}}-\sigma\gamma+\tilde{b}\right)+b_1\left(1+\frac{\partial\gamma}{\partial z}\right)+\bar{b}_1\frac{\partial\gamma}{\partial\bar{z}},$$

where

$$\tilde{b}(t,\mu,z,z)=b_0\left(1+\frac{\partial\gamma}{\partial z}\right)+b_0\frac{\partial\gamma}{\partial\bar{z}};$$

$$\frac{dY}{dt}=f_u(t,\mu\,|\,Y)+\left(\frac{\partial\Gamma}{\partial t}+\sigma z\frac{\partial\Gamma}{\partial z}+\bar{\sigma}\bar{z}\frac{\partial\Gamma}{\partial z}-f_u(t,\mu\,|\,\Gamma)+\tilde{B}\right)$$

$$+B_1\left(1+\frac{\partial\Gamma}{\partial z}\right)+\bar{B}_1\frac{\partial\Gamma}{\partial\bar{z}},$$

where

$$\tilde{B}(t,\mu,z,\bar{z})=B_0\left(1+\frac{\partial\Gamma}{\partial z}\right)+\bar{B}_0\frac{\partial\Gamma}{\partial\bar{z}}.$$

Expand

$$\begin{matrix}\tilde{b}\\ \tilde{B}\end{matrix}=\sum_{p+q\geq 2}^{N}\begin{matrix}\tilde{b}_{pq}\\ \tilde{B}_{pq}\end{matrix}(t,\mu)z^p\bar{z}^q+O(|z|^{N+1}),$$

where $\tilde{b}_{pq},\tilde{B}_{pq}$ are T periodic and $\tilde{B}_{pq}\perp\xi^*,\bar{\xi}^*$. Then

$$\frac{dy}{dt}=\sigma y+\sum_{p+q\geq 2}^{N}\left(\frac{\partial\gamma_{pq}}{\partial t}+[\sigma(p-1)+\bar{\sigma}q]\gamma_{pq}+\tilde{b}_{pq}\right)z^p\bar{z}^q$$

$$+O(|z|\,\|w\|+\|w\|^2+|z|^{N+1}),$$

$$\frac{\mathrm{d}Y}{\mathrm{d}t}=f_u(t,\mu\mid Y)+\sum_{p+q\geq 2}\left(\frac{\partial \Gamma_{pq}}{\partial t}-f_u(t,\mu\mid\Gamma_{pq})+[\sigma p+\bar{\sigma}q]\Gamma_{pq}+\tilde{B}_{pq}\right)z^p\bar{z}^q$$
$$+\mathrm{O}(|z|\,\|w\|+\|w\|^2+|z|^{N+1}).$$

Finally use eq. (4.8) on the right hand side to get

$$\frac{\mathrm{d}y}{\mathrm{d}t}=\sigma y+\sum_{p+q\geq 2}\left(\frac{\partial \gamma_{pq}}{\partial t}+[\sigma(p-1)+\bar{\sigma}q]\gamma+\bar{\bar{b}}_{pq}\right)y^p\bar{y}^p$$
$$+\mathrm{O}(|y|\,\|Y\|+\|Y\|^2+|y|^{N+1}), \tag{4.9a}$$
$$\frac{\mathrm{d}Y}{\mathrm{d}t}=f_u(t,\mu\mid Y)+\sum_{p+q\geq 2}^{N}\{-J_\mu(\Gamma_{pq})+[\sigma p+\bar{\sigma}q]\Gamma_{pq}+\bar{\bar{B}}_{pq}\}$$
$$+\mathrm{O}(|y|\,\|Y\|+\|Y\|^2+|y|^{N+1}), \tag{4.9b}$$

where $\bar{\bar{b}}_{pq}$ and $\bar{\bar{B}}_{pq}$ are functions of γ_{ij}, Γ_{ij} with $i+j<p+q$ with T-periodic coefficients and such that all terms in eq. (4.9b) are orthogonal to $\zeta_\mu^*, \bar{\zeta}_\mu^*$.

Now γ_{pq}, Γ_{pq} are chosen successively for $p+q=2,3,\ldots,N$ so as to simplify eq. (4.9). This is the key step. We choose Γ_{pq} to make $-J_\mu(\Gamma_{pq})+[\sigma p+\bar{\sigma}q]+\bar{\bar{B}}_{pq}\equiv 0$ for small μ. This is always possible since $\Gamma_{pq}, \bar{\bar{B}}_{pq}\in\{\xi\in\mathbb{P}_T:\xi\perp\zeta_\mu^*,\bar{\zeta}_\mu^*\}$ and the bifurcation assumptions mean that for small μ none of the eigenvalues of J_μ on this space has real part as small as $\operatorname{Re}(\sigma p+\bar{\sigma}q)$. This reduces eq. (4.9b) to

$$\frac{\mathrm{d}Y}{\mathrm{d}t}=f_u(t,\mu\mid Y)+\mathrm{O}(\|Y\|\,|y|+\|Y\|^2+|y|^{N+1}). \tag{4.10a}$$

In order to choose γ_{pq}, write

$$\bar{\bar{b}}_{pq}(t,\mu)=\sum_{l\in\mathbb{R}}b_{pql}(\mu)\exp(2\pi\mathrm{i}lt)/T,$$
$$\gamma_{pq}(t,\mu)=\sum_{l\in\mathbb{R}}\gamma_{pql}(\mu)\exp(2\pi\mathrm{i}lt)/T.$$

Then

$$\frac{\partial\gamma_{pq}}{\partial t}+[\sigma(p-1)+\sigma q]\gamma_{pq}+\bar{\bar{b}}_{pq}=\sum_{l\in\mathbb{Z}}\alpha_{pql}(\mu)\exp(2\pi\mathrm{i}lt)/T,$$

where

$$\alpha_{pql}(\mu)=\left(\frac{2\pi\mathrm{i}l}{\mathrm{T}}+[\sigma(p-1)+\bar{\sigma}q]\right)\gamma_{pql}(\mu)+b_{pql}(\mu),$$
$$\alpha_{pql}(0)=\frac{2\pi\mathrm{i}}{T}\{l+r[p-1-q]\}\gamma_{pql}(0)+b_{pql}(0).$$

We see that we can always choose γ_{pql} to make $\alpha_{pql}(\mu)\equiv 0$ for small μ unless $l+r[p-1-q]=0$. We call $\{(p,q,l,r): l+r[p-l-q]=0\}$ the *Exceptional Set*. It is the union of two disjoint subsets:

I the mean set: $(p,q,l,r)=(q+1,q,0,r) \qquad 2\le 2q+1\le N,$

II the resonant set: $(p,q,l,r)=(q+1+nk,q,-km,m/n)$

$$0\le m<n,\ k\ge 1,\ 2\le 2q+1+nk\le N.$$

The mean set is present for any r, but the resonant set arises only when r is rational.

When (p,q,l,r) *is in the exceptional set choose* $\gamma_{pql}(\mu)\equiv 0$; *otherwise choose* $\gamma_{pql}(\mu)$ *to make* $\alpha_{pql}(\mu)\equiv 0$. This reduces eq. (4.9a) to

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}t} &= \sigma(\mu)y+\sum_{q\ge 1}^{2q+1\le N} y^{q+1}\bar{y}^{q}b_{q+1,q,0}(\mu)\\ &+\sum_{k>0}\sum_{q\ge 0}^{2q-1+nk\le N}[y^{q+1+nk}\bar{y}^{q}b_{q+1+nk,q,-mk}(\mu)\mathrm{e}^{-2\pi imkt/T}\\ &+y^{q}\bar{y}^{q-1+nk}b_{q,q-1+nk,mk}(\mu)\mathrm{e}^{2\pi imkt/T}]\\ &+\mathrm{O}(|y|\,\|Y\|+\|Y\|^{2}+|y|^{N+1}).\end{aligned} \tag{4.10b}$$

The asymptotic representation is obtained by neglecting the order terms in eqs. (10a, b). *The truncation number* N *in eq.* (4.10b) *is arbitrary*. The justification of this approximation will not be attempted here; see refs [3–5]. We proceed to study the approximate problem.

It is clear that eq. (4.10a) gives $Y(t,\mu)\equiv 0$. To study eq. (10b) set

$$y=x\mathrm{e}^{i\omega_0 t}. \tag{4.11}$$

Substitution in eq. (4.10b) gives an autonomous equation of the form

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= \mu\hat{\sigma}(\mu)x+\sum_{q\ge 1}^{2q+1<N} +x|x|^{2q}a_q(\mu)\\ &+\sum_{k>0}\sum_{q\ge 0}|x|^{2q}\{x^{l+nk}a_{qk}+\bar{x}^{nk-1}a_{q,-k}\},\end{aligned}$$

where $\mu\hat{\sigma}(\mu)=\sigma(\mu)-\sigma(0)$ and $a_{q,k}(\mu)\equiv 0$ if r is irrational.

We shall look for the equilibrium solutions of eq. (4.12). We expect to find fixed points and closed curves. These will be cross sections of subharmonic trajectories and invariant tori for the original problem. The type of solution will depend on which terms on the right hand side of eq. (4.12) have lowest order in x after $\mu\hat{\sigma}(\mu)x$. If $n=3$ the term from the resonant set

$a_{0,-1}\bar{x}^{n-1}$ is the only term of order 2, and we shall find fixed points for eq. (4.12). If $n=4$, $a_{0,1}\bar{x}^{n-1}$ from the resonant set and $a_1 x|x|^2$ from the mean set both have order 3, and either fixed points or an invariant circle can occur. If $n \geq 5$ then terms from the mean set have lower order, and we expect a closed orbit of eq. (4.12). Normally it is traversed at a speed $O(\varepsilon^2)$, but if enough exceptional conditions hold this speed can be so low that the terms from the resonant set break up the closed oribt into fixed points. This is weak resonance.

All of the above remarks assume that various terms are $\neq 0$. The exceptional cases where this is not true are ignored here. Also it will be assumed for simplicity that $\hat{\sigma}, a_q, a_{q,k}, a_{q,-k} \cdots$ are independent of μ. This does not change the essence of the arguments.

5. Bifurcation from periodic solutions. Hopf bifurcation into a torus of subharmonic and asymptotically quasiperiodic solutions

This section outlines how to compute the trajectories on the torus when $n \geq 5$. We introduce an amplitude which is the mean radius of the invariant circle,

$$\varepsilon = \frac{1}{2\pi} \int_0^{2\pi} x(s) e^{-is} \, ds.$$

We assume the orbit can be written in the form

$$\begin{aligned} x(t, \mu) &= \varepsilon \, e^{is} \chi(s, \varepsilon), \\ \mu &= \varepsilon \tilde{\mu}(\varepsilon), \\ s &= \varepsilon \Omega(\varepsilon) t, \end{aligned} \tag{5.1}$$

where χ is 2π periodic in θ. Note that $2\pi/\varepsilon\Omega(\varepsilon)$ is the period of the closed orbit of eq. (4.12).

Substitution in eq. (4.12) gives

$$\begin{aligned} (i\Omega - \tilde{\mu}\hat{\sigma})\chi + \Omega \frac{d\chi}{ds} &= \sum \chi |\chi|^{2q} a_q \varepsilon^{2q-1} \\ &\quad + \sum_{k>0} \sum_{q \geq 0} |\chi|^{2q} [a_{q,k} e^{ink\theta} \chi^{1+nk} \varepsilon^{2q+nk-1} \\ &\quad + a_{q,-k} e^{-ink\theta} \chi^{nk-1} \varepsilon^{2q+nk-3}]. \end{aligned} \tag{5.2}$$

Expand in powers of ε:

$$\chi(s,\varepsilon)=\sum_{j=0}^{\infty}\chi_j(s)\varepsilon^j,$$

$$\tilde{\mu}(\varepsilon)=\sum_{j=0}^{\infty}\tilde{\mu}_j\varepsilon^j,$$

$$\Omega(\varepsilon)=\sum_{j=0}^{\infty}\Omega_j\varepsilon^j. \tag{5.3}$$

The functions $\chi_j(\cdot)$ are 2π-periodic; and

$$1=\frac{1}{2\pi}\int_0^{2\pi}\chi(s,\varepsilon)\,\mathrm{d}s,$$

so

$$\frac{1}{2\pi}\int_0^{2\pi}\chi_j(s,\varepsilon)\,\mathrm{d}s=1 \qquad j=0,$$
$$=0 \qquad j\geq 1. \tag{5.4}$$

We now solve by evaluating coefficients of successive powers of ε. From the terms of order 0,

$$(\mathrm{i}\Omega_0-\tilde{\mu}_0\hat{\sigma})\chi_0+\Omega_0\frac{\mathrm{d}\chi_0}{\mathrm{d}s}=0.$$

Taking the mean over $(0,2\pi)$ gives

$$\mathrm{i}\Omega_0=\tilde{\mu}_0\hat{\sigma}.$$

Now it follows from the bifurcation assumptions that $\hat{\sigma}$ has positive real part. Hence, since Ω_0 and $\tilde{\mu}_0$ are both real,

$$\Omega_0=\tilde{\mu}_0=0.$$

The terms of order 1 in ε now give

$$(\mathrm{i}\Omega_1-\tilde{\mu}_1\hat{\sigma})\chi_0+\Omega_1\frac{\mathrm{d}\chi_0}{\mathrm{d}s}=|\chi_0|^2\chi_0 a_1. \tag{5.5}$$

Taking the mean over $(0,2\pi)$ gives

$$\mathrm{i}\Omega_1-\tilde{\mu}_1\hat{\sigma}=a_1\frac{1}{2\pi}\int_0^{2\pi}|\chi_0|^2\chi_0\,\mathrm{d}s. \tag{5.6}$$

It can be shown from eqs. (5.4), (5.5) and (5.6) that $\chi_0(s)\equiv 1$. From eq.

(5.13) we now obtain

$$\mathrm{i}\Omega_1 - \tilde{\mu}_1\hat{\sigma} = a_1.$$

Taking real and imaginary parts gives

$$\tilde{\mu}_1\hat{\xi} + \alpha_1 = 0$$

$$\Omega_1 - \tilde{\mu}_1\hat{\eta} = \beta_1.$$

To continue we have to assume that $\Omega_1 \neq 0$. It will be seen in the next section that $\Omega_1 = 0$ is the first of the special conditions leading to weak resonance.

The terms in ε^2 give

$$\Omega_1 \frac{\mathrm{d}\chi_1}{\mathrm{d}s} - a_1(\chi_1 + \bar{\chi}_1) = g_1(s) - (\mathrm{i}\Omega_2 - \tilde{\mu}_2\hat{\sigma})$$

where

$$g_1(s) = a_{0,-1}\mathrm{e}^{-5\mathrm{i}s} \qquad n = 5$$

$$= 0 \qquad n \geq 5.$$

We see from eq. (5.4) that we must have

$$\int_0^{2\pi} [g_1(s) - (\mathrm{i}\Omega_2 - \tilde{\mu}_2\hat{\sigma})]\,\mathrm{d}s = 0.$$

This is true if and only if

$$\Omega_2 = \tilde{\mu}_2 = 0.$$

It is easily shown using Fourier series that the equation

$$\Omega_1 \frac{\mathrm{d}y}{\mathrm{d}s} - a_1(y + \bar{y}) = \hat{g}(s)$$

where $\hat{g}$ is 2π-periodic and $\int_0^{2\pi} \hat{g}(s)\,\mathrm{d}s = 0$ has a unique 2π-periodic solution. We see that

$$\chi_1/s) = A\mathrm{e}^{5\mathrm{i}s} + B\mathrm{e}^{-5\mathrm{i}s} \qquad n = 5$$

$$= 0 \qquad n \geq 5.$$

The analysis continues along these lines. It is found that $\tilde{\mu}(\cdot)$ and $\Omega(\cdot)$ are both odd functions, and the $\chi(\cdot, \varepsilon)$ is $2\pi/n$ periodic (constant if r is irrational). This is to be expected since eq. (4.12) is invariant under rotation through $2\pi/n$. By tracing back through the derivation in section 2, we see that our approximate solution is quasi-periodic with the two frequencies $2\pi/T$ and $\omega_0+\varepsilon^2\omega(\varepsilon^2)=\omega_0+\Omega_1+\varepsilon^2\Omega_3+\cdots$.

5.1. Subharmonic bifurcation

Suppose $x=\delta e^{e\varphi(\delta)}$ is a steady solution of eq. (4.12). Note that $\delta\exp[i\varphi(\delta)]\exp(2\pi ik/n)$, $0\le k\le n-1$, are all steady solutions of eq. (4.12). They are the n-piercing points of a single nT-periodic trajectory. We have

$$0=\mu\hat{\sigma}+\delta^2a_1+\delta^4a_2+\cdots+\delta^{n-2}e^{-in\varphi}a_{0,-1}+\cdots.$$

Assume

$$\varphi(\delta)=\varphi_0+\varphi_1\delta+\varphi_2\delta^2+\cdots$$
$$\mu=\mu^{(1)}\delta+\delta^{(2)}\delta^2+\cdots.$$

We evaluate the coefficients of increasing powers of δ.

For $n=3$: the terms in δ give

$$\mu^{(1)}\hat{\sigma}+a_{0,-1}e^{-3i\varphi_0}=0.$$

Hence

$$\mu^{(1)}=|a_{0,-1}/\hat{\sigma}|,$$
$$\varphi_0=\tfrac{1}{3}\arg(a_{0,-1}/\hat{\sigma})+\frac{2k-1}{3}\qquad k=0,1,2$$

(taking $\mu^{(1)}=-|a_{0,-1}/\hat{\sigma}|$ will give the same solution). The higher order terms can now be calculated. We obtain a single $3T$-periodic trajectory. The bifurcation is two sided since $\mu(\delta)=0(\delta)$.

If $n\ge 4$: the terms in δ give

$$\mu^{(1)}=0.$$

For $n=4$ the terms in δ^2 give

$$\mu^{(2)}\hat{\sigma}+a_1+e^{-4i\varphi_0}a_{0,-1}=0,$$

so

$$|\mu^{(2)}\hat{\sigma}+a_1|^2=|a_{0,-1}|^2.$$

This gives a quadratic equation for $\mu^{(2)}$. If the discriminant is positive we have two different values of $\mu^{(2)}$ which lead to two different $4T$-periodic trajectories.

If $n \geq 5$: the terms in δ^2 give

$$\mu^{(2)}\hat{\sigma} + a_1 = 0.$$

This is the first special condition for weak resonance; the requirement that $\mu^{(2)}$ be real restricts $\hat{\sigma}$ and a_1. It can be verified that this restriction is equivalent to the requirement that $\Omega_1 = 0$ which was used in section 3.

For $n = 5$: the terms in δ^3 give

$$\mu^{(3)}\hat{\sigma} + a_{0,-1}e^{-5i\varphi_0} = 0.$$

This determines $\mu^{(3)}$ and φ_0. Higher order terms can then be calculated. Since $\mu(\delta) = O(\delta^2)$ the bifurcation is one sided. Since $\mu^{(3)} \neq 0$, $\mu(\delta)$ is not even, and we obtain two $5T$-periodic trajectories.

If $n \geq 6$: the terms in δ^3 give

$$\mu^{(3)} = 0.$$

For $n = 6$: the terms in δ^4 give

$$\mu^{(4)}\hat{\sigma} + a_2 + a_{0,-1}e^{-6i\varphi_0} = 0.$$

This gives a quadratic equation for $\mu^{(4)}$; if the discriminant is positive two $6T$-periodic trajectories bifurcate.

If $n \geq 7$: the terms in δ^4 give

$$\mu^{(4)}\hat{\sigma}_0 + a_2 = 0.$$

This is the second special condition for weak resonance.

The results continue along these lines. As n increases subharmonic trajectories are possible only if more and more special conditions hold.

5.2. Rotation number and lock-ins

We conclude with a few remarks about the phenomenon of frequency locking when there is an invariant torus. This occurs when all the trajectories on the torus are captured by a single (subharmonic) trajectory.

Consider the Poincaré (first return) map. This is a map from the invariant circle to itself, this map takes a point on the circle to where the trajectory passing through it meets the circle again after going round the torus once (i.e. after time T). Consider its rotation number, ϱ (defined for example in ref. [4]; the reader may think of ϱ as a frequency ratio). If ϱ

is irrational there is a change of coordinates which makes the Poincaré map a rotation, and the flow on the torus is quadiperiodic. The Poincaré map has no periodic points. If $\varrho = p/q$ is rational, the Poincaré map must have periodic points of order q, to which correspond subharmonic trajectories. Generally there will be two such trajectories one attracting, the other repelling.

It is important to distinguish between the rotation number $\hat{\varrho}(\varepsilon)$ for the asymptotic representation and the rotation number $\varrho(\varepsilon)$ for the real flow. It is known that $\varrho(\varepsilon)$ is continuous but is is generally not differentiable. What happens is that if $\varrho(\varepsilon_0) = p/q$ then $\varrho(\varepsilon) \equiv p/q$ on an interval about ε_0. The rotation number locks on to each rational value. This happens because if θ_0 is a periodic point of order q of the Poincaré map, f_{ε_0}, then generically $\partial/\partial\theta(f_\varepsilon^q)|_{\varepsilon=\varepsilon_0, \theta=\theta_0} \neq 0$. This enables us to solve for a fixed point of f_ε^q when ε is near ε_0, so $\varrho(\varepsilon)$ cannot change near ε_0.

In particular the set of values of ε for which $\varrho(\varepsilon)$ is rational has positive measure. It is an important result of Herman [2] that the set on which $\varrho(\varepsilon)$ is irrational also has positive measure.

The results show that the approximate rotation number is of the form

$$\hat{\varrho}(\varepsilon) = \omega_0 + \varepsilon^2 \omega(\varepsilon^2).$$

It can be concluded from this that the true rotation number lies between two polynomials

$$\varrho(\varepsilon) = \hat{\varrho}(\varepsilon) \pm K\varepsilon^N,$$

where N is arbitrary. It follows that the lengths of the flat line segments on which lock-ins occur must tend to zero faster than any power of N as $\varepsilon \to 0$.

5.3. Experiments

The type of dynamics that I have discussed here is characteristic of the observed dynamics in some mechanical systems involving fluid motions. The fact that an analysis of the kind given here does seem to fit well the observations of motion in small boxes of liquid heated from below, and in flow systems like the Taylor problem may surprise some readers. The surprise is that an analysis in two dimensions, and low dimensions greater than 2 give results in agreement with observations of continuum systems with "infinitely" many dimensions. In fact, this kind of agreement is associated with the fact the spectrum of eigennodes in the small scale systems for which agreements is sought is widely separated and the dimension of active eigenvalues is actually small.

I do not want to give a too cryptic explanation of the relevance to real fluid mechanics of the kind of analysis sketched in these lectures. In fact this kind of analysis is recommended for actual computation of bifurcated objects in fluid mechanics near the point of bifurcation [4]. A not too cryptic explanation of relevance can be found in my two review papers (D.D. Joseph, Hydrodynamic Instability and Turbulence, eds. H. Swiney and J. Gollub, Topics in Physics (Springer, 1980)) or in Bifurcation in Fluid Mechanics, in the translation of the XIIIth Int. Congr. of Theoretical and Applied Mechanics (IUTAM), Toronto (1980).

References

[1] M. Herman, Sur la conjugaison différentiable des difféomorphismes du cercle à des rotations, Publ. I.H.E.S. 49 (1979) 5-234.

[2] M. Herman, Mesure de Lebesgue et nombre de rotation, Lecture Notes in Math., Vol. 597 (Springer, Berlin, 1977) pp. 271-293.

[3] G. Iooss, Bifurcation of maps and applications, Mathematics Studies No. 36 (North-Holland, Amsterdam, 1979).

[4] G. Ioss and D.D. Joseph, Elementary Stability and Bifurcation Theory, Undergraduate Textbook in Mathematics (Springer, New York, 1980).

[5] G. Ioss and D.D. Joseph, Bifurcation an stability of nT-periodic solutions at a point of resonance, Arch. Rat. Mech. Anal. 66 (1977) 135-172.

[6] Y.H. Wan, Bifurcation into invariant tori at points of resonance, Arch. Rat. Mech. Anal. 68 (1978) 343-357.