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Fading Memory

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§ 1. Fading Memory

Fading memory expresses the intuitive idea that the recent rather than the remote history of deformation of a material body should have a greater effect on the present stress. The problem of fading memory is to give a useful mathematical formulation of this intuitive idea. The restricted problem of fading memory deals with the forms of stress which arise when the class of allowed deformations are small or slow in a sense specified differently in different theories. In this paper we give two new solutions to the restricted problem of fading memory, hereafter called the problem of fading memory. Our first solution is in a weighted Sobolev space. Our second solution is on a locally convex topological vector space. Both solutions are framed on subsets of the weighted Hilbert space used in the celebrated theory of COLEMAN & NOLL (1961). In each of the new solutions we find as special cases, constitutive equations of differential type (including Newtonian fluids), constitutive equations of integral type (with "smooth" $\mathbb{L}_h^2$ kernels) and mixtures of these. The mixed constitutive equations are said to describe "materials of mixed type of order n and degree k." Dynamics associated with constitutive equations of mixed type are studied for shear flows in a linearized approximation. The dynamic equations have shock-like solutions in the COLEMAN-NOLL limit which are smoothed by rate terms (viscosity) in the materials of mixed type.

Theoretical studies of the dynamic response of materials require that one first write constitutive equations relating stress and deformation. Only then is it possible to form the equations which govern the deformation and motion of the material. The study of dynamics comes after constitutive theory and the type of dynamics which are allowed depends on what one first says about constitutive theory. There are many approaches to constitutive theory but they fall under two categories:

(I) In the first category one seeks constitutive equations for a restricted class of materials in an unrestricted class of deformations. Newtonian fluids are a good example in the first category. Material models generally fall under this category.

(II) In the second category one seeks constitutive equations for an unrestricted class of materials in a restricted class of deformations. An early example, perhaps the first, of the second approach is in the work of CRIMINALE, ERICKSEN & FILBEY (1958) who gave a constitutive equation supposed valid for all fluids in viscometric flow. One example of the second approach is in the work of COLEMAN & NOLL (1961). There are two parts to their theory; in the first part they choose a weighted $\mathbb{L}^2(0,\infty)$ topology on dom $\mathscr{T}$ to express the idea of fading memory and to get a mathematical expression for stress relaxation. In the second part they try to justify Fréchet expansions of the stress relative to the rest (elastostatic) history in the same topology. The approximate multinomial functional forms for the stress derived by them can be regarded as applying to all simple materials perturbing elastostatics with motions whose history lies in a weighted $\mathbb{L}_h^2(0,\infty)$ space with a suitably chosen weight $h(s)$.

The two approaches (I) and (II) to constitutive theory are not mutually distinct. In fact, the assumption that the stress in a simple material depends only on the history of the first spatial gradient of the deformation may be regarded either as restricting the allowed class of materials or the allowed class of deformation. In this paper we shall think that simple materials actually exist, and that they have in principle some constitutive equation depending on the history of the first spatial gradient of the deformation, which may in practice be unknown (in a sense to be explained) and even unknowable. We then seek a constitutive theory for such a material, not in all the deformations which it may undergo, but only in motions which perturb states of relative rest.

To give a mathematical form to the remarks just made, we recall that the constitutive equation for the stress in a simple material may be written as

$$T = \mathscr{F}\left[B(t), \mathop{G(s)}_{s=0}^{\infty}\right] \stackrel{\text{def}}{=} \mathscr{F}[B(t), G] \tag{1.1}$$

where T is the stress at a particle X presently at $x(X, t)$. $\mathscr{F}(B(t), G)$ is a symmetric tensor-valued function of the left-Cauchy-Green tensor

$$B(t) = F(t)\, F^T(t),$$

$$F(t) = \operatorname{grad}_X x(X, t)$$

and is a symmetric tensor-valued functional of the history

$$G(s) = C_t(t - s) - \mathbf{1}$$

of the right relative Cauchy-Green tensor

$$C_t(\tau) = F_t^T(\tau)\, F_t(\tau),$$

$$F_t(\tau) = \operatorname{grad}_x \chi_t(x, \tau)$$

where $\tau = t - s$, $0 \leqq s < \infty$ and $\chi_t(x, \tau) = x(X, \tau)$ is the position at past time of the particle $x = \chi_t(x, t)$ presently at x. In states of rest,

$$\chi_t(x, \tau) = x, \quad \tau \leqq t,$$

and

$$T = \mathscr{F}[B(t), \mathbf{0}] \tag{1.2}$$

defines an elastostatic response.

The principle of fading memory says that the present value of the stress should not be influenced by events in the infinitely distant past. The generality of the principle leaves room for many mathematical interpretations. A solution of the problem of fading memory is to give the principle a mathematical form by specifying precisely how the stress shall be only slightly dependent on deformations which occured long ago. Different solutions of the problem lead to different representations for the stress, to different constitutive equations.

In practice, mathematical theories of fading memory can be framed as theories of functional analysis with the following structure: The constitutive equation relating stress to deformation is a (tensor-valued) functional on deformation histories. Thus deformation histories form the domain of the functional and stresses form the range. To get fading memory, one makes a choice of topology on the domain such that two histories are close when they are close recently. The stress functional is assumed to be continuous in this topology and so is only weakly dependent on events that occurred in the distant past. Theories of fading memory have been given by Coleman & Noll (1961), Wang (1965) and Coleman & Mizel (1968).

Theories of fading memory may be associated with perturbation of states of rest (elastostatic deformations) with small deviations of the history from the value zero associated with rest histories. Rigid motions also have

$G(s) = 0$ for $s \geqq 0$ $(F_t^T(\tau)\, F_t(\tau) = 1)$ and the theory of $\mathscr{F}$ for "small" G is also a theory of perturbation of rigid motions. We include the case of rigid motions in our theory under the name motions of *relative rest*. In fact rigid motions are dynamically possible without inventing special body forces to balance the equations if and only if the material is a *fluid* and the rotation is *steady* (JOSEPH, 1977).

The perturbation of histories of relative rest is in the assigned topology. Fading memory requires that $G(s)$ be small in the recent past and the continuity of the stress in the assigned topology then implies definite results about the forms allowed for stresses.

Theories of fading memory also lead to forms allowed for stresses linearized on base motions, like viscometric flow. For the linearization we must suppose that the response function is differentiable (in some sense) near the base state and its derivative at the base state is a *linear* continuous (tensor-valued) functional. Given a space of deformation histories, we may characterize the linearized stresses as the set of continuous linear functionals on that space, the dual. Thus it is evident that solutions of the problem of fading memory include the choice of a space and an associated topology and the study of the topological dual. It goes nearly without saying (see § 5) that thorough knowledge of the topological dual may have strong explicit consequences for the representations of higher order functional derivatives of the stress (see § 14).

The theory we are going to present is a theory of category (II) which is in the spirit of the theory of COLEMAN & NOLL (1961). Our theory, like theirs, leads to constitutive equations for an unrestricted class of materials in a restricted class of deformations. The deformation histories which we admit are also small in $\mathbb{L}_h^2(0, \infty)$, but we further restrict the space of deformations (and enlarge the dual) by adjoining differentiability conditions that might be supposed true for histories allowed in "real" materials.

GREEN & RIVLIN (1960) have proposed a constitutive equation of the form

$$T = \mathscr{F}[G(s), B(t), A_1(t), \ldots, A_n(t)] \tag{1.3}$$

where $\mathscr{F}$ is a function of $B(t)$ and the first n derivatives $A_n(t)$ of $G(s)$ at the present time $\tau = t$ $(s = 0)$ and a functional of the history of $G(s)$. Equation (1.3) is supposed to hold for certain classes of materials.

We shall show that certain realizations of (1.3) are implied by (1.1) and a suitable choice for topology on dom $\mathscr{T}$ (Sobolev spaces and some LF spaces; for examples, see § 9–14).

In § 15 we consider some problems of dynamics associated with constitutive equations in these new spaces. The recent interesting work of KAZAKIA & RIVLIN (1981) on linearized dynamics can also be viewed as examples in our theory.

§ 2. Stress Relaxation

It is generally agreed that materials whose memory fades should possess the following primitive property: the stress in a material which is ultimately in a state of rest should be the same as the stress in a material which was always at rest. All the theories of fading memory so far laid down are assumed to possess this primitive property. COLEMAN & NOLL (1962) introduced the notion of static continuation of a given history to give a mathematical discussion of stress relaxation. The static continuation of a given history in the COLEMAN-NOLL theory implies that the underlying space of histories contain non-smooth elements which lack differentiability at the joining point (*cf.* (2.4)). Theories that require that higher time derivatives of the history of the deformation at a particle be continuous do not possess such a static continuation. WANG's theory (1965) and our Sobolev space theory do not allow such a static continuation. In these theories, we get a static continuation by "rounding" the corner.

A material is said to possess the property of stress relaxation if (1.2) holds asymptotically after the relative motion of the body has been stopped. Thus the demonstration is complete when it is shown that for each and every history

$$\boldsymbol{G}(s) \in \operatorname{dom} \mathscr{F} \tag{2.1}$$

such that

$$|\boldsymbol{G}(s)| \to 0 \text{ as } t \to \infty \tag{2.2}$$

where convergence to zero is in the assigned topology, we have, when $t \to \infty$,

$$\tilde{\mathscr{F}}(t) = \mathscr{F}\left[\boldsymbol{B}(t), \overset{\infty}{\underset{s=0}{\boldsymbol{G}(s)}}\right] - \mathscr{F}[\boldsymbol{B}(t), 0] \to 0 \tag{2.3}$$

in $\mathbb{R}^6$, uniformly in $\boldsymbol{B}$. It is enough for (2.3) to *assume* the continuity of $\mathscr{F}$ with respect to $\boldsymbol{G}(\cdot)$ near zero uniformly in $\boldsymbol{B}$ in the topology of dom $\mathscr{F}$. Thus the property of stress relaxation is in part an *assumed* property expressed in terms of the continuity of a mapping.

COLEMAN & NOLL (1962) demonstrated (2.3) in a special class under (2.2) which they call the static continuation $\boldsymbol{G}_{t'}(s)$ of a given history $\boldsymbol{G}(s)$

$$\boldsymbol{G}_{t'}(s) = \begin{cases} \boldsymbol{0} \text{ if } 0 \leq s \leq t' \\ \boldsymbol{G}(s - t') \text{ if } s > t' \end{cases} \tag{2.4}$$

where $\boldsymbol{G}$ is assumed to lie in a weighted $\mathbb{L}_h^2(0, \infty)$ Hilbert space with weight

$$h(s) > 0, h(0) = 1, \quad h(s) \to 0 \text{ as } s \to \infty. \tag{2.5}$$

For further details, see TRUESDELL & NOLL (1965).

§ 3. Functional Derivatives

Dynamical problems whose data perturbs elastostatics (in solids) or relative rest (in fluids) may be expected to give rise to histories which are small in some topology expressing fading memory. And such perturbations may be expected to induce small perturbations of stress in the form of functional derivatives of the stress $\mathscr{F}$ evaluated at $\boldsymbol{G}(s) = \boldsymbol{0}$. The functional derivatives are tensor-valued multilinear forms on dom $\mathscr{F}$ in which the material parameters which distinguish one material from another are made explicit. To justify functional derivatives it is necessary to assume some smoothness of $\mathscr{F}$ near $\mathbf{0}$ in the assigned topology. It is uncertain whether any of these smoothness assumptions can be verified in practice, but in the same breath, we note that there is no experience of practice which prevents one from making such assumptions and seeing where they lead.

The approximations we seek may be written

$$\begin{aligned}(3.1)\qquad \mathscr{F}\left[\mathbf{B}(t),\ \overset{\infty}{\underset{s=0}{\mathbf{G}(s)}}\right] \sim\ & \mathscr{F}_1\left[\mathbf{B}(t), \mathbf{0}\,\middle|\, \overset{\infty}{\underset{s=0}{\mathbf{G}(s)}}\right] \\ &+ \mathscr{F}_2\left[\mathbf{B}(t), \mathbf{0}\,\middle|\, \overset{\infty}{\underset{s_1=0}{\mathbf{G}(s_1)}}\,\middle|\, \overset{\infty}{\underset{s_2=0}{\mathbf{G}(s_2)}}\right] + \dots \\ &+ \mathscr{F}_n\left[\mathbf{B}(t), \mathbf{0}\,\middle|\, \overset{\infty}{\underset{s_1=0}{\mathbf{G}(s_1)}}\,\middle|\, \overset{\infty}{\underset{s_2=0}{\mathbf{G}(s_2)}}\,\middle|\, \dots \,\middle|\, \overset{\infty}{\underset{s_n=0}{\mathbf{G}(s_n)}}\right]\end{aligned}$$

where $\mathscr{F}_l$ is a function of $\mathbf{B}(t)$ and an l-linear, continuous tensor-valued form in $\mathbf{G}(\cdot)$, a functional derivative of order l evaluated on $\mathbf{G}(s) = 0$ (the second variable) whose argument functions are histories.

The functional derivatives we use are of two types, Gateaux derivatives and Fréchet derivatives. Fréchet derivatives may be defined when the topology of dom $\mathscr{F}$ is given by a norm. The Fréchet derivative approximates the functional near a given point ($\mathbf{0}$) uniformly in a neighborhood of this point. Fréchet derivatives are used in the weighted $\mathbb{L}^2$ theory of COLEMAN & NOLL and in our Sobolev space theory. Gateaux derivatives may be defined for elements belonging to linear topological vector spaces, even when these spaces are not normable. Gateaux derivatives approximate the functional at a given point in a non-uniform fashion on rays. There is no satisfactory Fréchet calculus for non-normable vector spaces. Gateaux derivatives are required in our LF theory (see § 12) and (implicitly) in the (1965) theory of WANG.

To compute Gateaux and Fréchet derivatives it is necessary that the domain of $\mathscr{F}$ be a linear space (or a manifold). The tensor $\mathbf{G}(s)$ cannot be in a linear space because the eigenvalues of $\mathbf{G} - \mathbf{1}$ are non-negative $(\det(\mathbf{G}(s) + 1) = 1,$ for incompressible materials). It is possible to embed these tensors in a space of symmetric-tensor valued functions which can form the basis of a linear space. The functional analysis is then worked in this extended space (for further details, see TRUESDELL & NOLL, p. 103). We shall suppose now and henceforth that $\mathbf{G}(s)$ lies in the extended space of symmetric tensors endowed with a topology suitable for a topological vector space which may or may not be normable. We call this space dom $\mathscr{F}$ (= domain of $\mathscr{F}$).

The same algorithms are used to compute Fréchet and Gateaux derivatives. Suppose $\mathbf{G}(\cdot)$, $\mathbf{J}(\cdot)$, $\mathbf{K}(\cdot)$ belong to dom $\mathscr{F}$. Then

$$(3.2)\qquad \frac{\partial}{\partial\lambda}\mathscr{F}\left[\mathbf{B}(t),\ \mathbf{G}(s) + \lambda \overset{\infty}{\underset{s=0}{\mathbf{J}(s)}}\right]\Bigg|_{\lambda=0} \overset{\text{def}}{=} \mathscr{F}_1\left[\mathbf{B}(t),\ \overset{\infty}{\underset{s=0}{\mathbf{G}(s)}}\,\middle|\, \overset{\infty}{\underset{s_1=0}{\mathbf{J}(s_1)}}\right]$$

is linear and continuous in $J(\cdot)$. The second derivative may be defined by

(3.3)

$$\frac{\partial^2}{\partial\lambda_1\,\partial\lambda_2}\mathscr{F}\,[B(t),\,G(s)+\lambda_1 J(s)+\lambda_2 K(s)]\,|_{\lambda_1=\lambda_2=0}\overset{\text{def}}{=}\mathscr{F}_2\left[B(t),\,\overset{\infty}{\underset{s=0}{G(s)}}\,\middle|\,\overset{\infty}{\underset{s_1=0}{J(s_1)}}\,\middle|\,\overset{\infty}{\underset{s_2=0}{K(s_2)}}\right]$$

where $\mathscr{F}_2$ is linear and continuous in $J(\cdot)$ and $K(\cdot)$. Higher derivatives are computed in the same way. We get derivatives on the zero history by putting $G=0$ in (3.2) and (3.3).

It is obvious that the functional derivatives are symmetric to all transpositions of their linear arguments.

Notation. In all that follows we shall omit reference to the zero history on which the $\mathscr{F}_n$ are evaluated. We also omit explicit reference to the range of lapse times over which $\mathscr{T}$ and $\mathscr{F}_n$ are evaluated. For example

$$\mathscr{F}_2\left[B(t),\mathbf{0}\,\middle|\,\overset{\infty}{\underset{s_1=0}{G(s_1)}}\,\middle|\,\overset{\infty}{\underset{s_2=0}{G(s_2)}}\right]\overset{\text{def}}{=}\mathscr{F}_2[B(t)\mid G\mid G]. \tag{3.4}$$

In fluids, the reference position of the particle is the present position $\boldsymbol{x}$ and the tensor $B(t)$ may be replaced by ϱ, the density of the fluid. In fact, we shall consider only incompressible fluids and for these we use the notation

$$\mathscr{T}[G],\ \mathscr{T}_1[G],\ \mathscr{T}_2[G\mid G],\ \text{etc.}$$

Some properties of fading memory depend strongly on the topology of dom $\mathscr{F}$ and some properties are only weakly dependent on this topology.

§ 4. Nearly Steady slow Motions

The form taken by the expansion of $\mathscr{F}[B(t),G]$ for nearly steady slow motions is a robust property of simple materials which does not depend strongly on the topology of dom $\mathscr{F}$. On the other hand, the restrictions which can be placed on the sign of the coefficients in the expansion do appear to depend in a strong way on the topology chosen (see § 13).

Nearly steady slow motions are equivalent to motions which Coleman & Noll (1960) call retarded. To explain the expansion induced by such motions we introduce the Rivlin-Ericksen tensors

$$A_n(t)=(-1)^n\frac{d^n}{ds^n}G(s)\,|_{s=0}\,. \tag{4.1}$$

In the theory of Coleman & Noll, $G(\cdot)$ is in a weighted $\mathbb{L}^2(0,\infty)$ space. Thus not all the elements of their space have derivatives at $s=0$. We consider all the motions which have such derivatives and can be approximated by a (possibly finite) Taylor series

$$G(s)=-sA_1(t)+\frac{s^2}{2}A_2+\dots+(-1)^n\frac{s^n}{n_1}A_n(t)+R_n(s,t) \tag{4.2}$$

where the remainder $R_n(s, t)$ is such that

$$R_n(s, t) = o(s^n)\, C(t) \text{ as } s \to 0.$$

Let $L = \nabla u(x, t)$, where $u(x, t)$ is the velocity at the present position x. Then we can determine A_n through the recursion relations

$$A_1 = L + L^T,$$

$$A_{n+1} = \left(\frac{\partial}{\partial t} + u \cdot \nabla\right) A_n + A_n L + L^T A_n, \quad n > 1,$$

$$\left.\begin{array}{l} \operatorname{Tr} A_1 = 0 \\ \operatorname{Tr} A_1^2 = \operatorname{Tr} A_2 \end{array}\right\} \text{ for incompressible fluids.}$$

The expansion we seek requires that we introduce retarded motions or nearly steady slow motions. We shall proceed as far as possible without introducing these special motions.

In all situations for which (4.2) holds we find, using the property of linearity, that

$$\text{(4.3)} \qquad \begin{aligned} \mathscr{F}_1[B(t) \mid G] &= \mathscr{F}_1\left[B(t), -sA_1(t) + \frac{s^2}{2} A_2(t) + \ldots + R_n(s, t)\right] \\ &= -f_1[B(t) \mid s]\, A_1 + \frac{1}{2} f_1[B(t) \mid s^2]\, A_2 + \ldots \\ &\quad + \frac{(-1)^n}{n!} f_1[B(t) \mid s^n]\, A_n + \mathscr{F}_1[B(t) \mid R_n(s, t)] \end{aligned}$$

where

$$\mathscr{F}_1[B(t) | \, s^n A_n(t)] = f_1[B, s^n]\, A_n(t)$$

and

$$\text{(4.4)} \qquad f_1[B(t) \mid q] \overset{\text{def}}{=} f_1\left[B(t) \,\middle|\, \overset{\infty}{\underset{s=0}{q(s)}}\right]$$

is a fourth order tensor-valued function of $B(t)$ and functional of the function $q(s)$. For isotropic materials, f_1 is an isotropic tensor-valued function of $B(t)$, expressible, using Cauchy's theorem, by scalar functions of B times dyadic products of the unit tensor. Since $\mathscr{F}_1$ and A_n are symmetric, we have, modulo terms proportional to $\mathbf{1}$,

$$\text{(4.5)} \qquad f_1[B(t) \mid s^n]\, A_n = F_1[B(t) \mid s^n]\, A_n,$$

where

$$\text{(4.6)} \qquad F_1[B(t) \mid q] \overset{\text{def}}{=} F_1\left[B(t) \,\middle|\, \overset{\infty}{\underset{s=0}{q(s)}}\right]$$

is a scalar function of $B(t)$, and a linear functional computed on the history of $q(s)$. Omitting terms proportional to $\mathbf{1}$ in the expansion for $\mathscr{F}_1$, we find that

(4.7)

$$\mathscr{F}_1[B(t) \mid G] = \mu A_1(t) + \alpha_1 A_2(t) + \beta_1 A_3(t) + \gamma_1 A_4(t) + \ldots + \mathscr{F}_1[B(t), R_n(s, t)],$$

where

$$\mu = -F_1[B(t) \mid s],$$

(4.8)
$$\alpha_1 = \frac{1}{2} F_1[B(t) \mid s^2],$$

$$\beta_1 = -\frac{1}{3!} F_1[B(t) \mid s^3],$$

$$\gamma_1 = \frac{1}{4!} F_1[B(t) \mid s^4], \quad \text{etc.}$$

(For incompressible fluids, we suppress $\boldsymbol{B}(t)$ in (4.8) and obtain the expressions (8.1).) The Cayley-Hamilton theorem can be used to reduce these functions of $\boldsymbol{B}(t)$ to quadratic polynominals with coefficients that depend on the invariants of $\boldsymbol{B}(t)$. For incompressible fluids, μ, α_1, β_1, γ_1, *etc.* are constants evaluated on one and the same functional $F_1[q]$ for

$$q(s) = -s, \frac{s^2}{2!}, \frac{-s^3}{3!}, \frac{s^4}{4!} \quad \text{etc., respectively.}$$

Obviously, μ, α_1, β_1, γ_1 need not be independent.

The same type of consideration applies to the higher order functional derivatives. For example, using (4.2) and the bilinearity of $\mathscr{F}_2$, we find that

$$(4.9) \qquad \mathscr{F}_2[\boldsymbol{B}(t) \mid \boldsymbol{G}(s_1)] \mid \boldsymbol{G}(s_2)] = f_2[\boldsymbol{B}(t) \mid s_1 \mid s_2]\, \boldsymbol{A}_1^2 - f_2\left[\boldsymbol{B}(t) \mid s_1 \,\middle|\, \frac{s_2^2}{2}\right] (\boldsymbol{A}_1\boldsymbol{A}_2 + \boldsymbol{A}_2\boldsymbol{A}_1) + f_2\left[\boldsymbol{B}(t) \,\middle|\, \frac{s_1^2}{2} \,\middle|\, \frac{s_2^2}{2}\right] \boldsymbol{A}_2^2 + \dots,$$

where

$$(4.10) \qquad f_2[\boldsymbol{B}(t) \mid q(s_1) \mid m(s_2)] \overset{\text{def}}{=} f_2\left[\boldsymbol{B}(t) \,\middle|\, \overset{\infty}{\underset{s_1=0}{q(s)_1}} \,\middle|\, \overset{\infty}{\underset{s_2=0}{m(s_2)}}\right]$$

is a sixth order tensor, symmetric in successive pairs of indices and to interchanges of q and m. For isotropic materials these tensor functions are expressible by products of the unit tensor and scalar functions of $\boldsymbol{B}(t)$. Omitting terms proportional to $\mathbf{1}$, we find that in the isotropic case

$$(4.11) \qquad \mathscr{F}_2[\boldsymbol{B}(t) \mid \boldsymbol{G}(s_1) \mid \boldsymbol{G}(s_2)] = \alpha_2\boldsymbol{A}_1^2 + \beta_2[\boldsymbol{A}_2\boldsymbol{A}_1 + \boldsymbol{A}_1\boldsymbol{A}_2] + \gamma_3\boldsymbol{A}_2^2 + \dots,$$

where

$$\alpha_2 = F_2[\boldsymbol{B}(t) \mid s_1 \mid s_2],$$

(4.12)
$$\beta_2 = -\frac{1}{2} F_2[\boldsymbol{B}(t) \mid s_1 \mid s_2^2],$$

$$\gamma_3 = \frac{1}{4} F_2[\boldsymbol{B}(t) \mid s_1^2 \mid s_2^2]\, dc,$$

are defined by the bilinear functional

$$F_2[B(t) \mid q(s_1) \mid m(s_2)] = F_2\left[B(t) \left| \underset{s_1=0}{\overset{\infty}{q(s_1)}} \right| \underset{s_2=0}{\overset{\infty}{m(s_2)}}\right].$$

(For incompressible fluids, suppress B in (4.12) and obtain the expressions (8.3).) Obviously, α_2, β_2, γ_3, *etc.* need not be independent.

We may form an asymptotic expression for $\mathscr{F}$ perturbing states of relative rest from (3.1), (4.1), (4.7) and (4.11):

(4.13)

$$\mathscr{F}[B(t), G(s)]$$

$$\sim \mathscr{F}[B(t), 0] + \mu A_1 + \alpha_1 A_2 + \beta_1 A_3 + \ldots + \alpha_2 A_1^2 + \beta_2(A_2 A_1 + A_1 A_2) + \gamma_3 A_2^2 + \ldots .$$

In general, (4.13) is not a good representation, because (4.2) is not a convenient way to represent the history when the higher order terms are not negligible. The series representation (4.2) of $G(s)$ is made useful for retarded motion (nearly steady slow motion) in which the first terms dominate.

COLEMAN & NOLL (1960) retard a given history $G(s)$ by replacing s with εs. The retarded history,

$$G(\varepsilon s) = -\varepsilon s A_1 + \frac{\varepsilon^2 s^2}{2} A_2 + ,$$

obviously diminishes the importance of the higher A_n when ε is small. Then following our scheme rather than theirs, we get for example

$$\begin{aligned} F_1[B(t) \mid \varepsilon^l s^l] &= \varepsilon^l F_1(B(t) \mid s^l], \\ F_2[B(t) \mid \varepsilon^n s_1^n \mid \varepsilon^m s_2^m] &= \varepsilon^{n+m} F_2[B(t) \mid s_1^n \mid s_2^m], \end{aligned} \tag{4.14}$$

and it follows from (4.13) and (4.14) that

(4.15)

$$\begin{aligned} \mathscr{F}[B(t), G(\varepsilon s)] = \mathscr{F}[B(t), 0] &+ \mu A_1 \varepsilon + (\alpha_1 A_2 + \alpha_2 A_1^2)\,\varepsilon^2 \\ &+ [\beta_1 A_3 + \beta_2(A_1 A_2 + A_2 A_1) + \beta_3 A_1 \operatorname{Tr} A_2]\,\varepsilon^3 + O(|\varepsilon|^3). \end{aligned}$$

Clearly, $A_m = A_n[u(x, t)]$ is determined by the velocity $u(x, t)$ whose history has been retarded. Constitutive equations for simple materials of grade n arise from identifying powers of ε in (4.15). The N^{th} approximation $\mathscr{F}^{(N)}$ of $\mathscr{F}$ for retarded motions is given by

$$\begin{aligned} \mathscr{F}^{(N)} &= S_1 + S_2 + \ldots + S_n + \mathscr{F}[B(t), 0], \\ S_1 &= \mu A_1, \\ S_2 &= \alpha_1 A_2 + \alpha_2 A_1^2, \\ S_3 &= \beta_1 A_3 + \beta_2(A_1 A_2 + A_2 A_1) + \beta_3 A_1 \operatorname{Tr} A_2, \text{ etc.} \end{aligned} \tag{4.16}$$

The same type of approximation can be obtained without retarding histories. Instead, we consider motions which are slow, $U(\boldsymbol{x}, t) = \varepsilon \boldsymbol{u}(\boldsymbol{x}, \tau)$, and slowly varying, $\frac{\partial}{\partial t} = \frac{1}{\varepsilon} \frac{\partial}{\partial \tau}$. Such histories may be generated from the equations of motion with slow, slowly varying (nearly steady) data.

For such motions,

$$A_n[U(\boldsymbol{x}, t)] = \varepsilon^n A_n[\boldsymbol{u}(x, \tau)], \tag{4.17}$$

and the approximations (4.16) represent $\mathscr{F}[B(t) \mid G(s)]$ for the nearly steady slow motion $U(\boldsymbol{x}, t) = \varepsilon \boldsymbol{u}(\boldsymbol{x}, \tau)$ without retarding its history.

TRUESDELL & NOLL ((1965) p. 111)) have observed that the approximations for slow motions are robust in the sense that they seem to be independent of the choice of topology for dom $\mathscr{F}$. The foregoing discussion suggests that such approximations are valid whenever the stress may be expressed by a series of Gateaux derivatives on states of relative rest.

In principle, constants which arise from the same Gateaux derivative need not be independent; for example, the constants (4.8) need not be independent and the constants (4.12) need not be independent. But groups of constants which arise from different Gateaux derivatives are, in principle, independent; the constants in the group (4.8) are independent of those in group (4.12). This type of independence is always abrogated in the construction of model constitutive equations, independent of whether such models are based on phenomenological or molecular considerations. All such models introduce special assumptions, explicit or implicit, which imply certain relations between properties of presumably independent derivatives. For example, a well known assertion due to WEISSENBERG and achieved in some molecular models is that the second normal stress vanishes in shear flows. This implies a relation

$$2\alpha_1 + \alpha_2 = 0 \tag{4.18}$$

among independent constants α_1 and α_2. Such special relations can in principle hold for some particular fluid, but not for all. Thus we may conclude that special models may apply only to special fluids. If there is some special model which is valid for all fluids and abrogates the independence of different Gateaux derivatives, then we should be forced to conclude that the special models incorporate a generally valid principle presently lost to continuum mechanics.

We can carry this line of thought further. First we may observe that though the relations between the constants in each independent group is not completely explicit, there are certain suggestive possibilities. For example, we know that the viscosity $\mu = -F_1[s]$ at zero shear is positive (and $0 \leqq s \leqq \infty$) so it is not absurd to imagine fluids (like those with fading memory of COLEMAN & NOLL) in which $F_1[s^n]$ is also negative. Then (4.8) shows

$$\mu > 0, \alpha_1 < 0, \varrho_1 > 0, \gamma_1 < 0, \text{ and so on.} \tag{4.19}$$

If we knew the sign of α_2, we could make a similar guess:

$$\alpha_2 > 0, \varrho_2 < 0, \gamma_3 > 0, \text{ and so on.}$$

It has been observed that polymeric liquids and melts climb rotating rods and that the free surface on polymers in liquids flowing down a tilted trough bulges up. Solutions of the boundary value problem governing these flows, using the aforementioned theory of slow flow, show climbing if

$$3\alpha_1 + 2\alpha_2 = \frac{3}{2} F_1[s^2] + 2F_2[s_1 \mid s_2] > 0$$

and bulging if

$$2\alpha_1 + \alpha_2 = F_1[s^2] + 2F_2[s_1 \mid s_2] < 0.$$

Since $\alpha_1 < 0$ for fluids satisfying (4.19), such fluids will climb a rod only if $\alpha_2 > 0$. If this same fluid also bulges in a tilted trough, then there is constant K such that

$$2\alpha_1 + K\alpha_2 = 0, \quad 1 < K < \frac{4}{3}. \tag{4.20}$$

Thus many polymeric liquids come close to satisfying the limiting implication (4.18) of the Weissenberg assertion. It would be astonishing if the relation (4.20) held for all simple fluids though it apparently does hold for the ones so far measured. Even though nearly all the simple fluids studied in experiments known to us climb near the rod when the rod is small there is no principle known to us which gives $\alpha_2 > 0$.

The theory of approximations of the stress in slow, nearly steady motions neglects entirely the role of dynamics. There is no theory of existence, uniqueness and stability for the dynamic equations which these approximations imply. Nonetheless, many problems of rheology have been solved using the slow flow approximations. Such solutions are necessarily in series of powers of ε. There are no theorems proving that the solutions which have and will be computed represent true solutions in some sense. The problem is that the recursion relation (4.2) for the $\boldsymbol{A}_n$ defines $(n-1)$ differentiations $(\boldsymbol{U} \cdot \nabla)^{n-1} \boldsymbol{A}_1[\boldsymbol{U}] = \varepsilon^n (\boldsymbol{u} \cdot \nabla)^{n-1} \boldsymbol{A}_1[\boldsymbol{u}]$. On perturbation solutions the divergence of this term is of lower order because of the ε^n and the solutions are expressed by applying the inverse of the Laplacian $(\nabla^2)^{-1}$ on this term. If $n - 1 > 2$, then we get an unbounded operator after inversion and it is necessary to balance the smallness of ε^n against the high order of differentiation. This type of theoretical problem has not yet been satisfactorily resolved. (A theorem of the Nash-Moser type would probably be of interest in this context.)

§ 5. Duals

Stress relaxation and slow motion approximations are properties of simple materials which do not depend strongly on the choice of topology of dom $\mathscr{F}$. On the other hand, representation theorems for the linearized stresses are completely determined by the topology of dom $\mathscr{F}$. Suppose that $X = \text{dom}\, \mathscr{F}$ is a topological vector space. Then

$$G \mapsto \mathscr{F}_1[\boldsymbol{B}(t) \mid G], \quad G \in X$$

is a continuous linear mapping from X into $\mathbb{R}^6$, a continuous tensor-valued linear functional; *i.e.*, an element of $(X^*)^6$ where X^* denotes the topological dual of X. Thus we may know all the possible forms for the linearized stresses if we can represent all the elements in X^*. If X is a Hilbert space, then by the Riesz theorem all the elements of X^* can be represented by scalar products of G against some fixed element in X. For instance, using their weighted L^2 theory, COLEMAN & NOLL obtained in this way an integral representation for $\mathscr{F}_1[B(t) \mid \cdot]$. Such direct use of Riesz's theorem is not possible in other Hilbert spaces. For example, in Sobolev spaces in which the function and its first derivative are in L^2, the correct application of Riesz's theorem gives the linear functional in terms of two scalar products (as in (10.5)), one of which can be expressed as the integral of a fixed element in L^2 against a first derivative. If we suppose that the fixed element is only piecewise continuously differentiable, we may integrate by parts, putting the derivative on the piecewise smooth functions. This leads directly then to derivatives of Heaviside functions; that is, to Dirac measures. To obtain integral representation of the dual in Sobolev space we must therefore allow kernels which are distributions on $\mathbb{R}$ with support in $[0, \infty)$ (for instance, the Dirac measure and its derivatives at 0). This method for obtaining integral representations works also for our theory in LF spaces (see § 12). The Fréchet space introduced by WANG (see § 16) has a dual which can be represented by Stieltjes integrals.

We shall give two new approaches to the problem of fading memory. To us this means that we characterize the duals of two spaces which among other properties have bounded weighted $L^2(0, \infty)$ norms of the COLEMAN-NOLL type. In both cases we give integral representations for the dual. The kernels of the integrals are essentially the elements of the dual space, but in our solutions these kernels may be "smooth" in the sense of COLEMAN & NOLL or singular, with derivatives of Dirac measures.

The appearance of Dirac distributions on our dual alows us to generate the stress implied by our theory directly from the integrals defined in the theory of COLEMAN & NOLL. It is perhaps of interest to add that the theory which we describe is a rigorous one for representing linearized stresses. For the higher order stresses, there are no theorems justifying representations in integral form but, if these be assumed, we can characterize the kernels as elements on the dual containing the Dirac distributions. This leads to interesting explicit forms for the stress in fluids in terms of *constitutive equations of mixed type of order n, grade k* which are discussed in § 13, § 14. Similar expressions hold for isotropic simple materials, including solids.

§ 6. Fading Memory of Coleman and Noll. First order Theory

The theory of COLEMAN & NOLL is generated by the assumption that

$$\text{dom}\, \mathscr{F} = \mathbb{L}_h^2(0, \infty), \tag{6.1}$$

where $\mathbb{L}_h^2(0, \infty)$ is a weighted Hilbert space associated with the scalar product

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle_h \overset{\text{def}}{=} \int_0^\infty h^2(s) \operatorname{Tr}[\boldsymbol{v}(s)\, \boldsymbol{v}(s)]\, ds, \tag{6.2}$$

where $\boldsymbol{u}(\,)$, $\boldsymbol{v}(\,)$ are symmetric tensors which can be identified with elements of $\mathbb{R}^6$. The weight $h(\cdot)$ is supposed to have the following properties:

$$h \text{ is smooth, say } C^1, \tag{6.3}$$

$$h(0) = 1, \quad h'(s) < 0, \quad h(s) \to 0 \text{ as } s \to \infty,$$

The norm

$$\|\boldsymbol{u}\|^2_{\mathbb{L}^2_{h(0,\infty)}} = \langle \boldsymbol{u}, \boldsymbol{u} \rangle_h \tag{6.4}$$

defines the topology and the weight $h(s)$ diminishes the importance of the values of $\boldsymbol{v}(s)$ in the distant past. In their theory, various rates of decay of $h(s)$ insuring the boundedness of functionals like (4.8) and (4.12) are important; this aspect of their theory is not of interest here.

By the Riesz theorem, all linear functionals, like the components of $\mathscr{F}_1[\boldsymbol{B}(t) \mid \cdot]$, can be represented by inner products (6.4) with a fixed element of $\mathbb{L}^2_h(0, \infty)$, that is

$$\begin{aligned} \mathscr{F}_1[\boldsymbol{B}(t) \mid \boldsymbol{G}] &= \langle \boldsymbol{v}(\boldsymbol{B}(t)), \boldsymbol{G} \rangle_h \\ &= \int_0^\infty h^2(s) \operatorname{Tr} [\boldsymbol{v}([(\boldsymbol{B}(t), s) \, \boldsymbol{G}(s)] \, ds, \end{aligned} \tag{6.5}$$

where $\boldsymbol{v}(\boldsymbol{B}(t), s)$ is a fourth order tensor function of $\boldsymbol{B}(t)$ defined almost everywhere and such that $\boldsymbol{v}\boldsymbol{G} = (\boldsymbol{v}\boldsymbol{G})^T$ has the symmetry of $\mathscr{F}_1$ and $\boldsymbol{v}(\boldsymbol{B}(t), \cdot) \in \mathbb{L}^2_h$.

In the case of incompressible isotropic viscoelastic solids material symmetry and isotropy imply that

$$\begin{aligned} \mathscr{F}_1[\boldsymbol{B}(t) \mid \boldsymbol{G}] = \int_0^\infty \{ &\boldsymbol{B} \operatorname{tr} [\phi_{10} \mathbf{1} + \phi_{11}\boldsymbol{B} + \phi_{12}\boldsymbol{B}^2) \, \boldsymbol{G}(s)] \\ &+ \boldsymbol{B}^2 \operatorname{tr} [(\phi_{20} \mathbf{1} + \phi_{21}\boldsymbol{B} + \phi_{22}\boldsymbol{B}^2) \, \boldsymbol{G}(s) \\ &+ \boldsymbol{G}(s) \, (\phi_{30} \mathbf{1} + \phi_{31}\boldsymbol{B} + \phi_{32}\boldsymbol{B}^2) \\ &+ \phi_{30} \mathbf{1} + \phi_{31}\boldsymbol{B} + \phi_{32}\boldsymbol{B}^2) \, \boldsymbol{G}(s) \} \, ds, \end{aligned} \tag{6.6}$$

where $\phi_{ij}(II, III, s)$ are functions of the invariants of $\boldsymbol{B}(t)$ and s.

When viewed as functions of s, the ϕ_{ij} are restricted to lie in the space implied by $\|\boldsymbol{G}\|_{L^2_h} < \infty$ and Schwarz inequality. For any tensor $\boldsymbol{C}(s)$ we have

$$\begin{aligned} \int_0^\infty \boldsymbol{C}(s) \, \boldsymbol{G}(s) \, ds &= \int_0^\infty \frac{1}{h(s)} \boldsymbol{C}(s) \, h(s) \, \boldsymbol{G}(s) \, ds \\ &\leq \left\{ \int_0^\infty \frac{\operatorname{tr} \boldsymbol{C}^2(s)}{h^2(s)} \, ds \right\}^{\frac{1}{2}} \left\{ \int_0^\infty h^2(s) \operatorname{tr} \boldsymbol{G}^2(s) \, ds \right\}^{\frac{1}{2}} \\ &\leq \left\{ \int_0^\infty \frac{\operatorname{tr} \boldsymbol{C}^2(s)}{h^2(s)} \, ds \right\}^{\frac{1}{2}} \|\boldsymbol{G}\|_{\mathbb{L}^2_h}; \end{aligned} \tag{6.7}$$

hence

(6.8) $$C/h \in \mathbb{L}^2(0,\infty).$$

The expression (6.7) and a very long one for the second derivatives are displayed in the paper by DIXIT, NARAIN & JOSEPH (1981).

The form of the integrals representing the Fréchet derivatives of the stress in $\mathbb{L}_h^2$ are simplest for incompressible fluids.

Here, rigorously, one has

(6.9) $$\mathscr{F}_1[G] = \int_0^\infty \mu(s)\, G(s)\, ds$$

where $\mu/h \in L^2(0,\infty)$.

Integral representations for the higher order derivatives are assumed in the COLEMAN-NOLL theory and will be assumed in our theory. But it is not known whether or not all of the higher derivatives, or even the class which may be interesting in mechanics can be so represented.

§ 7. Fading Memory of Coleman & Noll. Second order Theory

After all was said and done, COLEMAN & NOLL arrived at the following handsome expression for the second Fréchet derivative of $\mathscr{F}$ with respect to G at $G = 0$ in $\mathbb{L}_h^2$:

(7.1)
$$\mathscr{F}_2[G \mid G] = \int_0^\infty \int_0^\infty \{\hat{\nu}(s_1, s_2)\, G(s_1)\, G(s_2) + \hat{\alpha}(s_1, s_2)\, G(s_1) \operatorname{tr} G(s_2)\}\, ds_1,\, ds_2,$$

where $\hat{\nu}(s_1, s_2) = \hat{\nu}(s_2, s_1)$ and $\hat{\nu}$ and $\hat{\alpha}$ tend to zero when s_1 or s_2 tends to ∞. PIPKIN (1964) noticed that tr $G(s)$ is quadratic in $G(s)$ when the fluid is incompressible and (7.1) may be simplified:

(7.2) $$\mathscr{F}_2[G\, G] = \int_0^\infty \int_0^\infty \hat{\nu}(s_1, s_2)\, G(s_1)\, G(s_2)\, ds_1\, ds_2,$$

where

(7.3) $$\frac{\hat{\nu}(s_1, s_2)}{h(s_1)\, h(s_2)} \in L^2[(0,\infty) \times (0,\infty)].$$

The condition (7.3) follows from the application of Schwarz's inequality to (7.2).

Different assumptions may be used to motivate (7.2). One such motivation makes use of the Riesz theorem and the bilinearity of (7.1). We fix $G(s_1)$ and apply the theorem of Riesz to $G(s_2)$ in the integral expressing

(7.4) $$\mathscr{F}_2[G \mid G] = \left\langle \boldsymbol{\nu} \left[\mathop{G(s_1)}_{s_1=0}^{\infty} \right], \quad G \right\rangle_h$$
$$= \int_0^\infty h(s_2)\, \boldsymbol{\nu}[G(s_1)]\,(s_2) : G(s_2)\, ds_2,$$

where

$$(V : G)_{ij} = v_{ijkl} G_{kl}$$

and

$$v \left[\overset{\infty}{\underset{s_1=0}{G(s_1)}} \right] (\cdot) \in \mathbb{L}_h^2(0, \infty). \tag{7.5}$$

Since $\mathscr{F}_2[G(s_1) \mid G(s_2)]$ is linear and bounded in $G(s_1)$, $v[G(s_1)]\,(s_2)$ is a bounded linear operator,

$$v[G(s_1)]\,(s_2) = TG(s_2),$$

$$v_{ijkl} = T_{ijklpq} G_{pq}$$

from $\mathbb{L}_h^2$ into itself. COLEMAN & NOLL (private communication) have noted that (7.1) follows from (7.4) if $T(s_2)$ is an operator of Hilbert-Schmidt type. This type of mathematical assumption is not known to have a physical basis. It seems desirable to replace this assumption with the most general possible assumption compatible with integrals. We call this the *Integral operator assumption*: T is an integral operator

$$T(\cdot)\,(s_2) = \int_0^\infty k(s_1, s_2)\,(\cdot)\,(s_1)\, ds$$

(where k may be a distribution).

In the theory of COLEMAN & NOLL,

$$TG(s_2) \in \mathbb{L}_h^2(0, \infty)$$

and

$$TG(s_2) = \int_0^\infty k(s_1, s_2)\, G(s_1)\, ds_1, \tag{7.6}$$

where

$$\frac{k(s_1, s_2)}{h(s_1)\, h(s_2)} \in \mathbb{L}[(0, \infty) \times (0, \infty)]. \tag{7.7}$$

We may now proceed directly to (7.1) and (7.2) by incorporating the restrictions of $k(s_1, s_2)$ implied by the isotropy of the fluid and the symmetry of Fréchet derivatives.

The kernel condition (7.3) is rather too strong, even in the L_h^2 theory, because it excludes very popular and very useful single integral models, like the BKZ model. A statement better than (7.2) is

$$\begin{aligned} \mathscr{F}_2[G \mid G] = {} & \int_0^\infty \int_0^\infty \hat{v}_1(s_1, s_2)\, G(s_1)\, G(s_2)\, ds_1\, ds_2 \\ & + \left\{ \int_0^\infty \hat{v}_2(s)\, G(s)\, ds \right\} \left\{ \int_0^\infty \hat{v}_3(s)\, G(s)\, ds \right\} \\ & + \int_0^\infty \hat{v}_4(s)\, G(s)\, G(s)\, ds, \end{aligned} \tag{7.8}$$

where for $G \in \mathbb{L}_h^2(0, \infty)$, we have $\hat{v}_1(s_1, s_2) = \hat{v}_1(s_2, s_1)$ satisfying (7.3), $\hat{v}_2$ and $\hat{v}_3$ satisfying (6.9), $\dfrac{\hat{v}_4(s)}{h^2(s)} \in L^\infty$.

Of course, (7.8) follows from (7.2) when

$$\hat{\nu}(s_1, s_2) = \hat{\nu}_1(s_1, s_2) + \hat{\nu}_2(s_1)\,\hat{\nu}_3(s_2) + \hat{\nu}_4(s_1)\,\delta(s_1 - s_2), \tag{7.9}$$

where $\delta(s_1 - s_2)$ is the Dirac measure. But (7.9) does not satisfy (7.3). Though (7.8) can make sense in the theory of COLEMAN & NOLL, it cannot be deduced from (7.2).

PIPKIN (1964) wrote down a formula for the third Fréchet derivative:

$$\mathscr{F}_3[G \mid G \mid G] = \int_0^\infty \int_0^\infty \int_0^\infty \{\hat{\gamma}(s_1, s_2, s_3)\, G(s_1)\, G(s_2)\, G(s_3) + \beta(s_1, s_2, s_3)\, G(s_1) \operatorname{tr}[G(s_2)\, G(s_3)]\}\, ds_1\, ds_2\, ds_3 .$$

The remarks we have already made are about the kernels in $\mathscr{F}_2$ and we have clear analogues for $\mathscr{F}_3$. However, we note that though iterated triple integrals are defined in $\mathbb{L}_h^2(0, \infty)$, the cubic integral

$$\int_0^\infty \hat{\gamma}(s)\, G^3(s)\, ds$$

is not defined for all $G(\cdot) \in \mathbb{L}_h^2$ even for good $\hat{\gamma}(s)$ with compact support $[0, M]$ for some finite M. This is a serious defect of $\mathbb{L}_h^2(0, \infty)$ theory. MICHAEL RENARDY reminded us that many of the simplest rheological modes are framed as single integrals of good kernels against polynomial or analytic functions of $G(s)$; for example, models of the form

$$\int_{-\infty}^{t} e^{-\lambda(t-s)} f(G(s))\, ds$$

with

$$f(x) = f(0) + f'(0)\, x + \frac{1}{2} f''(0)\, x^2 + \dots,$$

are very often used, but they are not well-defined in $\mathbb{L}_h^2$.

In view of the importance of single integral models in rheological studies, it is perhaps useful to note that such models are indeed well-defined in Sobolev space (see the last paragraph of § 11).

§ 8. Fading Memory of Coleman & Noll. The Theory of Moments

COLEMAN & MARKOVITZ (1964) noted that the constants of the fluids of grade N (see (4.16)) could be defined in terms of moments of the kernels. They showed that

$$\mu = -\int_0^\infty s\hat{\mu}(s)\, ds$$

and

$$\alpha_1 = \int_0^\infty s^2 \hat{\mu}(s)\, ds$$

and indicated the general method by which the constants of fluids of grade N could be related to the moments of the kernels.

We have already noted in the more general context of § 4 that the constants of the fluids of grade N are functionals which may be defined generally, without necessary ties to the $\mathbb{L}^2_h(0, \infty)$ in the theory of COLEMAN & NOLL. When put in the integral $\mathbb{L}^2_h(0, \infty)$ theory, we find that the constants (4.8) of $\mathscr{F}_1$ (for incompressible fluids) may be expressed as

$$\mu = -F_1[s], \quad \alpha_1 = (1/2)\, F_1[s^2], \quad \beta_1 = -(1/3!)\, F_1[s^3], \quad \gamma_1 = (1/4!)\, F_1[s^4], \text{ etc.} \tag{8.1}$$

where

$$F_1[s^n] = \int_0^\infty s^n \hat{\mu}(s)\, ds. \tag{8.2}$$

Similarly, the constants (4.12) of G_2 may be expressed by

$$\alpha_2 = F_2[s_1 \mid s_2], \quad \beta_2 = -\left(\frac{1}{2!}\right) F_2[s_1 \mid s_2^2], \quad \gamma_3 = \left(\frac{1}{4!}\right) F_2[s_1^2 \mid s_2^2], \text{ etc.} \tag{8.3}$$

where

$$F_2[s_1^n \mid s_2^m] = \int_0^\infty \int_0^\infty \hat{\nu}(s_1, s_2)\, s_1^n, s_2^m\, ds_1\, ds_2. \tag{8.4}$$

The moment theory has some special properties. For example, integral inequalities among the constants associated with independent derivatives are immediate (JOSEPH 1980). For instance, by applying Schwarz's inequality to $F_1[s^2]$, assuming that $-\hat{\mu}(s)$ is not negative, we can show that

$$-\alpha_1 \leqq \mu^{\frac{1}{2}} \left(\frac{3}{2}\beta_1\right)^{\frac{1}{2}}. \quad (*) \tag{8.5}$$

The moments $F_1[s^n]$ also enter into the expansions of the complex viscosity

$$\eta^*(\omega) \overset{\text{def}}{=} \int_0^\infty \hat{g}(s)\, e^{-i\omega s}\, ds \tag{8.6}$$

for small frequencies, $e^{-i\omega s} = 1 - i\omega s - \frac{\omega^2 s^2}{2} + \frac{i\omega^3}{3!} s^3 + \dots$. We find that

$$\eta^*(\omega) = \mu + i\omega\alpha_1 - \omega^2\beta_1 + i\omega^3\gamma_1 + \dots. \tag{8.7}$$

(*) The inequality (8.5) is a slight improvement of the inequality $-\alpha_1 < \mu^{\frac{1}{2}}(2\beta)^{\frac{1}{2}}$, which was given by JOSEPH (1980). He wrote $\hat{\mu}(s) + d\hat{g}(s)/ds$, $\hat{g}(s) > 0$ and applied Schwarz's inequality to $-\alpha_1 = \int_0^\infty sg(s)\, ds$.

Theories of fading memory should be compatible with the notion that materials at rest are asymptotically stable, certainly to small disturbances. If the small disturbances can be described by a problem linearized on the rest state, we must consider the evolution problem implied by linearization with

$$\mathscr{F} \to \mathscr{F}_1$$

and

$$G(s) \to -sA_1(t-s).$$

We assume then that linearized stability can be obtained from spectral values $\sigma = \xi + i\eta$ with $u(x,t) = e^{-\sigma t}\,\xi(x)$. It is easy to show (JOSEPH 1974) that σ must satisfy

$$\frac{\sigma}{\Lambda_n} = \int_0^\infty \hat{g}(s)\, e^{\sigma s}\, ds, \tag{8.8}$$

where $\Lambda_n > 0$ are critical values of a Rayleigh quotient. It is known from work of CRAIK (1968), JOSEPH (1974), and SLEMROD (1976) that $\xi > 0$ (stability) when $g(s)$, $-g'(s) \geqq 0$ for $s \geqq 0$. On the other hand,

$$\int_0^\infty g(s)\, e^{\sigma s}\, ds = \mu - \alpha_1\sigma + \beta_1\sigma^2 + \ldots + \text{remainder} \tag{8.9}$$

may be defined by moments. JOSEPH (1980) showed that if at any order whatsoever the remainder be neglected and if the signs of the constants are those implied by $\hat{g}(s) > 0$ and the theory of moments, then the rest state is unstable. This is a queer result which shows that we can reach wrong results using Taylor series, even with many or even any finite number of terms.

This long exposition of the pioneering work of COLEMAN & NOLL perhaps clarifies certain commonly misunderstood features of their theory. In fact, $\mathbb{L}_h^2$ is probably too large a space for dom $\mathscr{F}$; it allows even discontinuous G's and infinite velocities. If we restrict dom $\mathscr{F}$ to a smaller and perhaps more realistic set, we can get a bigger dual which includes Newtonian fluids and other rate fluids, COLEMAN-NOLL fluids and mixtures of all three. Some form of the moment theory, however, will hold even under the big changes of topology which we shall introduce in the following sections.

§ 9. Sobolev Space

Notation: If H is a space of functions from $(0,\infty)$ in $\mathbb{R}$, then $\mathbb{H}$ is the corresponding symmetric tensor valued space with values in $\mathbb{R}^6$.

We define the scalar products

$$[\boldsymbol{u}, \boldsymbol{v}] \overset{\text{def}}{=} \sum_{j=0}^{k} \left\langle \frac{d^j\boldsymbol{u}}{ds^j}, \frac{d^j\boldsymbol{v}}{ds^j} \right\rangle_{\mathbb{L}_h^2}, \tag{9.1}$$

where the derivatives are in the sense of distributions in the Sobolev space

$$\mathbb{H}_h^k(0,\infty) = \left\{ \boldsymbol{u} \in \mathbb{L}_h^2(0,\infty), \frac{d^j\boldsymbol{u}}{ds^j} \in \mathbb{L}_h^2(0,\infty),\ 1 \leqq j \leqq k \right\}. \tag{9.2}$$

We assume that the influence function h satisfies the same hypotheses as in COLEMAN & NOLL's theory (*cf.* § 6). For simplicity we assume the same influence functions for different orders of differentiation. The Sobolev space

$$\mathring{\mathbb{H}}_h^1(0,\infty) = \{\boldsymbol{u} \in \mathbb{H}_h^1(0,\infty),\ \boldsymbol{u}(0) = 0\} \tag{9.3}$$

makes sense (and is closed in $\mathbb{H}_h^1(0,\infty)$) because of the continuous embedding

$$\mathbb{H}_h^1(0,\infty) \subset C[0,\infty), \tag{9.4}$$

where $C[0,\infty)$ is the (Fréchet) space of continuous functions from $[0,\infty)$ into $\mathbb{R}^6$, equipped with the topology of convergence on every compact subset. More precisely, every function in $\mathbb{H}_h^1(0,\infty)$ is equal almost everywhere to one in $C[0,\infty)$. In fact, the property of the influence function h guarantees that if $\boldsymbol{u} \in \mathbb{H}_h^1(0,\infty)$, for each and every $T > 0$, the restriction of $\boldsymbol{u}$ to $(0, T)$ belongs to the (usual) Sobolev space $\mathbb{H}^1(0, T)$ and therefore is equal almost everywhere to a continuous function on $[0, T]$.

We shall also use the Sobolev space

$$\mathring{\mathbb{H}}_h^1(0,\infty) \cap \mathbb{H}_h^k(0,\infty) \subset C^{k-1}[0,\infty],$$

with continuous imbedding into the space of $k - 1$ continuously differentiable functions equipped with the topology of C^{k-1} convergence on every compact subset of $[0,\infty)$.

Lemma 1. *$\mathscr{D}(0,\infty)$ is dense in $\mathring{\mathbb{H}}_h^1(0,\infty)$.*

Proof. The proof, as usual, is in two steps. (i) the functions of $\mathbb{H}_h^1(0,\infty)$ with compact support in $(0,\infty)$ are dense in $\mathring{\mathbb{H}}_h^1(0,\infty)$. For this we define first $\tilde{\boldsymbol{u}}_R$ by $\tilde{\boldsymbol{u}}_R(s) = \boldsymbol{0},\ 0 \leqq s \leqq \frac{1}{R},\ \tilde{\boldsymbol{u}}_R(s) = \boldsymbol{u}\left(s - \frac{1}{R}\right), s > \frac{1}{R}$. Then let $M \in C^\infty(0,\infty)$, $0 \leqq M \leqq 1$, $M(s) = 1$ for $0 \leqq s \leqq \frac{1}{2}$ and $M(s) = 0$ when $s \geqq 1$. Define $M_R(s) = M\left(\frac{s}{R}\right)$ and $\boldsymbol{u}_R = \tilde{\boldsymbol{u}}_R M_R$. Clearly $\boldsymbol{u}_R \in \mathring{\mathbb{H}}_h^1(0,\infty)$, with a compact support and $\boldsymbol{u}_R \to \boldsymbol{u}$ in $\mathbb{H}_h^1(0,\infty)$. In step (ii) we regularize $\boldsymbol{u}_R$ by convolution.

§ 10. The dual of $\mathring{\mathbb{H}}_h^1(0,\infty)$

Lemma 1 enables us to describe the dual $[\mathring{\mathbb{H}}^1(0,\infty)]^*$ of $\mathring{\mathbb{H}}^1[0,\infty)$ as a space of distributions in $(0,\infty)$. It will suffice to describe the dual of the space $\mathring{H}_h^1(0,\infty)$ of scalar functions. Primes denote differentiation in the sense of distributions.

Lemma 2. *Each and every element*

$$T \in \mathring{H}_h^1(0,\infty), \tag{10.1}$$

may be represented (not uniquely) as

$$\langle T, u\rangle = \langle f, u\rangle_{L_h^2} + \langle u, g'\rangle, \tag{10.2}$$

where $f \in L_h^2(0,\infty)$ *and* $g/h \in L^2(0,\infty)$.

The bracket $\langle u, g'\rangle$ *is an integral in the sense of distributions. If* $u \in \mathcal{D}(0,\infty)$, *then*

$$u \mapsto \langle u, g'\rangle \overset{\text{def}}{=} -\langle g, u'\rangle = -\int_0^\infty g u'\, ds$$

is a distribution on $(0,\infty)$ *which can be extended to a bounded linear functional* $u \mapsto \langle u, g'\rangle$ *on* $\mathring{H}_k^1(0,\infty)$.

Proof. The proof of Lemma 2 is analogous to the well-known proof for the classical Sobolev space $H_h^1(\Omega)$.

The mapping

$$\begin{gathered} u \overset{\tau}{\mapsto} (u, u'),\\ \mathring{H}_h^1(0,\infty) \overset{\tau}{\to} L_h^2(0,\infty)\times L_h^2(0,\infty)\end{gathered} \tag{10.3}$$

is an isometry between $\mathring{H}_h^1$ and $\tau(\mathring{H}_h^1)$ = a closed subspace of $L_h^2 \times L_h^2$. Let $T \in (\mathring{H}_h^1)^*$. Then T defines a continuous linear form M on $\tau(\mathring{H}_h^1)$. Suppose $(u, u') \in \tau(\mathring{H}_h^1)$; then

$$M(u, u') \leqq \|T\|\,\|u\| = \|T\|\,(|u|_{L_h^2}^2 + |u'|_{L_h^2}^2)^{\frac{1}{2}}. \tag{10.4}$$

It follows from (10.4) and the Hahn-Banach theorem that we can extend (in general, in a non-unique fashion) M to $L_h^2 \times L_h^2$. Let $\tilde{M}$ be this extension. By Riesz's theorem, there are functions f and β both in $L_h^2(0,\infty)$ such that

$$\tilde{M}(u, v) = \langle f, u\rangle_{L_h^2} + \langle \beta, v\rangle_{L_h^2}, \tag{10.5}$$

Hence

$$Tu = M(u, u') = \int_0^\infty f u h^2\, ds + \int_0^\infty \beta u' h^2\, ds. \tag{10.6}$$

Since $\mathcal{D}(0,\infty)$ is dense in $\mathring{H}_h^1(0,\infty)$, T is determined by its restriction on $\mathcal{D}(0,\infty)$ and we have

$$Tu = \int_0^\infty f u h^2\, ds - \int_0^\infty (\beta h^2)'\, u\, ds \qquad \forall u \in (0,\infty), \tag{10.7}$$

where the last integral is to be taken in the sense of distributions. Let $-g = \beta h^2$; then $\dfrac{g}{h} = \beta h$ belongs to $L^2(0,\infty)$ since $\beta \in L_h^2(0,\infty)$.

Conversely, every distribution of the type (10.7) defines a bounded linear form on $(0,\infty)$ equipped with the norm of $L_h^2(0,\infty)$ and the proof is complete.

Lemma 2 shows in particular that $(\mathring{\mathbb{H}}_h^1)^*$ contains Dirac measures at a point $s_0 \in (0, \infty)$ since

$$\delta(s - s_0) = g',$$

where $g(s) = 1$ when $0 \leqq s \leqq s_0$ and $g(s) = 0$ when $s > s_0$.

Of course this simple fact can be also shown in a direct fashion by using the continuous imbedding $\mathring{\mathbb{H}}_h^1(0, \infty) \subset C^0[0, \infty)$.

§ 11. The Sobolev Space $\mathring{\mathbb{H}}_h^1(0, \infty) \cap \mathbb{H}_h^k(0, \infty)$, $k \geqq 2$

This is any basic domain space in the Hilbertian framework. It is a smaller space than $\mathring{\mathbb{H}}_h^1$. By restricting the domain space, we enlarge the dual.

An argument like the one leading to (9.4) implies that

$$\mathring{\mathbb{H}}_h^1(0, \infty) \cap \mathbb{H}_h^k(0, \infty) \subset C^{k-1}[0, \infty),$$

with a continuous imbedding, where C^{k-1} is equipped with the topology of convergence on every compact subset of $[0, \infty)$. Moreover, $\mathring{\mathbb{H}}_h^1(0, \infty) \cap \mathbb{H}_h^k(0, \infty)$ is dense in $\mathring{\mathbb{H}}_h^1(0, \infty)$ and we can identify $(\mathring{\mathbb{H}}_h^1)^*$ with a subspace of $(\mathring{\mathbb{H}}_h^1 \cap \mathring{\mathbb{H}}_h^k)^*$.

In order to characterize completely the dual of $\mathring{\mathbb{H}}_h^1(0, \infty) \cap \mathbb{H}_h^k(0, \infty)$, we first consider the case of $\mathbb{H}_h^k(0, \infty)$. Define $\tilde{h}(\cdot)$ on $\mathbb{R}$ by $\tilde{h}(s) = h(s)$ for $s \geqq 0$ and $\tilde{h}(s) = 1$ for $s \leqq 0$ and consider the Hilbert space

$$\mathbb{H}_{\tilde{h}}^k(\mathbb{R}) = H_{\tilde{h}}^k(\mathbb{R})^6.$$

Lemma 3. *Each and every element* $T \in (\mathbb{H}_{\tilde{h}}^k(\mathbb{R}))^*$ *can be written (in a non-unique fashion) in the form*

$$T = f_0 + f_1' + \dots + f_k^{(k)},$$

where

$$\frac{f_p}{\tilde{h}} \in L^2(\mathbb{R}), \quad \frac{f_{p|[0,\infty)}}{h} \in L^2(0, \infty), \qquad 1 \leqq p \leqq k.$$

Proof. We follow the proof of Lemma 2 using the fact that $\mathscr{D}(\mathbb{R})$ is dense in $H_{\tilde{h}}^k(\mathbb{R})$ and that $\mathbb{H}_{\tilde{h}}^k(\mathbb{R})$ is isometric to a closed subspace of $(L_{\tilde{h}}^2(\mathbb{R}))^{k+1}$. Therefore, for $T \in (\mathbb{H}_{\tilde{h}}^k(\mathbb{R}))^*$, there exist $f_0, g_1, \dots, g_k$ in $L_{\tilde{h}}^2(\mathbb{R})$ such that

$$Tu = \int_{\mathbb{R}} u f_0 \tilde{h}^2\, ds + \int_{\mathbb{R}} u' g_1 \tilde{h}^2\, ds + \dots + \int_{\mathbb{R}} u^{(k)} g_k \tilde{h}\, ds \qquad \forall u \in H_h^k(\mathbb{R}).$$

Since $\mathscr{D}(\mathbb{R})$ is dense in $\mathbb{H}_{\tilde{h}}^k(\mathbb{R})$, T is determined by its restriction to $\mathscr{D}(\mathbb{R})$ and

$$Tu = \int_{\mathbb{R}} u f_0 \tilde{h}^2\, ds + \sum_{p=1}^{k} \int_{\mathbb{R}} (-1)^p\, (g_p \tilde{h}^2)^{(p)}\, ds, \tag{11.1}$$

where the last integrals are in the sense of distributions. Let $f_p = (-1)^p g/\tilde{h}^2$. Then $\frac{f_p}{\tilde{h}} \in L^2(\mathbb{R})$ and $\frac{f_{p|[0,\infty)}}{h} \in L^2(0,\infty)$. To state our results (Lemma 4) for the space $(H_h^k(0,\infty))^*$, we shall need the following notations:

$$H_{\tilde{h}}^{-k}(\mathbb{R}) \stackrel{\text{def}}{=} (H_{\tilde{h}}^k(\mathbb{R}))^*. \tag{11.2}$$

This is a space of distributions in $\mathbb{R}$ described in Lemma 3. Then we define

$$H_{\tilde{h},[0,\infty)}^{-k}(\mathbb{R}) \stackrel{\text{def}}{=} \{T \in H_{\tilde{h}}^{-k}(\mathbb{R}) \text{ with support in } [0,\infty)\}.$$

For instance, $\delta, \delta', \dots, \delta^{(k-1)}$ are in $H_{\tilde{h}[0,\infty)}^{-k}(\mathbb{R})$. The following lemma characterizes the dual of $H_{\tilde{h}}^k(0,\infty)$:

Lemma 4. *Let the mapping*

$$\pi: (H_h^k(0,\infty))^* \to H_{\tilde{h}}^{-k}(\mathbb{R})$$

be defined by

$$\langle \pi f, v \rangle = \langle f, v_{|(0,\infty)} \rangle .$$

This map is an isomorphism of $(H_h^k(0,\infty))^*$ *onto* $H_{h,[0,\infty)}^{-k}(\mathbb{R})$.

Proof. The proof is an adaptation of a well known one for the usual Sobolev space (*cf.* LIONS (1961)). The map

$$H_{\tilde{h}}^k(\mathbb{R}) \xrightarrow{r} H_h^k(0,\infty),$$

$$v \to v_{|(0,\infty)}$$

is continuous and onto. Its kernel is the space

$$\left\{ v = \tilde{\omega}; \quad \omega \in H_0^k(-\infty,0) \text{ where } \tilde{\omega} = \begin{bmatrix} \omega \text{ in } (-\infty,0) \\ 0 \text{ in } [0,+\infty) \end{bmatrix} \right\}$$

and $H_0^k(-\infty,0)$ is the closure of $\mathscr{D}(-\infty,0)$ in $H^k(-\infty,0)$. The transpose of $r, f \mapsto \pi f$ is an isomorphism from $(H_h^k(0,\infty))^*$ onto $X =$ the orthogonal complement of Ker r in $H_{\tilde{h}}^{-k}(\mathbb{R})$. To characterize this space we note that if $g \in X$, then $\langle g, \tilde{\omega} \rangle = 0 \ \forall \omega \in H_0^k(-\infty,0)$; *i.e.*, g has a support in $[0,\infty)$. Conversely, let $g \in \mathbb{H}_{\tilde{h}}^{-k}(\mathbb{R})$ with support in $[0,\infty)$. If $\omega \in H_0^k(-\infty,0)$, one has

$$\langle g, \tilde{\omega} \rangle = \lim_{j\to\infty} \langle g, \phi_j \rangle, \text{ where } \phi_j \in \mathscr{D}(-\infty,0)$$

and $\phi_j \to \omega$ in $H_0^k(-\infty,0)$. But $\langle g, \tilde{\phi}_j \rangle = 0$; hence $\langle g, \tilde{\omega} \rangle = 0$ and $g \in X$.

We remark that Lemma 4 characterizes the dual of $\mathring{H}_h^1(0,\infty) \cap H_h^k(0,\infty)$ (a closed subspace of $H_h^k(0,\infty)$), since the bounded linear forms on $\mathring{H}_h^1(0,\infty) \cap H_h^k(0,\infty)$ are exactly the restrictions on $\mathring{H}_h^1(0,\infty) \cap H_h^k(0,\infty)$ of bounded linear forms on $H_h^k(0,\infty)$. By Lemmas 4 and 3, $(H_h^k \cap \mathring{H}^1)^*$ contains a subspace whose elements can be represented as

$$T = \alpha_1 \delta^1 + \ldots + \alpha_{k-1} \delta^{(k-1)} + \langle f_j \cdot \rangle_h, \tag{11.3}$$

where $\alpha_1, \ldots, \alpha_{k-1} \in \mathbb{R}$ and $f \in L_h^2$ are uniquely determined by T. *

The distributions $\delta^{(n)}$, $1 \leqq n \leqq k-1$ in (11.3) may be interpreted in the following sense. Every function f in $\mathring{H}(k)$ is (almost everywhere) equal to a function which is $k-1$ right differentiable at $s = 0$ and then

$$\int_0^\infty \delta^{(n)}(s) f(s)\, ds = (-1)^n f^{(n)}(0), \quad n = 1, \ldots, k-1. \tag{11.4}$$

It is of definite interest that the single integral models which are so popular for rheological studies are in general not defined on all elements of $\mathbb{L}_h^2$ but are well defined in $\mathring{\mathbb{H}}^1 \cap \mathbb{H}^k$ for $k \geqq 1$. Consider an integral of the form

$$\int_0^\infty k(s)\, G^p(s)\, ds, \tag{11.5}$$

where, for simplicity, $G \in H_h^1$ is a scalar-valued function. Since $G(\cdot)\, h(\cdot)$ is in $L^2(0, \infty)$,

$$[G(\cdot)\, h(\cdot)]' = G'(\cdot)\, h(\cdot) + G(\cdot)\, h'(\cdot) \in L^2(0, \infty)$$

provided that

$$|h'(s)| \leqq C\, |h(s)| \tag{11.6}$$

for sufficiently large values of s (recall that $h'(s)$ is locally bounded and G^2 locally integrable). The inequality (11.6) holds for example when $h(s) = e^{-\beta s}$, $\beta > 0$ or $h(s) = \dfrac{1}{(1+s)^\beta}$, $\beta > 1$. Thus by the Sobolev imbedding theorem, we have

$$|G(s)\, h(s)|^p \leqq C\, \|Gh\|_{H^1(0,\infty)}^p \leqq C\, \|G\|_{H_h^1(0,\infty)}^p$$

and

$$\left| \int_0^\infty k(s)\, G^p(s)\, ds \right| \leqq C\, \|G\|_{H_h^1(0,\infty)}^p \left\| \frac{k}{h^p} \right\|_{L^1(0,\infty)}$$

for $\forall s \geqq 0$ provided that the decay of $k(s)$ is fast enough so that

$$\frac{k}{h^p} \in L^1(0, \infty). \tag{11.7}$$

It follows that (11.5) makes sense and is a p-linear continuous form on H_h^1 when (11.7) holds.

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* *Cf.* (12.1). Lemma 4 also shows that there are elements in the dual which cannot be represented as in (11.3).

§ 12. Some LF Spaces and their Duals

We now consider the totality of forms for the first Gateaux derivative $\mathscr{T}_1[0 \mid \boldsymbol{u}(s)]$ when the domain

$$\operatorname{dom} \mathscr{F} = \mathscr{H}(k) = \{\boldsymbol{u} \in \mathbb{L}_h^2(0, \infty), \quad u_{|(0,T)} \in \mathbb{H}^k(0, T) \text{ for some } T > 0\}.$$

We shall now describe the natural topology of $\mathscr{H}(k)$. This space is not a normable space; neither is it a Fréchet space. It is a strictly denumerable inductive limit of Fréchet (in fact, Hilbert) spaces, a so-called LF space (details on LF spaces are given for instance in the book of TRÊVES (1967), p. 126). The inductive limit we have in mind may be described as follows. Let $T_n > 0$ be a sequence of positive numbers, decreasing to 0 as $n \to \infty$, and define

$$\mathscr{H}_n(k) = \{\boldsymbol{u} \in \mathbb{L}_h^2(0, \infty), u_{|(0,T_n)} \in \mathbb{H}^k(0, T_n)\}.$$

This is a Hilbert space with norm

$$\|u\|_{\mathscr{H}_n(k)} = [\|u\|^2_{\mathbb{L}_h^2(0,\infty)} + \|u\|^2_{\mathbb{H}^k(0,T_n)}]^{\frac{1}{2}}.$$

Clearly $\mathscr{H}_n(k) \subset \mathscr{H}_{n+1}(k)\ \forall n$, and $\mathscr{H}(k) = \bigcup_n \mathscr{H}_n(k)$ equipped with its natural topology of LF space. A convex neighborhood[1] of 0 may be defined as follows: a convex set V in $\mathscr{H}(k)$ is a neighborhood of 0 if and only if $V \cap \mathscr{H}_n(k)$ is a neighborhood of 0 in $\mathscr{H}_n(k)$ for each and every n; that is, V contains a ball centered in 0, in $\mathscr{H}_n(k)$.

By this procedure, $\mathscr{H}(k)$ is a locally convex topological vector space (LCTVS) with the following properties:

1. The topology induced by $\mathscr{H}(k)$ on $\mathscr{H}_n(k)$ is exactly the Hilbert topology of $\mathscr{H}_n(k)$.
2. $\mathscr{H}(k)$ is complete: every Cauchy filter is convergent (we shall make no use of that).
3. A linear form on $\mathscr{H}(k)$ is continuous if and only if its restriction to $\mathscr{H}_n(k)$ is continuous for every n.
4. Every function in $\mathscr{H}(k)$ is (almost everywhere) equal to a function which is C^{k-1} in a (variable) right neighborhood of 0. In particular, a function in $\mathscr{H}(k)$ is right C^{k-1} at 0.
5. Although $\mathscr{D}(0, \infty)$ is dense in $\mathscr{H}(k)$, $\mathscr{D}(0, \infty)$ is not. Hence the dual of $\mathscr{H}(k)$ is not a space of distributions on $(0, \infty)$.
6. A sequence $\boldsymbol{u}_p \in \mathscr{H}(k)$ which tends to 0 in $\mathscr{H}(k)$ can be described as follows. We denote by $B_n(\varrho)$ the ball of center 0 and radius ϱ in the Hilbert space $\mathscr{H}_n(k)$. Then $u_p \to 0$ in $\mathscr{H}(k)$ if and only if for every $\varrho > 0$ there exists a number $N(\varrho)$ such that for $p \geqq N(\varrho)$, $u_p \in \bigcup_n B_n(\varrho)$.

We use property 3 to describe the dual space of $\mathscr{H}(k)$. First we note that the dual of $\mathscr{H}_n(k)$ is easily seen to be $L_h^2(0, \infty) + (H^k(0, T_n))^*$, where $(H^k(0, T_n))^*$ is isomorphic to the subspace of those distributions in $H^{-k}(\mathbb{R})$ ($=$ dual of $H^k(\mathbb{R})$) having support on $[0, T_n]$. (See the proof of Lemma 4.)

[1] They form a basis of neighborhoods of 0.

Thus by property 3, $(\mathscr{H}(k))^* = L_h^2(0,\infty) +$ the space of distributions in $H^{-k}(\mathbb{R})$ with $\{0\}$ as support. But this space of distributions is exactly $\mathbb{R}\,\delta \oplus \mathbb{R}\,\delta' \oplus \ldots \oplus \mathbb{R}\,\delta^{(k-1)}$.

Hence

$$[\mathscr{H}(k)]^* \simeq L_h^2(0,\infty) \oplus \mathbb{R}\,\delta \oplus \ldots \oplus \mathbb{R}\,\delta^{(k-1)}.$$

We are in fact interested in the space

$$\mathring{\mathscr{H}}(k) = \{u \in \mathscr{H}(k), u(0) = 0\}.$$

This closed subspace $\mathring{\mathscr{H}}(k)$ of $\mathscr{H}(k)$ has similar properties.

Theorem. *Every bounded linear form on* $\mathring{\mathscr{H}}(k)$ *can be represented (in a unique fashion) as*

$$T = \alpha_1\,\delta' + \alpha_2\,\delta'' + \ldots + \alpha_{k-1}\,\delta^{(k-1)} + \langle f, \cdot\rangle_h, \tag{12.1}$$

where

$$\alpha_1, \ldots, \alpha_{k-1} \in \mathbb{R}, \quad f \in L_h^2.$$

The formula (12.1) is to be interpreted in the same sense as (11.3). It follows from (12.1) that elements of $\mathring{\mathscr{H}}(k+1)$ are expressible by tensor-valued linear forms $\mathscr{F}_1\left[B(t)\,\Big|\,\overset{\infty}{\underset{s=0}{G(s)}}\right]$ whose components relative, say, to an orthonormal basis $\{e_i\}$ are given for $G \in \mathring{\mathscr{H}}(k+1)$ by

$$(\mathscr{F}_1)_{ij} = (\mathscr{F}_1)_{ji} = \int_0^\infty \hat{K}_{ijmn}(B(t), s)\, G_{mn}(s)\, ds, \tag{12.3}$$

where

$$\hat{K}_{ijmn}(B(t), s) = \tilde{K}_{ijmn}(B(t), s) + \sum_{l=1}^{k} \mu_{ijmn}(l, B(t))\, \delta^{(l)}(s) \tag{12.4}$$

and the $\mu_{ijmn}(l, B(t))$ are $k-1$ functions of $B(t)$ and

$$\tilde{K}_{ijmn}(B(t), s)/h(s) \in L^2(0,\infty).$$

It follows now from (12.2, 3, 4) that

$$(\mathscr{F}_1)_{ij} = \sum_{l=1}^{k} \mu_{ijmn}(l, B(t))\, [A_l(t)]_{mn} + \int_0^\infty \tilde{K}_{ijmn}(B(t), s)\, G_{mn}(s)\, ds. \tag{12.5}$$

Of course (12.5) may be simplified when the material is isotropic (see (6.6)).

GREEN & RIVLIN (1960) have also noted that terms involving the tensors $A_l(t)$ may be obtained from integrals with kernels expressed with Dirac measures. They interpret (12.2) as

$$\lim_{\varepsilon \downarrow 0} \int_{-\varepsilon}^{\infty} \delta^{(n)}(s) f(s)\, ds = (-1)^n f^{(n)}(0),$$

where $f(s)$ has been extended into $s < 0$. This extension is in the spirit of the extensions to $\mathbb{R}$ which we have discussed in Lemmas 3 and 4 and in the analysis leading to (12.1).

§ 13. Fluids of Mixed Type of Order n, degree k, $n = 1$

In all that follows we shall confine our attention to incompressible simple fluids. We are going to admit that there are real physical fluids which, like Navier-Stokes fluids, appear not to allow shocks. In fact, we mathematize our intuitions by assuming that in the recent past, near $s = 0$, there are no motions whose histories are less smooth than those for which

$$G \in \mathring{\mathscr{H}}(k+1). \tag{13.1}$$

In this case, after using isotropy to simplify the result and omitting terms proportional to $\mathbf{1}$, we find that

$$\begin{aligned} \mathscr{F}_1[G] &= \int_0^\infty \hat{\mu}(s)\, G(s)\, ds \\ &= \int_0^\infty [\mu_1\, \delta' + \mu_2\, \delta'' + \dots + \mu_k\, \delta^{(k)} + \tilde{\mu}(s)]\, G(s)\, ds \\ &= \mu_1 A_1(t) + \mu_2 A_2(t) + \dots + \mu_k A_k(t) + \int_0^\infty \tilde{\mu}(s)\, G(s)\, ds, \end{aligned} \tag{13.2}$$

where μ_l, $l = 1, 2, \dots, k$ are constants, trace $A_1 = 0$ and

$$\tilde{\mu}(\cdot)/h(\cdot) \in L^2(0, \infty).$$

The same representation (13.2) is valid for G in the Sobolev space $\mathring{\mathbb{H}}_h^1 \cap \mathbb{H}_h^{k+1}$. If $\tilde{\mu}(\cdot) = 0$, then (13.2) is a fluid of rate type, degree k. On the other hand, if all $\mu_l = 0$, then (13.2) reduces to an integral fluid of order 1. We have already explained that the μ_l are necessarily zero if $G(\cdot) \in \mathbb{L}_h^2(0, \infty)$.

A theory of slow motions and a theory of moments may be derived from constitutive equations of mixed type from the integrals given in § 8. To know the explicit content of this theory at first order, we need to investigate the consequences of the formula

$$\mu(s) = \mu_1\, \delta'(s) + \dots + \mu_k\, \delta^{(k)}(s) + \hat{\mu}(s). \tag{13.3}$$

Then, with

$$\begin{aligned} \varkappa_n &\overset{\text{def}}{=} \frac{(-1)^n}{n!} \int_0^\infty \mu(s)\, s^n\, ds, \\ [\varkappa_1, \varkappa_2, \varkappa_3, \varkappa_4, \dots] &\overset{\text{def}}{=} [\mu, \alpha_1, \beta_1, \gamma_1, \dots], \end{aligned} \tag{13.4}$$

we find, using (13.8), that

$$\begin{aligned} \varkappa_n &= \mu_n + \tilde{\varkappa}_n, \\ \varkappa_n &= \hat{\varkappa}_n, \quad n > k, \end{aligned} \tag{13.5}$$

where

$$\varkappa_n \stackrel{\text{def}}{=} \frac{(-1)^n}{n!} \int_0^\infty \mu(s)\, s^n\, ds.$$

Now we shall make some observations about the constants $\tilde{\mu}_l$ and the "smooth" kernel $\tilde{\mu}(s)$ which define "fluids of degree k, order 1".

(i) The theory of slow, nearly steady motions may still be defined in terms of moments of kernels, even though the kernels contain derivatives of Dirac measures. For analytic histories

$$\mathscr{F}_1 = \sum_{n=1}^{\infty} \varkappa_n A_n(t).$$

In fact, finite approximations of k terms

$$\mathscr{T}_1(k) = \sum_{n=1}^{k} \varkappa_n A_n(t), \tag{13.6}$$

where the $A_n(t)$, $n \leqq k$, are well defined in $\mathring{\mathscr{H}}(k+1)$ but need not exist in $\mathbb{L}_h^2$. Equation (8.7) giving the complex viscosity in terms of the moments $\varkappa_n$ also survives under the change of topology.

(ii) In the $\mathbb{L}_h^2$ theory the sign of the kernel $\hat{\mu}(s) < 0$ and its derivative $\mu'(s) > 0$ for $s \geqq 0$ is fixed by a stability argument leading to Equation (8.8) where $\hat{g}'(s) \stackrel{\text{def}}{=} \hat{\mu}(s)$. When dom $\mathscr{F}_1 = \mathring{\mathscr{H}}(k+1)$, Dirac measures appear in the kernels as in (13.3), or

$$\hat{g}(s) = \mu_1\, \delta(s) + \mu_2\, \delta'(s) + \dots + \mu_k\, \delta^{(k-1)} + \tilde{g}(s), \tag{13.7}$$

where $g'(s) = \tilde{\mu}(s)$. It follows then from (8.8) and (13.7) that the spectral values σ satisfy

$$\frac{\sigma}{\Lambda_n} = \mu_1 - \mu_2\sigma + \mu_3\sigma^2 + \dots + \mu_k(-1)^{k-1}\, \sigma^{k-1} + \int_0^\infty \tilde{g}(s)\, e^{\sigma s}\, ds. \tag{13.8}$$

The rest state will be stable in the spectral sense if the constants μ_l and the function $g(\cdot)$ are such that the roots of (13.7) have positive and only positive real parts for all $\Lambda_n > 0$.

(iii) The signs of the constants μ_n and the functions $\mu(s)$ determine the sign of the moments $\varkappa_n$. In the $L_h^2(0,\infty)$ theory of COLEMAN & NOLL, $\mu_l = 0$ and $\tilde{\mu}(s) = \hat{\mu}(s)$ is negative. Hence the $\varkappa_n$ defined in (13.4) are of alternating sign. This fact was used by JOSEPH to conclude the rest state of fluids of arbitrary grade or complexity is unstable. The indeterminancy of the signs of μ_n and $\varkappa_n$ in (13.5) shows that the conclusions there apply strictly only where they can be proved strictly; for example, in L_h^2 and not in $\mathring{\mathscr{H}}(k+1)$ (but see (v) below).

(iv) It may be assumed that $\varkappa_1 = \mu_1$, the shear viscosity at zero shear, is positive. If $\tilde{\mu}(s)$ is negative for $s \geqq 0$, then $\varkappa_n$ is of alternating sign and the integral inequalities like (8.15) are replaced by inequalities among the moments; for example,

$$-\varkappa_2 < \varkappa_1^{\frac{1}{2}}\left(\frac{3}{2}\varkappa_3\right)^{\frac{1}{2}}. \tag{13.9}$$

On the other hand if the $\varkappa_n$ have the sign implied by the $L_h^2(0,\infty)$ theory of COLEMAN & NOLL, the integral inequalities like (8.5) still hold and, for example, imply that

$$-(\mu_2+\varkappa_2) < (\mu_1+\varkappa_1)^{\frac{1}{2}}\left(\frac{3}{2}\mu_3+\frac{3}{2}\varkappa_3\right)^{\frac{1}{2}}. \tag{13.10}$$

(v) There are several reasons to give special consideration to the case where the signs of the moments $\varkappa_n$ are preserved under the change of topology which leads to fluids of mixed type. This procedure is consistent with the idea that rate terms can arise from integrals with good kernels in a continuous way. It is possible to define sequences of smooth kernels which have the required Dirac measures in the limit. For example,

$$\hat{\mu}_{(l)}(s) = \mu_1\,\delta_l'(s) + \mu_2\,\delta_l''(s) + \ldots + \mu_k\,\delta_l^{(k)}(s) + \tilde{\mu}(s), \tag{13.11}$$

where

$$\delta_l(s) = e^{-ls^2}(l/\pi)^{\frac{1}{2}}$$

leads to (13.3) as $l\to\infty$. The moments $\varkappa_n$ of $\mu_{(l)}(s)$ are expressible as $\varkappa_n = \mu_n + \tilde{\varkappa}_n$, $\mu_n = 0$ when $n > k$ for each and every integer $l > 0$. Clearly the $\varkappa_n$ are of oscillating sign; $\varkappa_1 > 0$, if $\mu(s) < 0$, $\mu'(s) > 0$ for $s \geqq 0$ and μ_n has the same sign as $\varkappa_n$. This same choice of sign preserves stability as the following argument shows:

Expand $e^{\sigma s}$ in (13.8) in powers of s. Then, using (13.5), we have

$$\frac{\sigma}{\Lambda} = \sum_{m=1}^{\infty}\varkappa_m\sigma^m = \int_0^{\infty}\nu(s)\,e^{\sigma s}\,ds$$

for some kernel $\nu(s)$ with moment

$$\varkappa_m = \frac{(-1)^{m-1}}{(m-1)!}\int_0^{\infty} s^{m-1}\,\nu(s)\,ds,$$

where

$$\varkappa_m = \mu_m + \tilde{\varkappa}_m$$

and

$$\tilde{\varkappa}_m = \frac{(-1)^{m-1}}{(m-1)!}\int_0^{\infty} s^{m-1}\tilde{g}(s)\,ds.$$

For stability we need $\nu(s) > 0$, $\nu'(s) < 0$. These conditions are satisfied by $\nu(s)$ given by (13.11) $\nu'(s) = \mu_{(l)}(s)$ for large values of l when $(\varkappa_m, \mu_n, \tilde{\varkappa}_m)$ have the same alternating sign.

(vi) The argument given in (v) is open to the following criticism. Put $\tilde{\mu}(s) = 0$ in (13.2) and take the constants $\mu_n = \varkappa_n$ of alternating sign, as in (v). This choice is not disallowed by our theory. It is known from a theorem of JOSEPH (1981) that the rest state of such a fluid is unstable (in the spectral sense of linear theory). Since there are no real fluids with unstable rest states, the assumption that μ_n is of alternating sign, which was discussed in (v) is open to criticism.

§ 14. Fluids of Mixed Type of order n, grade k. Higher Order Theory

Consider the COLEMAN-NOLL constitutive equation of integral type of order n (see § 7 and § 37 of TRUESDELL & NOLL (1965) for further discussion of order n). This constitutive equation is a possibility whenever

$$\text{dom}\,\mathscr{F} = \mathbb{L}_h^2(0,\infty). \tag{14.1}$$

We now define a fluid of mixed type of order n, degree k, as an integral fluid of order n such that

$$\text{dom}\,\mathscr{F} = \mathring{\mathscr{H}}(k+1) \tag{14.2}$$

or

$$\text{dom}\,\mathscr{F} = \mathring{\mathbb{H}}_h^1 \cap \mathbb{H}_h^{k+1}. \tag{14.3}$$

When (14.1) holds, there can be no Dirac measures in the kernels. But when (14.2) or (14.3) holds there may be Dirac measures in the kernels (see (12.1) and (13.2)). When the kernel function in the integral is zero the fluids of mixed type reduce to fluids of differential type which depend on the Rivlin-Ericksen tensors through order k and constants.

To obtain explicit formulas for the higher order theory, it is sufficient to express the kernels in the integral fluids of order n in terms of Dirac distributions. For example, if

$$\text{dom}\,\mathscr{T} = \mathring{\mathbb{H}}_h^1 \cap \mathbb{H}_h^3$$

then integral expressions for $\mathscr{T}_2[G \mid G]$ of the form

$$\int_0^\infty \hat{\nu}(s_1, s_2)\, G(s_1)\, G(s_2)\, ds_1\, ds_2$$

may be reduced to

$$\begin{aligned} &\nu_1 A_1^2 + \nu_2 A_2 + \nu_{12}(A_1 A_2 + A_2 A_1) \\ &\quad + \int_0^\infty \nu_1(s)\,[A_1(t)\,G(s) + G(s)\,A_1(t)]\,ds \\ &\quad + \int_0^\infty \nu_2(s)\,[A_2(t)\,G(s) + G(s)\,A_2(t)]\,ds \\ &\quad + \int_0^\infty \left[\nu_3(s)\,G(s)\,ds\right]\left[\int_0^\infty \nu_4(s)\,G(s)\,ds\right] \\ &\quad + \int_0^\infty \nu_5(s)\,G(s)\,G(s)\,ds \\ &\quad + \int_0^\infty \int_0^\infty \nu(s_1, s_2)\,G(s_1)\,G(s_2)\,ds_1\,ds_2, \end{aligned} \tag{14.3}$$

where the kernels are "$L_h^2(0, \infty)$ smooth". Since general integral representations for bounded multilinear forms of order $n > 1$ are unknown, (14.3) follows only if integrals are assumed.

§ 15. Dynamics

We have looked at constitutive theory from the point of view of functional analysis and we have discussed the implications of choosing different topologies for the domain of $\mathscr{F}[\cdot]$. But now we must add that it is not possible to know which are the good choices of topology without considering dynamics. Choosing a topology is equivalent to choosing an allowed class of deformations. From this choice we get constitutive equations and explicit equations of motion. Our original choice of topology is an assumption, a guess about what will be good. Now we have to check our guess and see if the solutions of the equations are compatible with the deformations we assumed. For example, discontinuous solutions and shocks are allowed in the $\mathbb{L}_2^h(0, \infty)$ theory of COLEMAN & NOLL but are disallowed in our $\overset{\circ}{\mathscr{H}}(k+1)$, $k \geqq 1$ theory. It is not however at first obvious that the equations of motion in $\mathbb{L}_2^h$ allow discontinuities and shocks, or that the equations of motion in $\overset{\circ}{\mathscr{H}}(k+1)$ disallow them. Thus in addition to everything else in the theory of simple materials we need to examine the consistency of constitutive equations with dynamics.

The analysis developed in this section is an example of the type of dynamic analysis which is important in the discussion of constitutive theory. We are going to consider two linearized problems of shearing flow. We are interested in the propagation into the interior of a fluid of shearing discontinuities which are prescribed initially at the boundary of the fluid (see Figure 15.1). We show that such discontinuities do propagate in some COLEMAN-NOLL fluids of integral type but are smoothed in fluids of mixed type, so the computed dynamics in both theories are not inconsistent with the assumed choice of topology.

We turn now to two problems of shearing flow in a semi-infinite domain $0 \leqq x \leqq \infty$ above a rigid wall at $x = 0$. The first problem is well-known as Stokes's first problem or Rayleigh's problem. In this problem one studies the response of the fluid to a step jump from rest in the *velocity* of the boundary at $x = 0$. The problem is classical for Newtonian fluids and also finds application in the theory of rheometry of viscoelastic fluids (see pages 154–157 of BIRD, ARMSTRONG & HASSAGER, 1977). The rheological theories are greatly oversimplified. They *assume* that a *uniform* strain is achieved instantly at the moment of inception of motion without studying dynamics. In fact, the dynamical problem was studied by TANNER (1962) for an Oldroyd fluid of type *B*. We reduce our problem to TANNER's when the smooth kernel $\hat{g}(s) = ke^{-\lambda s}$ is of exponential type. Similar problems for exponential kernels have been studied by KAZAKIA & RIVLIN (1981). Some recent results of NARAIN & JOSEPH (1982a) and RENARDY (1982) show that discontinuities will propagate with speed $c = \sqrt{G(0)/\varrho}$ and amplitude $a = \exp\left[x\varrho^{\frac{1}{2}}\, G'(0)/2G_0^{\frac{3}{2}}\right]$. If $G(0) < \infty$ and $G'(0) = -\infty$, the boundary of the support of the solution will propagate with speed C but the amplitude $a = 0$. In some cases the zero and non-zero parts of the solution are C^∞ continuous where they join. Nearly identical considerations are of interest in the theory of viscoelastic solids (see NARAIN & JOSEPH, 1982b).

In the second problem we study the response of fluid to a step jump in displacement from rest to rest. This problem has apparently not been studied even in the Newtonian case. It is an interesting and even important problem with good applications whose solution can be obtained cheaply as the partial time derivative of Stokes's first problem. The version of impulsive displacement problem in which the fluid is confined between parallel planes, with the step displacement prescribed at the bottom and zero displacement at the top, is the dynamical problem for the celebrated rheological test associated with step shear strains (see, for example, BIRD, ARMSTRONG & HASSAGER, pages 161–163). In the theory used for that experiment, it is assumed that a uniform strain can be obtained instantaneously after the step in displacement. But dynamics does not give instantaneous uniform strains except possibly in cases in which inertia is negligible (some polymer melts). In many real materials (polymer solutions) the strain and stress distributions are transient and the transient stresses one measures under the assumption of uniform strain, may actually decay long before the strain is uniform. Suppose that

$$\boldsymbol{v}(\boldsymbol{x}, t) = \boldsymbol{e}_y v(x, t)$$

is the fluid velocity in $0 \leqq x < \infty$, $-\infty < y < \infty$ and

$$v(x, t) = 0 \text{ for } t < 0$$

as in Figure 15.1 below.

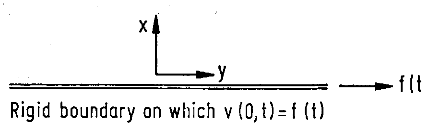

Fig. 15.1. Two shearing problems
(i) There is a step increase in velocity in which $f(t) = H(t)$ is a Heaviside function.
(ii) There is a step increase in displacement in which $f(t) = \delta(t) = dH/dt$ is a delta "function".

We shall suppose that $G \in \overset{\circ}{\mathscr{H}}(2)$. Then

$$(15.1) \qquad \mathscr{F}_1 = \mu_1 A_1(t) + \int_0^\infty \tilde{\mu}(s)\, G(s)\, ds = \mu_1 A_1(t) + \int_0^\infty \tilde{g}(s)\, \dot{G}(s)\, ds.$$

The last term of (15.1) arises after an integration by parts which is not allowed $\forall G \in \overset{\circ}{\mathscr{H}}(2)$ but is allowed for the special histories which we shall consider in shearing problems of this section. When $G \in \mathbb{L}^2_h(0, \infty)$, then $\mu_1 = 0$. Linearization of $\dot{G}$ for small velocities or displacements gives

$$\dot{G} \to A_1(t - s)$$

where

$$[A_1(t-s)]_{ij} = \frac{\partial v_i}{\partial x_j}(\boldsymbol{x}, t-s) + \frac{\partial v_j}{\partial x_i}(\boldsymbol{x}, t-s)$$

is twice the symmetric part of the velocity gradient at the past time $t - s$ but at the present place $\boldsymbol{x}$. Hence, under complete linearization

$$\mathscr{F}_1 = \mu_1 A_1(t) + \int_0^\infty \tilde{g}(s)\, A_1(t-s)\, ds. \tag{15.2}$$

For incompressible fluids, $\operatorname{div} v = 0$ and the linearized equations of motion are in the form

$$\frac{\partial \boldsymbol{v}}{\partial t} = -\nabla p + \mu_1 \Delta \boldsymbol{v} + \int_0^\infty \tilde{g}(s)\, \Delta \boldsymbol{v}(t-s)\, ds. \tag{15.3}$$

We want to consider (15.3) for motions $\boldsymbol{v} = e_x v(x, t)$ which are driven from the boundary by shears, without pressure forces, $\nabla p = 0$. We assume that

$$\left.\begin{aligned} v(0, t) &= 0 \\ v(x, t) &= 0 \end{aligned}\right\} \quad t \leqq 0. \tag{15.4}$$

Then

$$v(x, t-s) = 0 \text{ for } s = t - \tau \geqq t$$

and

$$\int_0^\infty \tilde{g}(s)\, \Delta v(x, t-s)\, ds = \int_0^t \tilde{g}(s) \frac{\partial^2 v}{\partial x^2}(x, t-s)\, ds$$

and (15.3) may be written as

$$\varrho \frac{\partial v}{\partial t}(x, t) = \mu_1 \frac{\partial^2 v}{\partial x^2}(x, t) + \int_0^t \tilde{g}(s) \frac{\partial^2 v}{\partial x^2}(x, t-s)\, ds. \tag{15.5}$$

$(15.4)_2$ and (15.5) imply that $v_t(x, 0) = 0$ so that we may consider an initial boundary-value problem consisting of (15.5) and

$$\begin{aligned} v(0, t) &= f(t), \quad t > 0, \\ v(x, 0) &= v_t(x, 0) = 0. \end{aligned} \tag{15.6}$$

We may study shearing flows due to a step in velocity

$$f(t) = H(t) = \begin{cases} 0, & t < 0, \\ 1, & t > 0 \end{cases} \tag{15.7}$$

or a step in displacement

$$f(t) = \delta(t) = \frac{dH(t)}{dt}, \tag{15.8}$$

where $\delta(t)$ is the delta "function".

Now we show that *if* $v(x,t)$ *satisfies* (15.5), (15.6) *and* (15.7), *then* $v_t(x,t)$ *satisfies* (15.5), (15.6) *and* (15.8). To show this we differentiate (15.5) partially with respect to t, holding x fixed, and find that

$$\varrho\frac{\partial v_t}{\partial t}=\mu_1\frac{\partial^2 v_t}{\partial x^2}+\int_0^t \tilde{g}(s)\frac{\partial^2 v_t}{\partial x^2}(x,t-s)\,ds+\tilde{g}(t)\frac{\partial^2 v_t}{\partial x^2}(x,0). \tag{15.9}$$

Since $(15.6)_2$ implies that $\frac{\partial^2 v_t}{\partial x^2}(x,0)=0$, (15.9) shows $v_t(x,t)$ satisfies (15.5) and $v_{tt}(x,0)=0$. This proves our assertion and shows that

$$u(x,t)=v_t(x,t)$$

is the solution of the problem of step displacement. If

$$\frac{dy(x,t)}{dt}=u(x,t)$$

is a velocity, and $u(0,t)=\delta(t)$, then

$$y(0,t)=H(t).$$

Now we shall show that the prescribed discontinuities at the boundary propagate into the interior $x>0$ when $\mu_1=0$ and are smoothed when $\mu_1>0$. Suppose that

$$\tilde{g}(s)=ke^{-\lambda s}=ke^{-\lambda(t-\tau)}.$$

Then we may write (15.5) as

$$\varrho\frac{\partial v}{\partial t}=\mu_1\frac{\partial^2 v}{\partial x^2}+k\int_0^t e^{-\lambda(t-\tau)}\frac{\partial^2 v}{\partial x^2}(x,\tau)\,d\tau. \tag{15.11}$$

Differentiation of (15.11) with respect to t leads to

$$\varrho\frac{\partial^2 v}{\partial t^2}=\mu_1\frac{\partial^3 v}{\partial x^2\,\partial t}+(k+\lambda\mu_1)\frac{\partial^2 v}{\partial x^2}-\varrho\lambda\frac{\partial v}{\partial t}, \tag{15.12}$$

where v satisfies (15.6) and $v_{tt}(x,0)=0$. We may write the problem in dimensionless form by introducing dimensionless variables x, t measured in units of $1/\lambda$, $\sqrt{(k+\lambda\mu_1)/\lambda\varrho}$. Thus

$$\frac{\partial^2 v}{\partial t^2}+\frac{\partial v}{\partial t}=\frac{\partial^2 v}{\partial x^2}+a\frac{\partial^3 v}{\partial t\,\partial x^2}, \tag{15.13}$$

where

$$a=\lambda\mu_1(k+\lambda\mu_1),\quad 0\leqq a\leqq 1,$$

$$a=0 \text{ when } \mu_1=0,$$

$$a=1 \text{ when } k=0.$$

TANNER (1962) solved (15.13) by Laplace transforms for the conditions

$$(15.14)\qquad \begin{cases} v_t(x,0)=0, \\ v_{tt}(x,0)=0, \end{cases}$$

$$(15.15)\qquad \begin{cases} v(\infty,t)=0, \\ v(0,t)=H(t) \end{cases}$$

associated with step changes in velocity. The same solution holds when (15.14) is replaced by

$$(15.16)\qquad \begin{cases} v(x,0)=0, \\ v_t(x,0)=0. \end{cases}$$

We want to solve (15.13), (15.15) and (15.16).

The Laplace transform of $v(x, t)$ is

$$(15.17)\qquad w(x,s)=\frac{1}{s}\exp\left\{-x\sqrt{\frac{s(1+s)}{1+as}}\right\}.$$

This expression has a pole at $s=0$, branch points at $s=0$, $s=-1$, and $s=-1/a$ and an essential singularity at $s=-1/a$. The inverse of this Laplace tranform is

$$(15.18)\qquad v(x,t,a)=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\exp\left\{st-x\sqrt{\frac{s(1+s)}{1+as}}\right\}\frac{ds}{s},$$

where $\gamma>0$.

The case $a=1$ corresponds to $k=0$. In this case (15.13) can be derived from Stokes's first problem for Newtonian fluids where

$$(15.19)\qquad \frac{\partial v}{\partial t}=\frac{\partial^2 v}{\partial x^2}$$

and

$$(15.20)\qquad v(x,t,1)=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\exp\{st-x\sqrt{s}\}\frac{ds}{s}=\operatorname{erfc}\left[\frac{x}{2\sqrt{t}}\right].$$

There is no discontinuity in this solution for $t>0$.

The case $a=0$ corresponds to COLEMAN-NOLL fading memory. In this case the solution of (15.13), (15.15) and (15.16) is known (CARSLAW & JAEGER, 1963, p. 201) and can be written in terms of Heaviside's function $H(t-x)$

$$(15.21)\qquad v(x,t,0)=\left\{e^{-x/2}+\frac{x}{2}\int_x^t\frac{e^{-\eta/2}}{(\eta^2-x^2)}I_1\left[\frac{1}{2}(\eta^2-x^2)^{\frac{1}{2}}\right]d\eta\right\}H(t-x).$$

The step change in velocity (the vortex sheet) propagates toward increasing x with speed one. The fluid ahead of the wave, $x>t$ is always at rest. The amplitude $v(x,x)=e^{-x/2}$ of the step change decays to zero as $x=t$ tends to infinity (see Figure 15.2).

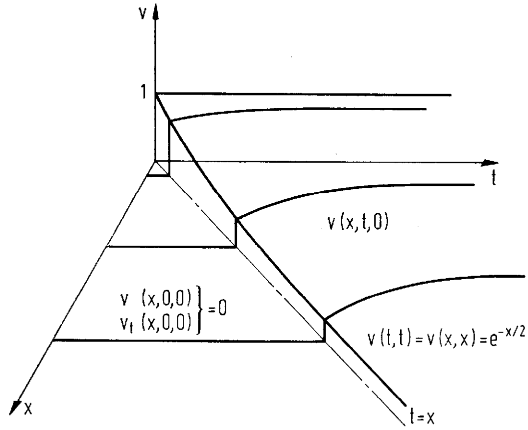

Fig. 15.2. Solution surface $v(x, t, 0)$ given by (15.20) for the problem of a step change in velocity. The solution is discontinuous at $x = t$ when $a = 0$ and it has a narrow propagating transition layer when a is small.

For $0 < a \leqq 1$ the solution in the interior is smoothed and the boundary data is felt instantly throughout the fluid, as in (15.19). TANNER calculated $v(x, t, a)$ from (15.18) for $a = 0.4$ and $a = 0.2$. His graphs suggest that the smooth solution approaches the discontinuous one with a narrow transition layer around the discontinuity at $x = t$, $a = 0$. In fact our analysis shows that the particular evaluation of (15.18) which was used by TANNER is valid for $0 \leqq a \leqq 1$, that is, even at $a = 0$.

Now we shall show that when $a > 0$ is small there is a transition layer of thickness a at $x = t$. To prepare our derivation we shall first consider several preliminary problems:

$$\hat{v}(x, 0) = \hat{v}_t(x, 0), \quad \hat{v}(0, t) = H(t), \tag{15.22}$$

$$\frac{\partial^2 \hat{v}}{\partial t^2} = \frac{\partial^2 \hat{v}}{\partial x^2}, \tag{15.23}$$

where v is bounded at $x \to \infty$. The solution of (15.22) and (15.23) is a propagating Heaviside function

$$\hat{v}(x, t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \exp[tu - xu] \frac{du}{u} = H(t - x). \tag{15.24}$$

Now compare (15.22) and (15.23) with (15.13), (15.15) and (15.16). These problems differ when $a = 0$ only by the dissipative term $\partial v/\partial t$ which is missing on the left of (15.23). In fact, this dissipative term is the one which causes the "shock" amplitude to decay (compare Figure 15.2 with a propagating Heaviside function (14.25)).

Now consider our original problem (15.13), (15.15) and (15.16) for small values of a and introduce new variables

$$\begin{cases} x = a\tilde{x}, \\ t = a\tilde{t}, \\ \tilde{v}(\tilde{x}, \tilde{t}, a) = v(x, t, a). \end{cases} \tag{15.25}$$

For $\tilde{v}$, we have (15.15), (15.16) where x, t and replaced with $\tilde{x}$ and $\tilde{t}$ and

$$\frac{\partial^2 \tilde{v}}{\partial \tilde{t}^2} + a\frac{\partial \tilde{v}}{\partial \tilde{t}} = \frac{\partial^2 \tilde{v}}{\partial \tilde{x}^2} + \frac{\partial^3 \tilde{v}}{\partial \tilde{t}\, \partial \tilde{x}^2}. \tag{15.26}$$

It is not hard to ascertain that

$$\tilde{v}(\tilde{x}, \tilde{t}, 0) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} \exp\left[\tilde{t}u - \frac{\tilde{x}u}{\sqrt{1+u}}\right]\frac{du}{u}. \tag{15.27}$$

In fact, we may derive (15.27) from the differential equations problem for $\tilde{v}$, using Laplace transforms, or from $v(x, t, a)$ of (15.18), by putting $as = u$ and letting $a \to 0$ for a fixed u. We can assert that (15.27) gives the transition layer solution for the propagating Heaviside function (15.24). To prove this, put $a = 0$ in (15.26) and reverse the transformation (15.25). We get

$$\frac{\partial^2 \tilde{v}}{\partial t^2} = \frac{\partial^2 \tilde{v}}{\partial x^2} + a\frac{\partial^3 \tilde{v}}{\partial t\, \partial x^2} \tag{15.28}$$

perturbing (15.23) for small a. More directly, we may observe that $\tilde{x} = x/a$, $\tilde{t} = t/a$ are large when a is small, so that the major contribution to the integral in (15.27) is obtained for small values of u. But for small u, $\sqrt{1+u} \sim 1$ and (15.27) reduces to $H(\tilde{t} - \tilde{x}) = H(t - x)$. The argument just given is not complete because

$$\tilde{\eta} = \tilde{t} - \tilde{x} = \frac{t - x}{a}$$

may be small even when $1/a$ is large. Therefore we replace $\tilde{t}$ with $\tilde{\eta} + \tilde{x}$ in (15.27) and define

$$\begin{aligned} \tilde{\tilde{v}}(\tilde{x}, \tilde{\eta}) &\overset{\text{def}}{=} \tilde{v}(\tilde{x}, \tilde{\eta} + \tilde{x}, 0) \\ &= \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} \exp(\tilde{\eta}u)\exp\left[\tilde{x}\left(u - u/\sqrt{1+u}\right)\right]\frac{du}{u}. \end{aligned} \tag{15.29}$$

Hence $\tilde{\tilde{v}}(\tilde{x}, \tilde{\eta})$ differs from $H(\tilde{\eta})$ for fixed values of $\eta = t - x \neq 0$ when $a \to 0$ only for small values of $\tilde{\eta} = \eta/a$.

Though small values of $\tilde{t}$ and $\tilde{x}$ are of no relevance in the transition layer analysis, it is perhaps of interest to note that (15.27) tends to $\operatorname{erfc}\left(\tilde{x}/2\sqrt{\tilde{t}}\right)$ for small $\tilde{x}$. Roughly, when $\tilde{x}$ is small we may replace $u/\sqrt{1+u}$ with $\sqrt{u}$ in (15.27). More precisely, we introduce variables $\bar{x} = a^{\alpha}\tilde{x}$ and $\bar{t} = a^{2\alpha}\tilde{t}$, $0 < \alpha < 1$ into (15.27), and setting $u/a^{2\alpha} = s$ we find, after putting $a \to 0$ for fixed s, $\bar{x}$, $\bar{t}$, that

$$\tilde{v}(\tilde{x}, \tilde{t}, 0) = \bar{v}(\bar{x}, \bar{t}) = \operatorname{erfc}\left[\bar{x}/2\sqrt{\bar{t}}\right] = \operatorname{erfc}\left[\tilde{x}/2\sqrt{\tilde{t}}\right].$$

The same result may be obtained directly from the differential equation (15.26) after putting $a = 0$ and then changing variables.

The function (15.27) may therefore be understood as follows. Fix rays $\tilde{x} = \beta\tilde{t}$, $\beta \neq 1$. On each ray, $\tilde{v}(\tilde{x}, \tilde{t}) \to H(\tilde{t} - \tilde{x})$ when t is large and $\operatorname{erfc}\left(\tilde{x}/2\sqrt{\tilde{t}}\right)$ when $\tilde{t}$ is small.

The damping effect associated with the second term $a\, \partial t/\partial v$ of (15.26) is entirely absent from (15.27) and (15.29). This damping effect is what produces the slow increase to $v = 1$ after the jump at $t = x$ from 0 to $e^{-x/2} = e^{-t/2}$ (see Figure 15.2). It folllows that the amplitude of the solution

$$\lim_{\tilde{\eta} = [\eta/a] \to \infty} \tilde{v}(\tilde{\eta}, \tilde{x}) \to 1 \; (\neq e^{-t/2})$$

is not correctly given. To get the correct limiting value we use limit-matching techniques of matched asymptotic expansions and write the inner (transition layer) solution as

$$\omega(\tilde{x}, \tilde{\eta}, x) = A(x)\, \tilde{v}(\tilde{x}, \tilde{\eta}), \tag{15.30}$$

where $A(x)$ is function of a parameter x which is regarded as independent of $\tilde{x} = x/a$ in the transition layer. The function $\omega(\tilde{x}, \tilde{\eta}, x)$ satisfies (15.26) with $a = 0$ and the boundary condition

$$\omega(0, \tilde{t}, x) = A(x)\, H(\tilde{t}). \tag{15.31}$$

We call the solution (15.20)

$$v(x, \eta + x, 0) = v(x, t, 0)$$

the outer or "shock" solution. The limit matching requires that the inner limit $\eta \downarrow 0$ of the outer solution

$$v(x, x, 0) = e^{-x/2}$$

equal the outer limit $\tilde{\eta} = \eta/a \to \infty$ of the inner solution

$$\omega(\tilde{x}, \infty, y) = A(x);$$

that is,

$$A(x) = e^{-x/2}.$$

Thus

$$\omega(\tilde{x}, \tilde{\eta}, x) = e^{-x/2}\, \tilde{v}(\tilde{x}, \tilde{\eta}) \tag{15.32}$$

is the required transition layer solution.

To get a uniformly valid asymptotic solution for small values of a, we use an additive composition in which the common part $e^{-x/2} H(\eta)$ of the inner and outer solution is subtracted from the sum. Thus

$$V(x, \eta, a) = v(x, \eta + x, 0) + e^{-x/2}\left\{\tilde{v}\left(\frac{x}{a}, \frac{\eta}{a}\right) - H(\eta)\right\}. \tag{15.33}$$

Recalling that $v(x, x, 0) = e^{-x/2}$, we note that

$$v(x, \eta + x, 0) - e^{-x/2} H(\eta) \to 0,$$

when $|\eta|$ is small. Hence

$$V(x, \eta, a) \to e^{-x/2}\, \tilde{v}\left(\frac{x}{a}, \frac{\eta}{a}\right)$$

for small η. For fixed, non-zero values of η, the bracket $\{\}$ in (15.33) vanishes as $a \to 0$ and

$$V(x, \eta, a) \to v(x, \eta + x, 0).$$

The problem of step displacement is like the one just discussed. We set

$$u(x, t) = v_t(x, t),$$

where $v(x, t)$ is any of the solutions just described. We think of $u(x, t)$ as a velocity and note that the condition $u(0, t) = \delta(t)$ is equivalent to a step displacement. When $0 < a \leqq 1$, the delta function does not persist and does not propagate into the interior.

When $a = 0$ a point $t = x$ of infinite velocity propagates into the interior of a fluid undergoing a step change in displacement. To see $\delta(t - x)$ we need only take the partial time derivative of (15.20)

$$u(x, t) = \left[e^{-x/2} + \frac{x}{2}\int_x^t \frac{e^{-\eta/2} I_1\left[\frac{1}{2}\sqrt{\eta^2 - x^2}\right] d\eta}{\eta^2 - x^2}\right]\delta(t - x)$$
$$+ \frac{x}{2}\,\frac{e^{-t/2} I_1\left[\frac{1}{2}\sqrt{t^2 - x^2}\right]}{(t^2 - x^2)}\, H(t - x).$$

It follows from this analysis that fluids satisfying the assumptions of Coleman & Noll's theory of fading memory may support the propagation of discontinuous and even infinite velocities. The addition of slight amounts of viscosity $a > 0$ smooths the solution. If a is small, this smoothing is confined to a small transition layer of thickness a.

§ 16. Discussion

In this paper we formed theories of fading memory following along lines laid out in the excellent work of Coleman & Noll (1963). The basic step in this type of theory is the choice of dom $\mathscr{F}$. This choice defines the topology in which $\mathscr{F}$ is continuous and determines a topological dual; that is, the collection of all bounded linear forms on dom $\mathscr{F}$ with values in $\mathbb{R}^6$. The topological dual coincides with the set of all possible forms for the linearization of $\mathscr{F}$. Thus the choice of topology in dom $\mathscr{F}$ determines the forms of the linearized stresses. To a certain extent, the various choices which one can make for dom $\mathscr{F}$ are mathematizations of physical intuition about what deformation histories might actually occur in nature. Dynamics, of course, determines what histories will actually occur. But we can not study dynamics without constitutive equations, so dynamics and representation theory are interwoven; like the chicken and the egg, no one can say which comes first. Ultimately the good choice of topology is one which gives predictions of dynamic response which agrees with experience.

We have developed two theories of fading memory which are meant to overcome some difficulties posed by the theory of COLEMAN & NOLL. The main defect of the $\mathbb{L}_h^2(0,\infty)$ theory of fading memory is that it perhaps allows too many motions, with discontinuous and even infinite velocities. The mathematical consequences of choosing dom $\mathscr{F} = \mathbb{L}_h^2(0,\infty)$ is that the topological dual is rather narrowly restricted with linearized stresses in the form of integrals with kernels in $L_h^2(0,\infty)$. Fluids of the differential type, especially Newtonian fluids, are not special cases in COLEMAN-NOLL $\mathbb{L}_h^2$ theory and they cannot be obtained as exact realizations when dom $\mathscr{F} = \mathbb{L}_h^2$. The COLEMAN-NOLL theory also implies that the nonlinear stresses, if integrals are assumed, must be expressed as iterated integrals with kernels in L_h^2 and not as a single integral in which powers (>2) of $G(s)$ are integrated against a kernel in L_h^2. Since single integral models are very popular and lead to reasonable predictions of dynamics in some cases, it is not desirable to exclude them by the choice of an overly large domain space for $\mathscr{F}$.

The topological vector space which is closest to $\mathbb{L}_h^2(0,\infty)$ is $\mathring{\mathscr{H}}(k)$. In this space we require that elements in $\mathbb{L}_h^2$ should also be C^{k-1} smooth near the origin (that is, they have some derivatives in the sense of distributions in a one-sided neighborhood of the present time). The topological dual for $\mathring{\mathscr{H}}(k)$ is uniquely determined as the direct sum of $\mathbb{L}_h^2$ and Dirac measures at zero. Single integral models in which powers of G (>2) are integrated against smooth kernels do not fall out as special cases of our theory in $\mathring{\mathscr{H}}(k)$.

The Sobolev space theory restricts the dom $\mathscr{F}$ even more; for the same k, elements of the Sobolev space $\mathring{\mathbb{H}}_h^1 \cap \mathbb{H}_h^k$ are k smooth, even away from the origin. These elements are continuous functions of s for all s when $k = 1$ and are continuously differentiable when $k = 2$. We get single integral models in $\mathring{\mathbb{H}}_h^1$, but no rates. Single integral models and Newtonian fluids are special cases in our $\mathring{\mathbb{H}}_h^1 \cap \mathbb{H}_h^k$ theory for $k \geqq 2$.

OLDROYD (1965) criticized the theory of simple fluids. He cites some statements to the effect that the stress in a fluid is determined by the history of the deformation up to and including the present time t. He notes that in Newtonian fluids we could have stress without deformation because the stress depends on rates of deformation (velocity) rather than on the deformations themselves. In the Rayleigh problem studied in § 15, the applied jump in velocity produces infinite stresses at $x = t = 0$, without deformation. HUILGOL (1975) noted that GREEN & RIVLIN's constitutive assumption (1.3) removes this objection of OLDROYD. These objections are also not relevant to the theories given in this paper.

Thus OLDROYD'S remarks do not apply to all simple fluids but they do apply to the $\mathbb{L}_h^2$ theory of COLEMAN & NOLL.

WANG'S theory (1965) and our Sobolev space and LF space theory all address the possible shortcomings of the $\mathbb{L}_h^2$ theory of COLEMAN & NOLL. WANG works with the space $C^k[0,\infty)$ of C^k functions on $[0,\infty)$ with the topology of uniform convergence on compact subsets. This space is a Fréchet space (not normable) and functional derivatives in it are Gateaux derivatives. There is no weighted integral in WANG'S theory; fading memory is obtained by demanding convergence only on compact subsets which don't include the distant past. Distributions do appear in the dual of WANG'S Fréchet space. Some properties of WANG'S theory are shared by our Sobolev space theory and there are also some differences. Elements of $\mathring{\mathbb{H}}_h^1 \cap \mathbb{H}_h^k$ are C^{k-1} smooth $(k \geqq 2)$ $\forall s \geqq 0$. The static continuation of a given history is not in general a C^1 function and hence cannot lie in $\mathring{\mathbb{H}}_h^1 \cap \mathbb{H}_h^k$ (but it does belong to $\mathring{\mathbb{H}}_h^1$) or in WANG'S Fréchet space. A convergent sequence of histories in the space $\mathring{\mathbb{H}}_h \cap \mathbb{H}_h^k$ is in C^k, hence is convergent in WANG'S Fréchet space of order k. On the other hand, linearized stresses in our Sobolev space theory are Fréchet derivatives leading to integrals of the COLEMAN-NOLL type, but with kernels in the sense of distributions. In contrast, WANG'S theory leads to a linearized theory generated by a Gateaux derivative on the state of relative rest expressible by Stieltjes integrals.

Theories of fading memory in Sobolev space and LF space avoid some defects of the older theories. Since derivatives of the Dirac measures lie in the topological duals of the Sobolev spaces and LF spaces, Newtonian fluids, rate fluids, integral fluids of the COLEMAN-NOLL type and mixtures of rates and integrals all live in these duals. In addition, single integral models make sense in Sobolev space.

Finally, we note that theories leading to integrals with smooth kernels support the interior propagation of discontinuous and even singular boundary data. In the non-linear case, not considered here, the recent results of SLEMROD suggest the possibility of shock-up, even of smooth data. These same features may be expected to hold for various models, like LODGE'S rubber-like fluid, the BKZ fluid, OLDROYD'S fluid B and other models, which may be framed as integral equations with smooth kernels.

On the other hand, the propagation of discontinuities in velocity and displacement, and perhaps the formation of shocks, is prevented by the rate terms in the constitutive equations. Thus the topology assumed in the representation theory in LF and Sobolev space is consistent with the dynamics it generates. For the linearized problems treated in § 15 we found that a small Newtonian contribution smoothed discontinuities. But smoothed solutions can be barely distinguished from discontinuous ones when the viscosity is small. The situation for such viscoelastic fluids with small Newtonian viscosity resembles the well-known one in gas dynamics. "Shock" solutions are given in the $\mathbb{L}_h^2$ theory and "shock structure" may be studied by the addition of Newtonian and other rate terms.

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