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THE APPLICATION OF BIFURCATION THEORY TO PHYSICAL PROBLEMS

D. D. Joseph
Department of Aerospace Engineering and Mechanics
University of Minnesota
Minneapolis, Minnesota

I am going to start my lecture with a citation by James Lighthill [1]. He says "There is one great complicating feature that introduces major difficulties into mechanics, physics, chemistry, engineering, astronomy and biology. This complicating feature is that an equilibrium can be stable but may become unstable; while, similarly, a process can take place continuously but may become discontinuous".

In recent years the mathematical treatment of such discontinuous changes has undergone deep and penetrating developments of great generality. In an astonishingly short period of time the modern theory of bifurcations has attained the status of an important independent branch of mathematical analysis. In fact bifurcation theory is a part of a somewhat broader subject known as singularity theory. Let me say here that it is perhaps obvious that one ought to try to treat singular events of nature with a mathematical theory of singularities. It would be easy to rattle off a long list of scientific disciplines and physical problems to which some aspect of singularity theory applies. But it is even easier to assert that there are almost no disciplines to which it does not apply. Many applications of singularity theory and bifurcation theory to physical problems, ranging from astrophysics to cell biology can be found in any sample of a list of references, say [1], [2], [3], [4], among many others.

In my lecture, I want to be simple and convincing. I hope it will not be said that I was frequently simple and rarely convincing. On the other hand, in my book with Gerard Iooss [5], Elementary Stability and Bifurcation Theory, there is a quote by Einstein which says that "Everything should be made as simple as possible, but not too simple". Ideas which I can only suggest in this lecture are more fully developed in my book with Iooss.

My first goal is to convince you that the complications of which Lighthill speaks occur already in the simplest problems. The reason is that a given physical system may have available many modes of operation, and the mathematical model of this system can have many solutions corresponding to the same prescribed data. In even moderately complicated physical problems the selection rules by which the actual realized solutions are determined are elusive. To illustrate this point, consider the simple scalar ordinary differential equation whose solution set is fully defined in Fig. 1.

$\frac{du}{dt} = F(\mu,u)$, $F(\mu,\varepsilon) = 0$ are steady solutions

$F = F_1 F_2 F_3 \ldots F_m$, m steady solutions $F_\ell(\mu,\varepsilon) = 0$

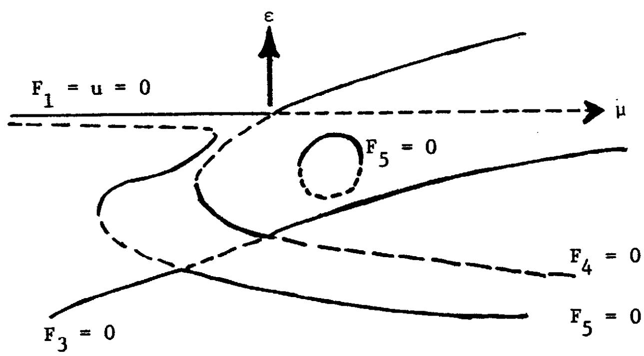

Fig. 1 Example of the solution set for a simple ordinary differential equation in $\mathbb{R}$ depending on a parameter μ. The dotted lines represent unstable solutions.

Already in this example we see that even in the simplest of problems we can have the highest degree of degeneracy with many solutions and many discontinuous changes as the control parameter μ is varied.

In the diagram, the solutions drawn with dotted lines are unstable. The curves are given by the solutions of $F(\mu,u) = 0$. The smooth portions of these curves are supposed to represent continuous changes of some physical process. Discontinuous changes are associated with the singular points of these plane curves, at which first derivatives also vanish,

$$F_\mu = 0, \quad F_\varepsilon = 0. \tag{1}$$

In the mathematical theory of bifurcation these types of singular points are the mathematization of the discontinuous events we actually observe in the physical applications.

Usually we study the local theory of bifurcation in which the singular points are of the simplest type. Suppose $u = \varepsilon$ is any steady solution of

$$\frac{du}{dt} = f(\mu,u), \quad f(\mu,\varepsilon) = 0 \tag{2}$$

and

$$f(\mu,0) = 0 \ \forall \mu \in \mathbb{R} \tag{3}$$

This problem is said to be reduced to local form and (0,0) is supposed to be a singular point which is least degenerate when

$$f_{\mu\varepsilon}(0,0) \neq 0 . \tag{4}$$

Thus we have

$$\left.\begin{aligned} f(\mu,0) &= 0 \ \forall \mu, \\ f_\varepsilon(0,0) &= 0 , \end{aligned}\right\} \tag{5}$$

$$f_{\varepsilon\mu}(0,0) \neq 0 \quad .$$

We want to find other solutions $\varepsilon \neq 0$ of $f(\mu,\varepsilon) = 0$. Then

$$\begin{aligned} f(\mu,\varepsilon) &= f(0,0) + f_\mu(0,0)\mu + f_\varepsilon(0,0)\varepsilon \\ &\quad + \tfrac{1}{2}[f_{\mu\mu}(0,0)\mu^2 + 2f_{\varepsilon\mu}(0,0)\varepsilon\mu \\ &\quad + f_{\varepsilon\varepsilon}(0,0)\varepsilon^2] + \ldots \\ &= \varepsilon[2\mu f_{\varepsilon\mu} + \varepsilon f_{\varepsilon\varepsilon}] + \ldots \end{aligned} \tag{6}$$

Solving $f(\mu,\varepsilon) = 0$ to lowest order, we find the old solution $\varepsilon = 0$ and the new solution

$$\mu = -\varepsilon \frac{f_{\varepsilon\varepsilon}}{f_{\varepsilon\mu}} \quad . \tag{7}$$

Of course we have only two tangents, as shown in Fig. 2.

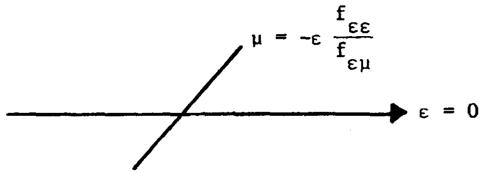

Fig. 2

We get real curves from higher order terms, using the implicit function theory. If $f_{\varepsilon\varepsilon} = 0$ the two curves are perpendicular and the higher order terms give pitchforks

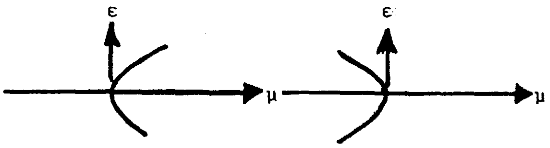

Fig. 3 Pitchfork bifurcation

Now we add the study of stability to the study of singular points of plane curves. We use linear stability theory

$$\begin{aligned} \frac{du}{dt} &= f(\mu,u) \quad , \\ u &= \varepsilon + v \quad , \\ \frac{dv}{dt} &= f_\varepsilon(\mu,\varepsilon)v, \\ v &= v_0 e^{\sigma t} \quad , \\ \sigma &= f_\varepsilon(\mu,\varepsilon) \quad . \end{aligned} \tag{8}$$

Suppose there is a solution $\mu = \mu(\varepsilon)$ of $f(\mu,\varepsilon) = 0$. Then

$$f_\mu(\mu,\varepsilon)\mu_\varepsilon + f_\varepsilon(\mu,\varepsilon) = 0 \quad .$$

Hence

$$\sigma = -\mu_\varepsilon \, f_\mu(\mu,\varepsilon) \tag{9}$$

and σ changes sign with μ_ε if $f_\mu \neq 0$. We say that there is a change of stability at a regular turning point

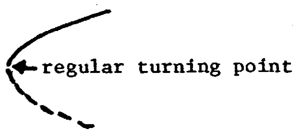

Fig. 4 Change of stability at a regular turning point

Now consider the solution $\varepsilon = 0$ and suppose that there is a strict loss of stability as μ passes through zero

$$\left.\begin{aligned} \sigma_\mu &> 0 \quad \text{at} \quad \mu = 0 \ , \\ \sigma_\mu &= f_{\varepsilon\mu}(0,0) > 0 \ . \end{aligned}\right\} \tag{10}$$

The condition that the loss of stability at $\mu = 0$ is strict (The Hopf condition) is the minimum hypothesis for nondegeneracy.

To study the stability of the bifurcating solution we use (9) and note that

$$f_\mu(\mu(\varepsilon),\varepsilon) = f_\mu(0) + f_{\varepsilon\mu}(0,0)\varepsilon + O(\varepsilon^2) \quad .$$

Hence, to lowest order

$$\sigma = -\mu_\varepsilon(\varepsilon)[f_{\varepsilon\mu}(0,0)\varepsilon + O(\varepsilon^2)] \tag{11}$$

Formula (11) shows that subcritical bifurcating solutions are unstable and supercritical ones are stable. This major result is shown graphically in Fig. 5.

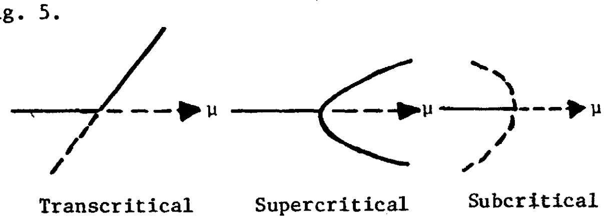

Fig. 5 Stability of bifurcating solutions

These results, suprisingly enough, hold for many complicated problems governed by partial differential equations when the spectral problem for the stability of the null solution has simple eigenvalues so that the stability near $\mu = 0$ is controlled by just one eigenfunction, and the other stable eigenfunctions play just a small, secondary role.

To fix our ideas in the simplest of physical problems we want to explain a demonstration of the buckling of a wire arch which was shown to me by Brooke Benjamin and discussed in greater detail [5]. In this example we imagine that

$$\frac{d\theta}{dt} = f(\mu,\theta) \tag{12}$$

where θ is the angle of deflection shown in Fig. 6, $\mu = \ell - \ell_c$ where ℓ is the length of the wire arch and ℓ_c is a critical length. In fact, $f(\mu,\theta)$ is even function of θ which can be solved for an even function $\mu = \mu(\theta)$ which is of the form shown in Fig. 6.

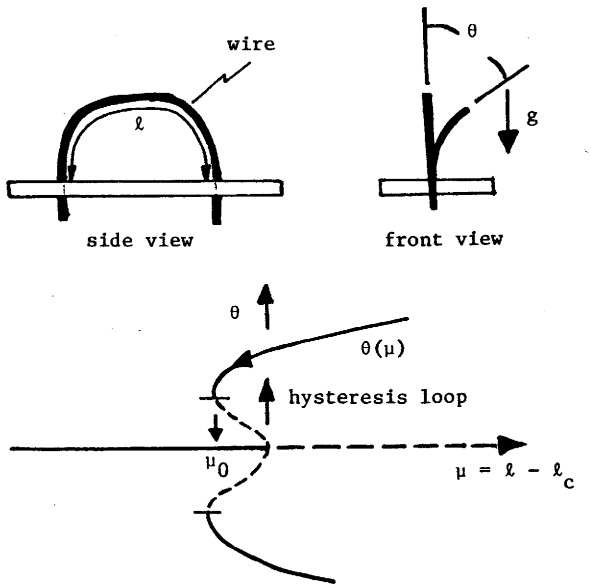

Fig. 6 Apparatus for demonstrating the buckling of the wire arch under gravity. When $\mu = \ell - \ell_c < 0$ the only solution is the upright one with $\theta = 0$. The upright solution loses stability at $\mu = 0$ and as new solution $\mu(\theta)$, $\theta \neq 0$ bifurcates subcritically and then turns back at a regular turning point. The system exhibits hysteresis. If μ is increased past zero, the solution $\theta = 0$ loses its stability and the system jumps to the bifurcating solution with $\theta \neq 0$. Then, if μ is decreased the system will stay in the deflected position until the turning point is reached. Then it drops back to the zero solution.

The equation (12) was invented in such a way that every point in the bifurcation diagram of Fig. 6 corresponds to an observed event in our demonstration, and vice versa.

Real applications are seldom described by simple scalar equations of the type so far considered. But in many applications it is possible to reduce problems even of partial differential equations into finite dimensions by the method of projections. I am now going to briefly describe the simplest of such possibilities in mathematical terms. Then I will show a movie [6] of bifurcations of motion of a rubber tube through which water is flowing. This movie shows even the simplest hypotheses may be applied in physical problems. The problem is analyzed in [7].

Let suppose that our system is governed by equations expressible as an evolution equation in Banach Space

$$\frac{d\underline{u}}{dt} = \underline{f}(\mu,\underline{u}) \ , \quad \underline{f}(\mu,0) = 0 \tag{13}$$

It does no harm to think of (1.5) as a system of ordinary differential equations in $\mathbb{R}^n$.

To study the stability of $\underline{u} = 0$ we linearize (13) and introduce exponential solutions to derive the associated spectral problem:

$$\frac{d\underline{v}}{dt} = \underline{f}_u(\mu|\underline{v}) \ ,$$

$$\underline{v} = e^{\sigma t}\underline{\zeta} \ , \tag{14}$$

$\sigma = \zeta(\mu) + i\eta(\mu)$ spectrum of the linear operator $\underline{f}_u(\mu|\cdot)$,

$\sigma\underline{\zeta} = \underline{f}_u(\mu|\underline{\zeta})$ is the spectral problem. (15)

We use the same set up as before; $\underline{u} = 0$ loses stability as μ passes through zero. We assume that the spectral values at the cross-over are isolated and simple (to each simple eigenvalue, there one and one eigenfunction $\underline{\zeta}$). The rest of the spectrum has $\zeta(\mu) < 0$, and as before, the crossing is strict.

$$\zeta'(0) > 0 \ . \tag{16}$$

If there is an eigenvalue σ at $\mu = 0$ with eigenfunction $\underline{\zeta}$, and

$$\eta(0) \overset{\text{def}}{=} \omega_0 \neq 0 \ ,$$

then there is another eigenvalue $\bar{\sigma}$, the conjugate of σ, with eigenfunction $\bar{\underline{\zeta}}$.

Given the above assumptions there are two cases. The first case in $\sigma(\mu)$ is a real eigenvalue ($\omega_0 = 0$) is just like the one leading to Fig. 5. All the lively action takes place in a space of one dimension even though the governing problem can be very complicated. In physical problems these one dimensional problems are sometimes associated with the break up of a uniform solution into a spatially periodic one.

The second possibility is that $\omega_0 \neq 0$. In this case the condition (16) insures that a complex pair of eigenvalues cross into the unstable part of the complex σ plane. Then there are two active eigenfunctions $\underline{\zeta}$, and $\bar{\underline{\zeta}}$ and the dynamics associated with these two eigenfunctions dominates the evolution of the entire nonlinear problem. At criticality the linearized solution is time periodic, and if all our hypotheses with $\omega_0 \neq 0$ are realized, a time periodic solution will bifurcate. This is called Hopf bifurcation. The main properties of the bifurcating periodic solutions are these. Let ε be the amplitude of the periodic bifurcating solution; then

$$\mu(\varepsilon) = \mu(-\varepsilon) \tag{17}$$

Subcritical and supercritical bifurcations are both possible but transcritical bifurcation is impossible

$$\omega(\varepsilon) = \omega(-\varepsilon) \tag{18}$$

The frequency depends on the amplitude. Apart from these features we have all the local properties that in the simplest case with stability properties of the type shown in Figs 4, 5 (but no transcritical), and 6. We will see these possibilities realized in the movie.

Now we ask what happens when a periodic solution bifurcates. There are, in general, two possibilities: another periodic solution with a longer period may bifurcate, or a doubly periodic solution with two frequencies may bifurcate. For simplicity, suppose that we have a periodic solution with velocity $\underline{u}(t,\mu) = \underline{u}(t+T,\mu)$, periodic with fixed period T. Typically such solutions arise from forced T-periodic data. In bifurcation problems the period $T = T(\mu)$ changes with μ. We suppose T is independent of μ at the expense of some fine points, but the qualitative results are nearly the same. A small disturbance $\underline{v}$ of $\underline{V}$ satisfies the linearized equation

$$\frac{d\underline{v}}{dt} = F_u(\mu,\underline{u}(t,\mu)|\underline{v})$$

which can be studied by the method of Floquet. We may represent $\underline{v}(t)$ solving the linearized equations as

$$\underline{v} = e^{\sigma(\mu)t}\underline{\zeta}(\mu,t) \ , \quad \underline{\zeta}(\mu,t) = \underline{\zeta}(\mu,t+T)$$

where $\sigma(\zeta) = \xi(\mu) + i\Omega(\mu)$, the Floquet exponent, and $\underline{\zeta}(\mu,t)$ are eigenvalues and eigenvector of the operator

$$\mathbf{J} = -\frac{d}{dt} + F_u(\mu,\underline{u}(t,\mu)\,|\ )$$

whose domain is of T-periodic functions and $\underline{\zeta} = \sigma\underline{\zeta}$. For each and every σ in the spectrum of $\mathbf{J}$ there is a Floquet multiplier

$$\lambda(\mu,T) = e^{\sigma(\mu)T}$$

If $\mu < 0$ where $\xi(0) = 0$ for some exponent σ, then, $\xi(\mu) < 0$ for σ and

$$\lambda(\mu,nT) = e^{\sigma(\mu)nT} < 1$$

and, in fact tends to zero as $n \to \infty$. This is the stable case. At criticality $\mu = 0$ and we assume that $\xi'(0) > 0$ for the eigenvalue for which $\xi(0) = 0$. This gives rise to the situation in which a pair of multipliers

$$\lambda_0 = e^{\pm i\Omega_0 T}$$

escapes from the unit disk.

Now we classify all the possibilities for bifurcation. The classification is parametrized by the points on the Floquet circle at which the conjugate multipliers escape. All points on circle

$$\lambda_0 = e^{i\Omega_0 T}$$

are given in terms of

$$r = \frac{\Omega_0}{(2\pi/T)} \ , \quad 0 \le r < 1 \ ,$$

the frequency ratio at criticality. At criticality solutions of the bifurcation problem are in the form

$$\hat{\underline{v}}_0(t_1,t_2) = \hat{\underline{v}}_0(\Omega_0 t, \tfrac{2\pi}{T}\, t) = e^{i\Omega_0 t}\, \underline{\zeta}_0(t)$$

where $\underline{\zeta}_0(t) = \underline{\zeta}_0(t+T)$ and $\Omega_0 = 2\pi r/T$. $\hat{\underline{v}}_0(t_1,t_2)$ is doubly periodic, periodic with period 2π, jointly in t_1 and t_2.

(I) $\hat{\underline{v}}_0(t_1,t_2)$ is quasi-periodic if r, $0 < r < 1$, is irrational.

(II) $\hat{\underline{v}}_0(t_1,t_2)$ is nT-periodic if $r = m/n < 1$, $0 \le m/n < 1$, is rational.

These properties (I) and (II) are not preserved when the linear problem is perturbed with the nonlinear terms. We construct an asymptotic (at least) approximation

$$\hat{\underline{v}}(\Omega(\varepsilon)t, \tfrac{2\pi}{T}\, t) \ ,$$

doubly periodic with period 2π in each place, with a smooth $\Omega(\varepsilon)$ which bifurcates for each fixed r, $0 \le r < 1$, even for $r = m/n$, provided only that $n \neq 1$, 2, 3, 4. At these special rational values we get subharmonic, nT periodic solutions with $\Omega(\varepsilon) = 2\pi mt/nT$, independent of ε. We call the points n = 1, 2, 3, 4 where subharmonic solutions bifurcate, points of strong resonance. The T-periodic bifurcating solution (n = 1) is transcritical; it bifurcates on both sides of criticality. The 2T-periodic solution (n = 2) bifurcates either entirely to one side or to the other. And in both cases supercritical bifurcating solutions ($\mu > 0$) are stable and subcritical solutions ($\mu < 0$) are unstable. The 3T-periodic solution is transcritical and it is unstable on both sides of criticality. Two 4T-periodic bifurcate (if a certain inequality holds) and if the two bifurcate supercritically, one of the two is unstable.

$$r = \frac{\Omega_0}{(2\pi/T)} \ , \quad 0 \le r < 1 \qquad \text{Frequency ratio of criticality}$$

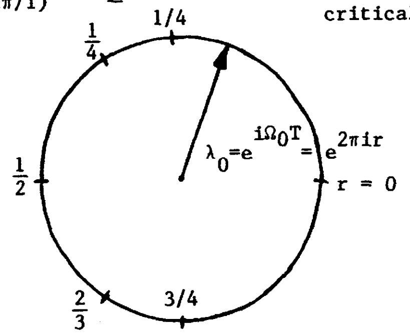

Fig. 7 Classification of points on the Floquet circle. The Floquet circle at criticality is given by $\lambda_0 = e^{2\pi ir}$. (I) r is irrational, (II) $r = m/n < 1$ is rational $\lambda_0^{\,n} = e^{2\pi im} = 1$.

Solutions which bifurcate at irrational point, are asymptotic to doubly periodic solutions with two independent frequencies. In some cases, like those involving rotational symmetry, the solutions which bifurcate are strictly doubly periodic. In other problems, lacking symmetry we get phase lacking, that is periodic solutions for small intervals of μ near and quasiperiodic solutions for the other values of $\mu = 0$. In all cases, except n = 1, 2, 3, 4, these solutions lie on an invariant torus.

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