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Linearized dynamics for step jumps of velocity and displacement of shearing flows of a simple fluid

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Abstract: We consider linearized dynamics associated with step jumps in the velocity or displacement of the boundary of a fluid in a shearing motion. The discontinuity will propagate into the interior with a speed $C = \sqrt{G(0)/\rho}$ (ρ is the density) if the initial values $G(0)$ and $G'(0)$ of the fading memory kernels are bounded, $0 < G(0) < \infty$, $-\infty < G'(0) < 0$. If $G(0) \neq \infty$ but $G'(0) = -\infty$, then the boundary of the support of the solution still propagates with the speed C. However, the solutions on both sides of the boundary match together in a C^∞-fashion. If $G(0) \neq \infty$ but $G'(0) = 0$, the amplitude of the discontinuity will not damp as in a purely elastic fluid. If $G(0) = \infty$, the step change is felt immediately throughout the fluid, without shocks, as in Navier-Stokes fluids. This same type of parabolic behavior can be achieved by a small Newtonian contribution added to the integral form of the stress but if this contribution is small, a smooth transition layer around the shock will propagate with the speed C. In the case of step displacement, from rest to rest, singular surfaces of infinite velocity can propagate into the interior with speed of propagation C. The singular surfaces undergo multiple reflections off bounding walls, but the final steady state reached asymptotically is in universal form independent of material.

Key words: Step jump, singular surface, reflection, shearing flow, simple fluid

1. Introduction

There is currently considerable interest in the way in which the dynamics of viscoelastic fluids depends on the choice of a constitutive equation. A rough classification of constitutive equations is into three groups depending on whether dynamic equations allow propagation of discontinuities as in the case of hyperbolic equations or smooth discontinuities as in the case of parabolic equations or there are smooth solutions with a finite speed of propagation of the information that the material was initially at rest. The first result of the classification of the first two types of dynamic response is due to Tanner [24] in his study of the Rayleigh problem (Stokes first problem) for an Oldroyd fluid of type B. Tanner's problem turns out to be a linear one and it falls into the frame of the most general classes of simple fluids (fluids of "mixed" type defined explicitly in the theory of fading memory of Saut and Joseph [21] and implicitly by the constitutive assumption for materials which we call of "mixed type" given on page 403 of Green and Rivlin [15] [1]). The type of constitutive equation which Tanner studied, and which we study here, and which has recently been studied for start up and spin-up problems by Kazakia and Rivlin [17] are in the form of an integral plus a Newtonian term. Our theory differs from the other two in that we allow a general kernel and not just one of exponential type (Maxwell model). We also get new results about reflections (similar to Böhme [3] and Kazakia and Rivlin [17]) and treat for the first time ever the exact linearized problem for the step jump in displacement of the wall of a parallel plate channel. (This was treated by us together with Saut in Saut and Joseph [21].) This second problem is currently popular in experimental studies of the rheology of polymer melts (see Bird, Armstrong and Hassager [2]) but it has not been

[1]) Huilgol [16] has given reasons why Green and Rivlin's fluids of mixed type are free of criticisms which could be raised by "smoother" theories of the fading memory type of Coleman and Noll [12]. Green and Rivlin ([15], p. 402) also pointed out the connection of their theory with integral constitutive equations with kernels in the sense of distributions.

studied theoretically. In our study the method of Laplace transforms, introduced by Tanner is basic.

We find that the speed of propagation of discontinuities is the same as the one Coleman, Gurtin and Herrera [8] found as a necessary condition for propagating shock waves. To be precise, suppose that the linearized stress in an incompressible fluid is given by

$$T = -p\mathbf{1} + \mu A_1(t) + \int_0^\infty \tilde{\mu}(s) G(s)\,ds$$

where $s = t - \tau$ is the lapse time, $G(s) = C_t(\tau) - 1$ is modified history of the relative Cauchy strain tensor $C_t(\tau)$ and $\mu A_1(t)$ is the stress in a Newtonian fluid with "viscosity" μ. Then if $\mu = 0$ the singular boundary data will propagate into the interior with a speed $C = \sqrt{G(0)/\rho}$, where ρ is the fluid density and $G(0)$ is the value of $G(s)$ at $s = 0$ where $G(s)$ is defined by $\tilde{\mu}(s) = G'(s)$. We also show that these discontinuities are smoothed if $\mu > 0$ and we follow our work reported in the paper of Saut and Joseph [21] in constructing a "shock" layer analysis. The analysis shows that when $\mu > 0$ is small the solution is almost like the one with $\mu = 0$, in fact, the solution is continuous in $\mu = 0$ at each point (x, t) and we show that the thickness of the transition layer scales with μ and propagates with the velocity C for finite x and t and sufficiently small μ. So when $\mu > 0$ is small the mathematical picture is qualitatively different than $\mu = 0$, but this rigorous mathematical property may be of apparent rather than real physical interest for the case of small μ in which the $\mu = 0$ problem is actually the leading order for the whole problem. This situation is like gas dynamics in which viscous fluids exhibit shock-like solutions with shock layers which are entirely ignored at the leading order. Then smooth kernel theories like Coleman and Noll's [12] or Lodge's [18] rubber-like liquids have a physical relevance for asymptotic motions resembling inviscid solutions of the equation of viscous fluids.

It appears to be the case from results obtained so far that if the linearized constitutive equation is written as an integral, with kernels in the sense of distributions, then discontinuities will propagate whenever $C = \sqrt{G(0)/\rho}$ is finite and $-\infty < G'(0) < 0$. We note that $G(0) = \infty$ if there is a derivative of the Dirac function in the kernel (Newtonian case) and for some other kernels in common use in rheological models, see Renardy [20].

The linearized theories of viscoelastic fluids we consider are hyperbolic but they have damping for $G'(0) < 0$. So Heaviside or Dirac data at the boundary can propagate as discontinuities if $\mu = 0$. But the amplitude of the discontinuity is damped. In nonlinear theories, like those considered by Coleman and Gurtin [7] and Selmrod [22] discontinuities may draw energy from non-linear terms and actually grow in amplitude. It could be conjectured that the existence of shocks in viscoelastic materials is intimately associated with the structure of the linearized operator and that the kind of constitutive models which allow propagating discontinuities which damp in linearized models may propagate without damping in some nonlinear models.

2. Shearing motion of an incompressible simple fluid

The position of a particle P of a fluid in motion is given by $\xi = \chi_t(x, \tau)$ where $\tau \leqslant t$ is the past time and $x = \chi_t(x, t)$. The strain history at P is given in terms of the right relative Cauchy-Green tensor

$$C_t(\tau) = F_t(\tau)^T F_t(\tau); \quad \tau \leqslant t$$

where

$$F_t(\tau) = \operatorname{grad}_x \chi_t(x, \tau).$$

It is convenient to define the history

$$G(s) = C_t(t - s) - \mathbf{1}, \quad G(0) = \mathbf{0}$$

where $s = (t - \tau) \in [0, \infty)$ is the lapse time. The constitutive equation of an *incompressible simple fluid* [9, 10] is given by

$$T = -p\mathbf{1} + \underset{s=0}{\overset{\infty}{\mathscr{G}}} [G(s)] \tag{2.1}$$

where

$$|\det C_t(\tau)| = 1 \tag{2.2}$$

and $\mathscr{G}$ satisfies the isotropy relation

$$\underset{s=0}{\overset{\infty}{\mathscr{G}}} [Q G(s) Q^T] = Q \underset{s=0}{\overset{\infty}{\mathscr{G}}} [G(s)] Q^T \tag{2.3}$$

for any second order orthogonal tensor Q.

By a shearing flow [9] (or generalized viscometric flow) we mean a motion in which at each particle P at any time t, there exists an orthonormal basis $e^{\langle 1 \rangle}$, $e^{\langle 2 \rangle}$, $e^{\langle 3 \rangle}$ (independent of s) such that the components of $C_t(t - s)$ with respect to $\{e^{\langle i \rangle}\}$ are given by

$$[e^{\langle i \rangle} \cdot C_t(t - s) e^{\langle j \rangle}] = \begin{bmatrix} 1 + \lambda^t(s)^2 & \lambda^t(s) & 0 \\ \lambda^t(s) & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.4}$$

where $\lambda^t : [0, \infty) \to \mathbb{R}$ is a well defined scalar valued function called the history up to t of the relative shear strain at P.

Various authors [6, 7, 10, 25] have shown that (2.1) – (2.4) imply that the stress tensor T for shearing flows obeys the following relations:

$$[e^{\langle i \rangle} \cdot T e^{\langle j \rangle}] = \begin{bmatrix} T^{\langle 11 \rangle} & T^{\langle 12 \rangle} & 0 \\ T^{\langle 12 \rangle} & T^{\langle 22 \rangle} & 0 \\ 0 & 0 & T^{\langle 33 \rangle} \end{bmatrix} \tag{2.5}$$

where

$$\begin{aligned} T^{\langle 12 \rangle}(t) &= \overset{\infty}{\underset{s=0}{t}} (\lambda^t(s)) = - \overset{\infty}{\underset{s=0}{t}} (-\lambda^t(s)), \\ T^{\langle 11 \rangle}(t) - T^{\langle 22 \rangle}(t) &= \overset{\infty}{\underset{s=0}{\eta_1}} (\lambda^t(s)) = \overset{\infty}{\underset{s=0}{\eta_1}} (-\lambda^t(s)), \\ T^{\langle 22 \rangle}(t) - T^{\langle 33 \rangle}(t) &= \overset{\infty}{\underset{s=0}{\eta_2}} (\lambda^t(s)) = \overset{\infty}{\underset{s=0}{\eta_2}} (-\lambda^t(s)). \end{aligned} \tag{2.6}$$

The shear stress and normal stress functional vanish on the zero history $\lambda^t(s) = 0^+(s) = 0, s \geqslant 0$,

$$\overset{\infty}{\underset{s=0}{t}} (0^+(s)) = \overset{\infty}{\underset{s=0}{\eta_1}} (0^+(s)) = \overset{\infty}{\underset{s=0}{\eta_2}} (0^+(s)) = 0. \tag{2.7}$$

Coleman and Noll [11 – 13] have shown that in their weighted L_2 theory of fading memory

$$\begin{aligned} T^{\langle 12 \rangle}(t) &= \overset{\infty}{\underset{s=0}{t}} (\lambda^t(s)) = \int_0^\infty \frac{dG}{ds}(s) \lambda^t(s) ds \\ &\quad + O(\|\lambda^t\|_h^3), \\ \overset{\infty}{\underset{s=0}{\eta_1}} (\lambda^t(s)) &= - \int_0^\infty \frac{dG}{ds}(s) \lambda^t(s)^2 ds + O(\|\lambda^t\|_h^4) \end{aligned} \tag{2.8}$$

where $\|\lambda^t\|_h^2 = \int_0^\infty \lambda^t(s)^2 h(s)^2 \, ds$ and G is strictly positive on $[0, \infty)$ and tends to zero "sufficiently fast" as $s \to \infty$.

3. Step change in velocity

An incompressible simple fluid fills the half-space $x > 0$, $-\infty < y, z < \infty$, and is bounded below at $x = 0$ by a rigid plate. The plate at rest for $t \leqslant 0$ is suddenly accelerated to a constant velocity $U (\forall t > 0)$ in the direction of increasing y. We assume a shearing motion for each and every particle $P(X^1, X^2, X^3)$ of the form

$$\begin{aligned} &\xi^1 = X^1, \quad \xi^2 = X^2 + \mu(X^1, \tau), \quad \xi^3 = X^3, \\ &x^1 = X^1, \quad x^2 = X^2 + \mu(X^1, t), \quad x^3 = X^3 \end{aligned} \tag{3.1}$$

where ξ^j are the cartesian coordinates of P at $\tau < t$ and x^j are the coordinates of P at $\tau = t$. We next define $\mu^t(X^1, s) \overset{\text{def}}{=} \mu(x^1, t - s) - \mu(x^1, t)$ and find that $\lambda^t(s)$ in (2.4) is given by

$$\lambda^t(s) = \frac{\partial \mu^t(x^1, s)}{\partial x^1} = \int_t^{t-s} \frac{\partial v^2(x^1, \sigma)}{\partial x^1} d\sigma \tag{3.2}$$

where

$$v^1 = \dot{\xi}^1 = 0, \quad v^2 = \dot{\xi}^2 = \frac{\partial \mu}{\partial \tau}(x^1, \tau), \quad v^3 = \dot{\xi}^3 = 0.$$

The momentum equations for shearing flows may be written as

$$\begin{aligned} &\frac{\partial \mathscr{S}^{\langle xx \rangle}}{\partial x} - \frac{\partial p}{\partial x} - \rho g = 0, \\ &\frac{\partial \mathscr{S}^{\langle xy \rangle}}{\partial x} - \frac{\partial p}{\partial y} = \rho \frac{\partial v}{\partial t}, \quad - \frac{\partial p}{\partial z} = 0. \end{aligned} \tag{3.3}$$

The flows which we consider have no pressure gradients in the y direction which follows from $p(x, y, t) = p(x, -y, t)$ and symmetry of the flow governed by $(3.3)_2$ with respect to changes in the sign of y. If the step jump of U is small enough so that $\|\lambda^t\|_h$ in (2.8) is uniformly small for all x and t, then (2.8) implies that

$$\mathscr{S}^{\langle xy \rangle} \sim \int_0^\infty \frac{dG}{ds}(s) \lambda^t(s) ds. \tag{3.4}$$

Eq. (3.4) gives a first approximation to $\mathscr{S}^{\langle xy \rangle}$. Noting that $G(\infty) = 0$, $\lambda^t(x, s) = 0$, $\forall s \geqslant t$ and $\frac{d\lambda^t}{ds}(x, s) = -\frac{\partial v}{\partial x}(x, s)$, (3.4) may be written as

$$\mathscr{S}^{\langle xy \rangle} = \int_0^t G(s) \frac{\partial v}{\partial x}(x, t - s) ds. \tag{3.5}$$

Combining the second equation in (3.3) with (3.5) we find the evolution problem

$$\int_0^t G(s)\frac{\partial^2 v}{\partial x^2}(x, t-s)\,ds = \rho\frac{\partial v}{\partial t}(x,t)\,,$$

$$v(0,t) = \begin{cases} U & \text{for } t > 0 \\ 0 & \text{for } t \leqslant 0\,, \end{cases} \tag{3.6}$$

$$v(x,0) = 0\,,$$

$v(x,t)$ is bounded for $x \geqslant 0$ and $t \geqslant 0$.

Before undertaking a detailed study of (3.6) we observe that, after differentiation, $(3.6)_1$ may be written as

$$G(0)\frac{\partial^2 v}{\partial x^2} - \rho\frac{\partial^2 v}{\partial t^2} = -\int_0^t \frac{dG}{ds}(s)\frac{\partial^2 v}{\partial x^2}(x,t-s)\,ds \tag{3.7}$$

where $G(0) > 0$. This equation is dominated by the wave operator on the left and it gives rise to hyperbolic behavior, propagation of discontinuities, with some damping produced by the "small" term on the right. We shall show that the speed of propagation of a discontinuity is $\sqrt{G(0)/\rho}$. Thus we see that $G(0)$ corresponds to the elastic part of the fluid response and G restricted to $(0,\infty)$ corresponds to viscous damping.

We are going to study (3.6) by the method of Laplace transform. If $\bar{v}(x,u)$ denotes the Laplace transform of $v(x,t)$ with respect to t, then a formal Laplace transform of (3.6) shows

$$\bar{v}(x,u) = \frac{U}{u}\exp\left(-x\sqrt{\frac{\rho u}{\bar{G}(u)}}\right);$$

$$u \in \mathbb{C} \ni Re\,u > 0\,. \tag{3.8}$$

Under suitable assumptions on G we can characterize the inverse transform of (3.8) and obtain the solution of (3.6).

4. Properties of the linear viscoelastic function G and its Laplace transform

Let

(i) $G\colon [0,\infty) \to \mathbb{R}^+ = \{x \in \mathbb{R} \mid x > 0\}$

(ii) G is strictly monotonically decreasing.
Condition (i) and (ii) imply that $G(0)$ exists and is finite.

(iii) $G \in C[0,\infty) \cap PC^1(0,\infty)$.
(G is continuous and piecewise continuously differentiable.)

(iv) $G(s)$ is $0(e^{-\bar{\lambda}s})$ for some $\bar{\lambda} > 0$ as $s \to \infty$.

(v) $\dfrac{dG}{ds} < 0$ is strictly monotonically increasing and $\lim\limits_{s\to\infty}\dfrac{dG(s)}{ds} = 0$. The implications of assumption (v) are used only in Section 8 to establish reflection.

The assumption (iv) guarantees that the Laplace transform

$$\bar{G}(u) = \int_0^\infty G(s)e^{-us}\,ds \tag{4.1}$$

is defined for any $u \in \mathbb{C}$ such that $Re\,u > -\bar{\lambda}$.

Using (4.1) we find that

$$|\bar{G}(u)| \leqslant \int_0^\infty G(s)e^{-(Re\,u)s}\,ds < \infty\,. \tag{4.2}$$

Hence $\bar{G}(u)$ has no poles in the half plane $Re\,u > -\bar{\lambda}$. Also, putting $u = x + iy$, we find that for $y = 0$

$$\bar{G}(u) = \int_0^\infty G(s)e^{-(Re\,u)s}\,ds > 0\,. \tag{4.3}$$

And if $y = Im\,u \neq 0$, then again $\bar{G}(u) \neq 0$. To prove the latter assertion, consider the case $y = Im\,u > 0$ and observe that

$$\begin{aligned} Im\,\bar{G}(u) &= -\int_0^\infty G(s)e^{-xs}\sin ys\,ds \\ &= -\sum_{n=0}^{\infty}\int_{n\pi/y}^{(n+1)\pi/y} G(s)e^{-xs}\sin ys\,ds \\ &= -\sum_{n=0,2,4,\ldots}\left[\int_{n\pi/y}^{(n+1)\pi/y}\sin ys\left\{G(s)e^{-xs} - G\left(s+\frac{\pi}{y}\right)e^{-x(s+\pi/y)}\right\}ds < 0\,.\right. \end{aligned} \tag{4.4}$$

Moreover, the same type of chain of inequalities can be made to show that if $y = Im\,u < 0$ then

$$Im\,\bar{G}(u) > 0\,. \tag{4.5}$$

Eqs. (4.3, 4.5) imply that $\bar{G}(u)$ has no zeroes in the half-plane $Re\,u < -\bar{\lambda}$.

We next determine *some asymptotic properties of* $\bar{G}(u)$. Since G is a bounded function and $\lim\limits_{\substack{s\to 0\\ s>0}} G(s) =$

$G(0^+)$ exists and is equal to $G(0)$ then by a standard theorem (e.g., 33.4 [4] or Theorem 4 [23, p. 185]) we may assert that if $u \in \mathbb{C}$ and in the sector $|\arg u| \leqslant \psi < \pi/2$ then

$$\lim_{|u| \to \infty} u\bar{G}(u) = G(0). \tag{4.6}$$

In our case the result given in (4.6) can be extended to:

$$\lim_{|u| \to \infty} u\bar{G}(u) = G(0) \quad \text{for} \quad |\arg u| \leqslant \pi/2. \tag{4.7}$$

To prove (4.7) we note that

$$\bar{G}(iy) = \int_0^\infty G(s)e^{-iys}ds \quad \text{and} \quad \int_0^\infty \frac{dG}{ds}(s)e^{-iys}ds$$

exist. Hence

$$iy\bar{G}(iy) = \int_0^\infty iyG(s)e^{-iys}ds$$
$$= G(0) + \int_0^\infty \frac{dG}{ds}(s)e^{-iys}ds.$$

Application of the Reimann-Lebesgue Lemma then yields

$$\lim_{y \to \infty} iy\bar{G}(iy) = G(0).$$

Thus (4.7) is a valid extension of (4.6).

We turn next to a characterization of the argument of $\bar{G}(u)$. Let $u \in \mathbb{C}$ such that $\arg u \in [-\pi, \pi]$ with a branch cut in the left half plane, $Re\ u < 0$, and $\arg u \equiv 0$ for $Re\ u > 0$ and $Im\ u = 0$. We also define $\arg \bar{G}(u) = 0$ if and only if $\arg u = 0$. Let $\arg u \in (0, \pi/2)$, hence $x = Re\ u > 0$ and $y = Im\ u > 0$. Now

$$Re\,\bar{G}(u) = \int_0^\infty G(s)e^{-xs}\cos ys\,ds$$
$$= \left.\frac{G(s)e^{-xs}\sin ys}{y}\right]_0^\infty - \int_0^\infty \frac{\sin ys}{y} \cdot \frac{d}{ds}(G(s)e^{-xs})ds$$
$$= \int_0^\infty \frac{\sin ys}{y}\left\{-\frac{d}{ds}(G(s)e^{-xs})\right\}ds > 0. \tag{4.8}$$

and

$$Im\,\bar{G}(u) = -\int_0^\infty G(s)e^{-xs}\sin ys\,ds < 0 \text{ (as in (4.4))}.$$

Hence, $\arg u \in (0, \pi/2) \Rightarrow \arg \bar{G}(u) \in (0, -\pi/2)$.

Similarly, $\arg u \in (0, -\pi/2) \Rightarrow \arg \bar{G}(u) \in (0, \pi/2)$.

Clearly, $\arg u = 0 \Leftrightarrow \arg \bar{G}(u) = 0$. (4.9)

We also observe that for $\arg u = \pi/2$, i.e.; $u = iy$, $Re\,\bar{G}(iy) = \int_0^\infty G(s)\cos ys\,ds$ and $Im\ \bar{G}(iy) = -\int_0^\infty G(s)\ \sin ys\,ds$. Arguing as in (4.4) and (4.8), we show that

$$Im\,\bar{G}(iy) < 0 \quad \text{and} \quad Re\,\bar{G}(iy) > 0.$$

Hence, $\arg \bar{G}(iy) \in (0, -\pi/2)$ for $y > 0$.

Similarly, $\arg \bar{G}(iy) \in (0, \pi/2)$ for $y < 0$. (4.10)

Combining (4.9) and (4.10) we get:

$$\arg u \in [0, \pi/2] \Rightarrow \arg \bar{G}(u) \in [0, -\pi/2)$$

and

$$\arg u \in [0, -\pi/2] \Rightarrow \arg \bar{G}(u) \in [0, \pi/2). \tag{4.11}$$

5. On the inverse of the formal Laplace transform

Lemma: Suppose that G satisfies the properties (i) – (vi) of Sect. 4. Then the inverse Laplace transform of the quantity $\bar{v}(x, u) = \frac{1}{u} e^{-x\sqrt{\rho u/\bar{G}(u)}}$; $u \in \mathbb{C}$, such that $Re\ u > 0$ and $x \in \mathbb{R}^+$, is given by

$$v(x, t) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{ut}\bar{v}(x, u)du$$
$$= f(x, t)H(t - \alpha x)$$

where $H(\xi) = \begin{cases} 1, & \xi > 0 \\ 0, & \xi < 0 \end{cases}$ is the Heaviside function, $\alpha \overset{\text{def}}{=} \sqrt{\rho/G(0)} = 1/C$ and $f(x, t)$ is a well defined function with the properties:

$$f(x, \alpha x^+) \neq 0.$$

In fact for $G'(0) \leqslant 0$, see Renardy [20],

$$f(x, \alpha x^+) = e^{\alpha G'(0)x/2G(0)} > 0 \qquad \forall x \geqslant 0.$$

Proof: $\bar{v}(x, u) = \frac{1}{u} e^{-x\sqrt{\rho u/\bar{G}(u)}}$ is analytic in the half-plane $Re\, u > 0$ because (4.2–4.5) show that $\bar{G}(u)$ has no poles or zeroes in the half-plane $Re\, u > 0$. Therefore the inverse

$$v(x,t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{ut}}{u} e^{-x\sqrt{\rho u/\bar{G}(u)}}du; \quad \gamma > 0 \tag{5.1}$$

can be obtained by contour integration in the complex u-plane with possible branch cut occurring only in the half-plane $Re\, u < 0$.

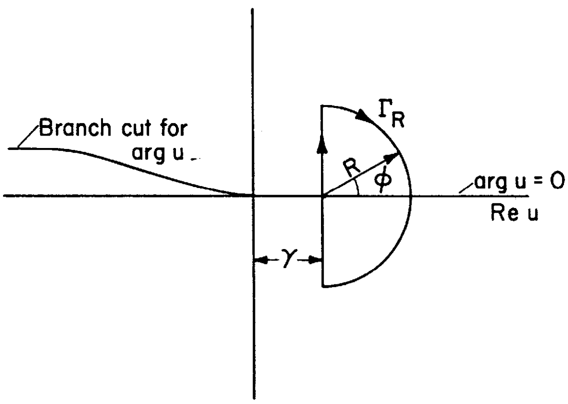

Fig. 5.1. Contour for the inversion integral (5.1)

Now for $t - \alpha x > 0$, we can chose the contour of figure 5.1 for evaluation of (5.1). We observe from (4.7) that on Γ_R of figure 5.1:

$$u\bar{G}(u) = G(0) + \varepsilon(R)$$

where

$$\varepsilon(R) \in \mathbb{C} \to 0 \in \mathbb{C} \quad \text{as} \quad R \to \infty.$$

Hence, if $t - \alpha x = -\beta \quad (\beta > 0)$

then

$$\int_{\Gamma_R} \frac{e^{ut-x\sqrt{\rho u/\bar{G}(u)}}}{u}du = \int_{\Gamma_R} \frac{1}{u} e^{u(t-x\sqrt{\rho/G(0)+\varepsilon(R)})}du$$

and

$$\left|\int_{\Gamma_R} \frac{e^{ut-x\sqrt{\rho u/\bar{G}(u)}}}{u}du\right| \leqslant \int_{-\pi/2}^{\pi/2} \frac{1}{R} e^{-(\beta+\delta(R))(\gamma+R\cos\phi)}R\,d\phi$$

where

$$\delta(R) \to 0 \quad \text{as} \quad R \to \infty$$

and therefore

$$\int_{\Gamma_R} \frac{1}{u} e^{ut-x\sqrt{\rho u/\bar{G}(u)}}du \to 0 \quad \text{as} \quad R \to \infty. \tag{5.2}$$

Applying Cauchy's theorem to the contour shown in figure 5.1, we get

$$\left\{\int_{\gamma-iR}^{\gamma+iR} + \int_{\Gamma_R}\right\}\left(\frac{e^{ut-x\sqrt{\rho u/\bar{G}(u)}}}{u}\right)du = 0. \tag{5.3}$$

We pass to the limit $R \to \infty$ in (5.3) and use (5.1) and (5.2) to establish that

$$v(x,t) = 0 \quad \text{if} \quad t - \alpha x < 0. \tag{5.4}$$

For $t - \alpha x > 0$, we cannot close the contour as in figure 5.1 but we can close the contour in both cases ($t - \alpha x > 0$ and $t - \alpha x < 0$) as in figure 5.2.

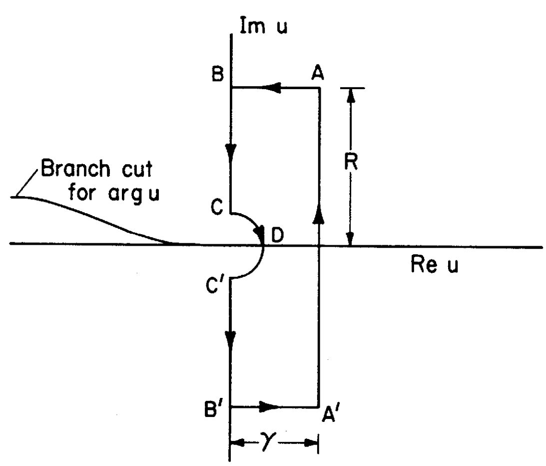

Fig. 5.2

Now let $\bar{G}(iy) \stackrel{\text{def}}{=} r(y)e^{ip(y)}$ where

$$r(y) = |\bar{G}(iy)|$$

and

$$p(y) = \arg \bar{G}(iy).$$

Then, using (4.10), we find that $y \geqslant 0 \Rightarrow p(y) \in [0, -\pi/2)$ and $y \leqslant 0 \Rightarrow p(y) \in [0, \pi/2)$.
Observing next that

$$\left|\int_{AB} \frac{1}{u} e^{ut-x\sqrt{\rho u/\bar{G}(u)}}du\right| \leqslant \int_0^{\gamma} \frac{1}{\sqrt{\xi^2+R^2}} e^{\xi(t-x\sqrt{\rho/G(0)+\varepsilon(R)})}d\xi \tag{5.5}$$

we get:

$$\int_{AB} (\cdot) \to 0 \quad \text{and} \quad \int_{B'A'} (\cdot) \to 0 \quad \text{as} \quad R \to \infty$$

where $(\cdot)$ stands for integrand in (5.5). Putting $u = \varepsilon e^{i\theta}$ on CDC', we find that

$$\lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{CDC'} \frac{1}{u} e^{ut - x\sqrt{\rho u/G(u)}} du$$

$$= \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{\pi/2}^{-\pi/2} i d\theta [1 + 0(\sqrt{\varepsilon})] = -\frac{1}{2}. \quad (5.6)$$

Evaluation on the contour shown in figure 5.2 by Cauchy's theorem gives

$$\{\int_{A'A} + \int_{AB} + \int_{BC} + \int_{CDC'} + \int_{C'B'} + \int_{B'A'}\}(\cdot) = 0. \quad (5.7)$$

Passing to the limit $R \to \infty$ in (5.7), we make use of (5.5) and (5.6) to find that

$$v(x,t) = \lim_{R \to \infty} \int_{\gamma - iR}^{\gamma + iR} \frac{1}{2\pi i} (\cdot)$$

$$= \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \frac{1}{y} \cdot e^{-x\sqrt{\rho y/r(y)}\cos(\pi/4 - p(y)/2)} \sin(yt - \theta(y))\, dy \quad (5.8)$$

where

$$\theta(y) = x \sqrt{\frac{\rho y}{r(y)}} \sin\left(\frac{\pi}{4} - \frac{p(y)}{2}\right). \quad (5.9)$$

Now (5.4) implies that (5.8) is identically zero for $t - \alpha x < 0$. For $t - \alpha x > 0$, (5.8) yields:

$$v(x,t) \equiv f(x,t)$$

$$= \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \frac{1}{y} \cdot e^{-x\sqrt{\rho y/r(y)}\cos(\pi/4 - p(y)/2)} \sin(yt - \theta(y))\, dy. \quad (5.10)$$

(5.4) and (5.10) imply a possible discontinuity at $t = \alpha x$. In this case the inversion integral gives the mean value

$$\frac{v(x, \alpha x^+) + v(x, \alpha x^-)}{2}$$

$$= \lim_{R \to \infty} \int_{\gamma - iR}^{\gamma + iR} \frac{1}{u} e^{ux\sqrt{\rho/G(0)} - x\sqrt{\rho/G(u)}} du$$

$$= \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \frac{1}{y} \cdot e^{-x\sqrt{\rho y/r(y)}\cos(\pi/4 - p(y)/2)} \sin(y\alpha x - \theta(y))\, dy. \quad (5.11)$$

Applying (5.4) and (5.10) to (5.11) we find that

$$f(x, \alpha x^+) = 1 + \frac{2}{\pi} \int_0^\infty \frac{1}{y} \cdot e^{-x\sqrt{\rho y/r(y)}\cos(\pi/4 - p(y)/2)} \sin(y\alpha x - \theta(y))\, dy$$

where

$$\theta(y) = x \sqrt{\frac{\rho y}{r(y)}} \sin\left(\frac{\pi}{4} - \frac{p(y)}{2}\right). \quad (5.12)$$

It follows now from (5.4) and (5.10) that

$$v(x,t) = L^{-1}[\bar{v}(x,u)] = f(x,t) H(t - \alpha x). \quad (5.13)$$

Renardy [20] has shown that (5.12) has the simple form:

$$f(x, \alpha x^+) = \exp(\alpha x G'(0)/2G(0)) \quad (5.14)$$

so that (5.13) is a shock solution with a velocity jump at $t = \alpha x$ for $G'(0) \leqslant 0$. (5.14) can be identified with (2.8) of Renardy's paper as follows:

$$a(s) = -\frac{dG}{ds}; \quad s \geqslant 0,$$

$$A = \int_0^\infty a(s) ds = G(0),$$

$$\rho = 1. \quad (5.15)$$

Thus we also conclude from (5.14) that $G'(0) = -\infty$ implies a smooth solution which has support on a half line for a fixed x or equivalently it has a compact support for fixed t. For $-\infty < G'(0) < 0$, we then have a damped shock wave. For $G'(0) = 0$, the fluid essentially behaves like an elastic solid with no damping[2]).

[2]) See Renardy [20] for examples and proofs of the above assertions.

We close this section by showing how to compute x and t derivatives of $f(x,t)$ at $t = \alpha x^+$. We first assume enough regularity to integrate $\bar{G}(u)$ by parts. Thus

$$\bar{G}(u) = \frac{G(0)}{u} + \frac{G'(0)}{u^2} + \frac{G''(0)}{u^3} + O\left(\frac{1}{u^4}\right). \tag{5.16}$$

Moreover,

$$\left[1 + \frac{\bar{y}'}{u} + \frac{\bar{y}''}{u^2} + O\left(\frac{1}{u^3}\right)\right]^{-1/2} = 1 - \frac{\bar{y}'}{2u} + \frac{3\bar{y}'}{8u^2} + \frac{\bar{y}''}{u^2} + O\left(\frac{1}{u^3}\right) \tag{5.17}$$

where

$$\bar{y}' \overset{\text{def}}{=} G'(0)/G(0),$$
$$\bar{y}'' \overset{\text{def}}{=} G''(0)/G(0).$$

After differentiating (5.13) we find that

$$\frac{\partial v}{\partial x}(x,t) = \frac{\partial f}{\partial x}(x,t)\,H(t-\alpha x) - \alpha f(x,\alpha x^+)\,\delta(t-\alpha x). \tag{5.18}$$

Moreover,

$$\frac{\partial v}{\partial x}(x,t) = -\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{ut}}{u}\sqrt{\frac{\rho u}{\bar{G}(u)}} \cdot e^{-x\sqrt{\rho u/\bar{G}(u)}}\,du \tag{5.19}$$

and, using (5.16) and (5.17) in (5.19), we get

$$\frac{\partial v}{\partial x}(x,t) = -\frac{\alpha}{2\pi i} \cdot \int_{\gamma-i\infty}^{\gamma+i\infty}\left\{\left[1 - \frac{\bar{y}'}{2u} + O\left(\frac{1}{u^2}\right)\right]e^{ut} \cdot e^{-\alpha x u\left[1-\frac{\bar{y}'}{2u}+\frac{3\bar{y}'^2}{8u^2}+\frac{\bar{y}''}{u^2}+O\left(\frac{1}{u^3}\right)\right]}\right\}du$$
$$= \frac{\alpha e^{\alpha x\bar{y}'/2}}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{u(t-\alpha x)} \cdot\left[-1 + \frac{\bar{y}'}{2u} + \frac{\alpha x}{u}\left(\frac{3}{8}\bar{y}'^2 + \bar{y}''\right) + O\left(\frac{1}{u^2}\right)\right]du$$
$$= \alpha e^{\alpha x\bar{y}'/2}\left\{-\delta(t-\alpha x) + \left[\frac{\bar{y}'}{2} + \alpha x\left(\frac{3}{8}\bar{y}'^2 + \bar{y}''\right)\right]H(t-\alpha x) + \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} O\left(\frac{1}{u^2}\right)e^{u(t-\alpha x)}\,du\right\}. \tag{5.20}$$

The last integral of (5.20) is uniformly convergent and hence continuous for all non-negative values of x and t. It follows then from (5.20) and (5.18) that

$$f(x,\alpha x^+) = e^{\alpha x\bar{y}'/2} \tag{5.21}$$

and

$$\frac{\partial f}{\partial x}(x,\alpha x^+) = \alpha f(x,\alpha x^+)\left[\frac{\bar{y}'}{2} + \alpha x\left(\frac{3}{8}\bar{y}'^2 + \bar{y}''\right)\right]. \tag{5.22}$$

We can use the same procedure to compute $\frac{\partial f}{\partial x}(x,\alpha x^+)$. Thus

$$\frac{\partial v}{\partial t}(x,t) = \frac{\partial f}{\partial t}(x,t)\,H(t-\alpha x) + f(x,\alpha x^+)\,\delta(t-\alpha x),$$

and

$$\frac{\partial v}{\partial t}(x,t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{ut-x\sqrt{\rho u/\bar{G}(u)}}.$$

Following the procedures and expansions generated from (5.16), we find that

$$\frac{\partial f}{\partial t}(x,\alpha x^+) = -\alpha x\,e^{\alpha x\bar{y}'/2}\left[\frac{3}{8}\bar{y}'^2 + \bar{y}''\right]. \tag{5.23}$$

Similar computations give the values of higher derivatives of f in terms of derivatives of $G(s)$ at $s = 0$.

6. Proof that the formal Laplace transform solves (3.6)

It is not obvious that (5.13) satisfies (3.6). The doubt arises from the fact that certain jumps associated with $v(x,t)$ are neglected when taking the "formal" Laplace transform. The formal Laplace transform is valid when $v(x,t)$ is twice continuously differentiable in $\mathscr{D} = \{x \geq 0,\ t \geq 0\}$. However, (5.13) is clearly discontinuous and the "true" transform contains additional terms which must vanish if the "true" transform is to coincide with the formal one.

We shall show that the formal transform coincides with the true one by demonstrating that the formal transform arises from the transform associated with a sequence of problems with a smooth data $q_n(t)$ tending to discontinuous data $v(0,t) = \begin{cases} 1 & \text{for} \quad t > 0 \\ 0 & \text{for} \quad t \leqslant 0 \end{cases}$.
We show that the resulting solution $v_n(x,t)$ converges to $v(x,t) = f(x,t)H(t - \alpha x)$. This proof establishes the continuity of the solutions of (3.6) with respect to the boundary data $v(0,t)$.

Proof: Let $v_n(x,t)$ be the solution of the problem

$$\begin{aligned} &\int_0^t G(s) \frac{\partial^2 v_n}{\partial x^2}(x, t-s)\, ds = \rho \frac{\partial v_n}{\partial t}(x,t), \\ &v_n(0,t) = q_n(t), \\ &v_n(x,0) = 0, \\ &v_n(x,t) \text{ is bounded as } x \to \infty \end{aligned} \tag{6.1}$$

where $\{q_n\}$ is a sequence of functions such that

$$\begin{aligned} &q_n \in C^\infty(-\infty, \infty), \\ &q_n(t) = 0 \;\forall\, t \in (-\infty, 0] \end{aligned} \tag*{(6.2)$_1$}$$

for every positive integer n and as $n \to \infty$,

$$q_n(t) \to \begin{cases} 1 & \text{for} \quad t > 0 \\ 0 & \text{for} \quad t \leqslant 0 \end{cases}. \tag*{(6.2)$_2$}$$

One sequence possessing the required properties is described in the Appendix. Since $q_n(t) \to 1$ uniformly in n as $n \to \infty$ on $t \in [\varepsilon, \infty)$ for any $\varepsilon > 0$. We have

$$\bar{q}_n(u) - \frac{1}{u} \overset{\text{def}}{=} \int_0^\infty (q_n(t) - 1) e^{-ut} dt \to 0 \tag*{(6.2)$_3$}$$

uniformly as $n \to \infty \;\forall\, u \in \mathbb{C}$ such that $Re\, u > 0$. Assuming that the solution $v_n(x,t)$ of (6.1) is twice continuously differentiable in $\mathscr{D}$ and its Laplace transform exists, we get

$$\begin{aligned} \bar{v}_n(x,u) &\overset{\text{def}}{=} \int_0^\infty v_n(x,t) e^{-ut} dt = \bar{q}_n(u) e^{-x\sqrt{\rho u/G(u)}} \\ &= u\bar{q}_n(u) \frac{1}{u} e^{-x\sqrt{\rho u/G(u)}} \end{aligned} \tag{6.3}$$

for all $u \in \mathbb{C}$ with $Re\, u > 0$. Moreover,

$$L[q_n'(t)] = u\bar{q}_n(u) - q_n(0) = u\bar{q}_n(u). \tag{6.4}$$

Applying the convolution theorem to (6.3), using (6.4) and (5.13) we get

$$v_n(x,t) = L^{-1}[\bar{v}_n(x,u)] = \int_{\alpha x}^t q_n'(t-s) f(x,s)\, ds. \tag{6.5}$$

The $v_n(x,t)$ given by (6.5) is continuous and bounded, so that the assumptions we made about its Laplace transform are justified. Using (6.2), we see

$$\int_{\gamma - iR}^{\gamma + iR} \bar{q}_n(u) e^{ut - x\sqrt{\rho u/G(u)}} du \to \int_{\gamma - iR}^{\gamma + iR} \frac{1}{u} e^{ut - x\sqrt{\rho u/G(u)}} du$$

as $\quad n \to \infty$. (6.6)

Convergence in (6.6) is uniform for any $R > 0$. The limit $R \to \infty$ on both sides of (6.6) exist; hence,

$$v_n(x,t) \to v(x,t) = f(x,t) H(t - \alpha x)$$

as $n \to \infty$ uniformly for t on either side of αx. Moreover interpreting $\frac{\partial^2 v}{\partial x^2}$ in the generalized sense:

$$\begin{aligned} \frac{\partial^2 v_n}{\partial x} &\to \frac{\partial^2 v}{\partial x^2} \\ &= f_{xx} H(t - \alpha x) - 2\alpha f_x(x,t) \delta(t - \alpha x) \\ &\quad + \alpha^2 f(x,t) \delta'(t - \alpha x). \end{aligned} \tag{6.7}$$

Now define

$$M\psi \overset{\text{def}}{=} \int_0^t G(s) \frac{\partial^2 \psi}{\partial x^2}(x, t-s)\, ds - \rho \frac{\partial \psi}{\partial t}$$

where ψ is piecewise continuous on $\mathscr{D}$ and the derivatives are defined in a generalized sense. Then, thanks to (6.7), we find that for $t > \alpha x$

$$\begin{aligned} M v_n \to M v \equiv &\int_{\alpha x}^t G(s) f_{xx}(x, t-s)\, ds \\ &- 2\alpha G(t - \alpha x) f(x, \alpha x^+) \\ &+ \alpha^2 G'(t - \alpha x) f(x, \alpha x^+) \\ &- \alpha^2 G(t - \alpha x) \frac{\partial f}{\partial t}(x, \alpha x^+) \\ &- \rho \frac{\partial f}{\partial t}(x,t). \end{aligned} \tag{6.8}$$

But (6.1) implies that

$$M v_n = 0 \;\forall\, n \geqslant 1 \tag{6.9}$$

and (6.8) shows that

$$M v = 0 \tag{6.10}$$

for $t > \alpha x$. When $t < \alpha x$, $v(x, t) = 0$ and $Mv = 0$. Therefore

$$v(x, t) = f(x, t) H(t - \alpha x)$$

solves (3.6) (write $U = 1$). Since (6.8) holds for any $t > \alpha x$, it follows that $f(x, t)$ satisfies the following compatibility condition:

$$2 \frac{\partial f}{\partial t}(x, \alpha x^+) + \frac{2}{\alpha} f(x, \alpha x^+) = \frac{G'(0)}{G(0)} f(x, \alpha x^+) . \tag{6.11}$$

We observe that (5.21), (5.22) and (5.23) reduce (6.11) to an identity.

Chadwick and Powdrill [5] have noted that the formal Laplace transform need not be justified in the discontinuous case. They require that some jump conditions should be satisfied as an apparently extra condition coming from physics or somewhere other than the governing problem. Our derivation here shows that these jump conditions are automatically satisfied and the formal Laplace transform is justified.

Finally we note (assuming $\lim_{t\to\infty} v(x, t)$ exists) that:

$$\lim_{t\to\infty} v(x, t) = \lim_{\substack{u \in \mathbb{R} \\ \text{and } u \to 0^+}} u \bar{v}(x, u) = \lim_{\substack{u \in \mathbb{R} \\ \text{and } u \to 0^+}} U e^{-x\sqrt{\rho u / G(u)}} = U$$

independent of the material function $G(s)$.

7. Step change in velocity when $G(s) = ke^{-\mu s}$

This is a solution of the problem (3.6) for a Maxwellian fluid ($G(s) = ke^{-\mu s}$). The explicit solution (see Tanner [24], Saut and Joseph [21]) is instructive to have as it gives features of the flow field behind the vortex-sheet for a "typical" fluid[3]).

[3]) "Typical" for kernels $G(s)$ such that $0 < G(0) < \infty$, $-\infty < G'(0) < 0$ where $G(s)$ is positive and monotone decreasing with $s > 0$.

We introduce dimensionless variables into (3.6) for the case $G(s) = ke^{-\mu s}$:

$$\{x, s, v(x, t)\} \stackrel{\text{def}}{=} \left\{ \sqrt{\frac{\kappa}{\rho \mu^2}}\,\hat{x},\;\; \frac{1}{\mu}\hat{s},\;\; U \hat{v}(\hat{x}, \hat{t}) \right\}.$$

The dimensionless variables satisfy

$$\begin{aligned} &\int_0^{\hat{t}} e^{-\hat{s}} \frac{\partial^2 \hat{v}}{\partial \hat{x}^2}(\hat{x}, \hat{t} - \hat{s})\, d\hat{s} = \frac{\partial \hat{v}}{\partial \hat{t}}(\hat{x}, \hat{t}) , \\ &\hat{v}(0, \hat{t}) = H(\hat{t} - 0) , \\ &\hat{v}(\hat{x}, 0) = 0 \;\forall\, \hat{x} > 0 , \\ &\hat{v}(\hat{x}, \hat{t}) \text{ is bounded as } \hat{x} \to \infty . \end{aligned} \tag{7.1}$$

Differentiating (7.1) and adding the extra condition $\frac{\partial \hat{v}}{\partial \hat{t}}(\hat{x}, 0) = 0$, we see that (7.1) is equivalent to the following problem studied by Tanner [24]:

$$\begin{aligned} &\frac{\partial \hat{v}}{\partial \hat{t}} + \frac{\partial^2 \hat{v}}{\partial \hat{t}^2} = \frac{\partial^2 \hat{v}}{\partial \hat{x}^2} , \\ &\hat{v}(0, \hat{t}) = H(\hat{t}) , \\ &\hat{v}(\hat{x}, 0) = 0 , \\ &\frac{\partial \hat{v}}{\partial \hat{t}}(\hat{x}, 0) = 0 , \\ &\hat{v}(\hat{x}, \hat{t}) \text{ is bounded as } \hat{x} \to \infty . \end{aligned} \tag{7.2}$$

The problem (7.2) is associated with the telegraph equation and may be solved by Laplace transform techniques. Carlslaw and Jaeger [4] give the solution of (7.2) in the form

$$\hat{v}(\hat{x}, \hat{t}) = \left[e^{-\hat{x}/2} + \frac{\hat{x}}{2} \int_{\hat{x}}^{\hat{t}} \frac{e^{-\sigma/2}}{\sqrt{\sigma^2 - \hat{x}^2}} \cdot I_1\left(\frac{1}{2}\sqrt{\sigma^2 - \hat{x}^2} \right) d\sigma \right] H(\hat{t} - \hat{x}) . \tag{7.3}$$

The jump across the vortex-sheet is given by:

$$[\![v(x, t)]\!] = U [\![\hat{v}(\hat{x}, \hat{t})]\!] = U e^{-\hat{x}/2} = U e^{-\mu/2\sqrt{\rho/\kappa}\, x} . \tag{7.4}$$

We may write (7.3) as

$$\hat{v}(\hat{x}, \hat{t}) = f(\hat{x}, \hat{t}) H(\hat{t} - \hat{x}) . \tag{7.5}$$

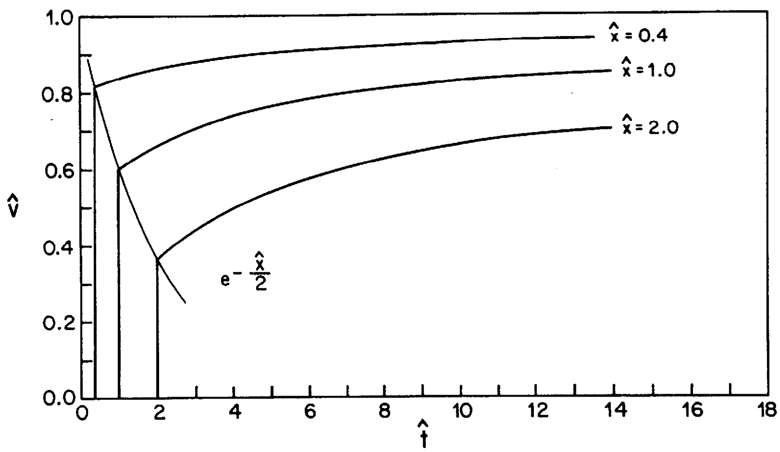

Fig. 7.1. Velocity $\hat{v}(\hat{x}, \hat{t})$ of the fluid above a plane undergoing a sudden acceleration from speed 0 to 1 [Eq. (7.3) satisfying (7.2)]

The time derivative of $f(x, t)$ at $\hat{t} = \hat{x}^+$ is, because of $I_1'(0) = 1/2$:

$$\frac{\partial \hat{f}(\hat{x}, \hat{x}^+)}{\partial \hat{t}} = \frac{\hat{x}}{4} e^{-\hat{x}/2} I_1'(0) = \frac{\hat{x}}{8} e^{-\hat{x}/2}. \tag{7.6}$$

This function of $\hat{x}$ is increasing for $0 \leqslant \hat{x} < 2$ and decreasing for $\hat{x} > 2$. So for large values of $\hat{x}$ the variation of the solution with $\hat{t}$ after $\hat{t} = \hat{x}^+$ is very gentle. The solution surface is shown in Fig. 7.1 for three different values of $\hat{x} \leqslant 2$.

8. Development of Couette flow from a sudden start from rest through reflections

This is the problem as described in Sect. 3 except that the fluid is confined between parallel plates separated by a distance l. Now $v(x, t)$ must satisfy:

$$\int_0^t G(s) \frac{\partial^2 v}{\partial x^2}(x, t - s) ds = \rho \frac{\partial v}{\partial t}(x, t),$$

$$v(0, t) = \begin{cases} U & \text{for} \quad t > 0 \\ 0 & \text{for} \quad t \leqslant 0 \end{cases}, \tag{8.1}$$

$$v(l, t) = 0.$$

Arguing as in Sect. 6 show that the solution of (8.1) is given by the inverse of the formal Laplace transform

$$\bar{v}(x, u) = \frac{U \sinh(l - x)\eta(u)}{u \sinh l\eta(u)}; \quad \forall u \in \mathbb{C} \ni Re\, u > 0 \tag{8.2}$$

where $\eta(u) \overset{\text{def}}{=} \sqrt{\rho u / \bar{G}(u)}$. Then, from (4.11) we see that if $Re\, u > 0$, i.e., $\arg u \in (-\pi/2, \pi/2)$, then $\arg \eta(u) \in (-\pi/2, \pi/2)$ and

$$Re\, u > 0 \Rightarrow Re\, \eta(u) > 0. \tag{8.3}$$

Rewriting (8.2) we find that

$$\begin{aligned} \bar{v}(x, u) &= \frac{U}{u} \left[\frac{e^{(l-x)\eta(u)} - e^{-(l-x)\eta(u)}}{e^{l\eta(u)} - e^{-l\eta(u)}} \right] \\ &= \frac{U}{u} e^{-x\eta(u)} (1 - e^{-2(l-x)\eta(u)}) \\ &\quad \cdot \left[\frac{1}{1 - e^{-2l\eta(u)}} \right]. \end{aligned} \tag{8.4}$$

A further reduction of (8.4) results from expanding

$$\frac{1}{1 - e^{-2l\eta(u)}} = 1 + e^{-2l\eta(u)} + e^{-4l\eta(u)} + \ldots \tag{8.5}$$

into a geometric series. The series on the right side of (8.5) is absolutely and uniformly convergent because $|e^{-2l\eta(u)}| = e^{-2l Re\eta(u)} < 1$. Using (8.5) in (8.4), we get

$$\begin{aligned} \bar{v}(x, u) = \frac{U}{u} [& e^{-x\eta(u)} + \{e^{-(x+2l)\eta(u)} - e^{-(2l-x)\eta(u)}\} \\ & + \{e^{-(x+4l)\eta(u)} - e^{-(4l-x)\eta(u)}\} + \ldots]. \end{aligned} \tag{8.6}$$

Now the right side of (8.6) is also absolutely uniformly convergent so term by term inversion is allowed. Returning to (5.13), we get

$$\begin{aligned} v(x, t) &= L^{-1}[\bar{v}(x, u); \quad u \to t] \\ &= U[f(x, t) H(t - \alpha x) \\ &\quad + \{f(x + 2l, t) H(t - \alpha(x + 2l)) \\ &\quad - f(2l - x, t) H(t - \alpha(2l - x))\} + \ldots]. \end{aligned} \tag{8.7}$$

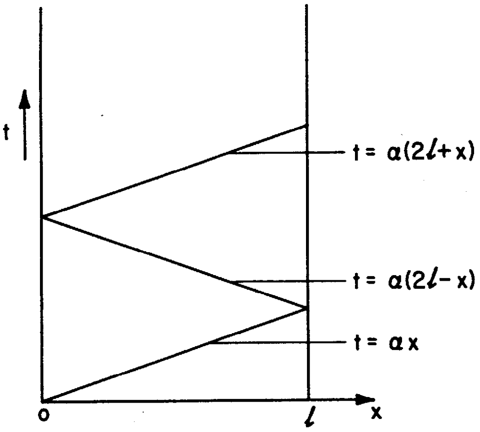

Fig. 8.1. Reflections at the walls following a step change in velocity

The relation (8.7) shows that vortex-sheets are reflected back and forth along the characteristic lines in the (x, t) plane shown in figure 8.1.

We may calculate $\lim_{t\to\infty} v(x,t)$ (if it exists) directly from the Laplace transform as follows.

$$\lim_{t\to\infty} v(x,t) = \lim_{\substack{u\in\mathbb{R}\\ \text{and } u\to 0^+}} u\bar{v}(x,u)$$

$$= \lim_{\substack{u\in\mathbb{R}\\ \text{and } u\to 0^+}} U\frac{\sinh(l-x)\sqrt{\rho u/\bar{G}(u)}}{\sinh\sqrt{\rho u/\bar{G}(u)}}$$

$$= U(l-x)/l. \tag{8.8}$$

So every solution of our transient problem tends to steady Couette flow independent of the density ρ or shear relaxation modulus G.

9. Development of Couette flow when $G(s) = ke^{-\mu s}$

Here we obtain an explicit solution of (8.1) when $G(s) = ke^{-\mu s}$. We again introduce dimensionless variable $\hat{x}$, $\hat{s}$, $\hat{t}$, $\hat{v}$ as

$$\{x, s, v(x,t)\} \stackrel{\text{def}}{=} \left\{\sqrt{\frac{\kappa}{\rho\mu^2}}\hat{x}, \frac{1}{\mu}\hat{s}, U\hat{v}(\hat{x},\hat{t})\right\} \tag{9.1}$$

into (8.1) and we solve the resulting problem as in (8.7). Thus

$$\begin{aligned}\hat{v}(\hat{x},\hat{t}) = [&\hat{f}(\hat{x},\hat{t})H(\hat{t}-\hat{x})\\ &+\{\hat{f}(\hat{x}+2\hat{l},\hat{t})H(\hat{t}-(\hat{x}+2\hat{l}))\\ &-\hat{f}(2\hat{l}-\hat{x},\hat{t})H(\hat{t}-(2\hat{l}-\hat{x}))\}\\ &+\{\ldots\}+\ldots]\end{aligned} \tag{9.2}$$

where $\hat{f}(\hat{x},\hat{t})$ in (9.2) is the function defined by (7.4). Representative graphs of (9.2) are shown in figure 9.1 (see also Böhme [3]).

10. Step displacement of the wall $x = 0$ bounding a fluid in the semi-infinite region $x > 0$. Momentum layers

This is again a problem in the class of shearing motions (see Sect. 3) except that the bottom plate $(x = 0)$ is given a step in displacement rather then a

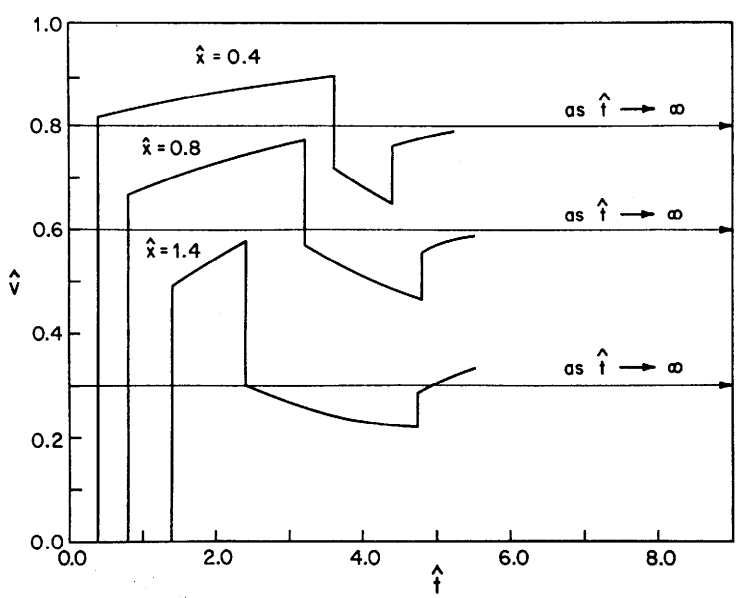

Fig. 9.1. Velocity $\hat{v}(\hat{x},\hat{t})$ of the fluid between parallel plates when the bottom is suddenly accelerated from speed 0 to 1. The wave undergoes multiple reflections at the boundaries [Eq. (9.2) satisfying (8.1)]

step in velocity. Thus, in the notation of (3.1), we require that

$$\mu(0,t) = UH(t-0)$$

or (10.1)

$$v(0,t) = U\delta(t-0)\,.$$

Thus the initial boundary value problem for evolution of the velocity is given as in (3.6):

$$\begin{aligned} &\int_0^t G(s)\frac{\partial^2 v}{\partial x^2}(x,t-s)\,ds = \rho\frac{\partial v}{\partial t}\,,\\ &v(0,t) = U\delta(t-0)\,,\\ &v(x,0) = 0\,,\\ &v(x,t) \text{ is bounded as } x\to\infty\,. \end{aligned} \tag{10.2}$$

The solution of (10.2) can be shown to be the inverse of the Laplace transform of (10.2):

$$\bar{v}(x,u) = Ue^{-x\eta(u)};\quad u\in\mathbb{C}\ni Re\,u>0\,. \tag{10.3}$$

Using (5.13) we find that:

$$L[f(x,t)H(t-\alpha x)] = \frac{1}{u}e^{-x\eta(u)} \tag{10.4}$$

where $\eta(u) \overset{\text{def}}{=} \sqrt{\rho u/G(u)}$.

The theory of the Laplace transform shows that the derivative of discontinuous functions ([23], p. 150) are in the form

$$L\left[\frac{\partial f}{\partial t}H(t-\alpha x)\right] = uL[f(x,t)H(t-\alpha x)] - [\![f(x,t)H(t-\alpha x)]\!]\,e^{-\alpha ux}$$

where

$$[\![\psi(x,t)]\!] = \psi(x,\alpha x^+) - \psi(x,\alpha x^-)\,. \tag{10.5}$$

Using (10.5) in (10.4) we get:

$$\begin{aligned} L\left[\frac{\partial f}{\partial t}H(t-\alpha x)\right] &= e^{-x\eta(u)} - f(x,\alpha x^+)e^{-\alpha ux}\\ &= e^{-x\eta(u)} - L[f(x,\alpha x^+)\delta(t-\alpha x)]\,. \end{aligned} \tag{10.6}$$

Using (10.6) we find that

$$\begin{aligned} v(x,t) &= UL^{-1}[e^{-x\eta(u)}]\\ &= U\left[\frac{\partial f}{\partial t}H(t-\alpha x) + f(x,\alpha x^+)\delta(t-\alpha x)\right]. \end{aligned} \tag{10.7}$$

The derivation leading to (10.7) justifies a calculation given in [21]. This formal calculation follows from differentiating (3.6) with respect to t. We get

$$\begin{aligned} &G(t)\frac{\partial^2 v}{\partial x^2}(x,0) + \int_0^t G(s)\frac{\partial^2 v_t}{\partial x^2}(x,t-s)\,ds\\ &= \rho\frac{\partial v_t}{\partial t}(x,t)\,,\\ &v_t(0,t) = U\delta(t-0)\,,\\ &v_t(x,0) = 0\,,\\ &v_t(x,t) \text{ is bounded as } x\to\infty. \end{aligned} \tag{10.8}$$

But in (10.8), $\partial^2 v(x,0)/\partial x^2 = 0$ and (10.8) is the same as (10.2). Therefore solutions of (10.2) can be obtained from the derivative of solutions of (3.6).

The important physical conclusion from (10.7) is that simple fluids with fading memory have the property that besides being able to sustain singular surfaces such as "vortex sheets", they can transmit and sustain even stronger singular surfaces which we call "momentum layers". Across these singular surface, the momentum/area of the fluid has a jump m_Σ given by

$$m_\Sigma \overset{\text{def}}{=} \lim_{\varepsilon\to 0}\int_{x=ct-\varepsilon}^{x=ct+\varepsilon} \rho v(x,t)\,dx\,,\quad c = \frac{1}{\alpha}. \tag{10.9}$$

(10.7) in (10.9) yields:

$$m_\Sigma = \rho c f(x,\alpha x^+)\,.$$

If $G'(0) < 0$, then because of (5.14) $m_\Sigma \to 0$ as $x\to\infty$.

Now we prove the "shift theorem" for the problem of the step displacement. As in the problem of the step jump in velocity, the transients associated with the step change decay to zero and the final steady state is material independent. Using the notation introduced in (3.1) we have

$$v(x,\tau) = \frac{\partial\mu}{\partial\tau}(x,\tau)\,. \tag{10.10}$$

The shift of a particle $\Delta(t) \equiv \mu(x,t) - \mu(x,0)$ at any instant t is given by:

$$\Delta(t) = \int_0^t v(x, \tau)d\tau. \qquad (10.11)$$

Hence

$$\bar{\Delta}(u) \overset{\text{def}}{=} \int_0^\infty e^{-ut}\Delta(t)dt$$
$$= \frac{1}{u}\bar{v}(x, u)$$
$$= \frac{U}{u}e^{-x\sqrt{\rho u/\bar{G}(u)}} \quad \forall u \in \mathbb{C} \ni Re\, u > 0.$$

Then if $\lim_{t\to\infty} \Delta(t)$ exists its value may be computed as in (6.2). We find that

$$\lim_{t\to\infty} \Delta(t) = U. \qquad (10.12)$$

The interpretation of (10.12) is that all particles which are initially in straight lines parallel to the x axis ultimately move a constant distance U in the direction of increasing y; that is, in the direction of the step displacement of the bottom plate.

11. The step displacement problem in Sect. 10 when $G(s) = ke^{-\mu s}$

The problem is governed by (10.2) with $G(s) = ke^{-\mu s}$ using the dimensionless variables introduced in (9.1) we may write the solution as

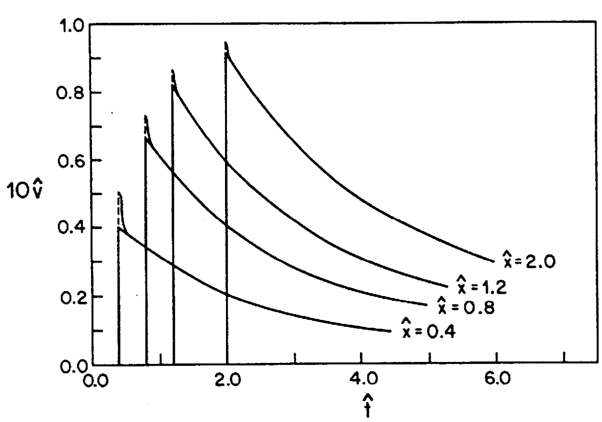

Fig. 11.1. Velocity $\hat{v}(\hat{x}, \hat{t})$ of the fluid above a plane undergoing a step displacement, from rest to rest. The velocity $\hat{v}(\hat{x}, \hat{t})$ is given by (11.1). The δ part of the solution is suggested by the peaklet. The magnitude of $\hat{v}(\hat{x}, \hat{t})$ *decreases* monotonically to zero with $\hat{t}$ for $\hat{t} \geqslant \hat{x}$ at each fixed $\hat{x}$. The jump also *decreases* to zero with $\hat{x}$ for each fixed $\hat{x} \geqslant 2$ (see Sect. 7). We have shown the increasing part of the jump for $\hat{x} \leqslant 2$. The "amplitude" of the delta functions tends to zero monotonically, $\hat{f}(\hat{x}, \hat{x}^+) = e^{-\hat{x}/2}$

$$v(x, t) = U\hat{v}(\hat{x}, \hat{t})$$
$$= U\left[\frac{\partial\hat{f}}{\partial\hat{t}}H(\hat{t} - \hat{x}) + \hat{f}(\hat{x}, \hat{x})\delta(\hat{t} - \hat{x})\right] \qquad (11.1)$$

where $\hat{f}(\hat{x}, \hat{t})$ in (11.1) is the function defined in (7.4). Representative graphs of (11.1) are shown in figure 11.1. We represent $\delta(\hat{t} - \hat{x})$ in the graph with a small peaklet around the step-jump. Since this δ part of solution vanishes for $\hat{t} \neq \hat{x}$, the graphs are essentially those of $\partial\hat{f}(\hat{x}, \hat{t})/\partial\hat{t}$ for fixed values of $\hat{x}$. We already have remarked in Sect. 7 that $\partial\hat{f}(\hat{x}, \hat{x}^+)/\partial\hat{t}$ is increasing for $\hat{x} < 2$ and decreases monotonically to 0 for $\hat{x} > 2$.

12. Step displacement of the wall $x = 0$ bounding a fluid in the channel $0 \leqslant x \leqslant l$. Reflections of propagating momentum layers. Universality of the final displacement

This problem is analogous to the problem in (8.1) except that the boundary data at $x = 0$ is a step displacement rather than a step velocity. The interior values of the velocity are determined by the problem

$$\int_0^t G(s)\frac{\partial^2 v}{\partial x^2}(x, t - s)ds = \rho\frac{\partial v}{\partial t},$$
$$v(0, t) = U\delta(t - 0),$$
$$v(l, t) = 0, \quad t \geqslant 0,$$
$$v(x, 0) = 0. \qquad (12.1)$$

Again we can show that solution of (12.1) is determined by the inverse of the transform

$$\bar{v}(x, u) = \frac{U\sinh(l - x)\eta(u)}{\sinh l\eta(u)}; \quad u \in \mathbb{C} \ni Re\, u > 0 \qquad (12.2)$$

and using an expansion as in (8.6), we get:

$$\bar{v}(x, u) = U[e^{-x\eta(u)} + \{e^{-(x+2l)\eta(u)} - e^{-(2l-x)\eta(u)}\} + \ldots]. \qquad (12.3)$$

using (10.7), we have

$$L^{-1}[e^{-x\eta(u)}] = \frac{\partial f}{\partial t}H(t - \alpha x) + f(x, \alpha x^+)\delta(t - \alpha x)$$
$$\overset{\text{def}}{=} g(x, t). \qquad (12.4)$$

We put (12.4) into (12.3) and invert the resulting expression to get

$$v(x,t) = U[g(x,t) + \{g(x+2l,t) - g(x-2l,t)\} + \ldots]. \qquad (12.5)$$

It follows from (12.5) that the characteristics in $x - t$ plane are the same as in figure 8.1. In the present case, however, we have "δ-discontinuities" or "momentum layers" travelling along characteristics. We know from (5.13) that the "δ-discontinuity" is damped.

Universality of the final displacement. Now we compute the shift of the particles from their initial position. Using the notations of (3.1), (10.10) and (10.11) we find that

$$\Delta(t) \overset{\text{def}}{=} \mu(x,t) - \mu(x,0) = \int_0^t v(x,\tau)d\tau: \qquad (12.6)$$

It follows from (12.2) and (12.6) that

$$\bar{\Delta}(u) \overset{\text{def}}{=} \int_0^\infty \Delta(t)e^{-ut}dt = \frac{U}{u}\left[\frac{\sinh(l-x)\eta(u)}{\sinh l\eta(u)}\right]$$
$$\forall\, u \in \mathbb{C} \ni Re\, u > 0 . \qquad (12.7)$$

Reasoning along the lines leading to (8.8), we compute

$$\lim_{t\to\infty} \Delta(t) = \lim_{\substack{u\in\mathbb{R}\\ u\to 0^+}} u\bar{v}(x,u) = \frac{U(l-x)}{l}. \qquad (12.8)$$

The interpretation of (12.8) is that all particles initially in vertical straight lines parallel to $y = 0$ ultimately move in the y-direction to a skew straight line given by $y = U(l-x)/l$. This final shift is independent of the material.

13. Reflection of momentum layers for $G(s) = \kappa e^{-\mu s}$

Here we seek an explicit solution of (12.1) for $G(s) = \kappa e^{-\mu s}$. Using (9.1) we find that

$$\hat{v}(\hat{x},t) = U[\hat{g}(\hat{x},\hat{t}) + \{\hat{g}(\hat{x}+2\hat{l},\hat{t}) - \hat{g}(\hat{x}-2\hat{l},\hat{t})\} + \ldots]$$

where (13.1)

$$\hat{g}(\hat{x},\hat{t}) = \frac{\partial \hat{f}}{\partial \hat{t}} H(\hat{t}-\hat{x}) + \hat{f}(\hat{x},\hat{x}^+)\delta(\hat{t}-\hat{x})$$

and $\hat{f}(\hat{x},\hat{t})$ is the function defined by (7.4). Representative graphs of (13.1) are shown in figure 13.1.

14. Sudden spin-up of a cylinder in a simple fluid at rest

A simple fluid occupies the region $r > a$, $-\infty < z < \infty$ outside a right circular cylinder. Let the position $\xi = \chi_t(x,\tau)$ of a particle $P(X^1, X^2, X^3)$ be given by coordinates $\{\xi^1, \xi^2, \xi^3\} \overset{\text{def}}{=} \{r, \theta, z\}$ and the position x by $\{x^1, x^2, x^3\} \overset{\text{def}}{=} \{R, \Theta, Z\}$. Let $\{e_i(\xi(\tau))\}$ be the covariant basis associated with the position $\xi = \chi_t(x,\tau)$. The spin-up problem is associated with the kinematics of shearing motions (see (2.4))

$$\xi^1 = X^1,\quad \xi^2 = X^2 + \mu(X^1,\tau),\quad \xi^3 = X^3$$
$$x^1 = X^1,\quad x^2 = X^2 + \mu(X^1,t),\quad x^3 = X^3. \qquad (14.1)$$

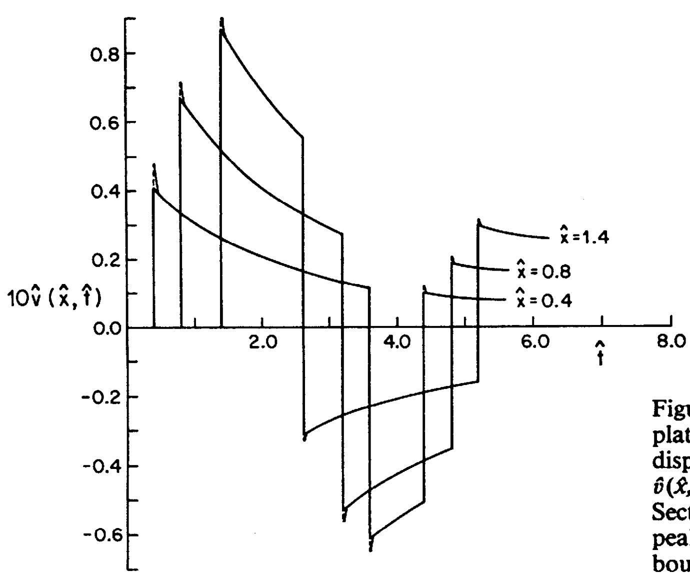

Figure 13.1. Velocity $\hat{v}(\hat{x},\hat{t})$ of the fluid between parallel plates when the bottom plate undergoes a step in displacement. The velocity is given by (13.1). The jumps in $\hat{v}(\hat{x},\hat{t})$ increase for $\hat{x} < 2$ and decrease to zero for $\hat{x} > 2$ (see Sect. 7). The infinite velocities at $\hat{t} = \hat{x}$ are suggested by the peaklets. The wave undergoes multiple reflections at the boundaries [Eq. (13.1) satisfying (12.1)]

The orthonormal basis for this shearing motion is $\{e^{\langle i\rangle}(x) \overset{\text{def}}{=} e_i(\xi(t))/|e_i|\}$ and

$$\lambda^t(s) = x^1 \frac{\partial \mu^t}{\partial x^1}(x^1, s)$$

where

$$\mu^t(x^1, s) \equiv \mu(x^1, t-s) - \mu(x^1, t). \tag{14.2}$$

The velocity v at any point $\xi(\tau) = \chi_t(x, \tau)$ is given by:

$$\begin{aligned} v &= \dot{\xi}^i e_i(\xi(\tau)) \\ &= \frac{\partial \mu}{\partial \tau}(x^1, \tau) e_2(\xi(\tau)) \\ &= x^1 \frac{\partial \mu}{\partial \tau}(x^1, \tau) e^{\langle 2\rangle}(\xi(\tau)). \end{aligned} \tag{14.3}$$

We define the shear rate $\omega(x^1, \tau) \overset{\text{def}}{=} x^1\, \partial\mu/\partial\tau(x^1, \tau)$ and use (14.2) and (14.3) to obtain

$$\lambda^t(s) = x^1 \frac{\partial \mu^t}{\partial x^1}(x^1, s) = x^1 \int_t^{t-s} \frac{\partial}{\partial x^1}\left(\frac{\omega(x^1, \sigma)}{x^1}\right) d\sigma$$

and, from $(2.8)_1$,

$$\begin{aligned} \mathscr{G}^{\langle \pi\theta\rangle}(t) &\sim \int_0^\infty \frac{dG}{ds}(s)\lambda^t(s)\,ds \\ &= -\int_0^\infty G(s) \frac{d\lambda^t(s)}{ds} ds \\ &= \int_0^\infty G(s)\left[\frac{\partial \omega}{\partial r}(r, t-s) - \frac{\omega(r, t-s)}{r}\right] ds. \end{aligned}$$

The equations of momentum in the basis $\{e^{\langle i\rangle}(x)\}$ are

$$\begin{aligned} &-\frac{\partial p}{\partial r} + \frac{\partial \mathscr{G}^{\langle rr\rangle}}{\partial r} + \frac{1}{r}\{\mathscr{G}^{rr\rangle} - \mathscr{G}^{\langle\theta\theta\rangle}\} = 0 \\ &-\frac{1}{r}\frac{\partial p}{\partial \theta} + \frac{\partial \mathscr{G}^{\langle r\theta\rangle}}{\partial r} + \frac{2\mathscr{G}^{\langle r\theta\rangle}}{r} = \rho \frac{\partial \omega}{\partial t}(r, t) \\ &-\frac{\partial p}{\partial z} = 0. \end{aligned} \tag{14.6}$$

A sudden spin-up from rest is defined by the following conditions:

$$\begin{aligned} &\omega(a, t) = \begin{cases} a\Omega & \text{for} \quad t > 0 \\ 0 & \text{for} \quad t \leqslant 0, \end{cases} \\ &\omega(r, 0) = 0 \qquad r \geqslant a > 0, \\ &\omega(r, t) \text{ is bounded}. \end{aligned} \tag{14.7}$$

Ω in (14.7) is the constant impulsive angular velocity of the inner cylinder. Using the invariance of (14.6) with respect to changes in the spin of θ we find that $\partial p/\partial\theta = 0$. Combining $(14.6)_2$, (14.7) and (14.5) we find that the governing evolution problem may be written as

$$\rho \frac{\partial \omega}{\partial t} = \int_0^t G(s)\left[\frac{\partial^2 \omega}{\partial r^2}(r, t-s) + \frac{1}{r}\frac{\partial \omega}{\partial r}(r, t-s) - \frac{\omega(r, t-s)}{r^2}\right] ds,$$

$$\omega(a, t) = \begin{cases} a\Omega & \text{for} \quad t > 0 \\ 0 & \text{for} \quad t \leqslant 0, \end{cases} \tag{14.8}$$

$$\begin{aligned} &\omega(r, 0) = 0 \quad \forall\, r \geqslant a > 0, \\ &\omega(r, t) \text{ is bounded as } r \to \infty. \end{aligned} \tag{14.8}$$

The problem (14.8) may be solved by Laplace transform as in Sect. 6. Suppose, that $L[\omega(r, t);\ t \to u] \overset{\text{def}}{=} \bar{\omega}(r, u)$. $\forall\, u \in \mathbb{C} \ni Re\, u > 0$. Then the transform of (14.8) is:

$$\begin{aligned} &\frac{d^2\bar{\omega}}{dr^2} + \frac{1}{r}\frac{d\bar{\omega}}{dr} - \left(\frac{1}{r^2} + \frac{\rho u}{\bar{G}(u)}\right)\bar{\omega}(r, u) = 0, \\ &\bar{\omega}(a, u) = \frac{a\Omega}{u}, \\ &\bar{\omega}(r, u) \text{ is bounded as } r \to \infty. \end{aligned} \tag{14.9}$$

The solution of (14.9) is given by

$$\bar{\omega}(r, u) = \frac{a\Omega}{u} \frac{K_1(r\sqrt{\rho u/\bar{G}(u)})}{K_1(a\sqrt{\rho u/\bar{G}(u)})}. \tag{14.10}$$

Therefore the solution of (14.8) is given by:

$$\begin{aligned} \omega(r, t) &= \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{ut}\bar{\omega}(r, u)\,du \\ &= \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} \frac{e^{ut}}{u}\frac{K_1(r\sqrt{\rho u/\bar{G}(u)})}{K_1(a\sqrt{\rho u/\bar{G}(u)})} du; \\ &\qquad u \in \mathbb{C} \ni Re\, u > 0. \end{aligned} \tag{14.11}$$

If $|\arg(z)| < 3\pi/2$, then $K_1(z) \neq 0$ ([26], p. 511). Since, by (8.3), $|\arg a\sqrt{\rho u/\bar{G}u}| < \pi/2$, we have $K_1(a\sqrt{\rho u/\bar{G}(u)}) \neq 0$. Moreover, since $\bar{G}(u)$ has no zeroes or poles for $Re\, u > 0$, the integrand in (14.11) is analytic in the half-plane $Re\, u > 0$. To make this integrand single valued we define a branch-cut in the half plane $Re\, u < 0$ extending from $|u| = 0$ to $|u| = \infty$. We also make use of the fact ([1], p. 375) that $K_1(z)$ behaves asymptotically like:

$$\text{(i) } K_1(z) \sim \frac{1}{z} \quad \text{for } z \text{ as } z \to 0, \qquad \text{(ii) } K_1(z) \sim \sqrt{\frac{\pi}{2z}} \quad \text{for } |z| \to \infty. \tag{14.12}$$

Using $(14.12)_2$, we find that

$$\frac{K_1(r\sqrt{\rho u/\bar{G}(u)})}{K_1(a\sqrt{\rho u/\bar{G}(u)})} \sim \sqrt{\frac{a}{r}}\, e^{-(r-a)u\alpha} \text{ as } |u| \to \infty. \tag{14.13}$$

We next use (14.13) in the inversion integral (14.11). When $t - \alpha(r - a) < 0$ we can evaluate the inversion integral on the contour of Fig. 5.1 in Sect. 5 and following the argument given there, we prove that $v(x, t) = 0$ when

$$t - (r - a)\alpha < 0. \tag{14.14}$$

When $t - (r - a)\alpha > 0$ we can evaluate the inversion integral on the contour shown in Fig. 5.2, Sect. 5. If,

$$\bar{G}(iy) \stackrel{\text{def}}{=} q(y)\, e^{ip(y)} \tag{14.15}$$

where,

$$q(y) = |\bar{G}(iy)|, \qquad p(y) = \arg \bar{G}(iy)$$

then the contour integration of Fig. 5.2, Sect. 5 gives:

$$v(r, t) = a\Omega \left[\frac{a}{2r} + \frac{1}{\pi} \cdot \int_0^\infty Im \left\{ \frac{e^{iyt} K_1(r\sqrt{\rho y/q(y)}\, e^{i(\pi/4 - p(y)/2)})}{y K_1(a\sqrt{\rho y/q(y)}\, e^{i(\pi/4 - p(y)/2)})} \right\} dy \right]. \tag{14.16}$$

$v(r,t)$ given by (14.16) is identically zero when $t - \alpha(r - a) < 0$. Putting $v(r,t) = g(r,t)$ for $t - \alpha(r - a) > 0$ we may write (14.16) as

$$v(r, t) = g(r, t) H(t - (r - a)\alpha). \tag{14.17}$$

Moreover

$$v(a, t) = g(a, t) = a\Omega > 0. \tag{14.18}$$

Hence $g(r, t) \neq 0$, at least in a small neighbourhood of $r = a$. The jump in $v(r, t)$ at $t = (r - a)\alpha$ is given, as in (5.12), by

$$g(r, \alpha(r - a)^+) = a\Omega \left[\frac{a}{r} + \frac{2}{\pi} \cdot \int_0^\infty Im \left\{ \frac{e^{iy(r-a)\alpha} K_1(r\sqrt{\rho y/q(y)}\, e^{i(\pi/4 - p(y)/2)})}{y K_1(a\sqrt{\rho y/q(y)}\, e^{i(\pi/4 - p(y)/2)})} \right\} dy \right]. \tag{14.19}$$

Moreover, using $(14.12)_1$, we find

$$\lim_{t\to\infty} \omega(r, t) = \lim_{\substack{u \in \mathbb{R} \\ \text{and } u \to 0^+}} u\bar{\omega}(r, u) = a\Omega \lim_{u\to 0} \frac{K_1(r\sqrt{\rho u/\bar{G}(u)})}{K_1(a\sqrt{\rho u/\bar{G}(u)})} = \frac{a^2\Omega}{r}. \tag{14.20}$$

The field $a^2\Omega/r$ is a potential flow solution for steady flow of a viscous fluid induced by steady rotation of a cylindrical rod of radius a.

15. Sudden spin-up of a cylinder when $G(s) = \kappa e^{-\mu s}$

We solve (14.8) when $G(s) = \kappa e^{-\mu s}$. First we introduce dimensionless variables $\hat{r}$, $\hat{s}$, $\hat{t}$

$$\{r, s, \omega(r, t)\} = \left\{ a\hat{r}, \frac{1}{\mu}\hat{s}, a\mu\hat{\omega}(\hat{r}, \hat{t}) \right\}. \tag{15.0}$$

The dimensionless form of (14.8) is

$$\begin{aligned} &\int_0^{\hat{t}} e^{-\hat{s}} \left[\frac{\partial^2 \hat{\omega}}{\partial \hat{r}^2}(\hat{r}, \hat{t} - \hat{s}) + \frac{1}{\hat{r}} \frac{\partial \hat{\omega}}{\partial \hat{r}}(\hat{r}, \hat{t} - \hat{s}) - \frac{\hat{\omega}}{\hat{r}^2}(\hat{r}, \hat{t} - \hat{s}) \right] d\hat{s} = \lambda^2 \frac{\partial \hat{\omega}}{\partial \hat{t}}, \\ &\hat{\omega}(1, \hat{t}) = \begin{cases} \delta & \text{for } \hat{t} > 0 \\ 0 & \text{for } \hat{t} \leqslant 0, \end{cases} \\ &\hat{\omega}(\hat{r}, 0) = 0 \qquad \hat{r} \geqslant 1, \\ &\hat{\omega}(\hat{r}, \hat{t}) \text{ is bounded as } \hat{r} \to \infty \end{aligned} \tag{15.1}$$

where

$$\lambda^2 \stackrel{\text{def}}{=} \frac{\rho a^2 \mu^2}{\kappa}$$

and

$$\delta \stackrel{\text{def}}{=} \frac{\Omega}{\mu}.$$

We solve (15.1) by Laplace transform

$$\bar{\bar{\omega}}(\hat{r}, u) = \delta \left\{ \frac{1}{u} \frac{K_1(\lambda\sqrt{u(1+u)}\,\hat{r})}{K_1(\lambda\sqrt{u(1+u)})} \right\}, \tag{15.2}$$

$$\hat{\omega}(\hat{r}, \hat{t}) = \frac{\delta}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{K_1(\lambda \hat{r}\sqrt{u(1+u)})}{K_1(\lambda\sqrt{u(1+u)})} \frac{du}{u}. \tag{15.3}$$

We again see (as in (14.14)) that

$$\hat{\omega}(\hat{r}, \hat{t}) = 0 \qquad \text{for} \qquad \hat{t} - \lambda(\hat{r} - 1) < 0.$$

For $\hat{t} - \lambda(\hat{r} - 1) > 0$, (15.3) may be evaluated along the contour shown in figure 15.1.
Evaluation of (15.3) along the contour in figure 15.1 using the identity:

$$K_1(ix) = -\frac{\pi}{2}[J_1(x) - iY_1(x)]$$
$$= \text{conjugate}\,(K_1(-ix))$$

leads us to the expression

$$\omega(\hat{r}, \hat{t}) = \delta \left[\frac{1}{\hat{r}} - \frac{1}{\pi} \int_0^1 \frac{e^{-\xi\hat{t}}}{\xi} \left\{ \frac{Y_1(\lambda\hat{r}\sqrt{\xi(1-\xi)}) J_1(\lambda\sqrt{\xi(1-\xi)}) - J_1(\lambda\hat{r}\sqrt{\xi(1-\xi)})\, Y_1(\lambda\sqrt{\xi(1-\xi)})}{J_1^2(\lambda\sqrt{\xi(1-\xi)}) + Y_1^2(\lambda\sqrt{\xi(1-\xi)})} \right\} d\xi \right]$$
$$\cdot H(\hat{t} - \lambda(\hat{r} - 1))$$
$$\stackrel{\text{def}}{=} \hat{g}(\hat{r}, \hat{t}) H(\hat{t} - \lambda(\hat{r} - 1)). \tag{15.4}$$

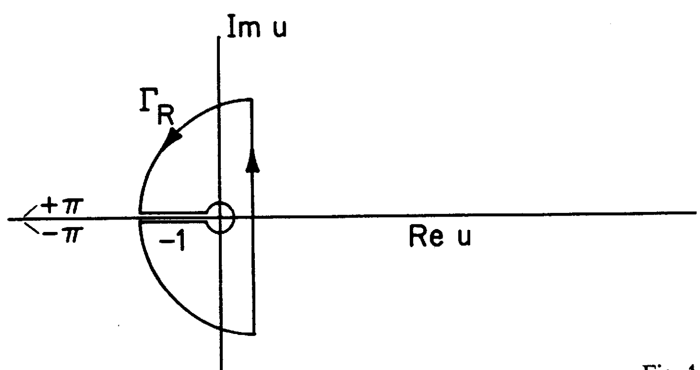

Fig. 15.1. There is a branch-cut along $Re\, u \leqslant 0$ and $Im\, u = 0$

16. Step jump in the angular displacement of the cylinder in a simple fluid at rest

This is the same problem as in Sect. 15 except that the cylinder is given a step jump in the angular displacement instead of step jump in angular velocity. We therefore require that

$$\mu(a, t) = \theta_0 H(t - 0)$$

where μ is the displacement function introduced in (14.1). Hence

$$\omega(a, t) = a\theta_0 \delta(t - 0) \tag{16.1}$$

Thus the problem governing $\omega(r, t)$ is:

$$\int_0^t G(s) \left[\frac{\partial^2 \omega}{\partial r^2}(r, t - s) + \frac{1}{r}\frac{\partial \omega}{\partial r}(r, t - s) - \frac{\omega(r, t - s)}{r^2} \right] ds = \rho \frac{\partial \omega}{\partial t},$$
$$\omega(a, t) = a\theta_0 \quad \delta(t - 0),$$
$$\omega(r, 0) = 0 \qquad \forall r \geqslant a > 0,$$
$$\omega(r, t) \text{ is bounded.} \tag{16.2}$$

We find the solution of the problem (16.2) of the step jump in angular displacement by taking the time derivative of the step jump in the angular velocity (14.17) (as in (10.1), (10.8)). Therefore $\omega(r,t)$ satisfying (16.2) is given by

$$\omega(r,t) = \frac{\partial g}{\partial t} H(t-(r-a)\alpha) + g(r,(r-a)^{+}\alpha)\,\delta(t-(r-a)^{+}\alpha) \tag{16.3}$$

where

$$g(r,t) = a\theta_0\left[\frac{a}{2r} + \frac{1}{\pi} \cdot \int_0^\infty Im\left\{\frac{e^{iyt}K_1(r\sqrt{\rho y/q(y)}\,e^{i(\pi/4-p(y)/2)})}{yK_1(a\sqrt{\rho y/q(y)}\,e^{i(\pi/4-p(y)/2)})}\right\}dy. \right. \tag{16.4}$$

The angular-shift in the position of a particle is given by:

$$\Delta(t) \overset{\text{def}}{=} \mu(\theta,t) - \mu(\theta,0) = \int_0^t \frac{\omega(r,\tau)}{r}\,d\tau. \tag{16.5}$$

Therefore,

$$\bar{\Delta}(u) \overset{\text{def}}{=} \int_0^\infty \Delta(t)e^{-ut}dt; \quad u\in\mathbb{C}\ni Re\,u>0 = \int_0^t \frac{\bar{\omega}(r,\tau)}{r}\,d\tau. \tag{16.5}$$

Therefore,

$$\bar{\Delta}(u) \overset{\text{def}}{=} \int_0^\infty \Delta(t)e^{-ut}dt; \quad u\in\mathbb{C}\ni Re\,u>0 = \frac{1}{u}\,\frac{\bar{\omega}(r,u)}{r}. \tag{16.6}$$

The Laplace transform $\bar{\omega}(r,u)$ of (16.2) is

$$\bar{\omega}(r,u) = a\theta_0 \frac{K_1\left(r\sqrt{\dfrac{\rho u}{\bar{G}(u)}}\right)}{K_1\left(a\sqrt{\dfrac{\rho u}{\bar{G}(u)}}\right)} \tag{16.7}$$

and

$$\lim_{t\to\infty}\Delta(t) = \lim_{\substack{u\in\mathbb{R}\\ \text{and } u\to 0^+}} \frac{a\theta_0}{r}\cdot\frac{K_1\left(r\sqrt{\dfrac{\rho u}{\bar{G}(u)}}\right)}{K_1\left(a\sqrt{\dfrac{\rho u}{\bar{G}(u)}}\right)} = \frac{a^2\theta_0}{r^2}. \tag{16.8}$$

Eq. (16.8) shows that the final angular displacement of particles initially on the radial line $\theta = 0$ is given by $\theta = (a^2\theta_0)/r^2$. This particle shift is independent of the material.

17. Step angular displacement of the cylinder when $G(s) = \kappa e^{-\mu s}$

In this case the problem is governed by (16.2) with $G(s) = \kappa e^{-\mu s}$. Using the dimensionless variables defined in (15.0) we find that

$$\hat{\omega}(\hat{r},\hat{t}) = \frac{\partial\hat{g}}{\partial\hat{t}} H(\hat{t}-\lambda(\hat{r}-1) + \hat{g}(\hat{r},\lambda(\hat{r}-1))\,\delta(\hat{t}-\lambda(\hat{r}-1)) \tag{17.1}$$

where $\hat{g}(\hat{r},\hat{t})$ in (17.1) is the function defined in (15.4) and Ω is replaced by θ_0.

18. Shock layers in fluids of small viscosity

Many rheological models are framed as combinations of integrals of the Coleman-Noll type and rate terms of Newtonian type. Such rate terms may be obtained from integrals if the kernels which are allowed can have singularities of the Dirac type at the origin (Green and Rivlin, [15]). Saut and Joseph [21] have recently given a functional analytic theory of fading memory which leads to such singular kernels. For example, if the history $G(s)$ lies in a certain Sobolev space then

$$G(s) = \mu\delta(s) + g(s) \tag{18.1}$$

where $g(s)$ is in a weighted $L^2(0,\infty)$ space, μ is a constant which we shall call "viscosity" and

$$\int_0^\infty f(s)\delta(s)\,ds = f(0)$$

for any function $f(s)$.

The problem of a step jump in velocity may be framed in terms of a constitutive model of mixed type. If a kernel of the type (18.1) is assumed, then the jump in velocity problem (3.6) may be formulated as follows:

$$\rho\frac{\partial v}{\partial t} = \mu\frac{\partial^2 v}{\partial x^2} + \int_0^t g(s)\frac{\partial^2 v}{\partial x^2}(x, t-s)\,ds,$$

$$v(0,t) = \begin{cases} U, & t > 0 \\ 0, & t \leqslant 0, \end{cases} \tag{18.2}$$

$$v(x,0) = 0,$$

$v(x,t)$ is bounded for all $t, x \in \mathbb{R}^+$.

We shall show that the solution of this problem is continuous at $\mu = 0$ and has an interior layer of thickness μ around the "shock" wave which exists when $\mu = 0$ at any fixed value of x and t. We have already given such a result in a special case of a Maxwell fluid (see Saut and Joseph [21]).

In the shock layer analysis we follow methods used in the theory of matched asymptotic expansions. The solution

$$v(x,t,\mu)|_{\mu=0} = f(x,t)H(t-\alpha x), \tag{18.3}$$

where $\alpha = \sqrt{\rho/g(0)}$, is called the shock or outer solution. Renardy [20] has shown that if $g'(0) \leqslant 0$ then

$$f(x, \alpha x) = e^{\frac{\alpha x}{2}\frac{g'(0)}{g(0)}}. \tag{18.4}$$

We can complete the analysis of (18.2) by the method of Laplace transforms. We must invert the transform

$$\bar{v}(x,u,\mu) = \frac{U}{u}e^{-x\sqrt{\rho u/(\mu+\bar{g}(u))}}$$

of (18.2). Then, by the theory of Laplace transforms

$$v(x,t,\mu) = \lim_{R\to\infty}\frac{1}{2\pi i}\int_{\gamma-iR}^{\gamma+iR} e^{ut}\bar{v}(x,u,\mu)\,du \tag{18.5}$$

for $\gamma > 0$ and $u \in \mathbb{C}$. We show that (18.5) gives smooth solution as follows:

Let $P_R \overset{\text{def}}{=} (\gamma - i\infty, \gamma + i\infty) - (\gamma - iR, \gamma + iR)$.

Now integrating by parts yields

$$\bar{g}(u) = \frac{g(0)}{u} + O\left(\frac{1}{u^2}\right), \tag{18.6}$$

$$\int_{P_R}\frac{e^{ut-x\sqrt{\rho u/\mu+\bar{g}(u)}}}{u}\,du$$

$$= \int_{P_R}\frac{1}{u}e^{ut-x\sqrt{\rho u/\mu[1+g(0)/\mu u+O(1/u^2)}}\,du$$

$$= \int_{P_R}\frac{1}{u}e^{ut-x\sqrt{\rho u/\mu}[1-1/2\cdot g(0)/\mu u+O(1/u^2)]}\,du$$

$$= \int_{P_R}\frac{1}{u}e^{ut-x\sqrt{\rho u/\mu}}\,du + \frac{1}{\mu}\int_{P_R}\frac{1}{u}O\left(\frac{1}{u}\right)e^{ut-x\sqrt{\rho u/\mu}}. \tag{18.7}$$

Now the first term in (18.7) is known to be arbitrarily small for large R because

$$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{1}{u}e^{ut-x\sqrt{\rho u/\mu}}\,du =$$

$$= \operatorname{erfc}\left(\frac{x}{2}\sqrt{\frac{\rho}{\mu t}}\right).$$

The second term in (18.7) can be made arbitrarily small for some μ_0 if $\mu \geqslant \mu_0 > 0$. Thus (18.8) implies that (18.5) is uniformly convergent for any fixed x, t if $\mu \geqslant \mu_0 > 0$. This implies that $v(x,t,\mu)$ is continuous in $x \geqslant 0$, $t \geqslant 0$. An identical argument for the derivatives of $v(x,t,\mu)$ with respect to x and t show that $v(x,t,\mu)$ is in fact smooth.

The shock layer analysis proceeds as follows: We first scale the variables so that

$$\tilde{x} \overset{\text{def}}{=} x/\mu,$$

$$\tilde{t} \overset{\text{def}}{=} t/\mu, \tag{18.8}$$

$$\tilde{v}(\tilde{x},\tilde{t},\mu) \overset{\text{def}}{=} v(x,t,\mu).$$

The equation satisfied by $\tilde{v}$ is

$$\frac{\partial^2\tilde{v}}{\partial\tilde{x}^2} + \int_0^{\tilde{t}} g(\mu\tilde{s})\frac{\partial^2\tilde{v}}{\partial\tilde{x}^2}(\tilde{x},\tilde{t}-\tilde{s})\,d\tilde{s} = \rho\frac{\partial\tilde{v}}{\partial\tilde{t}}. \tag{18.9}$$

Differentiation of (18.9) with respect to $\tilde{t}$ yields

$$\frac{\partial^3\tilde{v}}{\partial\tilde{x}^2\partial\tilde{t}} + g(0)\frac{\partial^2\tilde{v}}{\partial\tilde{x}^2} + \mu\int_0^{\tilde{t}}\frac{dg(\mu\tilde{s})}{d\mu\tilde{s}}\frac{\partial^2\tilde{v}}{\partial\tilde{x}^2}(\tilde{x},\tilde{t}-\tilde{s})\,d\tilde{s}$$

$$= \rho\frac{\partial^2\tilde{v}}{\partial\tilde{t}^2}.$$

In deriving this equation we used

$$\int_0^{\tilde t} \frac{dg}{d(\mu\tilde t)}(\mu(\tilde t - \tilde s))\frac{\partial^2 \tilde v}{\partial \tilde x^2}(\tilde x, \tilde t)\,d\tilde s$$

$$= \int_0^{\tilde t} \frac{dg(\mu\tilde s)}{d\mu\tilde s}\frac{\partial^2 \tilde v}{\partial \tilde x^2}(\tilde x, \tilde t - \tilde s)\,d\tilde s .$$

The boundary value problem satisfied by $\tilde v(\tilde x, \tilde t, 0)$ is

$$\frac{\partial^3 \tilde v}{\partial \tilde x^2 \partial \tilde t} + g(0)\frac{\partial^2 \tilde v}{\partial \tilde x^2} = \rho\frac{\partial^2 \tilde v}{\partial \tilde t^2}$$

$$\tilde v(0,\tilde t,0) = \begin{cases} U & \text{for } \tilde t > 0 \\ 0 & \text{for } \tilde t \leqslant 0 , \end{cases} \qquad (18.10)$$

$$\tilde v(\tilde x,0,0) = \frac{\partial \tilde v}{\partial \tilde t}(\tilde x,0,0) = 0 ,$$

$\tilde v(\tilde x, \tilde t, 0)$ is bounded for $\tilde x, \tilde t \in \mathbb{R}^+$.

We may also formulate the boundary value problem (18.10) for $\tilde v(\tilde x, \tilde t, 0)$ in terms of the second order equation

$$\frac{\partial^2 \tilde v}{\partial \tilde x^2} + g(0)\frac{\partial^2}{\partial \tilde x^2}\int_0^{\tilde t} \tilde v(\tilde x, \tilde t - \tilde s)\,d\tilde s = \rho\frac{\partial \tilde v}{\partial \tilde t} .$$

We may write using

$$\int_0^{\tilde t} \frac{\partial \tilde v}{\partial \tilde t}(\tilde x, \tilde t - \tilde s)\,d\tilde s = \int_0^{\tilde t} \frac{\partial \tilde v}{\partial \tau}(\tilde x, \tau)\,d\tau$$

$$= \tilde v(\tilde x, \tilde t) - \tilde v(\tilde x, 0) = \tilde v(\tilde x, \tilde t) .$$

The solution of (18.10) can be given in terms of Laplace transforms

$$\tilde v(\tilde x, \tilde t, 0) = \frac{U}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} \frac{e^{\left\{u\tilde t - \tilde x\sqrt{\frac{\rho u^2}{\mu + g(0)}}\right\}}}{u}\,du . \qquad (18.11)$$

It can be shown that for fixed finite values of x, t such that $t - \alpha x \neq 0$,

$$\lim_{\mu \to 0} \tilde v\left(\frac{x}{\mu}, \frac{t}{\mu}, 0\right) \to H(\tilde t - \alpha\tilde x) = H(t - \alpha x) . \qquad (18.12)$$

In fact, if $\tilde x$ is large, then the principal contribution to the integral (18.11) giving $\tilde v(\tilde x, \tilde t, 0)$ is obtained when u is relatively small. In fact, we may set $u = \zeta\mu$. Then

$$\tilde v\left(\frac{x}{\mu}, \frac{t}{\mu}, 0\right)$$

$$= \frac{U}{2\pi i}\int_{\gamma' - i\infty}^{\gamma' + i\infty} \exp\left(\zeta t - x\zeta\sqrt{\frac{\rho}{g(0) + \mu\zeta}}\right)\frac{d\zeta}{\zeta} \cdot$$

$$\xrightarrow{\mu \to 0^+} \frac{U}{2\pi i}\int_{\gamma' - i\infty}^{\gamma' + i\infty} \exp\{\zeta t - \alpha x\zeta\}\frac{d\zeta}{\zeta}$$

$$= H(t - \alpha x) . \qquad (18.13)$$

The same result follows directly from (18.10). To show this we return to x, t variables in (18.10) and find

$$\hat v(x, t, \mu) \stackrel{\text{def}}{=} \tilde v(\tilde x, \tilde t, 0)$$

satisfies

$$\mu\frac{\partial^3 \hat v}{\partial x^2 \partial t} + g(0)\frac{\partial^2 \hat v}{\partial x^2} = \rho\frac{\partial^2 \hat v}{\partial t^2} ,$$

$$\hat v(0, t, \mu) = \begin{cases} U & \text{for } t > 0 \\ 0 & \text{for } t \leqslant 0 , \end{cases}$$

$$\hat v(x, 0, \mu) = \frac{\partial \hat v}{\partial t}(x, 0, \mu) = 0 ,$$

$\hat v(x, t, \mu)$ bounded for $x, t \in \mathbb{R}^+$.

Clearly

$$\hat v(x, t, 0) = UH(t - \alpha x)$$

and the thickness of the shock layer for $\hat v(x, t, \mu)$ around $t = \alpha x$ is given by

$$\eta = t - \alpha x = \mu(\tilde t - \alpha\tilde x) = \mu\tilde\eta$$

for $\tilde t \neq \alpha\tilde x$.

We remark now that $\tilde v(\tilde x, \tilde t, 0) = \tilde v(x, t, \mu)$ is the shock structure solution for a propagating Heaviside function. In this solution there is no decay or damping; the smoothed out shock layer propagates without change of amplitude. The damping term is the term proportional to μ in the equation under (18.9). This term is omitted in the analysis of the shock layers. The effect of this term is to permit a smooth gentle increase of $f(x, t)$ for fixed x, increasing from the value $e^{+\alpha x g'(0)/2g(0)}$ at $t = \alpha x^+$ to 1 for $t \to \infty$.

The shock layer solution (the "inner" solution) for the step jump in velocity problem (18.2) can be obtained from the shock layer solution for the propagating Heaviside function by limit matching. We may take $A(x)\,\tilde{v}(\tilde{x}, \tilde{t}, 0)$ as the shock layer (inner) solution. The amplitude $A(x)$ is a slowly varying function in the small layer around $t - \alpha x = \mu\eta$ for $\eta = O(1)$ and $\mu \to 0$. The outer limit of this inner solution can be described as follows: let t tend to αx from above and $\tilde{x} = x/\mu \to \infty$. In this limit $\tilde{v} \to 1$ and the whole solution must match the inner limit $t \downarrow \alpha x$ of the outer solution; that is

$$A(x) = e^{\alpha x g'(0)/2g(0)}.$$

We have then the outer (shock) solution given by (18.5) and the inner shock layer solution $e^{+\alpha x g'(0)/2g(0)}$ $\tilde{v}(\tilde{x}, \tilde{t}, 0)$. Neither solution is uniformly valid for all non-negative values of x and t. We get a uniformly valid solution for bounded x and t using additive composition. We first identify the common part of the inner and outer solution as $e^{+\alpha x g'(0)/2g(0)} H(t - \alpha x)$. This is the inner limit of the outer (shock) solution and the outer limit of the inner shock layer solution. Then we subtract the common part from the outer (shock) solution and add the inner (shock layer) solution. This gives a uniformly valid approximation in the layer near $t = \alpha x$ and for all bounded non-negative values of x and t. Thus, the uniformly valid additive composition

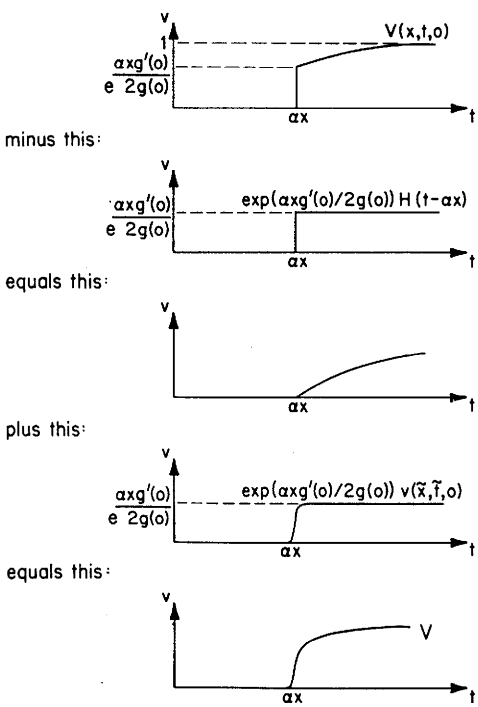

Fig. 18.1. Composite shock layer solution V constructed as an additive composition

$$V = [v(x, t, 0) - \exp(\alpha x g'(0)/2g(0)) H(t - \alpha x)] + \exp(\alpha x g'(0)/2g(0))\,\tilde{v}(\tilde{x}, \tilde{t}, 0)$$

may be represented graphically, for fixed x, as t varies, as in figure 18.1.

Appendix. Construction of a sequence $\{q_n(t)\}$ satisfying $(6.2)_{1,2}$

We first define

$$\psi\left(\frac{1}{2n}, t, \tau\right) \stackrel{\text{def}}{=} \begin{cases} \exp\left(\left[(t-\tau)^2 - \dfrac{1}{4n^2}\right]^{-1}\right) \\ \quad \text{for } |t-\tau| < \dfrac{1}{2n} \\ 0 \quad \text{for } |t-\tau| \geqslant \dfrac{1}{2n}, \end{cases}$$

$$h\left(\frac{1}{2n}\right) \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} \psi\left(\frac{1}{2n}, t, \tau\right) d\tau = \int_{-1/2n}^{1/2n} \exp\left(\frac{-1}{1/4n^2 - z^2}\right) dz,$$

$$\phi\left(\frac{1}{2n}, t, \tau\right) \stackrel{\text{def}}{=} \frac{\psi\left(\frac{1}{2n}, t, \tau\right)}{h\left(\frac{1}{2n}\right)}.$$

Clearly

$$\int_{-\infty}^{\infty} \phi\left(\frac{1}{2n}, t, \tau\right) d\tau = 1.$$

Define

$$I_{1/n}(t) \stackrel{\text{def}}{=} \begin{cases} 1 \quad \text{for} \quad t \geqslant \dfrac{1}{n} \\ 0 \quad \text{for} \quad t < \dfrac{1}{n}. \end{cases}$$

Then

$$q_n(t) \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} \phi\left(\frac{1}{2n}, t, \tau\right) I_{1/n}(\tau) d\tau$$

has the properties $(6.2)_{1,2}$.

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CORRECTIONS

Corrigendum to

Linearized dynamics for step jumps of velocity and displacement of shearing flows of a simple fluid

Rheol. Acta **21**, 228–250 (1982)

A. Narain and D. D. Joseph

1) The quantities $\boldsymbol{G}(s)$, $\boldsymbol{C}_t(t)$ and $\boldsymbol{A}_1(t)$ are tensors and should be in boldface.
2) The equations under (3.4) should read

$$\lambda'(x, s) = 0 \quad \text{for} \quad t \leqq 0$$

and

$$\frac{d\lambda'}{ds}(x, s) = -\frac{\partial v}{\partial x}(x, t-s).$$

3) Property (v) of $G(s)$ assumed in the beginning of § 4 on p. 231 can be dropped. This assumption was used in eq. (4.8) which is derivable from the property (ii) of $G(s)$ and $G'(s) = O(e^{-\bar{\lambda}s})$.
4) The sentence under eq. (4.5) should read "eqs. (4.3), (4.5) imply ... half-plane Re $u > -\lambda$".
5) The first sentence under Figure 5.1 should read "Now for $t - \alpha x < 0$ ...".
6) The eq. (5.17) should read

$$\left[1 + \frac{G'(0)}{G(0)\,u} + \frac{G''(0)}{G(0)\,u^2} + O\left(\frac{1}{u^3}\right)\right]^{-1/2}$$

$$= 1 - \frac{\bar{\gamma}'}{2u} + \frac{3}{8}\frac{\bar{\gamma}''}{u^2} + \frac{\bar{\gamma}''}{u^2} + O\left(\frac{1}{u^3}\right)$$

where

$$\bar{\gamma}' \stackrel{\text{def}}{=} \frac{G'(0)}{G(0)}, \quad \bar{\gamma}'' \stackrel{\text{def}}{=} -\frac{1}{2}\frac{G''(0)}{G(0)}.$$

7) The left side of eq. (6.7) should read:

$$\frac{\partial^2 v_n}{\partial x^2}.$$

8) Eq. (6.8) should be replaced by:

$$\begin{aligned} M v_n \to M v \equiv \int_{\alpha x}^{t} & G(t-s) f_{xx}(x, s)\, ds \\ & - 2\alpha\, G(t-\alpha x) f_x(x, \alpha x^+) \\ & + \alpha^2 G'(t-\alpha x) f(x, \alpha x^+) \\ & - \alpha^2 G(t-\alpha x) \frac{\partial f}{\partial t}(x, \alpha x^+) \\ & - \varrho \frac{\partial f}{\partial t}(x, t). \end{aligned}$$

9) Eq. (6.11) should be replaced by:

$$2\frac{\partial f}{\partial t}(x, \alpha x^+) + \frac{2}{\alpha}\frac{\partial f}{\partial x}(x, \alpha x^+) = \frac{G'(0)}{G(0)} f(x, \alpha x^+).$$

10) The definition of $\eta(u)$ underneath eq. (10.4) is

$$\eta(u) \stackrel{\text{def}}{=} \sqrt{\varrho\, u/\bar{G}(u)}.$$

11) The right side of eq. (12.5) should read:

$$v(x, t) = U[g(x, t) + \{g(x + 2l, t) - g(2l - x, t)\} + \{\ldots\} + \ldots].$$

12) The equations between (14.3) and (14.6) should be numbered (14.4) and (14.5).
13) The left side of the equations above (14.6) should read $\mathcal{I} < r\theta > (t)$ in place of $\mathcal{I} < \pi\theta > (t)$.
14) The left side of eq. (14.11) should be replaced by

$$\frac{\omega(r, t)}{a\,\Omega}.$$

15) Eq. $(14.12)_{(ii)}$ is:

$$K_1(z) \sim \sqrt{\frac{\pi}{2z}} \exp(-z) \quad \text{as} \quad |z| \to \infty.$$

16) Eq. (16.5) can be ignored.

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