
*Convergence of Biorthogonal
Series of Biharmonic Eigenfunctions by the
Method of Titchmarsh*

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0. Abstract

Canonical edge problems for the biharmonic equation can be solved by separating variables. The eigenvalues and eigenvectors arising in this separation are derived from a reduced system of ordinary differential equations along lines suggested in the excellent work of R. C. SMITH (1952). We study the reduced system which is governed by a vector ordinary differential equation. A solution of the biharmonic problem, governed by a partial differential equation, can be found only if the prescribed data is restricted to a subspace of the space spanned by the eigenfunctions of the reduced problem. The theory leads to problems in generalized harmonic analysis which seek conditions under which arbitrary vector fields $f(y)$ with values in \mathbb{R}^2 can be represented in terms of eigenvectors of the reduced problem. This paper adds new theorems and conjectures to the theory.

We extend SMITH's generalization to fourth-order problems of the methods introduced by TITCHMARSH (1946) to study eigenfunction expansions associated with second-order problems. We use this method to prove that, if $f(y) = [f_1(y), f_2(y)]$, $-1 \leq y \leq 1$, $f(y) \in C^1[-1, 1]$, $f'' \in L_2[-1, 1]$, then the series expressing $f(y)$ converges uniformly to $f(y)$ in the open interval $(-1, 1)$, uniformly in $[-1, 1]$ if $f_1(\pm 1) = 0$ and, in any case, to $[0, f_2(\pm 1) - f_1(\pm 1)]$ at $y = \pm 1$. This is unlike Fourier series, which converge to the mean value of the periodic extension of a function. The series exhibits a Gibbs phenomenon near the end points of discontinuity when $f_1(\pm 1) \neq 0$.

The Gibbs undershoot and overshoot for the step function vector $[1, 0]$ and ramp function vector $[y, 0]$ are computed numerically. The undershoot and overshoot are much larger than in the case of Fourier series and, unlike Fourier series, the Gibbs oscillations do not appear to be entirely suppressed by FÉJÉR's method of summing Cesaro sums. We show that, when $f(y)$ has interior points of discontinuity, the series for $f(y)$ diverges and we present numerical results which indicate that, in this divergent case, the Cesaro sums converge to $f(y)$ apparently with Gibbs oscillations near the point of discontinuity.

1. Introduction

A large number of problems arising in Stokes flow [3-7, 10, 13, 17-27, 29, 30, 34, 35, 39, 41, 42, 43] and elasticity [1, 2, 8, 9, 11-14, 16, 26, 28, 32, 33, 37, 40] can be solved in biorthogonal series of eigenfunctions generated by separating variables. Most of the applications of this elementary method and nearly all of the theorems are recent. We think that the potential for application of such an elementary method is virtually unlimited. It is therefore desirable to provide a theoretical basis for the method which may be readily adapted to the study of the validity of the method in different applications. We believe that the extended method of TITCHMARSH provides such a basis. We use this method to prove the theorem of convergence summarized in the abstract and precisely stated in § 7. Though this theorem is general and stands by itself, it is intimately connected with certain special biharmonic problems which we call canonically. The canonical biharmonic edge problem for the semi-infinite strip $0 \leq x, -1 \leq y \leq 1$ satisfies Dirichlet conditions $w(x, \pm 1) = w_y(x, \pm 1) = 0$ on the long walls and takes on prescribed values

$$(1.1) \quad f(y) = [f_1(y), f_2(y)] = [w_{xx}(0, y), w_{yy}(0, y)]$$

on the short wall (Fig. 1.1). The algorithm of SMITH (1952) leads from the canonical problem to derived problems (3.9-3.12) of ordinary differential equations for right and left eigenvectors $\varphi^{(n)}(y)$ and $\psi^{(n)}(y)$ with values in \mathbb{R}^2 . The expansion theorem for the edge data is given in terms of the eigenfunctions of these derived problems by SMITH's extension of the method of TITCHMARSH. SMITH showed that, if $f''(y)$ is of bounded variation and

$$(1.2) \quad f_1(\pm 1) = f_1'(\pm 1) = 0,$$

$$(1.3) \quad f_2(\pm 1) = f_2'(\pm 1) = 0,$$

then*

$$(1.4) \quad f(y) = c_0 \varphi^{(0)}(y) + \sum_{-\infty}^{\infty} c_n \varphi^{(n)}(y)$$

where $\langle \cdot \rangle$ is the integral defined by (3.11),

$$(1.5) \quad c_n = \frac{1}{k_n} \langle \psi^{(n)} A f \rangle, \quad k_n = \langle \psi^{(n)} A \varphi^{(n)} \rangle$$

and A is the 2×2 biorthogonality matrix (3.10). It follows that the eigenvectors $\{\varphi^{(0)}, \varphi^{(n)}\}$ are complete for expansions of $f(y)$ satisfying the stated conditions. On the other hand, compatibility conditions for the solution of the generating canonical biharmonic problem (Fig. 1.1) may be expressed by the condition

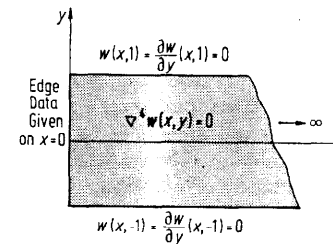


Fig. 1.1. The biharmonic edge problem. $w(x, y)$ is biharmonic in the semi-infinite strip, $w(x, y) \rightarrow 0$ as $x \rightarrow \infty$, $w(x, \pm 1) = w_y(x, \pm 1) = 0$. Two different linear combinations of w and x -derivatives of w are prescribed at the edge $x = 0$. The problem is canonical if w_{xx} and w (or w_{yy}) are prescribed at $x = 0$.

that $c_0 = 0$, so that solutions of the canonical partial differential equation lie on a subspace of solutions spanned by the eigenfunctions of the derived problem of ordinary differential equations.**

It appears that no mathematical results were published about SMITH's series between 1952 and 1977. This long period of inactivity may be associated with the fact that the conditions (1.2) and (1.3) posed by SMITH rule out nearly all the potential applications. SMITH stated that the details of his proof (see § 4-6) made it seem unlikely that these conditions could be very much relaxed. Fortunately SMITH

* Here and throughout this paper, the summation symbol over the range $(-\infty, \infty)$ does not include a term for $n = 0$. When such a term appears, it will be displayed explicitly, as in Equation (1.4).

** TAMARKIN (1928) gave a general theory of eigenfunction expansions generated by ordinary differential equations of arbitrary order. The eigenfunctions, like those in the work of TITCHMARSH (1946), arise as residues at the poles of a resolvent. TAMARKIN's theory might be expected to apply to the first component of (1.4) but in fact the governing problem $\phi_1^{IV} + 2s^2\phi_1'' + s^4\phi_1 = 0$, $\phi_1(\pm 1) = \phi_1'(\pm 1) = 0$, does not satisfy hypothesis (3) of TAMARKIN's Theorem 2.

was unnecessarily pessimistic. JOSEPH (1977) showed that SMITH's series converged absolutely and uniformly on the closed interval $[-1, 1]$ when (1.2) holds but (1.3) does not. In a later paper, JOSEPH (1979) showed that SMITH's demonstration does not require (1.3). Taken together, the two papers of JOSEPH establish convergence to $f(y)$ when (1.2) holds. JOSEPH & STURGES (1978) showed that if (1.2)₂ and (1.3) were dropped the series would still converge absolutely and uniformly on $[-1, 1]$. They showed that if (1.2)₁ was also dropped the series would converge conditionally in the open interval $(-1, 1)$ but they stated (case ii of theorem 3) a false theorem about convergence at an interior point of discontinuity. Their mistake was found by GREGORY (1980 A) and SPENCE (1980). In § 9 of this paper we are the first to show that the partial sums of biorthogonal series can diverge at an interior point of discontinuity and we give numerical results which show that the Cesaro sums converge even though the partial sums diverge.

GREGORY (1980 A) proved that the series (1.4) converges to the prescribed data when (1.6) and (1.3) hold and (1.2) and (1.3) are dropped, absolutely and uniformly on $[-1, 1]$ if (1.2)₁ is deleted, and conditionally on $(-1, 1)$ if (1.2)₁ is retained. GREGORY's proof is based on an explicit expression for the Green's function on the strip $-1 < y < 1, -\infty < x < \infty$ which has a representation in a series of biorthogonal eigenfunctions (GREGORY (1979)).

SPENCE (1980) uses Fourier transforms to prove convergence to even data for the series solving the canonical edge problem. His central result is the estimate (his Equation (1.7) in the notation of our Figure 1.1)

$$|w_{yy}(x, y) - f_2(y)| < x^\sigma M(\sigma, \varepsilon) \|f'\|_{L_2(-1, 1)}, \quad x > 0$$

where $M(\sigma, \varepsilon)$ is a constant, $\sigma \in (0, 1)$, $y \in [0, 1 - \varepsilon]$, $\varepsilon > 0$, f and f' are in L_2 and

$$w_{yy} = \sum c_n \phi_n''(y) e^{-\varepsilon_n x}.$$

Our (f_1, f_2) are SMITH's (f, g) and SPENCE'S $(f^{(4)} - f^{(2)}, f^{(2)})$. When coupled with convergence results of the type proved by JOSEPH (1977), this estimate proves that the series (1.4) converges to the data.

The methods used by GREGORY and SPENCE, unlike those used by SMITH, by JOSEPH (1979) and here, emphasize the partial differential equation and can be regarded as proving the existence of solutions in series for arbitrary data of a given class. Neither GREGORY nor SPENCE sum their series solution at $y = \pm 1$ when $f_1(\pm 1) \neq 0$.

In this paper we go "all the way" with SMITH's extension of the method of TITCHMARSH and prove that (1.4) holds when (1.3) is dropped (JOSEPH, 1979), when (1.2) is also dropped (GREGORY 1980A), and when $f''(y)$ is merely in L_2 and not necessarily of bounded variation. And we use this method to show that at an end point of discontinuity ($y = \pm 1$), the series (1.4) converges to

$$\begin{bmatrix} 0 \\ f_2 - f_1 \end{bmatrix} \text{ rather than to } \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}.$$

It is natural to wonder in what way this theorem of convergence might be improved. Since all the conditions on $f(y)$ at $y = \pm 1$ have been dropped, no further

improvement is possible here. It may however be possible to relax the requirements of regularity, $f \in C^1(-1, 1)$, $f'' \in L_2(-1, 1)$. It is however certain, as the example given in § 9 shows, that we cannot have convergence when $f(y)$ is merely of bounded variation, so we cannot possibly have convergence in classes as weak as those for which Fourier series converge.

The extended method of TITCHMARSH introduced by SMITH can be used to prove expansion theorems for eigenfunction bases other than the ones studied in this paper. These eigenfunction bases are generated by canonical problems of partial differential equations. Whatever may be the domain and long side wall boundary conditions, if the edge conditions are such as to allow inner product formulas for the coefficients, the generating partial differential equation can be called canonical. Though the canonical edge problem in the semi-infinite strip defined at the beginning of this Introduction is all that we shall need here, it is instructive to note that canonical problems can be defined for domains which are not strips and for equations which are not biharmonic, not even of order four, with other than Dirichlet conditions on the "long walls", and with other prescriptions of data on the "short walls". A few examples, among many, are: the problem of Stokes flow in rectangles [21] with second derivatives prescribed on the top and bottom of the rectangle; in sectors of circles and between circles [23]; Stokes flow between parallel circular disks of radius a [17], satisfying

$$(1.6) \quad \begin{cases} \psi = \psi_z = 0 & \text{on the disks,} \\ \left[r \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r} \right) + \frac{\partial^2}{\partial z^2} \right]^2 \psi = 0 & \text{between the disks,} \\ f = \left[\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial \psi}{\partial r} \right), \frac{\partial^2 \psi}{\partial z^2} \right] & \text{prescribed on } r = a; \end{cases}$$

Stokes flow between semi-infinite or finite concentric cylinders of radii a and $b > a$ [10, 41], satisfying

$$(1.7) \quad \begin{cases} \psi = \psi_r = 0 & \text{on } r = a, b, \\ \left[r \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r} \right) + \frac{\partial^2}{\partial z^2} \right]^2 \psi = 0 & \text{between the cylinders,} \\ f = \left[\frac{\partial^2 \psi}{\partial z^2}, \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial \psi}{\partial r} \right) \right] & \text{prescribed on } z = 0; \end{cases}$$

Stokes flow in a cone and between cones [24], satisfying

$$(1.8) \quad \begin{cases} \psi = \psi_\xi = 0 & \text{on the cones } \xi_0, \xi_1, \xi = \cos \theta, \\ \left[\frac{\partial^2}{\partial r^2} + \frac{(1 - \xi^2)}{r^2} \frac{\partial^2}{\partial \xi^2} \right]^2 \psi = 0 & \text{between the cones,} \\ f = \left[\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial \psi}{\partial r} \right), (1 - \xi^2) \frac{\partial^2 \psi}{\partial \xi^2} \right] & \text{prescribed on } r = 1; \end{cases}$$

unsteady Stokes flow generated by small amplitude vibrations of a fluid between oscillating parallel walls [34], satisfying

$$(1.9) \quad \left\{ \begin{array}{l} \psi = \psi_y = 0 \text{ on the sidewall,} \\ \nabla^4 \psi + \lambda^2 \nabla^2 \psi = 0 \text{ between the walls } (\lambda \text{ is a prescribed} \\ \text{complex number),} \\ f(y) = [\psi_{xx}, \psi_{yy}] \text{ prescribed on } x = 0; \end{array} \right.$$

the stream function $\psi(x, y)$ governing displacements of an incompressible elastic solid when the side walls are free of tractions [8], satisfying

$$(1.10) \quad \left\{ \begin{array}{l} \nabla^4 \psi = 0, \\ 3\psi_{xxy} + \psi_{yyy} = 0 \\ \psi_{xx} - \psi_{yy} = 0 \end{array} \right\} \text{ free surface conditions at } y = \pm 1, \\ f(y) = (\psi_{xx}, \psi_{yy}) \text{ prescribed on } x = 0;$$

and a similar fourth order problem for cylinders [9], to name a few.

Though the eigenfunction bases may differ from problem to problem, all such canonical problems can be studied with the same algorithms, the same type of reduction to ordinary differential equations, the same type of biorthogonality condition with the same matrix A , the same type of formulas for the biorthogonal coefficients and the same type of problems in the study of generalized harmonic analysis of vector-valued functions.

Many of the partial differential equations which arise in mechanics do not have canonical edge data. Apparently such problems may be solved in series formally using the bases generated in canonical problems but, of course, without scalar product formulas for the coefficients. The method of truncation seems always to lead to good approximations for the prescribed data, even when the matrix for the coefficients is not diagonally dominated [4, 5, 6, 7, 8, 9, 16, 22, 29, 30, 39, 40]. The classic example of a non-canonical problem in the semi-infinite strip is the traction problem of elasticity. Such non-canonical problems also arise in Stokes flow in cavities when the velocity is prescribed at the top or bottom of the cavity. GREGORY (1980B) has shown that such problems may be solved by series of bi-orthogonal eigenfunctions and SPENCE (1978) has given a construction which leads to a diagonally dominated matrix for the coefficients (see also TEODORESCU, 1960).

The eigenfunctions arising from separating variables which are used in series solutions can be said to be the natural ones; for example, the solution series in the semi-infinite strip, away from the edges, reduces to only *one term*, the slowest decaying one. The zero conditions at the long side walls will force even non-linear problems to look linear deep in the strip. The same remark obviously applies to the flow in a corner [3, 23, 25, 27, 29]. Thus the eigenfunctions we compute in domains where the linear problem separates may be good Galerkin bases for linear problems in which eigenfunction expansions belonging to different regions are matched [35, 39, 40] and even for nonlinear problems [25, 35]. There is no doubt that, in some problems, Galerkin analysis using natural eigenfunctions is the

most accurate and least expensive of all competing methods of numerical analysis [30].

We have reviewed some particular aspects of a special theory of eigenfunction expansions associated with linear partial differential equations of order four: It seems to us that this theory may be well-used in three ways:

- (1) as a branch of elementary applied analysis for solving problems of mechanics by separating variables;
- (2) as a branch of "generalized harmonic" pure analysis for the representation in series of prescribed and arbitrary vector-valued functions; and
- (3) as a branch of numerical analysis for Galerkin bases to be used in spectral methods for solving linear and nonlinear problems.

2. Completeness of Fourier Series by the Method of Titchmarsh

In this section, we rederive well-known results by the method of TITCHMARSH in order to display the method in its mathematically simplest context. We generate theorems of convergence and completeness of the eigenfunctions of the canonical edge problem for Laplace's equation. In this application of the method we establish

a Fourier series representation $f(x) = \sum_{n=1}^{\infty} c_n f_n(x)$ of a given function $f(\cdot)$ in terms of characteristic harmonic eigenfunctions $f_n(x)$ arising from separation of variables. The theorem of convergence establishes the solution of the original canonical harmonic edge problem in the subspace $(f, f_0) = 0$ orthogonal to the eigenfunction $f_0(x)$. The method of proof leads also to results about non-uniformity of convergence, essentially the Gibbs phenomenon, and to the value to which the series converges at an end point of discontinuity.

As we noted earlier, the results given in this section are not new (or even the best possible) but they do illustrate the main ideas of our application of the method of TITCHMARSH to higher order problems. In the next sections we establish a series representation $[f_1(x), f_2(x)] = \Sigma c_n [\phi_1^{(n)}(x), \phi_2^{(n)}(x)]$ for a given function $f(x) = [f_1, f_2]$ with values in \mathbb{R}^2 in terms of characteristic "biharmonic" eigenfunctions $[\phi_1^{(n)}, \phi_2^{(n)}] = \varphi^{(n)}$ arising from separating variables. We get completeness on the subspace of separable solutions of $\nabla^4 w = 0$ when the data vector $f(x)$ is orthogonal to a certain adjoint eigenvector.

For triharmonic problems, one wishes to establish a series representation

$$[f_1(x), f_2(x), f_3(x)] = \Sigma c_n [\phi_1^{(n)}, \phi_2^{(n)}, \phi_3^{(n)}]$$

of a given function $f(x) = [f_1(x), f_2(x), f_3(x)]$ with values in \mathbb{R}^3 in terms of characteristic "triharmonic" eigenfunctions $[\phi_1^{(n)}, \phi_2^{(n)}, \phi_3^{(n)}] = \varphi^{(n)}$ arising from separating variables in a canonical triharmonic $\nabla^6 w = 0$ edge problem. Such polyharmonic problems of higher order are easily formulated, and basic properties of their solutions are known. We hope to complete proofs for such solutions soon, and seek their applications in problems of mechanics.

Before closing this motivation for the work to follow we note that TITCHMARSH does not seem to have established the applicability of his method for "bad"

functions $f(x)$, like the ramp function and the step function. This justification does follow from his method as we shall show.

The method of TITCHMARSH is an explicit realization of the spectral theory of linear operators; one finds eigenvalues as poles of a resolvent operator and eigenfunctions as residues at those poles.

Consider the canonical edge problem for Laplace's equation on the strip

$$(2.1) \quad \begin{cases} \nabla^2 w = 0, & 0 \leq x, \quad -1 \leq y \leq 1, \\ w(x, \pm 1) = 0, \\ w(0, y) = f(y) \text{ is prescribed,} \\ w \rightarrow 0 \text{ as } x \rightarrow \infty, \text{ uniformly in } y. \end{cases}$$

The representation $w = \sum c_n e^{-s_n x} \phi^{(n)}(y)$ leads to the self-adjoint eigenvalue problem for $\phi(y)$

$$(2.2) \quad \begin{cases} \phi'' + s^2 \phi = 0, & -1 \leq y \leq 1, \\ \phi(\pm 1) = 0. \end{cases}$$

A set of solutions of this problem are

$$(2.3) \quad \phi^{(n)} = \sin [s_n(y-1)], \quad n \geq 1$$

where $s_n = n\pi/2$ are the non-zero roots of $\phi^{(n)}(-1) = -\sin 2s_n = 0$. Equation (2.1)₃ implies that the c_n should be chosen as Fourier coefficients of $f(y)$. That is,

$$(2.4)_1 \quad \begin{aligned} f(y) &= \sum_{n=1}^{\infty} c_n \sin \left[\frac{n\pi}{2} (y-1) \right] \\ &= \sum_{n=1}^{\infty} c_n \left[\sin \frac{n\pi}{2} y \cos \frac{n\pi}{2} - \cos \frac{n\pi}{2} y \sin \frac{n\pi}{2} \right] \end{aligned}$$

where the c_n are the Fourier coefficients

$$(2.4)_2 \quad \begin{aligned} c_n &= \frac{1}{2} \int_{-1}^1 f(u) \sin \frac{n\pi}{2} (u-1) du \\ &= \frac{1}{2} \left\{ \cos \frac{n\pi}{2} \int_{-1}^1 f(u) \sin \frac{n\pi}{2} u du \right. \\ &\quad \left. - \sin \frac{n\pi}{2} \int_{-1}^1 f(u) \cos \frac{n\pi}{2} u du \right\} \end{aligned}$$

It follows from (2.4) that

$$(2.5) \quad \begin{aligned} f(y) &= \sum_{n=2,4,\dots}^{\infty} \frac{1}{2} \sin \frac{n\pi}{2} y \left(\int_{-1}^1 f(u) \sin \left(\frac{n\pi}{2} u \right) du \right) \\ &\quad + \sum_{n=1,3,\dots}^{\infty} \frac{1}{2} \cos \frac{n\pi}{2} y \left(\int_{-1}^1 f(u) \cos \left(\frac{n\pi}{2} u \right) du \right). \end{aligned}$$

Thus the expansion is naturally decomposed into even and odd parts.

To prove that the set of eigenfunctions $\left\{ \sin \frac{n\pi}{2} (y-1) \right\}$ is complete by the method of TITCHMARSH we compute (2.3) as a residue at the poles of the resolvent associated with the problem

$$(2.6) \quad \begin{cases} m'' + s^2 m = f(y), \\ m(\pm 1) = 0. \end{cases}$$

To find the resolvent we invert (2.6) for all complex values of $s \neq n\pi/2$ using the method of variation of parameters. First set

$$(2.7) \quad \begin{aligned} m(y, s) &= X_1(y, s) F_1(y, s) + X_2(y, s) F_2(y, s), \\ X_1'' + s^2 X_1 &= 0, \quad X_1(1, s) = 0; \\ X_2'' + s^2 X_2 &= 0, \quad X_2(-1, s) = 0. \end{aligned}$$

Then

$$(2.8) \quad \begin{aligned} X_1 &= \sin s(y-1), \\ X_2 &= \sin s(y+1) \end{aligned}$$

and the functions F_1, F_2 are required to satisfy

$$(2.9) \quad X_1 F_1' + X_2 F_2' = 0, \quad F_1(-1, s) = F_2(1, s) = 0.$$

With this choice (2.7)₁ satisfies (2.6)₂ and it will also satisfy (2.6)₁ provided that

$$(2.10) \quad X_1' F_1 + X_2' F_2 = f.$$

Let W be the Wronskian

$$W = X_2 X_1' - X_1 X_2' = s \sin 2s \equiv 2s \sin s \cos s.$$

Then

$$(2.11) \quad \begin{aligned} F_1(y, s) &= \frac{1}{W} \int_{-1}^y f(u) X_2(u, s) du, \\ F_2(y, s) &= \frac{1}{W} \int_y^1 f(u) X_1(u, s) du, \\ m(y, s) W &= X_1(y, s) \int_{-1}^y X_2(u, s) f(u) du + X_2(y, s) \int_y^1 X_1(u, s) f(u) du \\ &= \sin s(y-1) \int_{-1}^y \sin s(u+1) f(u) du \\ &\quad + \sin s(y+1) \int_y^1 \sin s(u-1) f(u) du. \end{aligned}$$

We shall justify the expansion (2.4) and (2.5) when $f(\cdot)$, $f'(\cdot)$ are continuous and $f''(\cdot)$ is of bounded variation. To do this we consider a sequence of contour integrals in the complex s -plane:

$$(2.12) \quad \frac{1}{2\pi i} \oint_{D_n} sm(y, s) ds \stackrel{\text{def}}{=} I_n(f),$$

where D_n is the square in the complex s -plane with its four vertices at $\left(\pm \frac{4n+1}{4}\pi, \pm \frac{4n+1}{4}\pi\right)$. Our program is as follows. We first show, using residues, that

$$(2.13) \quad \lim_{n \rightarrow \infty} I_n(f) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi}{2} (y-1)$$

where the c_n are given by (2.5). Then we show

$$(2.14) \quad I_n(f) = f + \mathcal{R}_n$$

where

$$\mathcal{R}_n = \frac{1}{2\pi i} \oint_{D_n} s\{P(y, s) - G(y, s)\} ds$$

is a remainder. If we can establish that $\mathcal{R}_n \rightarrow 0$ as $n \rightarrow \infty$, then (2.13) and (2.14) imply (2.4); that is, the set of eigenfunctions $\left\{\sin \frac{n\pi}{2} (y-1)\right\}$ is complete; arbitrary data $f(y)$ may be represented by a series of sines.

In the demonstration sketched below we get convergence of the series to $f(y)$ in a sup norm, uniformly in y if $f(\pm 1) = 0$, but, in any case, at all interior points $y \in (-1, 1)$.

First we evaluate $I_n(f)$ by the method of residues, noting that the simple poles of $sm(y, s)$ are at the non-zero roots of $\sin 2s = 2 \cos s \sin s$ (that is, at $n\pi/2$) and that $s = 0$ is not a pole. This evaluation leads easily to (2.13).

To get (2.14), we note that

$$(2.15) \quad \begin{aligned} Wm(y, s) &= X_1(y) \int_{-1}^y \left(-\frac{X_2''(u)}{s^2}\right) f(u) du + X_2(y) \int_y^1 \left(-\frac{X_1''(u)}{s^2}\right) f(u) du \\ &= \frac{X_1(y)}{s^2} \left((-X_2'f + X_2f) \Big|_{-1}^y - \int_{-1}^y X_2 f'' du \right) \\ &\quad + \frac{X_2(y)}{s^2} \left((-X_1'f + X_1f) \Big|_y^1 - \int_y^1 X_1 f'' du \right) \end{aligned}$$

where we have assumed that $f(\cdot) \in C^1(-1, 1)$. Since $X_2(-1) = X_1(1) = 0$, two of the corner terms in the square brackets are zero and, since $X_2'(-1) = X_1'(1) = s$, the other two corner terms may be written as

$$\frac{1}{s} X_1(y) f(-1) - \frac{1}{s} X_2(y) f(1).$$

The coefficient of $f'(y)$ vanishes and the coefficient of $f(y)$ is

$$\frac{1}{s^2} [-X_1(y) X_2'(y) + X_2(y) X_1'(y)] = \frac{W}{s^2} = \frac{\sin 2s}{s}.$$

It follows that (2.15) may be written as

$$(2.16) \quad sm(y, s) = \frac{f(y)}{s} + sP(y, s) - sG(y, s)$$

where

$$(2.17)_1 \quad sP(y, s) = \frac{1}{s \sin 2s} [X_1(y) f(-1) - X_2(y) f(1)]$$

and

$$(2.17)_2 \quad \begin{aligned} sG(y, s) &= \frac{1}{s^2 \sin 2s} \left\{ X_1(y) \int_{-1}^y X_2(u) f''(u) du \right. \\ &\quad \left. + X_2(y) \int_y^1 X_1(u) f''(u) du \right\}. \end{aligned}$$

First we shall derive (2.13). Consider the integral

$$(2.18) \quad \begin{aligned} I_n(f) &= \frac{1}{2\pi i} \oint_{D_n} \frac{1}{\sin 2s} \left\{ \sin s(y-1) \int_{-1}^y \sin s(u+1) f(u) du \right. \\ &\quad \left. + \sin s(y+1) \int_y^1 \sin s(u-1) f(u) du \right\}. \end{aligned}$$

This integral is evaluated by computing residues at each zero $s = n\pi/2$ of $\sin 2s$. There is no residue at $s = 0$. Each contour D_n contains $4n$ poles. The computation is easy and straightforward and it leads first to (2.5); then, after some manipulation, to (2.13).

To complete the justification of (2.4) we must show that the contour integral \mathcal{R}_n in (2.14) tends to zero as $n \rightarrow \infty$.

Lemma 2.1. *Let D_n be the sequence of square contours with vertices at $\left[\pm \left(n + \frac{1}{4}\right)\pi, \pm \left(n + \frac{1}{4}\right)\pi\right]$ and let y lie in the open interval $-1 < y < 1$. Then, as $n \rightarrow \infty$*

$$(2.19) \quad \oint_{D_n} sP(y, s) ds \rightarrow 0.$$

Proof. We may rewrite the integral on the left of (2.19) as

$$(2.20) \quad - \oint_{D_n} \frac{1}{s} \left\{ \frac{\cos sy}{\cos s} f_e + \frac{\sin ys}{\sin s} f_o \right\} ds$$

where

$$f_e = \frac{1}{2} (f(1) + f(-1)), \quad f_o = \frac{1}{2} (f(1) - f(-1)).$$

Each of the two integrals converges to zero. We shall prove convergence to zero for the first integral. Let $s = \sigma + i\tau$. The contours D_n split into two horizontal lines $(\sigma, \pm a_n)$ along which σ varies and $\tau = \pm a_n \stackrel{\text{def}}{=} \pm \left(n + \frac{1}{4}\right)\pi$ and two vertical lines $(\pm a_n, \tau)$ on which τ varies and $\sigma = \pm a_n$. We establish an inequality bound on the magnitude of the first integral, and show that the bound goes to zero as $n \rightarrow \infty$. We have

$$\begin{aligned}
 (2.21) \quad & \left| \oint_{D_n} \frac{1}{s} \frac{\cos sy}{\cos s} ds \right| < \oint_{D_n} \frac{1}{|s|} \frac{|\cos sy|}{|\cos s|} |ds| \\
 &= \oint_{D_n} \left\{ \frac{\cos^2 \sigma y + \sinh^2 \tau y}{(\sigma^2 + \tau^2)(\cos^2 \sigma + \sinh^2 \tau)} \right\}^{\frac{1}{2}} ds \\
 &= 2 \int_{-a_n}^{a_n} \left\{ \frac{\cos^2 a_n y + \sinh^2 \tau y}{(a_n^2 + \tau^2)(\cos^2 a_n + \sinh^2 \tau)} \right\}^{\frac{1}{2}} d\tau \\
 &\quad + 2 \int_{-a_n}^{a_n} \left\{ \frac{\cos^2 \sigma y + \sinh^2 a_n y}{(a_n^2 + \sigma^2)(\cos^2 \sigma + \sinh^2 a_n)} \right\}^{\frac{1}{2}} d\sigma \\
 &< 2 \int_{-a_n}^{a_n} \frac{1}{a_n} \left\{ \frac{1 + \sinh^2 \tau y}{2 + \sinh^2 \tau} \right\}^{\frac{1}{2}} d\tau \\
 &\quad + 2 \int_{-a_n}^{a_n} \left(\frac{1 + \sinh^2 a_n y}{a_n^2 \sinh^2 a_n} \right)^{\frac{1}{2}} d\sigma \\
 &= \frac{4\sqrt{2}}{a_n} \int_0^{a_n} \frac{\cosh \tau |y|}{[\cosh 2\tau]^{\frac{1}{2}}} d\tau + \frac{4 \cosh a_n |y|}{\sinh a_n} \\
 &< \frac{8}{a_n} \int_0^{a_n} e^{\tau(|y|-1)} d\tau + \frac{4 \cosh a_n |y|}{\sinh a_n} \\
 &= \frac{8[e^{a_n(|y|-1)} - 1]}{a_n(|y|-1)} + \frac{4 \cosh a_n |y|}{\sinh a_n}.
 \end{aligned}$$

If this goes to zero as $a_n = \left(n + \frac{1}{4}\right)\pi \rightarrow \infty$, then the first part of (2.20) goes to zero. A similar proof holds for the second half, establishing (2.19) and the theorem.

Let $|y| - 1 < 0$ and $a_n \rightarrow \infty$; then the last line of (2.21) tends to

$$\frac{-8}{a_n(|y|-1)} + 4e^{a_n(|y|-1)} \rightarrow 0.$$

The convergence to zero of the integral (2.19) is not uniform because a Gibbs phenomenon occurs near points $y = \pm 1$. However, the convergence to zero of the integral (2.21) is uniform if $f''(y)$ is of bounded variation.

Lemma 2.2. *Let D_n be the contours mentioned in Lemma 2.1 and let y lie in the closed interval $-1 \leq y \leq 1$. Then as $n \rightarrow \infty$*

$$(2.22) \quad \oint_{D_n} sG(y, s) ds \rightarrow 0.$$

Proof. We must consider the line integral of

$$\begin{aligned}
 sG(y, s) &= \frac{1}{s^2 \sin 2s} \left\{ \sin s(y-1) \int_{-1}^y \sin s(u+1) f''(u) du \right. \\
 &\quad \left. + \sin s(y+1) \int_y^1 \sin s(u-1) f''(u) du \right\}
 \end{aligned}$$

on the contour D_n in the complex s -plane. If f'' is of bounded variation, it may be written as $f'' = \phi - \psi$ where ϕ and ψ are positive, bounded and non-decreasing. We assume that f'' is of bounded variation and use the second mean value theorem: if $F(x)$ is integrable and $\phi(x)$ is positive, bounded, and non-decreasing, then there is a ξ , $a < \xi < b$, such that

$$(2.23) \quad \int_a^b F(x) \phi(x) dx = \phi(b) \int_a^b F(x) dx$$

where $\phi(b) = \lim_{x \uparrow b} \phi(x)$. Split $\sin s(u \pm 1)$ into real and imaginary parts. Each of the integrals on u in $sG(y, s)$ is the sum of four integrals, to each of which the second mean value theorem may be applied. For example,

$$\begin{aligned}
 (2.24) \quad & \int_{-1}^y \phi(u) \operatorname{Re} \sin s(u+1) du = \phi(y) \int_{\xi_1}^y \operatorname{Re} \sin s(u+1) du \\
 &= \phi(y) \operatorname{Re} \left\{ \frac{1}{s} \cos s(\xi_1 + 1) - \frac{1}{s} \cos s(y+1) \right\}, \quad -1 < \xi_1 < y
 \end{aligned}$$

and (since $-\psi$ is non-increasing)

$$\begin{aligned}
 (2.25) \quad & - \int_y^1 \psi(u) \operatorname{Im} \sin s(u-1) du = -\psi(y) \int_{\xi_2}^1 \operatorname{Im} \sin s(u-1) du \\
 &= -\psi(y) \operatorname{Im} \left\{ -\frac{1}{s} - \frac{1}{s} \cos s(\xi_2 - 1) \right\}, \quad y < \xi_2 < 1.
 \end{aligned}$$

The terms in $sG(y, s)$ having the form (2.24) may be written

$$\begin{aligned}
 & \frac{\phi(y)}{s^3 \sin 2s} \{ \sin s(y-1) \cos s(\xi_1 + 1) - \sin s(y-1) \cos s(y+1) \} \\
 &= \frac{\phi(y)}{s^3 \sin 2s} \{ \sin s(y + \xi_1) + \sin s(y - \xi_1 - 2) - \sin 2sy + \sin 2s \}
 \end{aligned}$$

where $-1 < \xi_1 < y$. The dominant term is

$$\phi(y)/s^3$$

and

$$\lim_{n \rightarrow \infty} \oint_{D_n} \frac{ds}{s^3} \rightarrow 0.$$

Similar estimates for the other terms of $sG(y, s)$ prove (2.22).

Lemmas (2.1) and (2.2) allow one to evaluate the series (2.14) at $y = \pm 1$. This leads to

Theorem 2.1. *At each point $y \in [-1, 1]$ we have*

$$\sum_{n=1}^{\infty} c_n \sin \frac{n\pi}{2} (y-1) = \begin{cases} f(y), & -1 < y < 1 \\ 0, & y = \pm 1. \end{cases}$$

To prove this theorem we note that at $y = \pm 1$

$$\sum_{n=1}^{\infty} c_n \sin \frac{n\pi}{2} (y-1) = f(\pm 1) + \lim_{n \rightarrow \infty} \oint_{D_n} sP(\pm 1, s) ds.$$

Since from (2.20)

$$\frac{1}{2\pi i} \oint_{D_n} sP(y, s) ds = -\frac{1}{2\pi i} \oint \frac{1}{s} \left\{ \frac{\cos sy}{\cos s} f_e + \frac{\sin sy}{\sin s} f_o \right\} ds$$

we find that in the limit

$$\frac{1}{2\pi i} \oint_{D_n} sP(\pm 1, s) ds = -f_e \mp f_o = -f(\pm 1).$$

3. Formal Solution of the Canonical Biharmonic Edge Problem

We wish to find $w(x, y)$, $x \geq 0$, $-1 \leq y \leq 1$ satisfying

$$\nabla^4 w = 0, \quad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2},$$

$$(3.1) \quad w(x, \pm 1) = w_y(x, \pm 1) = 0,$$

$$\begin{bmatrix} w_{xx}(0, y) \\ w_{yy}(0, y) \end{bmatrix} = \begin{bmatrix} f_1(y) \\ f_2(y) \end{bmatrix} \stackrel{\text{def}}{=} f(y) \text{ is prescribed,}$$

$$w(x, y) \rightarrow 0 \text{ as } x \rightarrow \infty.$$

SMITH (1952) found a solution of (3.1) by separation of variables, which he justified under the overly restrictive conditions on the data vector $f(\pm 1) = f'(\pm 1) = 0$.

The solution given by SMITH is*

$$(3.2) \quad w(x, y) = \sum_{-\infty}^{\infty} \left\{ \frac{c_n}{s_n^2} \phi_1^{(n)}(y) e^{-s_n x} + \frac{\hat{c}_n}{\hat{s}_n^2} \hat{\phi}_1^{(n)}(y) e^{-\hat{s}_n x} \right\}.$$

Here $\phi_1^{(n)}(y)$ and $\hat{\phi}_1^{(n)}(y)$ are the even and odd eigenfunctions, respectively, associated with the eigenvalues s_n and \hat{s}_n . The eigenvalues are the non-zero roots of

$$(3.3) \quad \Delta(s) \hat{\Delta}(s) \stackrel{\text{def}}{=} (\sin 2s + 2s)(\sin 2s - 2s) = 0.$$

To each non-zero root s_n of $\Delta(s_n) \equiv \sin 2s_n + 2s_n = 0$, there corresponds the even eigenfunction

$$(3.4) \quad \phi_1^{(n)}(y) = s_n \sin s_n \cos s_n y - s_n y \cos s_n \sin s_n y$$

satisfying

$$(3.5) \quad \phi_1^{(n)}(\pm 1) = \phi_1^{(n)'}(\pm 1) = 0.$$

The product function $e^{-s_n x} \phi_1^{(n)}(y)$ is then biharmonic, satisfies the edge conditions (3.5) on the long sides of the strip, and, if $\text{Re } s_n > 0$, has the proper growth condition as $x \rightarrow \infty$. In the same way, to each non-zero root \hat{s}_n of $\hat{\Delta}(\hat{s}_n) \equiv \sin 2\hat{s}_n - 2\hat{s}_n = 0$ there corresponds the odd eigenfunction

$$(3.6) \quad \hat{\phi}_1^{(n)}(y) = \hat{s}_n \cos \hat{s}_n \sin \hat{s}_n y - \hat{s}_n y \sin \hat{s}_n \cos \hat{s}_n y$$

which also satisfies edge conditions (3.5). The product $e^{-\hat{s}_n x} \hat{\phi}_1^{(n)}(y)$ is then biharmonic with the right edge conditions and behaves properly as $x \rightarrow \infty$ if $\text{Re } (\hat{s}_n) > 0$.

All of the non-zero roots s_n and \hat{s}_n of Equation (3.3) are complex. We order the roots lying in the first quadrant of the complex s -plane according to the size of their real parts, i.e., $0 < \text{Re } s_1 < \text{Re } s_2 < \text{Re } s_3 < \dots$ with the same convention for \hat{s}_n . We identify roots of (3.3) in the fourth quadrant as $s_{-n} = \bar{s}_n$ (complex conjugate). Roots in the second quadrant are given by $-s_{-n}$ and in the third quadrant by $-s_n$. As we noted in the last paragraph, in the canonical problem (3.1), roots with negative real parts are unacceptable because the corresponding exponential functions are unbounded as $x \rightarrow \infty$. There is no non-trivial solution of (3.1) belonging to roots $s = 0$ of (3.3). It follows from our numbering convention that the summation given in (3.2) is over all the acceptable solutions of (3.1)₁, (3.1)₂ and (3.1)₄. Only (3.1)₃ is at issue.

To deal with (3.1)₃ SMITH introduced an algorithm for generating the coefficients c_n, \hat{c}_n . This algorithm requires that we treat a reduced problem in ordinary differential equations derived as follows. Suppose (3.1)₃ holds; then the data vector $f(y)$ must be represented as

$$(3.7) \quad f(y) \sim \sum_{-\infty}^{\infty} \{c_n \varphi^{(n)}(y) + \hat{c}_n \hat{\varphi}^{(n)}(y)\}$$

* Again we note that $\sum_{-\infty}^{\infty}$ contains no term for index $n = 0$ throughout this paper; any such term will be displayed explicitly.

where the vector eigenfunctions

$$(3.8)_1 \quad \phi^{(n)}(y) = \begin{bmatrix} \phi_1^{(n)}(y) \\ \phi_2^{(n)}(y) \end{bmatrix}, \quad \hat{\phi}^{(n)}(y) = \begin{bmatrix} \hat{\phi}_1^{(n)}(y) \\ \hat{\phi}_2^{(n)}(y) \end{bmatrix}$$

have first components (3.4), (3.6) and second components

$$(3.8)_2 \quad \phi_2^{(n)}(y) = \phi_1^{(n)''}(y)/s_n^2, \quad \hat{\phi}_2^{(n)}(y) = \hat{\phi}_1^{(n)''}(y)/\hat{s}_n^2.$$

The separation conditions guaranteeing that each term of (3.2) is biharmonic require that $\phi^{(n)}(y)$, $\hat{\phi}^{(n)}(y)$ satisfy

$$(3.9) \quad \begin{aligned} \phi^{(n)''} + s_n^2 A \phi^{(n)} &= 0, & \phi_1^{(n)}(\pm 1) &= \phi_1^{(n)' }(\pm 1) = 0; \\ \hat{\phi}^{(n)''} + \hat{s}_n^2 A \hat{\phi}^{(n)} &= 0, & \hat{\phi}_1^{(n)}(\pm 1) &= \hat{\phi}_1^{(n)' }(\pm 1) = 0 \end{aligned}$$

where

$$(3.10) \quad A = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}$$

is the biorthogonality matrix.

At this point the reader should suppress the fact that (3.7) and (3.9) were derived from separable solutions of the biharmonic. We want instead to represent (3.7) in terms of eigenvectors of the reduced problem (3.9). The problem (3.9), (3.7) is self-contained.

Now define

$$(3.11) \quad \langle \alpha \rangle \stackrel{\text{def}}{=} \int_{-1}^1 \alpha(y) dy$$

for any integrable function $\alpha(\cdot)$ defined on $[-1, 1]$. We define an adjoint problem relative to (3.9) in the usual way and find that the even and odd adjoint eigenvectors $\psi^{(n)}(y)$, $\hat{\psi}^{(n)}(y)$ must satisfy

$$(3.12) \quad \begin{aligned} \psi^{(n)''}(y) + s_n^2 A^T \psi^{(n)}(y) &= 0, \\ \psi_2^{(n)}(\pm 1) &= \psi_2^{(n)' }(\pm 1) = 0; \\ \hat{\psi}^{(n)''}(y) + \hat{s}_n^2 A^T \hat{\psi}^{(n)}(y) &= 0, \\ \hat{\psi}_2^{(n)}(\pm 1) &= \hat{\psi}_2^{(n)' }(\pm 1) = 0, \end{aligned}$$

where A^T is the transpose of A . The biorthogonality conditions

$$(3.13) \quad \begin{aligned} \langle \psi^{(m)} A \phi^{(n)} \rangle &= k_n \delta_{nm}, \\ \langle \hat{\psi}^{(m)} A \hat{\phi}^{(n)} \rangle &= \hat{k}_n \delta_{nm}, \\ \langle \hat{\psi}^{(m)} A \phi^{(n)} \rangle &= \langle \psi^{(m)} A \hat{\phi}^{(n)} \rangle = 0, \end{aligned}$$

where $\delta_{nm} = 0$ if $m \neq n$ and $\delta_{nm} = 1$ if $n = m$, follow directly from (3.9) and (3.12) in the usual way. The biorthogonality relations lead directly to inner

product formulas for the coefficients in (3.7):

$$(3.14) \quad \begin{aligned} c_n &= \frac{1}{k_n} \langle \psi^{(n)} A f \rangle, \\ \hat{c}_n &= \frac{1}{\hat{k}_n} \langle \hat{\psi}^{(n)} A f \rangle. \end{aligned}$$

All the quantities appearing in the formal solution of the reduced problem were given explicitly by SMITH (1952). For the even functions, including those corresponding to the zero roots of (3.3), we have

$$(3.15) \quad \begin{aligned} \sin 2s_n + 2s_n &= 0, & n &\geq 1, \\ s_0 &= 0, \\ \phi_1^{(0)} &= 0, \\ \phi_2^{(0)} &= 1, \\ \phi_1^{(n)}(y) &= s_n \sin s_n \cos s_n y - s_n y \cos s_n \sin s_n y, \\ \phi_2^{(n)}(y) &= -\phi_1^{(n)}(y) - 2 \cos s_n \cos s_n y, \\ \psi_1^{(0)} &= 1, \\ \psi_2^{(0)} &= 0, \\ \psi_1^{(n)}(y) &= \phi_1^{(n)}(y) - 2 \cos s_n \cos s_n y, \\ \psi_2^{(n)}(y) &= \phi_1^{(n)}(y), \\ k_0 &= -2, \\ k_n &= -4 \cos^4 s_n. \end{aligned}$$

For the odd functions we have

$$(3.16) \quad \begin{aligned} \sin 2\hat{s}_n - 2\hat{s}_n &= 0, & n &\geq 1, \\ \hat{s}_0 &= 0, \\ \hat{\phi}_1^{(0)} &= 0, \\ \hat{\phi}_2^{(0)} &= y, \\ \hat{\phi}_1^{(n)}(y) &= \hat{s}_n \cos \hat{s}_n \sin \hat{s}_n y - \hat{s}_n y \sin \hat{s}_n \cos \hat{s}_n y, \\ \hat{\phi}_2^{(n)}(y) &= -\hat{\phi}_1^{(n)}(y) + 2 \sin \hat{s}_n \sin \hat{s}_n y, \\ \hat{\psi}_1^{(0)} &= y, \\ \hat{\psi}_2^{(0)} &= 0, \\ \hat{\psi}_1^{(n)}(y) &= \hat{\phi}_1^{(n)}(y) + 2 \sin \hat{s}_n \sin \hat{s}_n y, \\ \hat{\psi}_2^{(n)}(y) &= \hat{\phi}_1^{(n)}(y), \\ \hat{k}_0 &= -\frac{2}{3}, \\ \hat{k}_n &= -4 \sin^4 \hat{s}_n. \end{aligned}$$

To get an expansion theorem of the form of (3.7) for unrestricted $f''(y) \in BV$ it is necessary to use *all* of the eigenfunctions of (3.9) including the eigenfunctions $\varphi^{(0)}, \hat{\varphi}^{(0)}$ belonging to $s_0 = 0$ and $\hat{s}_0 = 0$ which are *not* eigenfunctions of the generating canonical biharmonic problem. The biharmonic problem (3.1) is solvable only when the data vector $f(y)$ lies on a subspace defined by

$$(3.17) \quad \begin{aligned} c_0 &= \frac{1}{k_0} \langle \psi^{(0)} A f \rangle = 0, \\ \hat{c}_0 &= \frac{1}{\hat{k}_0} \langle \hat{\psi}^{(0)} A f \rangle = 0. \end{aligned}$$

Several types of decompositions of the data are useful in the theory and are very useful in applications. We collect them here. It is obvious that even and odd parts of the data can be treated separately. We have

$$(3.18) \quad \begin{aligned} f^e(y) &\sim c_0 \varphi^{(0)} + \sum_{-\infty}^{\infty} c_n \varphi^{(n)}, \\ f^o(y) &\sim c_0 \hat{\varphi}^{(0)} + \sum_{-\infty}^{\infty} c_n \hat{\varphi}^{(n)} \end{aligned}$$

where

$$(3.19) \quad \begin{aligned} f^e(y) &= \frac{1}{2} [f(y) + f(-y)], \\ f^o(y) &= \frac{1}{2} [f(y) - f(-y)] \end{aligned}$$

and

$$(3.20) \quad \begin{aligned} c_n &= \frac{1}{k_n} \langle \psi^{(n)} A f^e \rangle, \\ \hat{c}_n &= \frac{1}{\hat{k}_n} \langle \hat{\psi}^{(n)} A f^o \rangle. \end{aligned}$$

Using (3.12), we find, after integrating by parts, that

$$(3.21) \quad \begin{aligned} c_n k_n &= -\langle \psi^{(n)''} f^e \rangle / s_n^2 \\ &= 4f_1^e(1) - \frac{4 \cos^2 s_n}{s_n^2} f_1^e(1) - \langle \psi^{(n)} f^{e''} \rangle / s_n^2, \\ \hat{c}_n \hat{k}_n &= -\langle \hat{\psi}^{(n)''} f^o \rangle / \hat{s}_n^2 \\ &= -4f_1^o(1) + \frac{4 \sin^2 \hat{s}_n}{\hat{s}_n^2} f_1^o(1) - \langle \hat{\psi}^{(n)} f^{o''} \rangle / \hat{s}_n^2. \end{aligned}$$

We now introduce new data vectors, $F^e(y)$ and $F^o(y)$:

$$(3.22) \quad \begin{aligned} \begin{bmatrix} F_1^e(y) \\ F_2^e(y) \end{bmatrix} &\stackrel{\text{def}}{=} \begin{bmatrix} f_1^e(y) - f_1^e(1) + f_1^e(1)(1 - y^2)/2 \\ f_2^e(y) - c_0 \end{bmatrix}, \\ \begin{bmatrix} F_1^o(y) \\ F_2^o(y) \end{bmatrix} &\stackrel{\text{def}}{=} \begin{bmatrix} f_1^o(y) - f_1^o(1)y + [f_1^{o'}(1) - f_1^o(1)](y - y^3)/2 \\ f_2^o(y) - \hat{c}_0 y \end{bmatrix} \end{aligned}$$

where

$$(3.23) \quad F_1^e(1) = F_1^e(1) = F_1^o(1) = F_1^o(1) = 0$$

and

$$(3.24) \quad \begin{aligned} \langle \psi^{(n)} f^{e''} \rangle - \langle \psi^{(n)} F^{e''} \rangle &= f_1^e(1) \langle \psi_1^{(n)} \rangle = 8f_1^e(1), \\ \langle \hat{\psi}^{(n)} f^{o''} \rangle - \langle \hat{\psi}^{(n)} F^{o''} \rangle &= 3[f_1^{o'}(1) - f_1^o(1)] \langle \psi_1^{(n)} \rangle \\ &= 24 \tan^2 \hat{s}_n. \end{aligned}$$

After combining all these representations and the results given in § 7, we find that

$$(3.25) \quad -\sum_{-\infty}^{\infty} \frac{1}{\cos^4 s_n} \varphi^{(n)}(y) = \begin{cases} \begin{bmatrix} 1 \\ 0 \end{bmatrix} & \text{for } -1 < y < 1 \\ \begin{bmatrix} 0 \\ -1 \end{bmatrix} & \text{for } y = \pm 1, \end{cases}$$

$$(3.26) \quad -\sum_{-\infty}^{\infty} \left(\frac{2 \cos^2 s_n + 4}{s_n^2 \cos^4 s_n} \right) \varphi^{(n)}(y) = \begin{bmatrix} 1 - y^2 \\ 0 \end{bmatrix} \text{ for } -1 \leq y \leq 1,$$

$$(3.27) \quad -\sum_{-\infty}^{\infty} \frac{1}{k_n s_n^2} \langle \psi_1^{(n)} F_1^{e''} + \psi_2^{(n)} F_2^{e''} \rangle \varphi^{(n)}(y) = \begin{bmatrix} F_1^e(y) \\ F_2^e(y) \end{bmatrix} \text{ for } -1 \leq y \leq 1,$$

$$(3.28) \quad \sum_{-\infty}^{\infty} \left(\frac{1}{\sin^4 \hat{s}_n} - \frac{6}{\hat{s}_n^4} \right) \hat{\varphi}^{(n)}(y) = \begin{cases} \begin{bmatrix} \frac{3}{2}y - \frac{1}{2}y^3 \\ 0 \end{bmatrix} & \text{for } -1 < y < 1 \\ \begin{bmatrix} 0 \\ \mp 1 \end{bmatrix} & \text{for } y = \pm 1, \end{cases}$$

$$(3.29) \quad \sum_{-\infty}^{\infty} \left(\frac{1}{\hat{s}_n^2 \sin^2 \hat{s}_n} - \frac{6}{\hat{s}_n^4} \right) \hat{\varphi}^{(n)}(y) = \begin{bmatrix} \left[\frac{1}{2}y - \frac{1}{2}y^3 \right] \\ 0 \end{bmatrix} \text{ for } -1 \leq y \leq 1,$$

$$(3.30) \quad -\sum_{-\infty}^{\infty} \frac{1}{\hat{k}_n \hat{s}_n^2} \langle \hat{\psi}_1^{(n)} F_1^{o''} + \hat{\psi}_2^{(n)} F_2^{o''} \rangle \hat{\varphi}^{(n)}(y) = \begin{bmatrix} F_1^o(y) \\ F_2^o(y) \end{bmatrix} \text{ for } -1 \leq y \leq 1.$$

JOSEPH (1977) and JOSEPH & STURGES (1978) showed that the series (3.25)–(3.30) converge and they obtained the rates of convergence using asymptotic formulas for large n :

$$\begin{aligned}
 2s_n &\rightarrow \left(2n - \frac{1}{2}\right) \pi + i \log(4n - 1) \pi, \\
 2\hat{s}_n &\rightarrow \left(2n + \frac{1}{2}\right) \pi + i \log(4n + 1) \pi, \\
 \sin s_n y &= \frac{i}{2} [(4n - 1) \pi]^{y/2} e^{-i(n-1/4)\pi y} + O(n^{-y/2}), \\
 \sin \hat{s}_n y &= \frac{i}{2} [(4n + 1) \pi]^{y/2} e^{-i(n+1/4)\pi y} + O(n^{-y/2}), \\
 s_n &= O(n), \quad \hat{s}_n = O(n), \\
 k_n &= O(n^2), \quad \hat{k}_n = O(n^2), \\
 \phi^{(n)}, \hat{\phi}^{(n)}, \psi^{(n)}, \hat{\psi}^{(n)} &= O(n^{(3+|y|)/2}).
 \end{aligned}
 \tag{3.31}$$

JOSEPH (1977) showed that the coefficients in the series (3.27) and (3.30) are $O(n^2)$ and that these series may be majorized by a convergent numerical series

$$c \sum_{n=1}^{\infty} 1/n^{(5-|y|)/2}
 \tag{3.32}$$

where c is independent of n . The series (3.28) and (3.29) converge absolutely (and uniformly) on $-1 \leq y \leq 1$ but more slowly than (3.32) with a majorant equal to

$$c \sum_{n=1}^{\infty} 1/n^{(3-|y|)/2}.$$

JOSEPH & STURGES (1978) showed that the series (3.25) and (3.28) were conditional and not uniformly convergent.

In the sequel we shall forget about the generating canonical biharmonic problem (3.1). It is perhaps useful to note here that whenever the formal solution of (3.1) just given is justified, we have an immediate and precise mathematical realization of St. Venant's principle. Stating this principle in an informal way, we note the solutions of (3.1) in the strip decay very rapidly to

$$w(x, y) \sim c_1 \phi_1^{(1)}(y) \exp(-s_1 x) + \hat{c}_1 \hat{\phi}_1^{(1)}(y) \exp(-\hat{s}_1 x) + \text{complex conjugate}.$$

The decay is fast:

$$\begin{aligned}
 s_1 &= 2.106196 + i 1.125365, \\
 s_2 &= 5.356269 + i 1.551575, \\
 \hat{s}_1 &= 3.748838 + i 1.384339, \\
 \hat{s}_2 &= 6.949980 + i 1.676105.
 \end{aligned}$$

It follows that the interior form of the solution is independent of the details of the edge data. Only the constants c_1 and \hat{c}_1 depend on $f(y)$ and in fact are projections of f . The interior solution is a decaying system of closed eddies with a fixed spatial period.

4. Proof of Completeness of Biorthogonal Series by the Method of Titchmarsh

The method of proof is the same as the one used in § 2 for Fourier series. We begin the proof by constructing a resolvent operator. The computations are more involved here because we are interested in justifying the expansion formula (3.7) for a prescribed data vector $f(y)$ where $f'(\cdot) \in BV$ and BV is the collection of all $C^1(-1, 1)$ vectors in \mathbb{R}^2 whose second derivative is of bounded variation. This means that our resolvent is a tensor-valued operator

$$(s^2 \mathbf{1} - T)^{-1}, \quad T = -A^{-1} \frac{d^2}{dy^2},
 \tag{4.1}$$

carrying $f'' \in BV$ into $m(\cdot, s) = \begin{bmatrix} m_1 \\ m_2 \end{bmatrix} = (s^2 \mathbf{1} - T)^{-1} f$ where

$$\begin{cases} m'' + s^2 A m = A f, & A = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}, \\ m_1(\pm 1) = m_1'(\pm 1) = 0. \end{cases}
 \tag{4.2}$$

We follow SMITH (1952) and construct the expansion formula (3.7) as residues at the poles of $\tilde{m}(y, s^2) \stackrel{\text{def}}{=} s m(y, s)$; that is, we consider

$$I_N(f) = \frac{1}{2\pi i} \oint_{D_N} s m(y, s) ds$$

where D_N is a sequence of closed contours of increasing "radius" which do not pass through the singularities of $s m(s, y)$. We take D_N as the square with vertices at $(\pm 2N\pi, \pm 2N\pi)$ and get the expansion theorem by showing that

$$I_N(f) = f + \mathcal{R}_N = \sum_{-N}^N (c_n \phi^{(n)}(y) + \hat{c}_n \hat{\phi}^{(n)}(y))
 \tag{4.3}$$

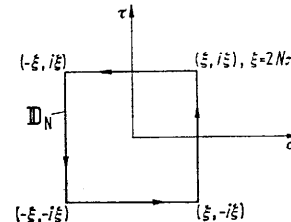


Fig. 4.1. Contour D_N in the complex $s = \sigma + i\tau$ plane

where c_n, \hat{c}_n are defined by (3.14) and

$$\mathcal{R}_N = \frac{1}{2\pi i} \oint_{\mathcal{D}_N} r(y, s) ds \rightarrow 0 \text{ as } N \rightarrow \infty.$$

The solution of (4.2) was given by SMITH (1952) as

$$(4.4) \quad m(y, s) = X_1(sy) W_{1,2}^{-1}(2s) \int_{-1}^y Y_2(su) Af(u) du \\ - X_2(sy) W_{2,1}^{-1}(2s) \int_y^1 Y_1(su) Af(u) du$$

where the X, Y , and W are also 2×2 matrices. The matrices X_i and Y_i satisfy

$$(4.5)_1 \quad \begin{cases} X_1(sy) = X[s(y-1)], & Y_1(sy) = Y[s(y-1)], \\ X_2(sy) = X[s(y+1)], & Y_2(sy) = Y[s(y+1)] \end{cases}$$

where X and Y are the solutions of the matrix differential equations

$$(4.5)_2 \quad \begin{aligned} X'' + s^2 AX &= 0, & EX(0) + \hat{E}X'(0) &= 0, \\ X(0) &= \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}, & E &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \\ X'(0) &= s \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}, & \hat{E} &= \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \end{aligned}$$

and

$$(4.5)_3 \quad \begin{aligned} Y'' + s^2 YA &= 0, & Y(0)\hat{E} + Y'(0)\bar{E} &= 0, \\ Y(0) &= \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, & Y'(0) &= X'(0), & \bar{E} &= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

The conditions (4.5) determine $X(\alpha), Y(\alpha)$ uniquely where $\alpha = sy$ and

$$(4.6) \quad \begin{aligned} X(\alpha) &\equiv \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} = \begin{bmatrix} \sin \alpha - \alpha \cos \alpha & \alpha \sin \alpha \\ \sin \alpha + \alpha \cos \alpha & 2 \cos \alpha - \alpha \sin \alpha \end{bmatrix} \\ &= \begin{bmatrix} X_{11} & X_{12} \\ -X_{11} + 2 \sin \alpha & -X_{12} + 2 \cos \alpha \end{bmatrix}, \\ Y(\alpha) &\equiv \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \begin{bmatrix} 2 \cos \alpha + \alpha \sin \alpha & \alpha \sin \alpha \\ 3 \sin \alpha - \alpha \cos \alpha & \sin \alpha - \alpha \cos \alpha \end{bmatrix} \\ &= \begin{bmatrix} Y_{12} + 2 \cos \alpha & Y_{12} \\ Y_{22} + 2 \sin \alpha & Y_{22} \end{bmatrix} = \begin{bmatrix} X_{12} + 2 \cos \alpha & X_{12} \\ X_{11} + 2 \sin \alpha & X_{11} \end{bmatrix}. \end{aligned}$$

The generalized Wronskian $W(a, b)$ is defined as follows:

$$\begin{aligned} \alpha &= s(y-a), & \beta &= s(y-b), & \hat{\alpha} &= s(b-a), \\ W(a, b) &= Y(\beta) \frac{dX(\alpha)}{dy} - \frac{dY(\beta)}{dy} X(\alpha), \\ &= s[Y(\beta) X'(\alpha) - Y'(\beta) X(\alpha)], \\ (4.7) \quad &= 2s \begin{bmatrix} \hat{\alpha} \sin \hat{\alpha} & -(\hat{\alpha} \cos \hat{\alpha} + \sin \hat{\alpha}) \\ \sin \hat{\alpha} - \hat{\alpha} \cos \hat{\alpha} & -\hat{\alpha} \sin \hat{\alpha} \end{bmatrix} \\ &\stackrel{\text{def}}{=} \hat{W}(s, \hat{\alpha}) \\ &= 2s \begin{bmatrix} X'_{11}(\hat{\alpha}) & -X''_{11}(\hat{\alpha}) \\ X_{11}(\hat{\alpha}) & -X'_{11}(\hat{\alpha}) \end{bmatrix}. \end{aligned}$$

The inverse Wronskian W^{-1} is determined by the equation $W^2 = I \det \hat{W}$. Then

$$(4.8) \quad \begin{aligned} \hat{W}^2(s, \hat{\alpha}) &= 4s^2(\hat{\alpha}^2 - \sin^2 \hat{\alpha}) I = I \det \hat{W}, \\ W^{-1}(a, b) &= W(a, b)/4s^2(\hat{\alpha}^2 - \sin^2 \hat{\alpha}), \\ W(a, a) &= 0, \\ W_{1,2} &\stackrel{\text{def}}{=} W(1, -1) = \hat{W}(s, 2s), \\ W_{2,1} &\stackrel{\text{def}}{=} W(-1, 1) = \hat{W}(s, -2s), \\ W_{1,2}^{-1} &= \frac{1}{2s(4s^2 - \sin^2 2s)} \begin{bmatrix} -2s \sin 2s & 2s \cos 2s + \sin 2s \\ -\sin 2s + 2s \cos 2s & 2s \cos 2s \end{bmatrix}, \\ W_{2,1}^{-1} &= \frac{1}{2s(4s^2 - \sin^2 2s)} \begin{bmatrix} -2s \sin 2s & -2s \cos 2s - \sin 2s \\ \sin 2s + 2s \cos 2s & 2s \cos 2s \end{bmatrix}. \end{aligned}$$

To derive $m(y, s)$ given by (4.4) set

$$m(y, s) = X_1(sy) F_1(y, s) + X_2(sy) F_2(y, s)$$

where F_1 and F_2 are to be determined. Calculate $m'(y, s), m''(y, s)$ and put $X_1 F_1' + X_2 F_2' = 0$ in m' . To satisfy $m'' + s^2 A m = Af$ we must determine F_1 and F_2 such that

$$(4.9) \quad \begin{cases} X_1 F_1' + X_2 F_2' = 0, \\ X_1' F_1' + X_2' F_2' = Af. \end{cases}$$

The boundary conditions $m(\pm 1, s) = 0$ lead to $F_1(-1, s) = F_2(1, s) = 0$. Equations (4.9) can be solved by premultiplication, first by $[Y_2', -Y_2']$ and then

by $[-Y'_1, Y_1]$. From the first premultiplication we get

$$[-W(1, -1), -W(-1, -1)] \begin{bmatrix} F'_1 \\ F'_2 \end{bmatrix} = [-W_{1,2}, 0] \begin{bmatrix} F'_1 \\ F'_2 \end{bmatrix} = -W_{1,2}F'_1 = -Y_2Af.$$

Hence

$$F_1(y, s) = W_{1,2}^{-1}(2s) \int_{-1}^y Y_2(su) Af(u) du.$$

Similarly, from the second premultiplication

$$F_2(y, s) = -W_{2,1}^{-1} \int_y^1 Y_1(su) Af(u) du.$$

An alternative form of $m(y, s)$ which exhibits the even and odd eigenfunctions and adjoint eigenfunctions explicitly is useful in the residue computation of the expansion formula (3.7). The alternative form is

$$(4.10) \quad m(y, s) = \frac{\varphi(y, s) \int_{-1}^1 \psi(u, s) \cdot g(u) du}{2s(\cos^2 s) (\sin 2s + 2s)} + \frac{\hat{\varphi}(y, s) \int_{-1}^1 \hat{\psi}(u, s) \cdot g(u) du}{2s(\sin^2 s) (\sin 2s - 2s)} + Q(y, s)$$

where

$$(4.11)_1 \quad Q(y, s) = \frac{\varphi(y, s)}{4s \cos^2 s} \left(\int_{-1}^y \sin s(u+1) (g_1(u) + g_2(u)) du - \int_y^1 \sin s(u-1) (g_1(u) + g_2(u)) du \right) + \frac{\hat{\varphi}(y, s)}{4s \sin^2 s} \left(\int_{-1}^y \sin s(u+1) (g_1(u) + g_2(u)) du + \int_y^1 \sin s(u-1) (g_1(u) + g_2(u)) du \right) - \frac{\sin s(y-1)}{2s \sin 2s} \int_{-1}^y \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} Y_2(su) g(u) du - \frac{\sin s(y+1)}{2s \sin 2s} \int_y^1 \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} Y_1(su) g(u) du$$

and

$$(4.11)_2 \quad \begin{aligned} g &= Af = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}, \\ \varphi(y, s) &= \begin{bmatrix} \phi_1(y, s) \\ \phi_2(y, s) \end{bmatrix}, \\ \psi(y, s) &= [\psi_1(y, s), \psi_2(y, s)] \end{aligned}$$

where ϕ_1, ϕ_2, ψ_1 and ψ_2 are the even eigenfunctions (3.15) with the particular s_n replaced by the variable s and

$$(4.11)_3 \quad \begin{cases} \hat{\varphi}(y, s) = \begin{bmatrix} \hat{\phi}_1(y, s) \\ \hat{\phi}_2(y, s) \end{bmatrix} \\ \hat{\psi}(y, s) = [\hat{\psi}_1(y, s), \hat{\psi}_2(y, s)], \end{cases}$$

where $\hat{\phi}_1, \hat{\phi}_2, \hat{\psi}_1$ and $\hat{\psi}_2$ are the odd eigenfunctions (3.16), also with \hat{s}_n replaced by s . The function $sQ(y, s)$ makes no contribution to the integrals $I_N(f)$ because it has zero residue at all s_n, \hat{s}_n .

To derive (4.10) we decompose the inverse Wronskians into two parts which will correspond to residues at even and odd eigenfunctions:

$$(4.12) \quad \begin{aligned} W_{12}^{-1}(2s) &= \Omega(2s) \xi(2s) = \omega_-(2s) H_1(2s) + \omega_+(2s) H_2(2s), \\ W_{21}^{-1}(2s) &= \Omega(2s) \xi(-2s) = \omega_-(2s) H_3(2s) + \omega_+(2s) H_4(2s) \end{aligned}$$

where $\omega_{\pm}(\beta) = (\sin \beta \pm \beta)^{-1}$, $\Omega(\beta) = \omega_+(\beta) \omega_-(\beta)/\beta$, and

$$(4.13) \quad \begin{aligned} \xi(\beta) &= \begin{bmatrix} -X_{12}(\beta) & X_{21}(\beta) \\ -X_{11}(\beta) & X_{12}(\beta) \end{bmatrix} = \begin{bmatrix} -X'_{11}(\beta) & X'_{11}(\beta) \\ -X_{11}(\beta) & X'_{11}(\beta) \end{bmatrix}, \\ H_1(\beta) &= \frac{1}{2\beta} \begin{bmatrix} -\sin \beta & 1 + \cos \beta \\ -1 + \cos \beta & \sin \beta \end{bmatrix}, \quad H_3(\beta) = H_1(-\beta), \\ H_2(\beta) &= \frac{1}{2\beta} \begin{bmatrix} \sin \beta & 1 - \cos \beta \\ -1 - \cos \beta & -\sin \beta \end{bmatrix}, \quad H_4(\beta) = H_2(-\beta). \end{aligned}$$

In the next step we compute

$$\begin{aligned} X_1(sy) W_{1,2}^{-1}(2s) &= \Omega(\beta) X(x) \xi(\beta) \Big|_{\substack{x=s(y-1) \\ \beta=2s}}, \\ X_2(sy) W_{2,1}^{-1}(2s) &= \Omega(\beta) X(x) \xi(-\beta) \Big|_{\substack{x=s(y+1) \\ \beta=2s}}, \end{aligned}$$

in the decomposed H_i form (4.12). This is a tedious computation. A typical intermediate result is of the form

$$2\mathbf{X}(\alpha) \mathbf{H}_1(\beta) = \begin{bmatrix} h_{11}(\alpha, \beta) & -\frac{1 + \cos \beta}{\sin \beta} h_{11}(\alpha, \beta) \\ h_{21}(\alpha, \beta) & -\frac{1 + \cos \beta}{\sin \beta} h_{21}(\alpha, \beta) \end{bmatrix} \\ = \begin{bmatrix} h_{11}(\alpha, \beta) \\ h_{21}(\alpha, \beta) \end{bmatrix} \begin{bmatrix} 1, & -\frac{1 + \cos \beta}{\sin \beta} \end{bmatrix}$$

where

$$h_{11}(\alpha, \beta) = \alpha \sin(\alpha + \beta) - (\alpha + \beta) \sin \alpha + (\beta - \sin \beta) \sin \alpha,$$

$$h_{21}(\alpha, \beta) = -h_{11}(\alpha, \beta) + 2 \cos(\alpha + \beta) - 2 \cos \alpha.$$

When the substitutions $\alpha = s(y - 1)$, $\beta = 2s$, $\alpha + \beta = s(y + 1)$ are made, combinations of the eigenfunctions $\varphi(y, s)$ and $\hat{\varphi}(y, s)$ appear. The results are

$$(4.14)_1 \quad \mathbf{X}_1 \mathbf{W}_{1,2}^{-1} = \frac{1}{2s(\sin 2s)(\sin 2s + 2s)} \begin{bmatrix} \phi_1(y, s) \\ \phi_2(y, s) \end{bmatrix} [\sin 2s, 1 - \cos 2s] \\ - \frac{1}{2s(\sin 2s)(\sin 2s - 2s)} \begin{bmatrix} \hat{\phi}_1(y, s) \\ \hat{\phi}_2(y, s) \end{bmatrix} [\sin 2s, -1 - \cos 2s] \\ - \frac{\sin s(y - 1)}{2s \sin 2s} \begin{bmatrix} -1 \\ 1 \end{bmatrix} [0, 1],$$

$$(4.14)_2 \quad \mathbf{X}_2 \mathbf{W}_{2,1}^{-1} = \frac{1}{2s(\sin 2s)(\sin 2s + 2s)} \begin{bmatrix} \phi_1(y, s) \\ \phi_2(y, s) \end{bmatrix} [-\sin 2s, 1 - \cos 2s] \\ - \frac{1}{2s(\sin 2s)(\sin 2s - 2s)} \begin{bmatrix} \hat{\phi}_1(y, s) \\ \hat{\phi}_2(y, s) \end{bmatrix} [\sin 2s, 1 + \cos 2s] \\ + \frac{\sin s(y + 1)}{2s \sin 2s} \begin{bmatrix} -1 \\ 1 \end{bmatrix} [0, 1].$$

The next step is to postmultiply these forms by \mathbf{Y} matrices. We proceed by multiplying the row vectors of the product \mathbf{XW}^{-1} into \mathbf{Y} and manipulating that row before multiplying by the column components. The computation brings out the even and odd adjoint eigenfunctions $\psi(y, s)$ and $\hat{\psi}(y, s)$ after a tedious computation which can be organized by paying attention to the (α, β) combination in the products.

The results are

$$(4.15)_1 \quad \mathbf{X}_1(sy) \mathbf{W}_{1,2}^{-1}(2s) \mathbf{Y}_2(su) = \\ - \frac{1}{2s(\cos^2 s)(\sin 2s + 2s)} \begin{bmatrix} \phi_1(y, s) \\ \phi_2(y, s) \end{bmatrix} [\psi_1(u, s), \psi_2(u, s)] \\ + \frac{1}{2s(\sin^2 s)(\sin 2s - 2s)} \begin{bmatrix} \hat{\phi}_1(y, s) \\ \hat{\phi}_2(y, s) \end{bmatrix} [\hat{\psi}_1(u, s), \hat{\psi}_2(u, s)] \\ + \frac{\sin s(u + 1)}{4s} \left(\frac{1}{\cos^2 s} \varphi(y, s) + \frac{1}{\sin^2 s} \hat{\varphi}(y, s) \right) [1, 1] \\ - \frac{\sin s(y - 1)}{2s \sin 2s} \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix} \mathbf{Y}_2(su),$$

$$(4.15)_2 \quad \mathbf{X}_2(sy) \mathbf{W}_{2,1}^{-1}(2s) \mathbf{Y}_1(su) = \\ + \frac{1}{2s(\cos^2 s)(\sin 2s + 2s)} \begin{bmatrix} \phi_1(y, s) \\ \phi_2(y, s) \end{bmatrix} [\psi_1(u, s), \psi_2(u, s)] \\ - \frac{1}{2s(\sin^2 s)(\sin 2s - 2s)} \begin{bmatrix} \hat{\phi}_1(y, s) \\ \hat{\phi}_2(y, s) \end{bmatrix} [\hat{\psi}_1(u, s), \hat{\psi}_2(u, s)] \\ + \frac{\sin s(u - 1)}{4s} \left(\frac{1}{\cos^2 s} \varphi(y, s) - \frac{1}{\sin^2 s} \hat{\varphi}(y, s) \right) [1, 1] \\ + \frac{\sin s(y + 1)}{2s \sin 2s} \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix} \mathbf{Y}_1(su).$$

To obtain (4.10) we substitute (4.15) into (4.4).

5. Computation of Eigenvectors as Residues at Poles of the Resolvent

In this section we show that

$$(5.1) \quad \lim_{N \rightarrow \infty} I_N(f) = \sum_{-\infty}^{\infty} \{c_n \varphi^{(n)}(y) + \hat{c}_n \hat{\varphi}^{(n)}(y)\}$$

by evaluating $I_N(f)$ at the poles of $sm(y, s)$. Since $\mathbf{X}(sy)$ and $\mathbf{Y}(sy)$ are entire functions, equation (4.4) shows that the poles of $sm(y, s)$ can only occur at zeros of the inverse Wronskians; that is, at the zeros of

$$2s(-4s^2 + \sin^2 2s) = 2s(2s + \sin 2s)(-2s + \sin 2s).$$

The second form of $m(y, s)$ given by (4.10) superficially suggests that $sm(y, s)$ has poles on the real axis at the zeros of $\sin s$, $\cos s$ and $\sin 2s$. These poles are apparent but not real; they have zero residue.

Integrals around the square contour D_N with vertices at $[\pm 2N\pi, \pm 2N\pi]$ do not pass through poles of $sm(y, s)$ (see SMITH, 1952). They do pick up contributions from the residue at the pole at the origin and from the $8N$ poles s_n and \hat{s}_n inside the contour D_N . There are N poles \hat{s}_n satisfying $\sin 2s + 2s = 0$ and N poles s_n satisfying $\sin 2s - 2s = 0$ in the first quadrant of the complex s -plane, and there are $2N$ poles in each of three other quadrants. The function $sQ(y, s)$ has no residue at $s = 0$ and $sm(y, s)$ has no residues at zeros of $\sin 2s$. The residues in the first and fourth quadrants split into even eigenfunctions at s_n satisfying $\sin 2s_n + 2s_n = 0$:

$$(5.2) \quad \lim_{s \rightarrow s_n} (s - s_n) sm(y, s) = -\frac{c_n k_n}{8 \cos^4 s_n} \varphi(y, s_n) = \frac{1}{2} c_n \varphi^{(e)}(y),$$

and odd eigenfunctions at \hat{s}_n satisfying $\sin 2\hat{s}_n - 2\hat{s}_n = 0$:

$$(5.3) \quad \lim_{s \rightarrow \hat{s}_n} (s - \hat{s}_n) sm(y, s) = -\frac{\hat{c}_n \hat{k}_n}{8 \sin^4 \hat{s}_n} \hat{\varphi}(y, \hat{s}_n) = \frac{1}{2} \hat{c}_n \hat{\varphi}^{(o)}(y),$$

where c_n and \hat{c}_n are the coefficients in the expansion formula (3.7).

To compute the residue of $sm(y, s)$ at $s = 0$ we make use of the following formulas:

$$\varphi(s, y) = 2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + O(s^2) = k_0 \varphi^{(0)} + O(s^2),$$

$$\psi(s, y) = -2[1, 0] + O(s^2) = k_0 \psi^{(0)} + O(s^2),$$

$$\hat{\varphi}(s, y) = 2s^2 \begin{bmatrix} 0 \\ y \end{bmatrix} + O(s^4) = 2s^2 \hat{\varphi}^{(0)} + O(s^4),$$

$$\tilde{\psi}(s, y) = 2s[y, 0] + O(s^4) = 2s^2 \tilde{\psi}^{(0)} + O(s^4),$$

$$\lim_{s \rightarrow 0} \frac{s}{\sin 2s + 2s} = \frac{1}{4},$$

$$\lim_{s \rightarrow 0} \frac{\sin s(y \pm 1)}{\sin 2s} = \frac{1}{2} (y \pm 1),$$

$$\frac{s}{2 \sin^2 s (\sin 2s - 2s)} \rightarrow -\frac{3}{8s^4} \text{ as } s \rightarrow 0.$$

Using these formulas, we find that $sQ(y, s)$ has a zero residue at $s = 0$ and

$$(5.4) \quad \begin{aligned} \lim_{s \rightarrow 0} s^2 m(y, s) &= \frac{1}{2} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \int_{-1}^1 f_2(u) du \\ &+ \left(-\frac{3}{8s^4}\right) \left\{ (2s^2) \begin{bmatrix} 0 \\ y \end{bmatrix} (-2s^2) \int_{-1}^1 ug(u) du \right\} \\ &= c_0 \varphi^{(0)}(y) + \hat{c}_0 \hat{\varphi}^{(0)}(y). \end{aligned}$$

Equations (5.2), (5.3) and (5.4) taken together prove (5.1).

We note that the eigenvalues s_n and \hat{s}_n are distinct when $n \neq 0$ but $s_0 = \hat{s}_0 = 0$ is a double, semi-simple eigenvalue.

6. Computation of the Data Vector Plus a Remainder from Another Form of the Resolvent

Now we shall show that

$$(6.1) \quad \begin{aligned} sm(y, s) &= (f(y)/s) - sG(y, s) + sP(y, s), \\ sG(y, s) &= \frac{1}{s} X_1(sy) W_{1,2}^{-1} \int_{-1}^y Y_2(su) f'(u) du \\ &\quad - \frac{1}{s} X_2(sy) W_{2,1}^{-1} \int_y^1 Y_1(su) f''(u) du, \\ sP(y, s) &= \frac{2}{s} X_2(sy) W_{2,1}^{-1} \begin{bmatrix} -f'_1(1) \\ sf_1(1) \end{bmatrix} \\ &\quad + \frac{2}{s} X_1(sy) W_{1,2}^{-1} \begin{bmatrix} -f'_1(-1) \\ sf_1(-1) \end{bmatrix}. \end{aligned}$$

Given (6.1), it follows that

$$(6.2) \quad \begin{aligned} I_N(f) &\stackrel{\text{def}}{=} \frac{1}{2\pi i} \oint_{D_N} sm(y, s) ds \\ &= f(y) + \frac{1}{2\pi i} \oint_{D_N} sG(y, s) ds + \frac{1}{2\pi i} \oint_{D_N} sP(y, s) ds. \end{aligned}$$

The last two integrals in (6.2) are the remainder \mathcal{R}_N defined in (4.3).

To derive (6.1) we set $Y_j A = -Y'_j/s^2$ ($j = 1, 2$) in (4.4) and integrate twice by parts using the boundary conditions satisfied by Y_1 and Y_2 . We then note, using (4.8) and (4.9), that

$$(6.3) \quad \begin{aligned} X_1 F'_1 + X_2 F'_2 &= (X_1 W_{1,2}^{-1} Y_2 + X_2 W_{2,1}^{-1} Y_1) Af = 0, \\ X'_1 F'_1 + X'_2 F'_2 &= (X'_1 W_{1,2}^{-1} Y_2 + X'_2 W_{2,1}^{-1} Y_1) Af = Af \end{aligned}$$

and, since (6.3) holds for all f ,

$$(6.4) \quad \begin{aligned} X_1 W_{1,2}^{-1} Y_2 + X_2 W_{2,1}^{-1} Y_1 &= 0, \\ X'_1 W_{1,2}^{-1} Y_2 + X'_2 W_{2,1}^{-1} Y_1 &= 1. \end{aligned}$$

Equations (6.4) may also be derived by direct computation and they imply that

$$X_1 W_{1,2}^{-1} Y'_2 + X_2 W_{2,1}^{-1} Y'_1 = -1.$$

Now we give an explicit and convenient form for the integrands appearing in the definition of the remainder \mathcal{R}_N . A useful form of $sG(y, s)$ can be obtained

from (4.10). Comparing (6.1)₂ with (4.4) we find that the form of $sG(y, s)$ can be obtained from that for $m(y, s)$ by replacing the vector $Af(u)$ in the two integrals by the vector $\frac{1}{s}f''(u)$. Since equation (4.10) gives an alternative form for $m(y, s)$, we can obtain a corresponding form for $sG(y, s)$ by replacing $g = Af$ by f''/s in (4.10), (4.11):

$$(6.5) \quad sG(y, s) = \frac{\phi(y, s) \int_{-1}^1 \psi(u, s) \cdot f''(u) du}{2s^2 \cos^2 s(\sin 2s + 2s)} \\ + \frac{\hat{\phi}(y, s) \int_{-1}^1 \hat{\psi}(u, s) \cdot f'' du}{2s^2 \sin^2 s(\sin 2s - 2s)} + \frac{1}{s} Q(y, s)$$

and, since $Q(1, s) = 0$,

$$(6.6) \quad sG(1, s) = \frac{1}{s^2(\sin 2s + 2s)} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \int_{-1}^1 \psi(u, s) \cdot f''(u) du \\ + \frac{1}{s^2(\sin 2s - 2s)} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \int_{-1}^1 \hat{\psi}(u, s) \cdot f''(u) du.$$

It is convenient to decompose the expression (6.1)₃ for $sP(y, s)$ into even and odd parts. From (3.19) we have

$$f_1(1) = f_e + f_0, \quad f'_1(1) = f'_e + f'_0, \\ f_1(-1) = f_e - f_0, \quad f'_1(-1) = -f'_e + f'_0$$

where $f_e = f^e(1)$, $f_0 = f^0(1)$, $f'_e = f'^e(1)$, $f'_0 = f'^0(1)$. Then (6.1)₃ may be written as

$$sP(y, s) = 2[X_2W_{2,1}^{-1} + X_1W_{1,2}^{-1}] \begin{bmatrix} -f'_0/s \\ f_e \end{bmatrix} + 2[X_2W_{2,1}^{-1} - X_1W_{1,2}^{-1}] \begin{bmatrix} -f'_e/s \\ f_0 \end{bmatrix}.$$

The matrices multiplying the data vectors may be reexpressed using (4.14):

$$2[X_2W_{2,1}^{-1} + X_1W_{1,2}^{-1}] = \{4f(1 - \cos 2s) + 2(l - \bar{l})\} [0, 1] - 4\hat{f} \sin 2s [1, 0],$$

$$2[X_2W_{2,1}^{-1} - X_1W_{1,2}^{-1}] = \{-4\hat{f}(1 + \cos 2s) + 2(l + \bar{l})\} [0, 1] - 4f \sin 2s [1, 0]$$

where

$$f = \frac{1}{2s(\sin 2s)(2s + \sin 2s)} \begin{bmatrix} \phi_1(y, s) \\ \phi_2(y, s) \end{bmatrix}, \\ \hat{f} = \frac{1}{2s(\sin 2s)(\sin 2s - 2s)} \begin{bmatrix} \hat{\phi}_1(y, s) \\ \hat{\phi}_2(y, s) \end{bmatrix}, \\ l = \frac{\sin s(y+1)}{2s \sin 2s} \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \\ \bar{l} = \frac{\sin s(y-1)}{2s \sin 2s} \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Hence

$$sP(y, s) = 4f'_e \{f(1 - \cos 2s) + \frac{1}{2}(l - \bar{l})\} + \frac{4}{s} f'_0 \sin 2s \hat{f} \\ - 4f_0 \{\hat{f}(1 + \cos 2s) - \frac{1}{2}(l + \bar{l})\} + \frac{4}{s} f'_e \sin 2s f.$$

That is,

$$(6.7) \quad sP(y, s) = \frac{f_e J_e(y, s)}{s(\sin 2s + 2s)} + \frac{2f'_0}{s^2(\sin 2s - 2s)} \begin{bmatrix} \hat{\phi}_1(y, s) \\ \hat{\phi}_2(y, s) \end{bmatrix} \\ + \frac{f_0 J_0(y, s)}{s(\sin 2s - 2s)} + \frac{2f'_e}{s^2(\sin 2s + 2s)} \begin{bmatrix} \phi_1(y, s) \\ \phi_2(y, s) \end{bmatrix}$$

where

$$J_e = \begin{bmatrix} a_1 - a_2 - a_3 + a_4 \\ -a_1 + a_2 - a_3 + a_4 \end{bmatrix} \\ = 2 \begin{bmatrix} -s(y \sin sy \sin s + \cos sy \cos s) - \sin s \cos sy \\ s(y \sin sy \sin s + \cos sy \cos s) - \sin s \cos sy \end{bmatrix}, \\ J_0 = \begin{bmatrix} a_1 + a_2 - a_3 - a_4 \\ -a_1 - a_2 - a_3 - a_4 \end{bmatrix} \\ = 2 \begin{bmatrix} s(y \cos sy \cos s + \sin sy \sin s) - \sin sy \cos s \\ -s(y \cos sy \cos s + \sin sy \sin s) - \sin sy \cos s \end{bmatrix}, \\ a_1 = s(y-1) \cos s(y+1), \quad a_2 = s(y+1) \cos s(y-1), \\ a_3 = \sin s(y+1), \quad a_4 = \sin s(y-1).$$

Finally we note that

$$(6.8) \quad sP(\pm 1, s) = f_e p_1(s) + f'_e p_2(s) \pm f_0 p_3(s) \pm f'_0 p_4(s)$$

where

$$(6.9) \quad p_1(s) = \frac{1}{s} \begin{bmatrix} -1 \\ 2s - \sin 2s \\ \sin 2s + 2s \end{bmatrix} = -\frac{1}{s} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{4}{\sin 2s + 2s} \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \\ p_2(s) = \frac{-4 \cos^2 s}{s^2(\sin 2s + 2s)} \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \\ p_3(s) = \frac{1}{s} \begin{bmatrix} -1 \\ -2s - \sin 2s \\ \sin 2s - 2s \end{bmatrix} = -\frac{1}{s} \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{4}{\sin 2s - 2s} \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \\ p_4(s) = \frac{4 \sin^2 s}{s^2(\sin 2s - 2s)} \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

7. Theorem of Convergence

We now show that $\mathcal{R}_N \rightarrow 0$ as $N \rightarrow \infty$ when $-1 < y < 1$ and we find the values assumed by \mathcal{R}_N on $y = \pm 1$. Since all of the functions of y defined in § 6 are either even or odd, it will suffice to study \mathcal{R}_N when $0 \leq y \leq 1$. To simplify our study of the contour integrals on the square contour \mathbb{D}_N of the complex $s = \sigma + i\tau$ plane (Fig. 4.1), we use the following computational

Lemma 7.1. *Let*

$$(7.1) \quad \Delta(s, y) = \Omega(\sigma, \tau, y) + i\chi(\sigma, \tau, y)$$

be an even function of s when $0 \leq y \leq 1$. Let \mathbb{D}_N be the square in the $s = \sigma + i\tau$ plane with vertices $(\pm\xi, \pm i\xi)$, $\xi = 2\pi N$. Then

$$(7.2) \quad \oint_{\mathbb{D}_N} \Delta(s, y) ds = i \left\{ \int_0^\xi \Omega(\xi, \tau, y) d\tau - \int_0^\xi \chi(\sigma, \xi, y) d\sigma \right\}.$$

The proof of this lemma is straightforward and is omitted.

Lemma 7.2 (SMITH, 1952). *Suppose $f''(y) \in BV$ for $0 \leq y \leq 1$. Then*

$$(7.3)_1 \quad \lim_{N \rightarrow \infty} \frac{1}{2\pi i} \oint_{\mathbb{D}_N} sG(y, s) ds \rightarrow 0$$

exponentially with the asymptotic order

$$(7.3)_2 \quad O(e^{-2\pi N(1-y)}).$$

SMITH'S proof does not cover the case in which $y = 1$. We get fast convergence at $y = 1$ (see lemmas 7.5 and 7.6).

Now we are going to prove the most delicate result of this paper. In fact this result implies all that we need for our theorem of convergence.

Lemma 7.3. *Let $0 \leq y < 1$. Then*

$$(7.4)_1 \quad \lim_{N \rightarrow \infty} \frac{1}{2\pi i} \oint_{\mathbb{D}_N} \frac{\sin sy \sin s}{\sin 2s - 2s} ds \rightarrow 0$$

with asymptotic order

$$(7.4)_2 \quad O\left(\frac{\log 2\pi N}{(2\pi N)^{(1-y)/2}}\right).$$

We note that the integral in (7.4)₁ does not tend to zero when $y = 1$ and when y is near to one it converges very slowly. The slow convergence here is associated with Gibbs phenomenon in the series which are generated as residues of (7.4)₁ at the zeros of $\sin 2s - 2s$.

Proof. The $\Delta(s, y)$ of (7.2) is given here by

$$(7.5) \quad \begin{aligned} \Delta(s, y) &= \frac{\sin sy \sin s}{\sin 2s - 2s} = \Omega(\sigma, \tau, y) + i\chi(\sigma, \tau, y), \\ \Omega(\sigma, \tau, y) &= \alpha(\sigma, \tau, y)/|\gamma(\sigma, \tau)|^2, \\ \chi(\sigma, \tau, y) &= \beta(\sigma, \tau, y)/|\gamma(\sigma, \tau)|^2, \\ |\gamma(\sigma, \tau)|^2 &= |\sin 2s - 2s|^2 = (\sin 2\sigma \cosh 2\tau - 2\sigma)^2 \\ &\quad + (\cos 2\sigma \sinh 2\tau - 2\tau)^2, \\ \alpha(\sigma, \tau, y) &= \mu(\sigma, \tau, y) [\sin 2\sigma \cosh 2\tau - 2\sigma] \\ &\quad + \nu(\sigma, \tau, y) [\cos 2\sigma \sinh 2\tau - 2\tau], \\ \beta(\sigma, \tau, y) &= \nu(\sigma, \tau, y) [\sin 2\sigma \cosh 2\tau - 2\sigma] \\ &\quad - \mu(\sigma, \tau, y) [\cos 2\sigma \sinh 2\tau - 2\tau], \\ \mu(\sigma, \tau, y) &= \sin \sigma \cosh \tau \sin \sigma y \cosh \tau y \\ &\quad - \cos \sigma \sinh \tau \cos \sigma y \sinh \tau y, \\ \nu(\sigma, \tau, y) &= \sin \sigma \cosh \tau \cos \sigma y \sinh \tau y \\ &\quad + \cos \sigma \sinh \tau \sin \sigma y \cosh \tau y. \end{aligned}$$

We first show that the integral of $\chi(\sigma, \xi, y)$ on the horizontal sides of \mathbb{D}_N tends to zero exponentially. For this part of the demonstration we note that

$$(7.6) \quad \begin{aligned} \int_0^\xi \chi(\sigma, \xi, y) d\sigma &\leq \xi \text{Max}_{0 \leq \sigma \leq \xi} |\chi(\sigma, \xi, y)|, \\ |\nu(\sigma, \xi, y)| &< e^{\xi(1+y)}, \quad |\mu(\sigma, \xi, y)| < e^{\xi(1+y)}, \quad |\beta(\sigma, \xi, y)| < e^{\xi(3+y)}, \\ |\gamma(\sigma, \xi)|^2 &> (\sinh 2\xi - 2\xi)^2 - 4\xi \cosh 2\xi > \frac{e^{4\xi}}{4} (1 - 16\xi e^{-2\xi}). \end{aligned}$$

The last expression is positive when $\xi > 1.64$. Therefore

$$(7.7) \quad \xi \text{Max} |\chi| < \frac{4\xi e^{\xi(y-1)}}{1 - 16\xi e^{-2\xi}}, \quad \xi > 1.64.$$

It follows that the integral on the left of (7.6) with $\xi = 2\pi N$ tends to zero as $N \rightarrow \infty$ with asymptotic order $Ne^{2\pi N(y-1)}$, $0 \leq y < 1$.

The delicate part of the proof of lemma 7.3 is associated with the fact that the path for the line integral

$$(7.8) \quad \int_0^\xi \Omega(\xi, \tau, y) d\tau = \int_0^\xi \frac{\sin \xi y \sinh \tau \cosh \tau y (\sinh 2\tau - 2\tau)}{4\xi^2 + (\sinh 2\tau - 2\tau)^2} d\tau$$

on the vertical side of \mathbb{D}_N passes through the sector of the complex s -plane containing the eigenvalues s_n and \hat{s}_n ; that is, the sector in the first quadrant con-

taining the roots of $(\sin^2 2s - 4s^2) = 0$. The path on $\sigma = \xi = 2\pi N$ passes between the real parts $(n - 1/4)\pi$ and $(n + 1/4)\pi$ of s_n and \bar{s}_n . The eigenvalue band lies in a "sector" whose intersection with the line $\sigma = \xi$ is an interval $(\frac{1}{2} \log \xi - \pi/2\xi, \frac{1}{2} \log \xi + \pi/2\xi)$ defined by the imaginary parts of s_n and \bar{s}_n (see (3.31)).

Since

$$(7.9) \quad |\Omega(\xi, \tau, y)| < \frac{e^{(3+y)\tau}}{4\xi^2 + (\sinh 2\tau - 2\tau)^2} \stackrel{\text{def}}{=} \Theta(\xi, \tau, y),$$

we have

$$(7.10) \quad \int_0^{\xi} \Omega(\xi, \tau, y) d\tau < \int_0^{\frac{1}{2} \log \xi} \Theta(\xi, \tau, y) d\tau + \int_{\frac{1}{2} \log \xi}^{\log \xi} \Theta(\xi, \tau, y) d\tau + \int_{\log \xi}^{\xi} \Theta(\xi, \tau, y) d\tau.$$

The first integral on the right of (7.10) clearly converges to zero because

$$(7.11) \quad \int_0^{\frac{1}{2} \log \xi} \Theta(\xi, \tau, y) d\tau \leq \int_0^{\frac{1}{2} \log \xi} \frac{e^{(3+y)\tau}}{4\xi^2} d\tau = \frac{1}{4(\xi + \varphi)} \left[\frac{1}{\xi^{(1-y)/2}} - \frac{1}{\xi^2} \right], \quad 0 \leq y < 1, \quad \xi = 2\pi N.$$

To obtain the asymptotic order of the other two integrals on the right of (7.10) we define a monotonically increasing function

$$(7.12) \quad M(\tau) \stackrel{\text{def}}{=} \left(\frac{\sinh 2\tau - 2\tau}{\sinh 2\tau} \right)^2$$

for $0 \leq \tau < \infty$, $M(0) = 0$, $M(\infty) = 1$. Clearly

$$(7.13) \quad M^* = M(\frac{1}{2} \log \xi) < M(\tau) \text{ if } \frac{1}{2} \log \xi < \tau.$$

Moreover, since $\xi = 2\pi N \rightarrow \infty$, $M^* \rightarrow 1$. It follows that

$$(7.14) \quad (\sinh 2\tau - 2\tau)^2 > M^* \sinh^2 2\tau > M^* e^{4\tau}/4 - M^*/2$$

when $\tau > \frac{1}{2} \log \xi$ and, since $4\xi^2 - M^*/2 > 0$,

$$(7.15) \quad 4\xi^2 + (\sinh 2\tau - 2\tau)^2 > M^* e^{4\tau}/4.$$

We may use (7.15) to majorize the last integral of (7.10). Thus

$$(7.16) \quad \int_{\log \xi}^{\xi} \Theta(\xi, \tau, y) d\tau < 4 \int_{\log \xi}^{\xi} \frac{e^{(3+y)\tau}}{M^* e^{4\tau}} d\tau = \frac{4e^{5(y-1)}}{M^*(y-1)} - \frac{4\xi^{y-1}}{M^*(y-1)}.$$

For the second integral on the right of (7.10) we note that

$$(7.17) \quad \int_{\frac{1}{2} \log \xi}^{\log \xi} \Theta(\xi, \tau, y) d\tau < \int_{\frac{1}{2} \log \xi}^{\log \xi} \frac{4e^{(3+y)\tau}}{M^* e^{4\tau} + 16\xi^2 - 2M^*} d\tau \leq 2 \log \xi \max_{0 \leq \tau < \infty} \left(\frac{e^{(3+y)\tau}}{16\xi^2 - 2M^* + M^* e^{4\tau}} \right).$$

This maximum occurs at the point

$$\tau = \frac{1}{4} \log \left(\frac{(3+y)(16\xi^2 - 2M^*)}{M^*(1-y)} \right).$$

Hence, the inequality (7.17) may be continued as

$$\leq \frac{2}{M^*} \log \xi \left(\frac{(3+y)(16\xi^2 - 2M^*)}{(1-y)M^*} \right)^{\frac{y-1}{4}} \leq K(y) \xi^{\frac{y-1}{2}} \log \xi,$$

where

$$K(y) = 2^y [M(\frac{1}{2} \log 2\pi)]^{-\frac{(3+y)}{4}} \left[\left(\frac{32\pi^2}{32\pi^2 - 1} \right) \left(\frac{1-y}{3+y} \right) \right]^{\frac{(1-y)}{4}}.$$

This inequality gives the asymptotic order asserted in (7.4)₂ and it establishes lemma 7.3.

Now we define integrands of the type

$$(7.18) \quad \Delta_0(s, y) \stackrel{\text{def}}{=} h(s, y)/(\sin 2s \pm 2s)$$

where $h(s, y)$ is a product of $\sin s$ or $\cos s$ with $\sin sy$ or $\cos sy$.

Lemma 7.4. *Lemma 7.3 holds for all integrands of the type (7.18).*

The proof of this lemma is exactly the same as the one we just gave for lemma 7.3.

We next define integrands of the type

$$(7.19) \quad \Delta_l(s, y) \stackrel{\text{def}}{=} \frac{1}{s^l} \frac{h(s, y)}{\sin 2s \pm 2s}$$

and note that on \mathbb{D}_N

$$\frac{1}{|s|} \geq \frac{1}{\xi}, \quad \xi = 2\pi N.$$

Hence, using our previous computation, we find

Lemma 7.5. *Let $0 \leq y < 1$. Then*

$$(7.20)_1 \quad \lim_{N \rightarrow \infty} \frac{1}{2\pi i} \oint_{\mathbb{D}_N} \Delta_l(s, y) ds \rightarrow 0, \quad l = 0, 1, 2, \dots$$

with asymptotic order

$$(7.20)_2 \quad O \left(\frac{\log 2\pi N}{(2\pi N)^{(2l+1-y)/2}} \right).$$

If $l \geq 1$ the integral (7.20)₁ converges to zero when $y = 1$ with asymptotic order given by (7.20)₂ with $y = 1$.

Lemma 7.6. Let $f''(y) \in BV$. Then $sG(1, s)$ is an integrand of the type (7.19) with $l = 2$.

The proof of lemma 7.6 is straightforward. We split (6.6) into real and imaginary parts following the recipe given in lemma 7.1. The integrands of (7.2) so defined are real functions of a real variable and, if $f''(\cdot) \in BV$, we may use the second mean value theorem as in (2.23), (2.24) to reduce (6.6) to an integrand of the type (7.19). We get an s in the numerator from ψ and $\bar{\psi}$ and an s in the denominator from integrating after using the second mean value theorem. This shows that $l = 2$. Of course, we get exactly the same result for $sG(-1, s)$.

Lemma 7.7. Let $-1 < y < 1$. Then $sP(y, s)$ is an integrand of the type (7.18) and

$$\lim_{N \rightarrow \infty} \oint_{\mathbb{D}_N} sP(y, s) ds \rightarrow 0.$$

To evaluate \mathcal{R}_N as $N \rightarrow \infty$ at $y = \pm 1$, we need to evaluate contour integrals which do not depend on y . Lemma 7.6 treats an integral of this type and the expression (6.8), which we evaluate below, leads to other integrals of the same type.

Lemma 7.8. $p_2(s)$ and $p_4(s)$ are integrands of the type (7.19) with $y = 1$ and $l = 2$.

$$\lim_{N \rightarrow \infty} \oint_{\mathbb{D}_N} p_j(s) ds \rightarrow 0, \quad j = 2 \text{ or } 4.$$

Lemma 7.9.
$$\lim_{N \rightarrow \infty} \oint_{\mathbb{D}_N} \frac{ds}{\sin 2s \pm 2s} \rightarrow 0$$

with asymptotic order

$$O\left(\frac{\log 2\pi N}{2\pi N}\right).$$

The proof of lemma 7.9, using the minus sign, leads us to evaluate $\Theta(\xi, \tau, y)$ given by (7.9) with $y = -1$. This evaluation gives the estimate (7.4)₂ with $y = -1$.

Lemma 7.10.

$$(7.21) \quad \lim_{N \rightarrow \infty} \frac{1}{2\pi i} \oint_{\mathbb{D}_N} sP(\pm 1, s) ds = -f_e \begin{bmatrix} 1 \\ 1 \end{bmatrix} \pm f_o \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Lemma 7.10 follows directly from lemmas 7.8 and 7.9.

Remark. The contour integrals studied in this section lead to series representations of numbers in terms of functions of s_n and \hat{s}_n . For example, using the method of

residues we may obtain the following results from integrals associated with (6.9):

$$\begin{aligned} -1 &= \sum_{-\infty}^{\infty} \frac{1}{\cos^2 s_n}, \\ -\frac{3}{10} &= \sum_{-\infty}^{\infty} \frac{1}{\sin^2 s_n}, \\ \frac{2}{10} &= \sum_{-\infty}^{\infty} \frac{1}{s_n^2} \end{aligned}$$

to name a few. All such results can be obtained directly and more easily from our theorem 7.1.

Lemma 7.11. Suppose that $\langle \psi F \rangle = O(h(|s|))$ when $F \in BV$. Then $\langle \psi F \rangle = O(|s| h(|s|))$ when $F \in L_2(-1, 1)$.

Proof. The proof depends on the fact that the L_2 -norm of ψ and the sup-norm of an integral of ψ have the large $|s|$ behavior stated, independently of the behavior of F . From equation (3.15), we have

$$\psi = \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} = \phi_1(y, s) \begin{bmatrix} 1 \\ 1 \end{bmatrix} - 2 \cos s \cos sy \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Recombining the trigonometric functions, we can write

$$\begin{aligned} \psi_2 &= \phi_1 = \frac{1}{2} [s(1-y) \sin s(1+y) + s(1+y) \sin s(1-y)], \\ \psi_1 &= \phi_1 - [\cos s(1+y) + \cos s(1-y)]. \end{aligned}$$

Each term in these expressions is certainly

$$O(|s| \sqrt{1 + \sinh^2 2|\tau|}), \quad s = \sigma + i\tau,$$

independently of y ; some terms do not grow this fast, of course. Thus

$$\|\psi\|_{L_2} = \left(\int_{-1}^1 |\psi|^2 dy \right)^{\frac{1}{2}} \leq K_1 |s| \sqrt{1 + \sinh^2 2|\tau|}.$$

Introduce $\Psi(y, s) = s \int_0^y \psi(y, s) dy$;

$$\Psi = \begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix} = \Psi_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} - 2 \cos s \sin sy \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$\Psi_2 = s \int_0^y \phi_1 dy = s \sin s \sin sy + sy \cos s \cos sy - \cos s \sin sy.$$

Terms in Ψ have the same behavior as terms in ψ , so

$$\sup_y |\Psi(y, s)| \leq K_2 |s| \sqrt{1 + \sinh^2 2|\tau|}.$$

Now calculate $\langle \psi F \rangle$ when $F \in BV$. Do this by representing each of the two components of F as the difference of two vector functions $\lambda(y)$, $\mu(y)$ whose components are nondecreasing and bounded. Separate ψ into real and imaginary parts. At most eight scalar integrals must be examined, each of which can be treated by the Second Mean Value Theorem. A typical result is

$$\begin{aligned} \int_{-1}^1 \lambda_2(y) \operatorname{Re} \psi_2(y, s) dy &= \lambda_2(1^-) \int_{y_2}^1 \operatorname{Re} \psi_2(y, s) dy \\ &= \lambda_2(1^-) \operatorname{Re} \left(\int_{y_2}^1 \frac{1}{s} \frac{\partial \Psi_2}{\partial y} dy \right) \\ &= \lambda_2(1^-) \operatorname{Re} \left(\frac{1}{s} \Psi_2(1, s) - \frac{1}{s} \Psi_2(y_2, s) \right), \quad -1 \leq y_2. \end{aligned}$$

Since there are two components of Ψ and eight integrals, then certainly at worst

$$|\langle \psi F \rangle| \leq \frac{16}{|s|} \operatorname{Max} (|\lambda_i(1)|, |\mu_i(-1)|) \operatorname{Max}_{-1 \leq y \leq 1} |\Psi|.$$

Now suppose $F \in L_2(-1, 1)$; then

$$|\langle \psi F \rangle| \leq \|\psi\|_{L_2} \|F\|_{L_2}.$$

We now see that the dependence on $|s|$ is that stated in the lemma, since $\operatorname{Max} |\Psi|$ and $\|\psi\|_{L_2}$ have at worst the same $|s|$ -behavior and the other factors involve F and not s .

Lemma 7.12. $\frac{1}{2\pi i} \oint_{\mathfrak{D}_N} sG(1, s) ds \rightarrow 0$ as $N \rightarrow \infty$ when $f'' \in L_2(-1, 1)$.

Proof. By lemma 7.6, $sG(1, s)$ is an integrand of type (7.19) with $l = 2$ when $f'' \in BV$. By lemma 7.11, $sG(1, s)$ is thus an integrand of type (7.19) with $l = 1$ when $f'' \in L_2$. Therefore, by the last sentence of lemma 7.5, $\frac{1}{2\pi i} \oint_{\mathfrak{D}_N} sG(1, s) ds \rightarrow 0$.

Theorem 7.1. Let $f(y) \in C^1(-1, 1)$ and $f''(y) \in L_2(-1, 1)$. Define

$$(7.22) \quad S(y) = c_0 \phi^{(0)}(y) + \hat{c}_0 \hat{\phi}^{(0)}(y) + \sum_{-\infty}^{\infty} \{c_n \phi^{(n)}(y) + \hat{c}_n \hat{\phi}^{(n)}(y)\},$$

where the constants c_n , \hat{c}_n are given by (3.14) for $n = 0, \pm 1, \pm 2, \dots$. Then

$$(7.23) \quad S(y) = \begin{cases} f(y), & -1 < y < 1 \\ \left[\begin{matrix} 0 \\ f_2(\pm 1) - f_1(\pm 1) \end{matrix} \right], & y = \pm 1. \end{cases}$$

Proof. Suppose first that $f'' \in BV$. Then equation (5.1) implies that

$$\begin{aligned} \lim_{N \rightarrow \infty} I_N(f) &= c_0 \phi^{(0)}(y) + \hat{c}_0 \hat{\phi}^{(0)} + \sum_{-\infty}^{\infty} \{c_n \phi^{(n)}(y) + \hat{c}_n \hat{\phi}^{(n)}(y)\} \\ &= S(y) \end{aligned}$$

where, by (6.2),

$$I_N(f) = f(y) + \frac{1}{2\pi i} \left\{ \oint_{\mathfrak{D}_N} [sG(y, s) + sP(y, s)] ds \right\}$$

and, by (7.3)₁ and lemma 7.7,

$$\lim_{N \rightarrow \infty} I_N(f) = f(y).$$

To show that $f'' \in L_2(-1, 1)$ is sufficient, use lemma 7.11 and SMITH's result (lemma 7.2) to prove that the integral of $sG(y, s)$ still goes to zero in the limit.

For the second part of the theorem, we set $y = \pm 1$ in (5.1), (6.2) and use lemmas (7.6), (7.10), (7.12).

SMITH (1952) proved theorem 7.1 with $c_0 = \hat{c}_0 = 0$ and unnatural conditions for f and f' at ± 1 . SMITH's proof works well when the unnatural conditions $f_2 = f'_2 = 0$ at $y = \pm 1$ are relaxed (JOSEPH, 1979). GREGORY (1980) proved theorem (7.1) without restriction on f_2, f'_2 or on f_1 and f'_1 at $y = \pm 1$ by a different method. SPENCE (1980) proves a similar result as noted in Section 2. The present proof goes "all the way" with SMITH's extension of the method of TITCHMARSH.

8. Gibbs Phenomenon and Cesaro Sums for the Step and Ramp Function Vectors at an End Point of Discontinuity

The Gibbs phenomenon deals with "overshoot" in the partial sums of certain Fourier series. For example, consider the expansion of the periodic extension of the ramp function

$$(8.1) \quad \frac{1}{2} y = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin ny}{n} \quad (-\pi < y < \pi).$$

The series on the right of (8.1) converges slowly, conditionally and not uniformly. At the point $y = k\pi$ of discontinuity the series converges to the mean value 0 of the function $f(y)$ which periodically extends $\frac{1}{2} y$ on $-\pi < y < \pi$ to \mathbb{R} . The Gibbs phenomenon refers to a property of non-uniformity in the convergence of the partial sums

$$(8.2) \quad S_N(y) = \sum_{n=1}^N (-1)^{n+1} \frac{\sin ny}{n}$$

near the points $k\pi$ of discontinuity. For each integer N , the partial sum $S_N(y)$ oscillates around the function $y/2$ many times. The overshoot and undershoot are largest near the point of discontinuity $y = \pm\pi$. As N is increased, the oscilla-

tions begin to crowd the points $y = \pm\pi$. The amplitude of these crowded oscillations also decays as N increases. As $N \rightarrow \infty$ the oscillations are all pushed into the points $y = \pm\pi$ from the interior as might be expected of "convergence", but the magnitudes of the oscillations do not go to zero; instead, they attain a finite limiting value. In general, the limiting value depends on the function $f(\cdot)$ being expanded. In the case (8.1) where $f(y) = \frac{1}{2}y$, the limiting values of the overshoot and undershoot at $y = \pi/2$ are

$$\begin{aligned} &1.1789797 (\pi/2), \\ &0.9028233 (\pi/2), \end{aligned}$$

respectively.

The Gibbs phenomenon may be eliminated by summing Fourier series in other ways. The arithmetic mean method of summing Cesaro sums

$$(8.3) \quad \bar{S}_M(y) = \frac{1}{M} \sum_{N=1}^M S_N(y)$$

eliminates Gibbs phenomenon, smooths the partial sums and improves convergence to $f(y)$.

HEWITT & HEWITT (1979) have written an interesting historical and mathematical paper about the Gibbs phenomenon. The problem interested many famous mathematicians and aspects of it were controversial for many years. In their paper, HEWITT & HEWITT exhibit graphs of convergence in the presence of Gibbs phenomenon which represent numerical computations of as many as 100,000 terms. Their study shows that it is possible to obtain precise hypotheses about convergence of Fourier series from numerical studies.

There are no theorems about the Gibbs phenomenon or summation of Cesaro sums for biorthogonal series. In this section and in the next we give numerical results and present conjectures based on these numerical results. The Gibbs phenomenon arises in biorthogonal eigenfunctions expansions whenever $f_1(\pm 1) \neq 0$. The difference between a bad data vector $f(y)$ for which $f_1(\pm 1) \neq 0$ and a good data vector $F(y)$ for which $F_1(\pm 1) = 0$ can be expressed as

$$f(y) - F(y) = f_e \begin{bmatrix} 1 \\ 0 \end{bmatrix} + f_o \begin{bmatrix} y \\ 0 \end{bmatrix}$$

where $f_1(1) = f_e + f_o$, $f_1(-1) = f_e - f_o$ and

$$(8.4) \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \lim_{N \rightarrow \infty} S_N(y), \quad S_N(y) \stackrel{\text{def}}{=} \sum_{-N}^N \left\{ -\frac{1}{\cos^4 s_n} \phi^{(n)}(y) \right\}$$

is the unit step function vector and

$$(8.5) \quad \begin{bmatrix} y \\ 0 \end{bmatrix} = \lim_{N \rightarrow \infty} \hat{S}_N(y), \quad \hat{S}_N(y) \stackrel{\text{def}}{=} \sum_{-N}^N \left\{ -\frac{1}{s_n^2} \hat{\phi}^{(n)}(y) \right\}$$

is the unit ramp function vector. (The components of the partial sums S_N are denoted by

$$(8.6) \quad S_N(y) = \begin{bmatrix} S_{N1}(y) \\ S_{N2}(y) \end{bmatrix}$$

with an identical convention for \hat{S}_N .) It follows that a "Gibbs phenomenon" for biorthogonal series is associated with (8.4) and (8.5).

JOSEPH & STURGES (1978) showed that the partial sums (8.4) and (8.5) converge conditionally on the open interval $(-1, 1)$ and not uniformly. They exhibited results which appeared to exhibit a Gibbs phenomenon and also appeared to be amenable to "smoothing" by Féjer's method of averaging Cesaro sums

$$(8.7) \quad CS_N(y) \stackrel{\text{def}}{=} \frac{1}{N} \sum_{M=1}^N S_M(y).$$

(The components of CS_N are CS_{N1} and CS_{N2} and the average of the odd partial sums is denoted by $\hat{C}S_N$.)

The numerical results of this section are displayed in Tables 8.1 and 8.2 and in Figs. 8.1-8.7 and are described in the captions. The following points deserve emphasis.

(i) There is a Gibbs phenomenon associated with the biorthogonal series representing the unit step function vector and the unit ramp function vector. The oscillations are much larger than those which occur for Fourier series.

(ii) The overshoot and undershoot for the unit step function vector, given by (8.4), is given in Table 8.1 as

$$(8.8) \quad \begin{aligned} &\max_{y \in (-1, 1)} \lim_{N \rightarrow \infty} \left\{ S_N(y) - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\} \cong \begin{bmatrix} 1.666 \\ 0.63 \end{bmatrix}, \\ &\min_{y \in (-1, 1)} \lim_{N \rightarrow \infty} \left\{ S_N(y) - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\} \cong \begin{bmatrix} 0.355 \\ -0.63 \end{bmatrix} \end{aligned}$$

where the maximum and minimum values are attained as $y \rightarrow \pm 1$.

The overshoot and undershoot for the unit ramp function vector, given by (8.5), is given in Table 8.2 as

$$(8.9) \quad \begin{aligned} &\max_{y \in (-1, 1)} \lim_{N \rightarrow \infty} \left\{ \hat{S}_N(y) - \begin{bmatrix} y \\ 0 \end{bmatrix} \right\} \cong \begin{bmatrix} 1.672 \\ 0.63 \end{bmatrix}, \\ &\min_{y \in (-1, 1)} \lim_{N \rightarrow \infty} \left\{ \hat{S}_N(y) - \begin{bmatrix} y \\ 0 \end{bmatrix} \right\} \cong \begin{bmatrix} 0.355 \\ -0.63 \end{bmatrix}. \end{aligned}$$

(iii) The method of averaging using Cesaro sums greatly improves convergence and reduces the magnitude of the undershoot and overshoot. Figures 8.1-8.7 do not appear to indicate that Gibbs oscillations are entirely suppressed by averaging. In fact, Figure 8.7 suggests there is a limiting Gibbs oscillation in the representation of 0 by the averaged sum $\hat{C}S_{N2}(y)$ with an undershoot and overshoot of about -0.10 to $+0.18$.

Table 8.1. Partial sum $S_{N_1}(y)$ and Cesaro sum $CS_{N_1}(y)$ for the step function vector (8.4) after $N = 10,000$ terms for $0.9995 \leq y \leq 1$. The overshoot and undershoot are italicized.

y	$S_{N_1}(y)$	$CS_{N_1}(y)$
0.99950	1.04045	0.87837
0.99955	0.36195	0.90694
0.99960	0.94973	0.94998
0.99965	1.64035	0.88945
0.99970	1.06635	0.79629
0.99974	0.43355	0.80744
0.99975	<i>0.35478</i>	0.82847
0.99976	0.37128	0.85340
0.99980	0.90273	0.90287
0.99985	1.66000	0.75394
0.99986	<i>1.66561</i>	0.67058
0.99987	1.57779	0.57468
0.99990	1.17904	0.36838
0.99995	0.23612	0.06212
0.99999	0.00394	0.00089
1.00000	0.00000	0.00000

Table 8.2. Partial sum $\hat{S}_{N_1}(y)$ and Cesaro sum $\hat{CS}_{N_1}(y)$ for the ramp function vector (8.5) after $N = 10,000$ terms for $0.9995 \leq y \leq 1$. The overshoot and undershoot are italicized.

y	$\hat{S}_{N_1}(y)$	$\hat{CS}_{N_1}(y)$
0.99950	1.04042	0.87794
0.99955	0.36152	0.90653
0.99960	0.94896	0.94965
0.99965	1.63998	0.88921
0.99970	1.06632	0.79605
0.99974	0.41037	0.81092
0.99975	<i>0.35456</i>	0.82826
0.99976	0.35997	0.84810
0.99980	0.90233	0.90272
0.99985	1.65981	0.75388
0.99986	<i>1.67198</i>	0.68836
0.99987	1.62356	0.61417
0.99990	1.17901	0.36870
0.99995	0.23622	0.06220
1.00000	0.00000	0.00000

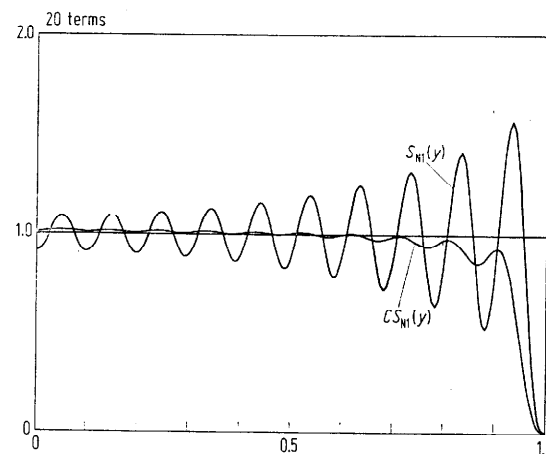


Fig. 8.1. Partial sum $S_{N_1}(y)$ and Cesaro sum $CS_{N_1}(y)$ for the first component of the unit step function vector (8.4) after $N = 20$ terms for $0 \leq y \leq 1$.

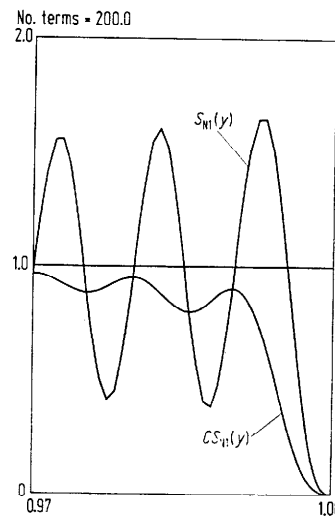


Fig. 8.2. $N = 200$, $0.97 \leq y \leq 1$. The magnitude of the overshoot and undershoot is nearly at their limiting $N \rightarrow \infty$ value. As N is increased, the position of these extreme values tends to $y = 1$ (cf. Fig. 8.3, Table 8.1).

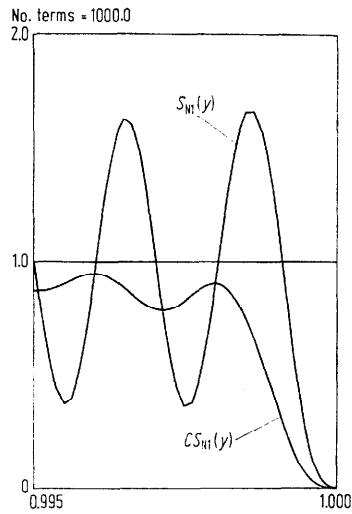


Fig. 8.3. $N = 1000$, $0.995 \leq y \leq 1$. The undershoot and overshoot are much closer to their limiting position $y \rightarrow 1$ as $N \rightarrow \infty$ (cf. Fig. 8.2). This graph should also be compared with the value for $N = 10,000$ given in Table 8.1.

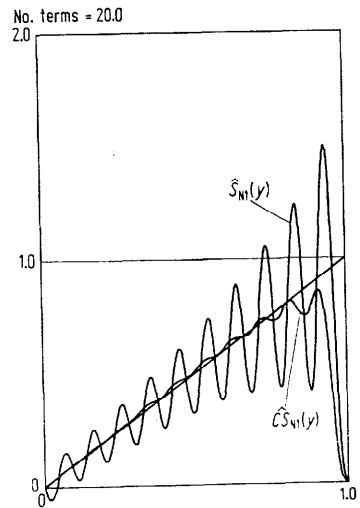


Fig. 8.4. Partial sum $\hat{S}_{N1}(y)$ and Cesaro sum $\hat{CS}_{N1}(y)$ for the first component of the ramp function vector (8.5) after $N = 20$ terms for $0 \leq y \leq 1$.

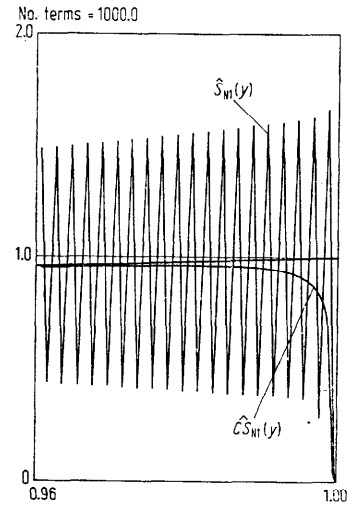


Fig. 8.5. Partial sum $\hat{S}_{N1}(y)$ and Cesaro sum $\hat{CS}_{N1}(y)$ for the first component of the ramp function vector (8.5) after $N = 1000$ terms for $0.96 \leq y \leq 1$.

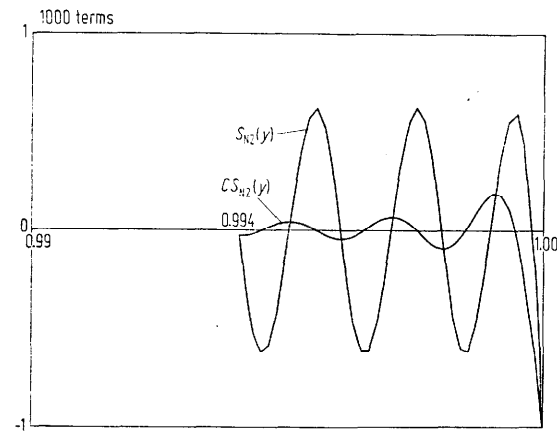


Fig. 8.6. Partial sum $S_{N2}(y)$ and Cesaro sum $CS_{N2}(y)$ for the second component of the step function vector (8.4) after $N = 1000$ terms for $0.994 \leq y \leq 1$.

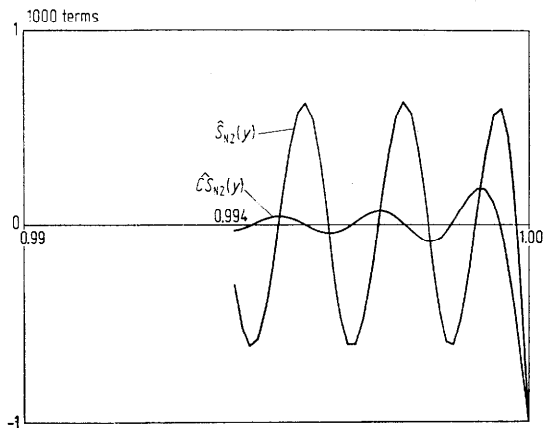


Fig. 8.7. Partial sum $\hat{S}_{N_2}(y)$ and Cesaro sum $\hat{C}\hat{S}_{N_2}(y)$ for the second component of the ramp function vector (8.5) after $N = 1000$ terms for $0.994 \leq y \leq 1$.

9. "Super" Gibbs Phenomenon; the Divergence of Partial Sums and the Convergence of Cesaro Sums at an Interior Point of Discontinuity

Fourier series converge to the mean value of a function at an interior point of discontinuity and have Gibbs oscillations near these points. For Fourier series, end point discontinuities and interior point discontinuities are much the same; for example, the method of averaging with Cesaro sums wipes out the Gibbs oscillations at an interior point of discontinuity.

The aforementioned properties of Fourier series do not hold for biorthogonal series. *We shall show that biorthogonal series can diverge everywhere when $f(y) \cdot (f''(y) \notin BV)$ has an interior point of discontinuity.* The example used in the demonstration is fairly typical and it is probable that the partial sums representing discontinuous functions are usually divergent. For the same example, *we give numerical results which show that the Cesaro sums converge, with Gibbs oscillations near the points of discontinuity.* It seems to us that the convergence of the Cesaro sums of divergent partial sums of an eigenfunction expansion is an extraordinarily interesting point of analysis.

The divergence of partial sums at an interior point of discontinuity interests us all the more because two of us claimed to have proved convergence under case 2 of theorem (iii) of the paper by JOSEPH & STURGES (1978). A flaw in the proof of case 2 was discovered by R. D. GREGORY (1980A, p. 74) and by D. SPENCE (1980). In the proof a mean value called $\bar{\tau}$ is assumed to be constant, but it may

oscillate with n . GREGORY exhibited a continuous data vector

$$(9.1) \quad f(y) = \begin{bmatrix} 0 \\ g(y) \end{bmatrix}, \quad g(y) = \begin{cases} y - \frac{1}{2}, & (0 \leq y \leq 1), \\ -\frac{1}{2} - y, & (-1 \leq y \leq 0) \end{cases}$$

which has discontinuous first derivative at $y = 0$. The series for $f(y)$ can be split into two series formed by summing over the eigenvalues in the first and fourth quadrants. These two series both diverge, although their real parts do converge. GREGORY requires that the two series mentioned above should converge separately, so the series for $f(y)$ diverges in GREGORY's sense but converges in the sense used here.

With these preliminary remarks aside, we turn now to our example:

$$(9.2) \quad f(y) = \begin{bmatrix} f(y) \\ 0 \end{bmatrix} = f(-y),$$

$$(9.3) \quad f(y) = \begin{cases} 1 & (-\frac{1}{2} \leq y \leq \frac{1}{2}), \\ 0 & (-1 \leq y < -\frac{1}{2}, \frac{1}{2} < y \leq 1). \end{cases}$$

We first show that the partial sums

$$(9.4) \quad \begin{aligned} S_{N1}(y) &= \sum_{-N}^N c_n \phi_1^{(n)}(y), \\ c_n &= \frac{1}{k_n} \langle \psi^{(n)} A f \rangle \\ &= \frac{\sin(s_n/2)}{2s_n \cos^3 s_n} - \frac{\cos(s_n/2)}{4 \cos^3 s_n} - \frac{\sin s_n \sin(s_n/2)}{2 \cos^4 s_n}. \end{aligned}$$

do not converge to $f(y)$, but diverge.

The asymptotic (large n) forms of the eigenvalues s_n and the eigenfunctions $\phi^{(n)}(y)$ were used by JOSEPH (1977) and by JOSEPH & STURGES (1978, see lemma 1) to prove that biorthogonal series for data in BV , with end point discontinuities, converge conditionally and not uniformly when $c_n = O\left(\frac{1}{n^2}\right)$. When n is large $\phi_1^{(n)}(y) = O(n^{(3+|y|)/2})$ and $c_n \phi_1^{(n)} = O(n^{-\alpha})$, $\alpha = \frac{1}{2}(1 - |y|)$. It follows that, if

$$(9.5) \quad c_n = O(n^\beta), \quad \beta > -\frac{3}{2},$$

the series on the right of (9.4) diverges for all y . By use of the same asymptotic estimates it is easy to show that the first term on the right of (9.4)₂ is $O(n^{-9/4})$ and the second two terms are $O(n^{-5/4})$. Hence, the c_n given by (9.4)₂ are $O(n^{-5/4})$ and the series on the right of (9.4) diverges.

Table 9.1. Divergence of partial sums (9.4) and convergence to $f(y) = 1$ at $y = 0.2$ of the Cesaro sums (9.6) with increasing N .

N	$S_N(0.2)$	$CS_N(0.2)$
10	0.90571	1.02743
11	0.69855	0.99753
12	1.09884	1.00597
13	0.74885	0.98619
14	1.07343	0.99242
15	1.49948	1.02623
16	0.82760	1.01381
17	0.95436	1.01032
18	1.02069	1.01089
19	0.48700	0.98332
1001	2.13748	1.00047
1002	0.03474	0.99951
1003	2.20128	1.00070
1004	0.66499	1.00037
1005	-0.88771	0.99850
1006	2.17302	0.99966
1007	0.96034	0.99962
1008	0.60859	0.99924
1009	2.90860	1.00113
5001	2.77057	1.00221
5002	-0.91526	1.00183
5003	3.10129	1.00225
5004	0.60339	1.00217
5005	-2.06240	1.00156
5006	3.14847	1.00199
5007	0.79857	1.00195
5008	0.06361	1.00176
5009	4.19523	1.00240
5010	-0.58266	1.00208
9090	-0.92886	1.00730
9091	-1.08558	1.00707
9092	3.45654	1.00734
9093	-1.58162	1.00706
9094	1.42031	1.00710
9095	4.68762	1.00751
9096	-1.70468	1.00721
9097	1.30874	1.00724
9098	2.25757	1.00738
9099	-2.87300	1.00695
9100	2.93368	1.00716

Now consider the behavior of the Cesaro sums

$$(9.6) \quad CS_N(y) = \frac{1}{N} \sum_{M=1}^N S_{M1}(y).$$

Numerical calculations of the partial sums $S_N(y)$ given by (9.4) and of the averaged sums given by (9.6) are displayed in Table 9.1 and Figures 9.1-9.4. Of course, these results show that the $S_N(y)$ diverge. On the other hand, they show that the $CS_N(y)$ converge with apparently small Gibbs oscillations near the point $y = 1/2$ of discontinuity.

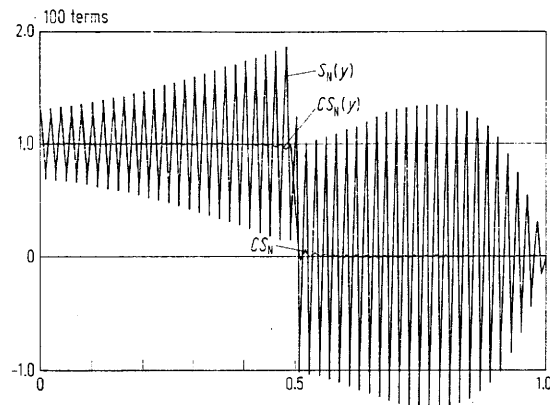


Fig. 9.1. Graphs of $f(y)$ given by (9.3), the partial sums $S_N(y)$ given by (9.4) and the Cesaro sums $CS_N(y)$ given by (9.6) for $N = 100$, $0 < y < 1$.

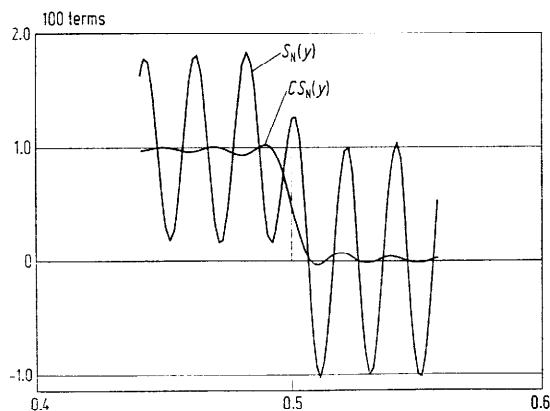


Fig. 9.2. Graphs of $f(y)$ given by (9.3), the partial sums $S_N(y)$ given by (9.4) and the Cesaro sums $CS_N(y)$ given by (9.6) for $N = 100$, $0.44 \leq y \leq 0.56$.

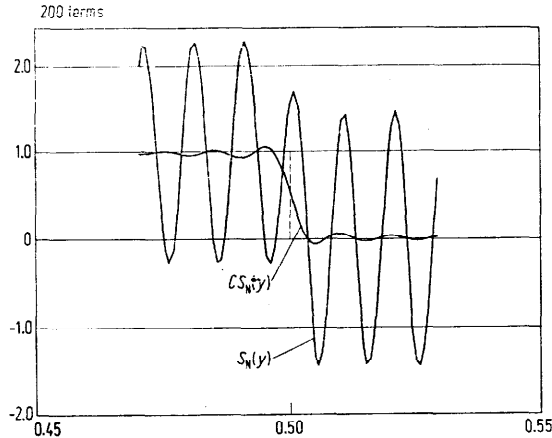


Fig. 9.3. Graphs of $f(y)$ given by (9.3), the partial sums $S_N(y)$ given by (9.4) and the Cesaro sums $CS_N(y)$ given by (9.6) for $N = 200$, $0.47 \leq y \leq 0.53$.

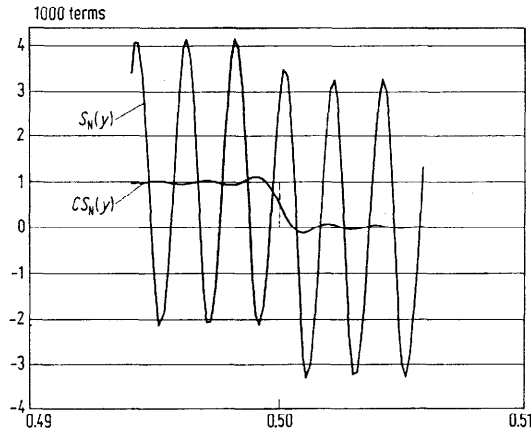


Fig. 9.4. Graphs of $f(y)$ given by (9.3), the partial sums $S_N(y)$ given by (9.4) and the Cesaro sums $CS_N(y)$ given by (9.6) for $N = 1000$, $0.494 \leq y \leq 0.506$.

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