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THE SHAPE OF STRESS-FREE SURFACES ON A SHEARED BLOCK*

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Abstract. We obtain solutions for the shape of the free surface on the upper and lower boundaries of an initially rectangular, incompressible linearly viscoelastic block when the block is sheared at the vertical sidewall. To solve the problem when the vertical displacement of the sidewall is prescribed we use biorthogonal series of the Fadle–Papkovich type. The series has a point of novelty in that zero is an algebraically double but geometrically simple eigenvalue. To expand arbitrary vector fields with two components it is necessary to include in the series the proper and generalized eigenvectors belonging to zero.

1. Introduction. In recent works (Dixit and Joseph (1979); Dixit, Narain and Joseph (1980)), we developed a perturbation theory for solving problems of stress, deformation and the shape of free boundaries in nonlinear viscoelastic solids for small deformations. The theory defines characterizing parameters for a general class of solids in a restricted class of deformations, small deformations perturbing the natural state. When the deformations are static the material responds as if it were elastic, and when the deformations are unsteady the material responds as if it were viscoelastic. The simplest theory, the theory with the smallest number of material parameters, is for incompressible materials. The leading, first order operator depends on one elastic constant and one viscoelastic function, the shear modulus and the shear relaxation modulus. In the second order theory there is another elastic constant and two more viscoelastic functions. Our aim in the two papers last mentioned was to predict shapes of stress-free surfaces on a cylinder undergoing torsional oscillations and in a rod climbing problem ("Weissenberg" effects in solids). In these problems the deformation of the traction free surface occurs first at second order. Following ideas which work for fluids, we seek to learn something about the second order parameters by fitting the observed shapes to the predicted ones.

In this paper we treat a problem in which the deformation of the free surface takes place already at first order. In this problem we induce deformation of the cross section of an infinitely long incompressible block (Fig. 1.1) by prescribing (I) the shearing stress and a zero normal displacement on the side walls at $X = \pm L$ or (II) the vertical displacement and a zero normal displacement at $X = \pm L$. Our aim is to determine the stress and displacement field in the block and the shape of the free surface on the top and bottom of the block at first order. We could not find any literature on these two problems even in the elastic case. In fact, problem (II) is evidently not elementary, at least in our stream function formulation of it, but problem (I) is elementary. Problem (II) is more difficult than (I) because (A) the boundary conditions do not fit Fourier series solutions and instead require biorthogonal series which are like the Fadle–Papkovich ones, and (B) the spectrum of the linear operator for the eigenfunctions solving (II) contain a zero eigenvalue of Riesz index two; that is, zero is an algebraically double but geometrically simple eigenvalue. So to solve problems of type (II) we have to include the proper eigenvector and the generalized one corresponding to the double eigenvalue zero. We could not find any theory or example of generalized eigenvectors for biorthogonal series of the Papkovich–Fadle type, but they are evidently required for completeness of the sets of eigenfunctions

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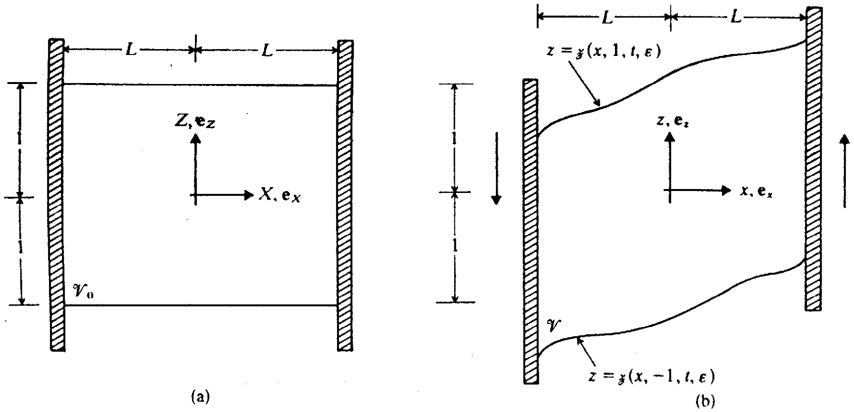

FIG. 1.1. *A viscoelastic block is sheared by the motion of two rigid plates at $X = \pm L$.* (a) *is the natural state:* $\mathscr{V}_0 = \{X, Z: -L < X < L, -1 < Z < 1\}$; (b) *is the distorted state:* $\mathscr{V} = \{x, z: -L < x < L, \mathfrak{z}(x, -1, t, \varepsilon) < z < \mathfrak{z}(x, 1, t, \varepsilon)\}$, *where $\mathfrak{z}(x, \pm 1, t, \varepsilon)$ are the free surfaces at the top and bottom. In problem* (I) *the shearing stress at $X = \pm L$ is prescribed. In problem* (II) *the vertical displacement at $X = \pm L$ is prescribed.*

used here. The problem therefore has a certain mathematical interest and, for example, invites rigorous analysis of problems of completeness of the new set of eigenfunctions along the lines recently used by Gregory (1980a), Spence (1981) and Joseph, Sturges and Warner (1981) to treat the problem of completeness of the Papkovich–Fadle eigenfunctions.

The solutions given in this paper are, therefore, proposed as contributions (1) to the theory of small motions perturbing the natural state of a viscoelastic solid, (2) to the analysis of free surface problems in this theory and (3) to the theory of biorthogonal eigenfunction expansions.

The paper is organized as follows: In § 2 we give the governing equations in the first (linearized) approximation, in § 3 we prove that the solutions of the equations are uniquely determined by prescribed data, in § 4 we formulate the problem in terms of a stream function, in § 5 we give the general algorithm for determining the shape of traction free surfaces, in § 6 and § 7 we give the elementary solutions of problem (I), in § 8 we solve the problem solved in § 7 by the method of biorthogonal series. In § 9 we solve the elastic problem in which the sidewall displacements are prescribed using the biorthogonal series with a generalized eigenvector. The mathematical properties of biorthogonal series and generalized eigenvectors are discussed in Appendix A. In Appendix B we discuss the nature of convergence of the series for data (step functions, ramp functions, etc.) which lead to slow convergence, and we give end regularity conditions for more rapid convergence. Appendix C treats the problem of singularities at the corner points of the block.

2. Equations of motion. We write the stress in an incompressible linear viscoelastic solid as

(2.1) $$\mathbf{T} = -p\mathbf{1} + 2\beta\mathbf{E} + 2\int_0^\infty \zeta(s)[\mathbf{E}(t-s) - \mathbf{E}(t)]\,ds,$$

where p is the reaction pressure, β is the shear modulus, $\zeta(s)$ is the shear relaxation

modulus,

$$E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right)$$

is the strain relative to the natural state of the body which occupies the region of space $\mathscr{V}_0(\mathbf{X})$ with coordinates X_i. The natural state of the body is unstressed and undeformed. The components u_i and the displacement $\mathbf{u}$ are reckoned relative to the undeformed state. This form of the stress for "simple" materials was derived by Coleman and Noll (1961), and it also arises at leading order in our general theory (Dixit and Joseph (1979), Dixit, Narain and Joseph (1980)).

In static problems, $\mathbf{E}(t)$ is independent of t and (2.1) reduces to the constitutive equation for linearly elastic incompressible solids $\mathbf{T} = -p\mathbf{1} + 2\beta\mathbf{E}$.

The equations governing the deformation of a material satisfying (2.1) hold in $\mathscr{V}_0(\mathbf{X})$:

(2.2) $$\operatorname{div}\mathbf{u} = 0, \qquad \rho\ddot{\mathbf{u}} = -\nabla p + \gamma\nabla^2\mathbf{u} + \int_0^\infty \zeta(s)\nabla^2\mathbf{u}(t-s)\,ds,$$

where ρ is the density,

$$\gamma = \beta - \int_0^\infty \zeta(s)\,ds.$$

Various combinations of components of $\mathbf{u}$ and $\mathbf{T}\cdot\mathbf{n}$, where $\mathbf{n}$ is the outward normal on $\partial\mathscr{V}_0$, may be prescribed on different parts of the boundary $\partial\mathscr{V}_0$ of $\mathscr{V}_0$.

In the problem studied here we consider small (two-dimensional) deformations $\mathbf{u}$ of an infinitely long rectangular cylinder (Fig. 1) of incompressible viscoelastic material satisfying (2.1). The top and bottom of the rectangle ($Z = \pm 1$) are traction free ($\mathbf{T}\cdot\mathbf{e}_z = \mathbf{0}$) and the normal component $u = \mathbf{u}\cdot\mathbf{e}_x$ of the displacement of the sidewalls ($X = \pm L$) vanishes. The deformation is driven by prescribing the shear stress T_{zx} or the vertical displacement w of the sidewall. The unsteady problems we treat are periodic, so the driving data is proportional to something small and sinusoidal, that is, $\varepsilon\cos\omega t$. Either

(2.3)$_1$ $$(\mathbf{T}\cdot\mathbf{e}_x)\cdot\mathbf{e}_z \overset{\text{def}}{=} T_{zx} = +\varepsilon H(z)\cos\omega t$$

or

(2.3)$_2$ $$\mathbf{u}\cdot\mathbf{e}_z \overset{\text{def}}{=} w = \pm\varepsilon h(z)\cos\omega t$$

at $X = \pm L$. For steady shearing, $\omega = 0$. The driving data of possibly greatest interest is when the shear stress at the sidewall is uniform, $H(z) = 1$, or when the vertical displacement of the sidewall is uniform, $h(z) = 1$. We shall carry out calculations for these uniform cases.

Since the governing equations are all linear, we may write $\cos\omega t = (e^{i\omega t} + e^{-i\omega t})/2$ and set

(2.4) $$\begin{pmatrix}\mathbf{u}(X, Z, t, \varepsilon)\\ p(X, Z, t, \varepsilon)\\ \mathbf{E}(X, Z, t, \varepsilon)\\ \mathbf{T}(X, Z, t, \varepsilon)\end{pmatrix} = \varepsilon\begin{pmatrix}\hat{\mathbf{u}}(X, Z)\\ \hat{p}(X, Z)\\ \hat{\mathbf{E}}(X, Z)\\ \hat{\mathbf{T}}(X, Z)\end{pmatrix} e^{i\omega t} + \text{conjugate}$$

into the governing equations. This leads to

$$\hat{\mathbf{T}} = -\hat{p}\mathbf{1} + 2\chi(i\omega)\hat{\mathbf{E}},$$

where $\hat{\mathbf{E}}$ is the symmetric part of the gradient of $\hat{\mathbf{u}}(X, Z)$ and

$$\chi(i\omega) = \beta + \int_0^\infty \zeta(s)[e^{-i\omega s} - 1]\, ds. \tag{2.5}$$

The displacement equations for $\hat{\mathbf{u}} = (\hat{u}, \hat{w})$ are

$$\begin{aligned} &\operatorname{div} \hat{\mathbf{u}} = 0, \\ &-\rho\omega^2\hat{\mathbf{u}} = -\nabla\hat{p} + \chi(i\omega)\nabla^2\hat{\mathbf{u}} = \operatorname{div}\hat{\mathbf{T}} \end{aligned} \tag{2.6}$$

in $\mathscr{V}_0$, and the boundary conditions are

$$\begin{aligned} &\hat{\mathbf{T}}\cdot\mathbf{e}_z = (\hat{T}_{xz}, \hat{T}_{zz}) = (0, 0) \quad \text{at } Z = \pm 1, \\ &\hat{\mathbf{u}}\cdot\mathbf{n} = \hat{u} = 0 \quad \text{at } X = \pm L \end{aligned} \tag{2.7$_1$}$$

and either

$$\hat{T}_{zx} = +H(Z) \tag{2.7$_2$}$$

or

$$\hat{\mathbf{u}}\cdot\mathbf{e}_z = \hat{w} = \pm h(Z) \tag{2.7$_3$}$$

at $X = \pm L$.

3. Uniqueness. An energy integral for $\hat{\mathbf{u}}$ may be derived from $(2.6)_2$ using $\operatorname{Tr}\hat{\mathbf{E}} = 0$ and the boundary conditions; after multiplying $(2.6)_2$ by the complex conjugate of $\hat{\mathbf{u}}$ and integrating over $\mathscr{V}_0$, we get

$$\rho\omega^2\langle|\hat{\mathbf{u}}|^2\rangle = \chi(i\omega)\langle|\hat{\mathbf{E}}|^2\rangle - \int_{-1}^{1} \mathscr{k}(Z), \tag{3.1}$$

where $\langle\cdot\rangle$ designates the integral over $\mathscr{V}_0$, and

$$\mathscr{k}(Z) = [\bar{\hat{w}}(L, Z) - \bar{\hat{w}}(-L, Z)]H(Z)$$

is defined relative to $(2.7)_2$ or

$$\mathscr{k}(Z) = \bar{h}(Z)[\hat{T}_{zx}(L, Z) + \hat{T}_{zx}(-L, Z)]$$

is defined relative to $(2.7)_1$. The overbar denotes complex conjugate.

Equation (3.1) can be used to show that the *solution of* (2.6) *and* (2.7) *is unique.* The difference $\mathbf{v} = \mathbf{u}_1 - \mathbf{u}_2$ of two different solutions $\mathbf{u}_1$ and $\mathbf{u}_2$ of (2.6) and (2.7) with the same $h(Z)$ or $H(Z)$ satisfies (2.6) and (2.7) with zero data. Hence,

$$\rho\omega^2\langle|\mathbf{v}|^2\rangle = \chi(i\omega)\langle|\mathbf{E}(\mathbf{v})|^2\rangle. \tag{3.2}$$

When $\omega = 0$, (3.2) shows that $\mathbf{v}$ is a rigid body displacement, which is impossible because of the boundary conditions. When $\mathbf{v} \neq 0$ and $\omega \neq 0$, it is impossible to satisfy (3.2) because $\chi(i\omega)$ is complex. The same type of argument may be used to show that the natural state ($\varepsilon = 0$) of linear viscoelastic solids is stable and that perturbation equations governing nonlinear viscoelastic solids are uniquely invertible (see Dixit and Joseph (1979), (1980)).

4. The stream function. It will be convenient to solve (2.6) and (2.7) with a stream function

$$\hat{u} = -\psi_z, \qquad \hat{w} = \psi_x, \tag{4.1}$$

where

$$\Psi = \varepsilon\frac{e^{i\omega t}\psi + e^{-i\omega t}\bar{\psi}}{2} \tag{4.2}$$

is the stream function for $\mathbf{u}$ and

$$\begin{aligned} &\text{(a)} \quad \nabla^4\psi - \Lambda^2\nabla^2\psi = 0, \\ &\text{(b)} \quad \left.\begin{aligned} &3\psi_{XXZ} + \psi_{ZZZ} - \Lambda^2\psi_Z = 0 \\ &\psi_{XX} - \psi_{ZZ} = 0 \end{aligned}\right\} \quad \text{at } Z = \pm 1, \\ &\text{(c)} \quad \left.\begin{aligned} &L(\psi) = \gamma(Z) \\ &\psi_Z = 0 \end{aligned}\right\} \quad \text{at } X = \pm L, \end{aligned} \tag{4.3}$$

where

$$\Lambda^2 = -\frac{\rho\omega^2}{\chi(i\omega)}$$

and $L(\psi) = \gamma(Z)$ stands for

$$\psi_{XX} - \psi_{ZZ} = \frac{H(Z)}{\chi(i\omega)} \tag{4.3$_{\rm I}$}$$

when the sidewall shear stress is prescribed and

$$\psi_X = h(Z) \tag{4.3$_{\rm II}$}$$

when the vertical displacement of the sidewalls is prescribed. The boundary condition (4.3b) follows by eliminating $\hat{p}$ from $\hat{T}_{XX} = 0$, using $\partial T_{zz}/\partial X = 0$ and the X-component of $(2.6)_2$.

5. The shape of traction free surface. First, we find ψ solving (4.3). Then we form the physical stream function Ψ using (4.2); the components (u, w) of $\mathbf{u}$ are then determined, $(u, w) = (-\partial\Psi/\partial Z, \partial\Psi/\partial X)$ and p may be computed from (2.2) up to an arbitrary constant which may be determined from the boundary condition

$$-\hat{p} + z\chi(i\omega)\frac{\partial\hat{w}}{\partial Z} = 0 \quad \text{at } Z = \pm 1. \tag{5.1}$$

The shape of the free surfaces in the deformed coordinates is given by

$$z = \mathscr{z}(x, \pm 1, t, \varepsilon), \tag{5.2}$$

where the plus sign refers to the surface which in the natural state is at $Z = 1$,

$$1 = \mathscr{z}(x, 1, t, 0),$$

and, in the same way,

$$-1 = \mathscr{z}(x, -1, t, 0).$$

The free surface may be obtained parametrically, with the parameter X by the relations

(5.3) $$x = X + u(X, \pm 1, t, \varepsilon), \qquad z = \pm 1 + w(X, \pm 1, t, \varepsilon).$$

The condition of conservation of the area of the cross-section of the block may be expressed as

(5.4)
$$\begin{aligned}
4L &= \int_{-L}^{L} [z(x, 1, t, \varepsilon) - z(x, -1, t, \varepsilon)]\, dx \\
&= \int_{-L}^{L} \left(z\frac{\partial x}{\partial X}\bigg|_{Z=1} - z\frac{\partial x}{\partial X}\bigg|_{Z=-1} \right) dX \\
&= \int_{-L}^{L} \left[(1+w)\left(1+\frac{\partial u}{\partial X}\right)\bigg|_{Z=1} - (-1+w)\left(1+\frac{\partial u}{\partial X}\right)\bigg|_{Z=-1} \right] dX \\
&= 4L + \int_{-L}^{L} (w|_{Z=1} - w|_{Z=-1})\, dX + \int_{-L}^{L} \left(w\frac{\partial u}{\partial X}\bigg|_{Z=1} - w\frac{\partial u}{\partial X}\bigg|_{Z=-1} \right) dX = 4L,
\end{aligned}$$

where the integrals vanish because w is an odd and u is an even function of X. Therefore area conservation of the incompressible block is automatically satisfied.

6. Elementary solution of $(4.3)_I$ when $\omega \neq 0$. Now we can consider problem $(4.3)_I$ in which the prescribed shear stress at the sidewall oscillates with frequency ω.

For clarity it is useful to treat even data $H(Z) = H(-Z)$ and odd data $H(-Z) = -H(-Z)$ separately. Then arbitrary data may be treated by superposition. Let us first consider the even data.

We assume that it is possible to find the "particular" solution $\hat{\Psi}$ such that the difference

(6.1) $$\tilde{\psi} = \hat{\psi} - \psi$$

satisfies the following problem:

(6.2)
(a) $$\nabla^4 \tilde{\psi} - \Lambda^2 \nabla^2 \tilde{\psi} = 0 \quad \text{in } \mathscr{V}_0,$$
(b) $$\left.\begin{aligned} 3\tilde{\psi}_{XXZ} + \tilde{\psi}_{ZZZ} - \Lambda^2 \tilde{\psi}_Z &= \pm F(X) \\ \tilde{\psi}_{XX} - \tilde{\psi}_{ZZ} &= G(X) \end{aligned}\right\} \quad \text{at } Z = \pm 1,$$
(c) $$\left.\begin{aligned} \tilde{\psi}_{XX} - \tilde{\psi}_{ZZ} &= 0 \\ \tilde{\psi}_Z &= 0 \end{aligned}\right\} \quad \text{at } X = \pm L,$$

where F and G depend on $\hat{\Psi}$ through the relation (6.1).

In the most important case, when $H(Z)$ is a constant, we find that

(6.3) $$\hat{\psi} = \frac{HX^2}{2\chi(i\omega)} + \phi(X),$$

where

$$\phi'' - \Lambda^2 \phi = \frac{H\Lambda^2}{2\chi(i\omega)}(X^2 - L^2), \qquad \phi''(\pm L) = 0,$$

$$\phi = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi X}{L}$$

and

$$A = \frac{1}{L((2n-1)^2\pi^2/4L^2 - \Lambda^2)} \int_{-L}^{L} \frac{H\Lambda^2}{2\chi(i\omega)} \cos \frac{(2n-1)\pi X}{2L}\, \partial X.$$

In this case $G(X) = \phi'' + H/\chi$ and $F(X) = 0$.

The solution of (6.2) is

(6.4) $$\tilde{\psi} = \sum_{n=1}^{\infty} (c_n \cosh \alpha_n Z + d_n \cosh \beta_n Z) \cos \alpha_n X,$$

where

(6.5) $$\alpha_n = \frac{(2n-1)\pi}{2L}, \qquad \beta_n = (\alpha_n^2 + \Lambda^2)^{1/2}.$$

The constants c_n and d_n are to be chosen to satisfy the boundary conditions at $Z = \pm 1$ (6.2b). Equations (6.4) and (6.2b) imply

(6.6) $$\sum_{n=1}^{\infty} \gamma_n \cos \alpha_n X = F(X), \qquad \sum_{n=1}^{\infty} \delta_n \cos \alpha_n X = G(X),$$

where

(6.7)
$$\begin{aligned}
\gamma_n &= -\{c_n \alpha_n [(\alpha_n^2 + \beta_n^2) \sinh \alpha_n] + d_n \beta_n [2\alpha_n^2 \sinh \beta_n]\}, \\
\delta_n &= -\{c_n [2\alpha_n^2 \cosh \alpha_n] + d_n [(\alpha_n^2 + \beta_n^2) \cosh \beta_n]\}.
\end{aligned}$$

Application of the orthogonality condition

(6.8) $$\int_{-L}^{L} \cos \alpha_n X \cos \alpha_m X\, dX = L\delta_{nm}$$

to (6.6) leads to

(6.9) $$\gamma_m L = \int_{-L}^{L} F(X) \cos \alpha_m X\, dX, \qquad \delta_m L = \int_{-L}^{L} G(X) \cos \alpha_m X\, dX.$$

Since

(6.10)
$$\begin{aligned}
\det &\begin{pmatrix} \alpha_n(\alpha_n^2 + \beta_n^2) \sinh \alpha_n & \beta_n (2\alpha_n^2 \sinh \beta_n) \\ 2\alpha_n^2 \cosh \alpha_n & (\alpha_n^2 + \beta_n^2) \cosh \beta_n \end{pmatrix} \\
&= \alpha_n (\alpha_n^2 + \beta_n^2)^2 \sinh \alpha_n \cosh \beta_n - \beta_n (4\alpha_n^4) \cosh \alpha_n \sinh \beta_n \neq 0,
\end{aligned}$$

c_n and d_n can be determined uniquely from (6.7) and (6.9). Convergence of the series in (6.6) to the functions F and G is guaranteed by standard theorems from the theory of Fourier series.

When the data $H(Z)$ is an odd function of Z, the problem governing $\tilde{\psi}$ is the same as (6.2) except that the conditions (6.2b) now are

$$\left. 3\tilde{\psi}_{XXZ} + \tilde{\psi}_{ZZZ} - \Lambda^2 \tilde{\psi}_Z = F(X) \right\} \quad \text{at } Z = \pm 1.$$

The corresponding solution is

$$\tilde{\psi} = \sum_{n=1}^{\infty} [\hat{c}_n \sinh \alpha_n Z + \hat{d}_n \sinh \beta_n Z] \cos \alpha_n X,$$

where α_n and β_n are given by (6.5). The coefficients $\hat{c}_n$ and $\hat{d}_n$ are determined in a similar fashion.

7. Elementary solution of $(4.3)_{\mathrm{I}}$ when $\omega=0$. When the prescribed shear stress is steady, i.e., $\omega=0$, the solution ψ of the problem $(4.3)_{\mathrm{I}}$ is independent of time t. The method of obtaining the solution is similar to the one described in the last section.

If we set $\Lambda=0$ and replace $H(Z)/\chi(i\omega)$ by $H(Z)/\beta$ in $(4.3)_{\mathrm{I}}$, we get the problem satisfied by ψ:

$$
(7.1)\qquad
\begin{aligned}
&\text{(a)} && \nabla^4\psi=0 \quad \text{in } \mathscr{V}_0,\\
&\text{(b)} && \left.\begin{aligned}3\psi_{XXZ}+\psi_{ZZZ}&=0\\ \psi_{XX}-\psi_{ZZ}&=0\end{aligned}\right\} \quad \text{at } Z=\pm 1,\\
&\text{(c)} && \left.\begin{aligned}\psi_{XX}-\psi_{ZZ}&=\frac{H(Z)}{\beta}\\ \psi_Z&=0\end{aligned}\right\} \quad \text{at } X=\pm L.
\end{aligned}
$$

We again assume that it is possible to find the "particular" solution $\hat{\psi}$ such that the difference

$$
(7.2)\qquad \tilde{\psi}=\hat{\psi}-\psi
$$

satisfies

$$
(7.3)\qquad
\begin{aligned}
&\text{(a)} && \nabla^4\tilde{\psi}=0 \quad \text{in } \mathscr{V}_0,\\
&\text{(b)} && \left.\begin{aligned}3\tilde{\psi}_{XXZ}+\tilde{\psi}_{ZZZ}&=\pm F(X)\\ \tilde{\psi}_{XX}-\tilde{\psi}_{ZZ}&=G(X)\end{aligned}\right\} \quad \text{at } Z=\pm 1,\\
&\text{(c)} && \left.\begin{aligned}\tilde{\psi}_{XX}-\tilde{\psi}_{ZZ}&=0\\ \tilde{\psi}_Z&=0\end{aligned}\right\} \quad \text{at } X=\pm L.
\end{aligned}
$$

We have assumed here that $H(Z)$ is an even function of Z. When $H=H(Z)$ is independent of Z, $\hat{\psi}=HX^2/2\beta$, $F(X)=0$ and $G(X)=H/\beta$.

The solution of (7.3) is

$$
(7.4)\qquad \tilde{\psi}=\sum_{n=1}^{\infty}\left[c_n\cosh\alpha_n Z+d_n Z\sinh\alpha_n Z\right]\cos\alpha_n X,
$$

where α_n is given by $(6.5)_1$. The coefficients c_n and d_n are determined, as before, from the conditions (7.3b) and the orthogonality relation (6.8). The coefficients satisfy

$$
(7.5)\qquad
\begin{aligned}
c_n[\sinh\alpha_n]+d_n[\cosh\alpha_n]&=\frac{\gamma_n}{(-2\alpha_n^3)},\\
c_n[\alpha_n\cosh\alpha_n]+d_n[\cosh\alpha_n+\alpha_n\sinh\alpha_n]&=\frac{\delta_n}{(-2\alpha_n)},
\end{aligned}
$$

where

$$
(7.6)\qquad \gamma_m L=\int_{-L}^{L}F(X)\cos\alpha_m X\,dX,\qquad \delta_m L=\int_{-L}^{L}G(X)\cos\alpha_m X\,dX.
$$

At each n it is possible to invert (7.5) uniquely as

$$
(7.7)\qquad \det\begin{pmatrix}\sinh\alpha_n & \cosh\alpha_n\\ \alpha_n\cosh\alpha_n & \cosh\alpha_n+\alpha_n\sinh\alpha_n\end{pmatrix}=\cosh\alpha_n\sinh\alpha_n-\alpha_n.
$$

This is never zero because α_n is given by $(6.5)_1$, $\alpha_n\neq 0$, and the other roots of (7.7) are complex. Theorems from the theory of Fourier series guarantee that the solution (7.4) does attain the boundary values at $Z=\pm 1$.

When H is an odd function of Z, the reduction (7.2) leads to

$$
\left.\begin{aligned}3\tilde{\psi}_{XXZ}+\tilde{\psi}_{ZZZ}&=F(X)\\ \tilde{\psi}_{XX}-\tilde{\psi}_{ZZ}&=\pm G(X)\end{aligned}\right\} \quad \text{at } Z=\pm 1.
$$

The corresponding solution is

$$
\tilde{\psi}=\sum_{n=1}^{\infty}\left[\hat{c}_n\sinh\alpha_n Z+\hat{d}_n Z\cosh\alpha_n Z\right]\cos\alpha_n X,
$$

where $\hat{c}_n$ and $\hat{d}_n$ are determined in the usual way.

8. Biorthogonal series solution of $(4.3)_{\mathrm{I}}$ in the static (elastic) case ($\omega=0$). Now we shall obtain a representation of the elementary linearly elastic solution given in § 7 in terms of biorthogonal series. Our aim is didactic. We want to show how biorthogonal series with generalized eigenvectors can arise in the elementary problem $(4.3)_{\mathrm{I}}$. The problem $(4.3)_{\mathrm{II}}$, in which the vertical displacement of the sidewalls is prescribed instead of the shear stress, is less elementary; it is not conveniently solved by Fourier series, and biorthogonal series with generalized eigenvectors are required and not just convenient. Moreover, the computation of the coefficients in the biorthogonal series used in § 9 is implicit, through truncation of infinite sets of linear equation, whereas here, relative to $(4.3)_{\mathrm{II}}$, the computation is explicit and the coefficients can be obtained directly as inner products.

The solutions given here and in § 7 are different representations of the same solution. In fact, the solution given here converges more rapidly, to the same functions, than the one given in § 7.

The solution of (7.1a) and (7.1b) is

$$
(8.1)\qquad
\begin{aligned}
\psi&=a_0 Z+a[X^2+Z^2]+b[X^4+6X^2Z^2-3Z^4-24X^2]\\
&\quad+\lim_{N\to\infty}\sum_N\left\{\frac{c_n\phi_1^n(Z)\cosh(p_nX)}{p_n^2\cosh(p_nL)}+\frac{d_n\hat{\phi}_1^n(Z)\cosh(s_nX)}{s_n^2\cosh(s_nL)}\right\},
\end{aligned}
$$

where the coefficients of a_0 and a represent deformations of a rigid body, the coefficient of b represents flexure and shearing due to forces at the sidewalls, ϕ_1^n, $\hat{\phi}_1^n$, p_n and s_n are given by (A.5)–(A.8) and $\sum_N$ denotes the sum over n from $-N$ to N, excluding $n=0$. For $n\geqq 1$, p_n and s_n are the first quadrant eigenvalues and are numbered according to the size of their real part. To make ψ real-valued, we need to choose

$$
c_{-n}=\bar{c}_n,\quad p_{-n}=\bar{p}_n,\quad d_{-n}=\bar{d}_n,\quad s_{-n}=\bar{s}_n.
$$

It is not necessary to consider the second and third quadrant eigenvalues because

$$
\phi_1^n(-p_n,Z)=-\phi_1^n(p_n,Z),\qquad \hat{\phi}_1^n(-s_n,Z)=-\hat{\phi}_1^n(s_n,Z).
$$

The coefficients a_0, a, b, c_n and d_n are to be chosen to satisfy the boundary conditions at $X = \pm L$ (7.1c). The procedure to determine them is as follows: First, we write the boundary conditions (7.1c) as:

$$\left.\begin{aligned}\psi_{XX} &= \frac{H(Z)}{\beta}\\ \psi_{ZZ} &= 0\end{aligned}\right\} \text{ at } X = \pm L. \tag{8.2}$$

Substitution of (8.1) into (8.2) leads to

$$\mathbf{h} = c_0\mathbf{\Phi}^0 + \tilde{c}_0\tilde{\mathbf{\Phi}}^0 + \lim_{N\to\infty}\sum_N [c_n\mathbf{\Phi}^n + d_n\hat{\mathbf{\Phi}}^n], \tag{8.3}$$

where

$$\mathbf{h} = \begin{pmatrix} H(Z)/\beta \\ 0 \end{pmatrix}, \quad c_0 = 2a + 12bL^2, \quad \tilde{c}_0 = 12b, \tag{8.4}$$

and $\mathbf{\Phi}^0$, $\tilde{\mathbf{\Phi}}^0$, $\mathbf{\Phi}^n$ and $\hat{\mathbf{\Phi}}^n$ are given by (A.16)–(A.19) of Appendix A. Now apply the biorthogonality conditions (A.30)–(A.31) to (8.3). This gives rise to

$$\begin{aligned} c_0 &= \langle \mathbf{h}, \tilde{\mathbf{\Phi}}^{*0}\rangle_A,\\ \tilde{c}_0 &= \langle \mathbf{h}, \mathbf{\Phi}^{*0}\rangle_A,\\ c_m &= \frac{\langle \mathbf{h}, \mathbf{\Phi}^{*m}\rangle_A}{k_m} \qquad (m \neq 0),\\ d_m &= \frac{\langle \mathbf{h}, \hat{\mathbf{\Phi}}^{*m}\rangle_A}{\hat{k}_m} \qquad (m \neq 0), \end{aligned} \tag{8.5}$$

where the constants k_m and $\hat{k}_m$, the inner product $\langle \cdot, \cdot \rangle_A$ and the adjoint eigenfunctions $\mathbf{\Phi}^{*0}$, $\tilde{\mathbf{\Phi}}^{*0}$, $\mathbf{\Phi}^{*m}$ and $\hat{\mathbf{\Phi}}^{*m}$ are defined in Appendix A.

After determining the coefficients, we have to verify numerically that the series in (8.3) converges to the data $\mathbf{h}$. Presently there are no theorems about completeness of the eigenfunctions $\{\mathbf{\Phi}^0, \tilde{\mathbf{\Phi}}^0, \mathbf{\Phi}^m, \hat{\mathbf{\Phi}}^m\}$. In proving such a theorem, the results of Gregory (1980a, b), Joseph, Sturges and Warner (1981) and Spence (1981) may be helpful. In Appendix B, we have presented numerical results showing convergence for the worst type of data (step functions, ramp functions, etc.).

Convergence of the series in (8.3) to $\mathbf{h}$ means that the conditions (8.2) and $(7.1c)_1$ are satisfied. To check the boundary condition $(7.1c)_2$, we integrate the second component of (8.3) and find that

$$k = c_0Z - \tilde{c}_0Z^3 + \lim_{N\to\infty}\sum_N \left[c_n\frac{\phi_1^{n'}}{p_n^2} + d_n\frac{\hat{\phi}_1^{n'}}{s_n^2}\right], \tag{8.6}$$

where k is the constant of integration. But

$$\psi_Z|_{X=\pm L} = a_0 + c_0Z - \tilde{c}_0Z^3 + \lim_{N\to\infty}\sum_N \left[c_n\frac{\phi_1^{n'}}{p_n^2} + d_n\frac{\hat{\phi}_1^{n'}}{s_n^2}\right]. \tag{8.7}$$

Comparing (8.6) and (8.7), we find that it is possible to satisfy $(7.1c)_2$ by choosing

$$a_0 = -k. \tag{8.8}$$

Thus the boundary condition $(7.1c)_2$ is satisfied and (8.1) with coefficients given by (8.5), solves (7.1) uniquely.

9. Biorthogonal series solution of $(4.3)_{II}$ in the static (elastic) case ($\omega = 0$). When the vertical shearing displacement of the sidewall is prescribed rather than the shear stress, the governing equations of linear elasticity do not admit an elementary Fourier series solution and biorthogonal series with generalized eigenvectors are required. The problem satisfied by ψ is

$$\begin{aligned} &\text{(a)} && \nabla^4\psi = 0 \quad \text{in } \mathcal{V}_0,\\ &\text{(b)} && \left.\begin{aligned}3\psi_{XXZ} + \psi_{ZZZ} &= 0\\ \psi_{XX} - \psi_{ZZ} &= 0\end{aligned}\right\} \text{ at } Z = \pm 1,\\ &\text{(c)} && \left.\begin{aligned}\psi_X &= \pm h(Z)\\ \psi_Z &= 0\end{aligned}\right\} \text{ at } X = \pm L. \end{aligned} \tag{9.1}$$

The solution of (9.1a) and (9.1b) is given by (8.1). The coefficients a_0, a, b, c_n and d_n are again chosen to satisfy the boundary conditions at $X = \pm L$. The procedure to determine them is almost the same as the one described in the last section. First we rewrite (9.1c) in the form:

$$\left.\begin{aligned}\psi_X &= \pm h(Z)\\ \psi_{ZZ} &= 0\end{aligned}\right\} \text{ at } X = \pm L. \tag{9.2}$$

Substitution of (8.1) into (9.2) leads to

$$\begin{aligned} \mathbf{h}(Z) \equiv \begin{pmatrix} h(Z)\\ 0\end{pmatrix} &= c_0\mathbf{\Phi}^0 + \left[c_0(L-1) - \frac{2\tilde{c}_0L^3}{3}\right]\begin{pmatrix}1\\0\end{pmatrix} + \tilde{c}_0\tilde{\mathbf{\Phi}}^0 + \tilde{c}_0(L-1)\begin{pmatrix}4-Z^2\\0\end{pmatrix}\\ &\quad + \lim_{N\to\infty}\sum_N\left\{c_n\left[\mathbf{\Phi}^n + \left(\frac{\tanh p_nL}{p_n} - 1\right)\begin{pmatrix}\phi_1^n\\0\end{pmatrix}\right]\right\}\\ &\quad + \lim_{N\to\infty}\sum_N\left\{d_n\left[\hat{\mathbf{\Phi}}^n + \left(\frac{\tanh s_nL}{s_n} - 1\right)\begin{pmatrix}\hat{\phi}_1^n\\0\end{pmatrix}\right]\right\} \end{aligned} \tag{9.3}$$

where c_0 and $\tilde{c}_0$ are given by (8.4) and the eigenfunctions $\mathbf{\Phi}^0$, $\tilde{\mathbf{\Phi}}^0$, $\mathbf{\Phi}^n$ and $\hat{\mathbf{\Phi}}^n$ are given by (A.16)–(A.19). Application of the biorthogonality condition (A.30)–(A.31) to (9.3) leads to

$$\begin{aligned} \langle\mathbf{h}, \tilde{\mathbf{\Phi}}^{*0}\rangle_A &= (1+x_1)c_0 + y_1\tilde{c}_0 + \lim_{N\to\infty}\sum_N[(-\tfrac{3}{40})A_n - (\tfrac{3}{16})B_n]\left(\frac{\tanh p_nL}{p_n} - 1\right)c_n,\\ \langle\mathbf{h}, \mathbf{\Phi}^{*0}\rangle_A &= x_2c_0 + (1+y_2)\tilde{c}_0 + \lim_{N\to\infty}\sum_N(-\tfrac{3}{16})A_n\left(\frac{\tanh p_nL}{p_n} - 1\right)c_n,\\ \langle\mathbf{h}, \mathbf{\Phi}^{*m}\rangle_A &= x_mc_0 + y_m\tilde{c}_0 + k_mc_m + \lim_{N\to\infty}\sum_N B_{mn}\left(\frac{\tanh p_nL}{p_n} - 1\right)c_n,\\ \langle\mathbf{h}, \hat{\mathbf{\Phi}}^{*m}\rangle_A &= \hat{k}_md_m + \lim_{N\to\infty}\sum_N\hat{B}_{mn}\left(\frac{\tanh s_nL}{s_n} - 1\right)d_n, \end{aligned} \tag{9.4}$$

where the constants k_m and $\hat{k}_m$, the inner product $\langle \cdot, \cdot \rangle_A$ and the adjoint eigenfunctions Φ^{*0}, $\hat{\Phi}^{*0}$, Φ^{*m} and $\hat{\Phi}^{*m}$ are defined in Appendix A and

$$x_1 = -\frac{11(L-1)}{40}, \qquad x_2 = -\frac{3(L-1)}{8},$$

$$y_1 = \frac{39(L-1)}{40} + \frac{11L^3}{60}, \qquad y_2 = \frac{11(L-1)}{8} + \frac{L^3}{4},$$

(9.5) $$A_n = \int_{-1}^{1} \phi_1^n \, dZ, \qquad B_n = \int_{-1}^{1} Z^2 \phi_1^n \, dZ,$$

$$B_{mn} = \int_{-1}^{1} \phi_1^m \phi_1^n \, dZ, \qquad \hat{B}_{mn} = \int_{-1}^{1} \hat{\phi}_1^m \hat{\phi}_1^n \, dZ,$$

$$x_m = (L-1)A_m, \qquad y_m = -2\left[\frac{L^3}{3} + 2(L-1)\right]A_m + (L-1)B_m,$$

so the coefficients satisfy an infinite set of linear equations[1].

Up to now our problem has not been completely specified. The simplest specification of a shearing displacement leads to

(9.6) $$\mathbf{h}(Z) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

This data corresponds to the case where the plate at $X = L$ moves up by ε and the one at $X = -L$ moves down by the same amount. For this choice of $\mathbf{h}$ the solution is an even function of Z and

(9.7) $$d_n = 0 \quad \text{for all } n.$$

The truncated coefficients $c_0^{(N)}$, $\tilde{c}_0^{(N)}$ and $c_n^{(N)}$ were found by solving the system (9.4) by truncation for various values of the truncation number N. It was found that for $N = 30$ and $N = 50$, $c_0^{(N)}$, $\tilde{c}_0^{(N)}$ and the first few $c_n^{(N)}$ were insensitive to change of N. But they were different at $N = 40$. So the coefficients seem not to converge monotonically. However, the series

(9.8) $$\psi_X|_{X=\pm L} = c_0^{(N)} L + \tilde{c}_0^{(N)} L\left[\left(-\frac{2L^2}{3}\right) + (-4 + Z^2)\right] + \sum_N c_n^{(N)} \phi_1^n \frac{\tanh p_n L}{p_n}$$

converges to ± 1 and

(9.9) $$\psi_Z|_{X=\pm L} \simeq c_0^{(N)} Z - \tilde{c}_0^{(N)} Z^3 + \sum_N c_n^{(N)} \frac{\phi_1^{n'}}{p_n^2}$$

[1] The exact solution $\{b_n\}$ of the system $\sum_n A_{mn} b_n + b_m = f_m$ is possible only in a few special cases. However, one may be able to get an approximate solution either by truncation (reduction) or by iteration (successive approximation). Kantorovich and Krylov (1958, Chap. 2) have shown that for regular systems $\sum_n |A_{mn}| < 1$ for all m the principal solution (i.e., the solution for which $b_m \to 0$ as $m \to \infty$) is unique and the approximate solution obtained by truncation does converge to the principal solution if

$$|f_m| < K\rho_m \quad \text{for all } m, \quad K > 0, \quad \rho_m = 1 - \sum_n |A_{mn}|.$$

They have also shown that the principal solution of such systems can be found by the method of iteration no matter what the initial guesses are. Spence (1978) has discussed optimal weighting functions to improve the convergence of the method of truncation.

converges to 0 (see Table 9.1). Hence $a_0 = 0$. Recall that in the last section a_0 was used to balance the constant of integration. Once $c_0^{(N)}$ and $\tilde{c}_0^{(N)}$ are determined, $a^{(N)}$ and $b^{(N)}$ can be found from the relations

(9.10) $$b^{(N)} = \frac{\tilde{c}_0^{(N)}}{12}, \qquad a^{(N)} = \frac{c_0^{(N)} - 12b^{(N)}L^2}{2}$$

(compare (9.10) and (8.4)).

TABLE 9.1

Convergence of the biorthogonal series to prescribed constant displacements (one and zero) when $L = 2$. The truncation number N is 50. $\psi(X, Z)$ is an even function of both variables. The truncated series give a good representation of $\psi_X(L, Z) = 1$ except at $Z = 1$.

$\psi_X(L, Z)$	$\psi_Z(L, Z)$	Z
.9987E+00	0	0
.9999E+00	.1186E−03	.50E−01
.1001E+01	−.8098E−04	.10E+00
.1000E+01	−.5506E−04	.15E+00
.9990E+00	.1162E−03	.20E+00
.9992E+00	−.4080E−04	.25E+00
.1001E+01	−.7881E−04	.30E+00
.1001E+01	.1150E−03	.35E+00
.1000E+01	−.2113E−04	.40E+00
.9988E+00	−.1110E−03	.45E+00
.9994E+00	.1237E−03	.50E+00
.1001E+01	.8063E−05	.55E+00
.1001E+01	−.1288E−03	.60E+00
.9997E+00	.1235E−03	.65E+00
.9985E+00	−.3209E−04	.70E+00
.9985E+00	−.9408E−04	.75E+00
.1000E+01	.1947E−03	.80E+00
.1002E+01	−.1711E−03	.85E+00
.1003E+01	.1581E−03	.90E+00
.1008E+01	−.2320E−03	.95E+00
.9638E+00	−.5125E−03	.10E+01

We have constructed an approximate solution. It is given by the expression (8.1); with the coefficients a, b and c_n replaced by the truncated coefficients $a^{(N)}$, $b^{(N)}$ and $c_n^{(N)}$; $a_0 = d_n = 0$ for all n. The approximate solution satisfies (9.1a) and (9.1b) exactly. The degree of exactness of the approximate solution is to be judged by convergence with truncation number to the prescribed edge values

$$\hat{u}(\pm L, Z) = -\psi_Z(\pm L, Z) = 0, \qquad \hat{w}(\pm L, Z) = \psi_X(\pm L, Z) = \pm 1.$$

The convergence shown in Table 9.1 is representative. We show in Table 9.2 that the displacement of the free surface is essentially independent of the truncation number when the truncation number is high.

In Figs. 9.1, 9.2 we have plotted the components of the displacement of boundary points which were at $Z = 1$ in the natural state. The free surface is symmetric with respect to lines drawn through origin; the distance to the free surface on the top and bottom is the same when measured from the origin along lines passing through the origin. Otherwise the surface does not possess obvious symmetry properties.

TABLE 9.2

Displacements of the surface $Z=1$ when $L=2$ and $N=50$ and 60. The horizontal displacement $\hat{u}=\hat{\mathbf{u}}\cdot\mathbf{e}_x$ is an even function of X and the vertical displacement is an odd function of X. Recall that to 1st order $\mathbf{u}=\varepsilon\hat{\mathbf{u}}$.

$N=50$		$N=60$		
$\hat{u}(X,1)$	$\hat{w}(X,1)$	$\hat{u}(X,1)$	$\hat{w}(X,1)$	X
-.371E+00	0	-.370E+00	0	0
-.370E+00	.556E-01	-.369E+00	.557E-01	.10E+00
-.367E+00	.111E+00	-.367E+00	.111E+00	.20E+00
-.362E+00	.166E+00	-.362E+00	.166E+00	.30E+00
-.356E+00	.221E+00	-.355E+00	.221E+00	.40E+00
-.347E+00	.274E+00	-.347E+00	.275E+00	.50E+00
-.357E+00	.327E+00	-.337E+00	.328E+00	.60E+00
-.325E+00	.379E+00	-.325E+00	.379E+00	.70E+00
-.311E+00	.429E+00	-.311E+00	.430E+00	.80E+00
-.296E+00	.478E+00	-.295E+00	.479E+00	.90E+00
-.279E+00	.526E+00	-.278E+00	.526E+00	.10E+01
-.260E+00	.571E+00	-.259E+00	.572E+00	.11E+01
-.239E+00	.615E+00	-.238E+00	.616E+00	.12E+01
-.217E+00	.657E+00	-.216E+00	.658E+00	.13E+01
-.193E+00	.697E+00	-.193E+00	.698E+00	.14E+01
-.169E+00	.736E+00	-.168E+00	.736E+00	.15E+01
-.142E+00	.773E+00	-.142E+00	.774E+00	.16E+01
-.115E+00	.811E+00	-.114E+00	.811E+00	.17E+01
-.871E+01	.851E+00	-.857E-01	.851E+00	.18E+01
-.566E-01	.898E+00	-.548E-01	.899E+00	.19E+01
.512E-03	.964E+00	.144E-03	.977E+00	.20E+01

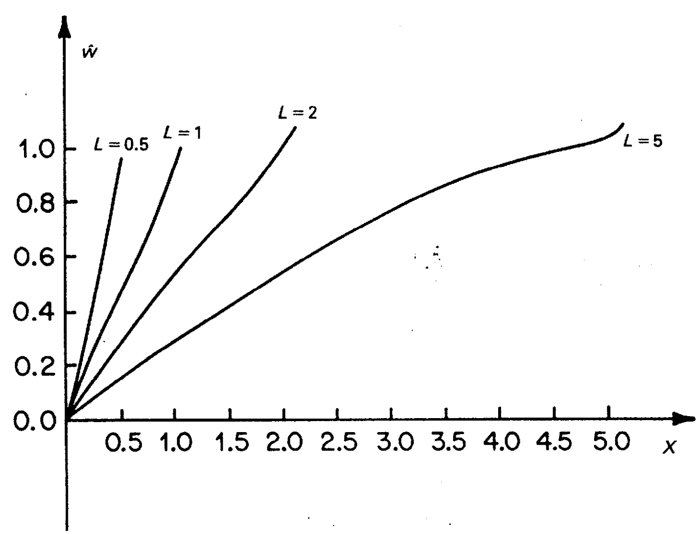

FIG. 9.1. *Vertical displacement of the top free surface for $L=.5$, $.1$, 2 and 5. $\hat{w}(X,1)$ is an odd function of X. The interior values of $\hat{w}(X,1)$, are well represented by the series when X is not close to $\pm L$. It is less good near $X=\pm L$ (see Table 9.1).*

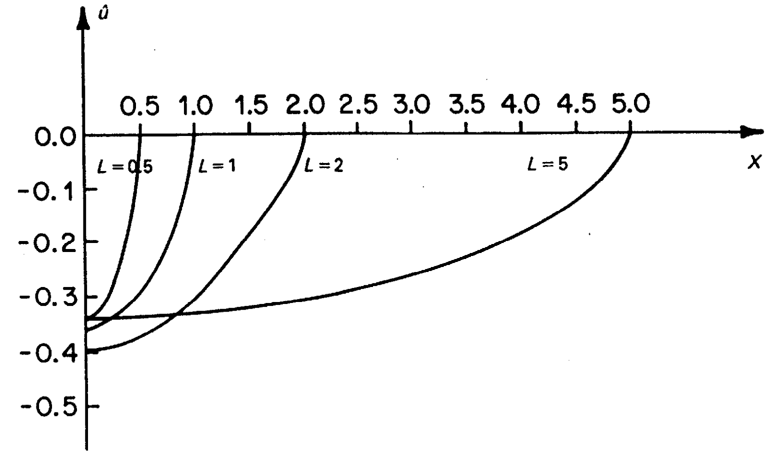

FIG. 9.2. *Horizontal displacement of the top free surface for $L=.5$, 1, 2 and 5. $u(X,1)$ is an even function.*

Appendix A. Biorthogonality of traction-free proper and generalized eigenvectors of the biharmonic stream function. Separable solutions

(A.1) $$\psi(X,Z)=X(X)\phi_1(Z)$$

of the biharmonic equation are governed by the reduced equation

(A.2) $$\frac{X^{iv}}{X'}+2\left(\frac{X''}{X}\right)\left(\frac{\phi_1''}{\phi_1}\right)+\left(\frac{\phi_1^{iv}}{\phi_1}\right)=0.$$

Setting

(A.3) $$\frac{X''}{X}=p^2, \qquad (p^2 \text{ complex}),$$

we find that (A.2) implies that

(A.4)$_1$ $$\phi_1^{iv}+2p^2\phi_1''+p^4\phi_1=0,$$

and (7.1) implies that

(A.4)$_2$ $$\left.\begin{aligned}3p^2\phi_1'+\phi_1'''&=0\\ p^2\phi_1-\phi_1''&=0\end{aligned}\right\} \text{ at } Z=\pm 1.$$

Solutions of the eigenvalue problem (A.4) may be expressed in terms of:

(a) Even eigenfunctions,

(A.5) $$\phi_1^n=p_n[\cos p_n \cos p_nZ+Z\sin p_n \sin p_nZ],$$

corresponding to the even eigenvalues p_n, where

(A.6) $$\sin 2p_n-2p_n=0;$$

(b) Odd eigenfunctions

(A.7) $$\hat{\phi}_1^n=s_n[\sin s_n \sin s_nZ+Z\cos s_n \cos s_nZ]$$

corresponding to the odd eigenvalues, where

(A.8) $$\sin 2s_n + 2s_n = 0.$$

(A.6) and (A.8) have a countably infinite number of roots which are symmetrically located in the complex plane. These eigenvalues are the same as those corresponding to the Papkovich–Fadle eigenfunctions with the exception that evenness and oddness is reversed here (see Joseph (1977)).

The even eigenfunction corresponding to the eigenvalue $p=0$ is

(A.9) $$\tilde{\phi}_1^0 = 1.$$

The corresponding odd eigenfunction is

(A.10) $$\hat{\tilde{\phi}}_1^0 = Z.$$

The scalar eigenvalue problem (A.4) for $p \neq 0$ may be put into the vector form by defining

(A.11) $$\mathbf{\Phi} = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix},$$

where

(A.12) $$\phi_2 = \frac{\phi_1''}{p^2}.$$

The problem satisfied by $\mathbf{\Phi}$ is

(A.13)
(a) $$\mathbb{L}\mathbf{\Phi} + p^2\mathbf{\Phi} = \mathbf{0},$$
(b) $$\left.\begin{array}{l} 3\phi_1' + \phi_2' = 0 \\ \phi_1 - \phi_2 = 0 \end{array}\right\} \quad \text{at } Z = \pm 1,$$

where

(A.14) $$\mathbb{L}(\cdot) = \mathbf{A}^{-1}\frac{d^2(\cdot)}{dZ^2}$$

and

(A.15) $$\mathbf{A} = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}$$

is the biorthogonality matrix. This matrix is associated with biorthogonality conditions of many different sets of eigenfunctions (see Joseph (1979)).

Eigenvalues of (A.13) are given by (A.6) and (A.8). The even eigenvectors are

(A.16) $$\mathbf{\Phi}^n = \begin{pmatrix} \phi_1^n \\ \phi_2^n \end{pmatrix},$$

where ϕ_1^n is given by (A.5) and

$$\phi_2^n \equiv \frac{\phi_1^{n''}}{p_n^2} = -\phi_1^n + 2\sin p_n \cos p_n Z.$$

The odd eigenvectors are denoted by

(A.17) $$\hat{\mathbf{\Phi}}^n = \begin{pmatrix} \hat{\phi}_1^n \\ \hat{\phi}_2^n \end{pmatrix},$$

where $\hat{\phi}_1^n$ is given by (A.7) and

$$\hat{\phi}_2^n \equiv \frac{\hat{\phi}_1^{n''}}{s_n^2} = -\hat{\phi}_1^n - 2\cos s_n \sin s_n Z.$$

An interesting feature of the problem (A.13) is that $p=0$ is an algebraically double and geometrically simple eigenvalue of $\mathbb{L}$. There is no odd eigenfunction belonging to $p=0$; there is an even proper eigenfunction

(A.18) $$\mathbf{\Phi}^0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

and a generalized eigenfunction

(A.19) $$\tilde{\mathbf{\Phi}}^0 = \begin{pmatrix} -4 + Z^2 \\ -3Z^2 \end{pmatrix},$$

where

(A.20) $$\mathbb{L}\mathbf{\Phi}^0 = \mathbf{0}, \qquad \mathbb{L}\tilde{\mathbf{\Phi}}^0 = k\mathbf{\Phi}^0, \quad k = -2, \qquad \mathbb{L}^2\tilde{\mathbf{\Phi}}^0 = \mathbf{0}.$$

The eigenvalue $p=0$ is not semisimple; it has a Riesz index 2 (see Iooss and Joseph (1979, Chap. 4) for an explanation of the Riesz index).

It is clear that the problems (A.4) and (A.13) are equivalent only when $p \neq 0$. At $p=0$ their eigenfunctions are not related.

The adjoint operator $\mathbb{L}^*$ and the adjoint boundary conditions are found from the relation

(A.21) $$\langle \mathbb{L}\mathbf{\Phi}, \mathbf{\Phi}^* \rangle_A = \langle \mathbf{\Phi}, \mathbb{L}^*\mathbf{\Phi}^* \rangle_A,$$

where the inner product is defined by

(A.22) $$\langle \mathbf{a}, \mathbf{b} \rangle_A = \int_{-1}^{1} (\mathbf{A}\mathbf{a}) \cdot \bar{\mathbf{b}}\, dZ.$$

In (A.21) $\mathbf{\Phi}$ satisfies the conditions (A.13b), while $\mathbf{\Phi}^*$ is required to satisfy the adjoint boundary conditions.

The adjoint vector eigenvalue problem is

(A.23)
(a) $$\mathbb{L}^*\mathbf{\Phi}^* + \bar{p}^2\mathbf{\Phi}^* = \mathbf{0},$$
(b) $$\left.\begin{array}{l} \phi_1^{*\prime} + \phi_2^{*\prime} = 0 \\ \phi_1^* - 3\phi_2^* = 0 \end{array}\right\} \quad \text{at } Z = \pm 1,$$

where

(A.24) $$\mathbb{L}^*(\cdot) = (\mathbf{A}^T)^{-1}\frac{d^2(\cdot)}{dZ^2}.$$

The adjoint eigenvalues satisfy (A.6) and (A.8). For $p \neq 0$ the even adjoint eigenvectors are

(A.25) $$\mathbf{\Phi}^{*m} = \begin{pmatrix} \phi_1^{*m} \\ \phi_2^{*m} \end{pmatrix},$$

where

$$\phi_1^{*m} = \overline{\phi_1^m} + 2\sin \bar{p}_m \cos \bar{p}_m Z, \qquad \phi_2^{*m} = \overline{\phi_1^m}$$

and the odd eigenvectors are

$$\hat{\Phi}^{*m}=\begin{pmatrix}\hat{\phi}_1^{*m}\\ \hat{\phi}_2^{*m}\end{pmatrix}, \tag{A.26}$$

where

$$\hat{\phi}_1^{*m}=\overline{\hat{\phi}_1^m}-2\cos\bar{s}_m\sin\bar{s}_m Z,\qquad \hat{\phi}_2^{*m}=\overline{\hat{\phi}_1^m}.$$

There is no odd adjoint eigenvector with eigenvalue $p=0$, but there are two even adjoint eigenvectors:

$$\Phi^{*0}=c\begin{pmatrix}3\\1\end{pmatrix} \tag{A.27}$$

is the proper adjoint eigenvector, and

$$\tilde{\tilde{\Phi}}^{*0}=-c\begin{pmatrix}-4+Z^2\\ -Z^2\end{pmatrix} \tag{A.28}$$

is the generalized eigenvector. We find it convenient to work with

$$\tilde{\Phi}^{*0}=c_1\begin{pmatrix}3\\1\end{pmatrix}+c_2\begin{pmatrix}-4+Z^2\\ -Z^2\end{pmatrix}, \tag{A.29}$$

instead of $\tilde{\tilde{\Phi}}^{*0}$. The constants c, c_1 and c_2 are to be determined later.

The biorthogonality conditions for the even eigenfunctions are

$$\text{(I)}\qquad \begin{aligned}&\langle\Phi^0,\tilde{\Phi}^{*0}\rangle_A=1,\\ &\langle\tilde{\Phi}^0,\tilde{\Phi}^{*0}\rangle_A=0,\\ &\langle\Phi^n,\tilde{\Phi}^{*0}\rangle_A=0\qquad (n\neq 0).\end{aligned}$$

$$\text{(II)}\qquad \begin{aligned}&\langle\Phi^0,\Phi^{*0}\rangle_A=0,\\ &\langle\tilde{\Phi}^0,\Phi^{*0}\rangle_A=1,\\ &\langle\Phi^n,\Phi^{*0}\rangle_A=0\qquad (n\neq 0).\end{aligned}$$

$$\text{(III)}\qquad \begin{aligned}&\langle\Phi^0,\Phi^{*m}\rangle_A=0\qquad (m\neq 0),\\ &\langle\tilde{\Phi}^0,\Phi^{*m}\rangle_A=0\qquad (m\neq 0),\\ &\langle\Phi^n,\Phi^{*m}\rangle_A=k_n\delta_{nm}\qquad (m\neq 0,\ n\neq 0),\end{aligned}$$

where

$$k_n=-4\sin^4 p_n.$$

For the odd eigenfunctions, we have

$$\langle\hat{\Phi}^n,\hat{\Phi}^{*m}\rangle=\hat{k}_n\delta_{nm}\qquad (m\neq 0,\ n\neq 0), \tag{A.31}$$

where

$$\hat{k}_n=-4\cos^4 s_n.$$

To derive $(\text{II})_3$ we note that for $m\neq 0$ and $n\neq 0$ a combination of (A.13a), (A.21) and (A.23a) leads to

$$p_n^2\langle\Phi^n,\Phi^{*m}\rangle_A=-\langle\mathbb{L}\Phi^n,\Phi^{*m}\rangle_A=-\langle\Phi^n,\mathbb{L}^*\Phi^{*m}\rangle_A=p_m^2\langle\Phi^n,\Phi^{*m}\rangle_A.$$

Therefore,

$$(p_n^2-p_m^2)\langle\Phi^n,\Phi^{*m}\rangle_A=0.$$

This leads to $(\text{III})_3$. $(\text{III})_1$, $(\text{II})_3$ and (A.31) follow in the same fashion. For $m\neq 0$: (A.23a), (A.21) and $(\text{A.20})_2$ imply

$$\langle\tilde{\Phi}^0,\Phi^{*m}\rangle_A=-\frac{\langle\tilde{\Phi}^0,\mathbb{L}^*\Phi^{*m}\rangle_A}{p_m^2}=-\frac{\langle\mathbb{L}\tilde{\Phi}^0,\Phi^{*m}\rangle_A}{p_m^2}=-k\frac{\langle\Phi^0,\Phi^{*m}\rangle_A}{p_m^2}. \tag{A.32}$$

Equations (A.32) and $(\text{III})_1$ then lead to $(\text{III})_2$. $(\text{I})_3$ is derived in a similar way. The remaining conditions $(\text{I})_1$, $(\text{I})_2$, $(\text{II})_1$ and $(\text{II})_2$ are derived by direct computations. The biorthogonality condition $(\text{II})_1$ between the proper eigenfunction and the corresponding proper adjoint eigenfunction holds whenever the corresponding eigenvalue is not semisimple (Iooss and Joseph (1979)). The conditions $(\text{I})_1$, $(\text{I})_2$ and $(\text{II})_2$ determine the values of the constants c, c_1 and c_2:

$$c=-\tfrac{3}{16},\quad c_1=-\tfrac{3}{40},\quad c_2=\tfrac{3}{16}. \tag{A.33}$$

Appendix B. Convergence of biorthogonal series associated with the biharmonic equation. Consider the expansion of a vector-valued function $\mathbf{f}$:

$$\mathbf{f}\equiv\begin{pmatrix}f\\g\end{pmatrix}=c_0\Phi^0+\tilde{c}_0\tilde{\Phi}^0+\lim_{N\to\infty}\sum_{\substack{n=-N\\ n\neq 0}}^{N}[c_n\Phi^n+d_n\hat{\Phi}^n]\equiv\lim_{N\to\infty}\begin{pmatrix}S_N^{(1)}\\ S_N^{(2)}\end{pmatrix}, \tag{B.1}$$

where the eigenfunctions Φ^0, $\tilde{\Phi}^0$, Φ^n and $\hat{\Phi}^n$ are given by (A.16)–(A.19). For $n\geqq 1$, Φ^n and $\hat{\Phi}^n$ correspond to the first quadrant eigenvalues p_n and s_n and are numbered according to the size of their real part. The coefficients c_0, $\tilde{c}_0$, c_n and d_n are obtained from (8.5) by replacing $\mathbf{h}$ with $\mathbf{f}$. For real-valued $\mathbf{f}$,

$$c_{-n}=\bar{c}_n,\quad p_{-n}=\bar{p}_n,\quad d_{-n}=\bar{d}_n,\quad s_{-n}=\bar{s}_n.$$

Table B.1 shows the convergence of the Cesàro sums

$$\lim_{N\to\infty}\begin{pmatrix}\dfrac{1}{N}\displaystyle\sum_{M=1}^{N}S_M^{(1)}\\ \dfrac{1}{N}\displaystyle\sum_{M=1}^{N}S_M^{(2)}\end{pmatrix} \tag{B.2}$$

corresponding to the expansions (B.1) of the vector-valued functions:

$$\begin{aligned}&\mathbf{f}_1=\begin{pmatrix}1\\0\end{pmatrix},\qquad \mathbf{f}_2=\begin{pmatrix}0\\1\end{pmatrix},\\ &\mathbf{f}_3=\begin{pmatrix}Z^2\\0\end{pmatrix},\qquad \mathbf{f}_4=\begin{pmatrix}0\\Z^2\end{pmatrix},\\ &\mathbf{f}_5=\begin{pmatrix}Z\\0\end{pmatrix},\qquad \mathbf{f}_6=\begin{pmatrix}0\\Z\end{pmatrix}.\end{aligned} \tag{B.3}$$

For $\mathbf{f}^{(1)}$, $\mathbf{f}^{(2)}$, $\mathbf{f}^{(3)}$ and $\mathbf{f}^{(4)}$, $d_n=0$ for all n. For $\mathbf{f}^{(5)}$ and $\mathbf{f}^{(6)}$, $c_0=\tilde{c}_0=c_n=0$ for all n. Convergence of the partial sums is very slow but convergence of the Césaro sums is more rapid. The boundary conditions (A.13b) on the eigenfunctions at $Z=\pm 1$ are such that both the components of the series should converge to the same value at $Z=\pm 1$. It is found that both the components converge to $f(\pm 1)$ at $Z=\pm 1$.

When $\mathbf{f}$ satisfies

$$\left.\begin{aligned}3f'+g'=0\\ f-g=0\end{aligned}\right\}\quad \text{at } Z=\pm 1, \tag{B.4}$$

TABLE B.1a

Convergence of the Cesaro sums of the biorthogonal series (B.1) *for* $N = 20$

$$\mathbf{f}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$f_1(Z)$	$\frac{1}{N}\sum_{M=1}^{N} S_M^{(1)}$	$g_1(Z)$	$\frac{1}{N}\sum_{M=1}^{N} S_M^{(2)}$	Z
.100E+01	.997E+00	0	.409E−02	0
.100E+01	.995E+00	0	.623E−02	.50E−01
.100E+01	.997E+00	0	.363E−02	.10E+00
.100E+01	.995E+00	0	.540E−02	.15E+00
.100E+01	.998E+00	0	.228E−02	.20E+00
.100E+01	.998E+00	0	.373E−02	.25E+00
.100E+01	.999E+00	0	.141E−03	.30E+00
.100E+01	.997E+00	0	.117E−02	.35E+00
.100E+01	.100E+01	0	−.256E−02	.40E+00
.100E+01	.999E+00	0	−.233E−02	.45E+00
.100E+01	.100E+01	0	−.542E−02	.50E+00
.100E+01	.100E+01	0	−.690E−02	.55E+00
.100E+01	.100E+01	0	−.764E−02	.60E+00
.100E+01	.101E+01	0	−.127E−01	.65E+00
.100E+01	.101E+01	0	−.739E−02	.70E+00
.100E+01	.102E+01	0	−.202E−01	.75E+00
.100E+01	.101E+01	0	.434E−03	.80E+00
.100E+01	.104E+01	0	−.300E−01	.85E+00
.100E+01	.100E+01	0	.391E−01	.90E+00
.100E+01	.117E+01	0	−.300E−01	.95E+00
.100E+01	.970E+00	0	.970E+00	.10E+01

TABLE B.1b

$$\mathbf{f}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$f_2(Z)$	$\frac{1}{N}\sum_{M=1}^{N} S_M^{(1)}$	$g_2(Z)$	$\frac{1}{N}\sum_{M=1}^{N} S_M^{(2)}$	Z
0	.322E−02	.100E+01	.996E+00	0
0	.536E−02	.100E+01	.994E+00	.50E−01
0	.297E−02	.100E+01	.996E+00	.10E+00
0	.497E−02	.100E+01	.995E+00	.15E+00
0	.225E−02	.100E+01	.998E+00	.20E+00
0	.415E−02	.100E+01	.996E+00	.25E+00
0	.109E−02	.100E+01	.100E+01	.30E+00
0	.278E−02	.100E+01	.999E+00	.35E+00
0	−.439E−03	.100E+01	.100E+01	.40E+00
0	.608E−03	.100E+01	.100E+01	.45E+00
0	−.220E−02	.100E+01	.101E+01	.50E+00
0	−.282E−02	.100E+01	.101E+01	.55E+00
0	−.398E−02	.100E+01	.101E+01	.60E+00
0	−.852E−02	.100E+01	.101E+01	.65E+00
0	−.536E−02	.100E+01	.101E+01	.70E+00
0	−.190E−01	.100E+01	.102E+01	.75E+00
0	−.555E−02	.100E+01	.100E+01	.80E+00
0	−.438E−01	.100E+01	.103E+01	.85E+00
0	−.225E−02	.100E+01	.961E+00	.90E+00
0	−.169E+00	.100E+01	.103E+01	.95E+00
0	.297E−01	.100E+01	.297E−01	.10E+01

TABLE B.1c

$$\mathbf{f}_3 = \begin{pmatrix} Z^2 \\ 0 \end{pmatrix}$$

$f_3(Z)$	$\frac{1}{N}\sum_{M=1}^{N} S_M^{(1)}$	$g_3(Z)$	$\frac{1}{N}\sum_{M=1}^{N} S_M^{(2)}$	Z
0	−.583E−02	0	.433E−02	0
.250E−02	−.555E−02	0	.662E−02	.50E−01
.100E−01	.478E−02	0	.378E−02	.10E+00
.225E−01	.156E−01	0	.558E−02	.15E+00
.400E−01	.366E−01	0	.220E−02	.20E+00
.625E−01	.579E−01	0	.352E−02	.25E+00
.900E−01	.894E−01	0	−.259E−03	.30E+00
.122E+00	.121E+00	0	.466E−03	.35E+00
.160E+00	.163E+00	0	−.330E−02	.40E+00
.203E+00	.206E+00	0	−.355E−02	.45E+00
.250E+00	.257E+00	0	−.637E−02	.50E+00
.303E+00	.312E+00	0	−.851E−02	.55E+00
.360E+00	.371E+00	0	−.850E−02	.60E+00
.422E+00	.439E+00	0	−.144E−01	.65E+00
.490E+00	.502E+00	0	−.771E−02	.70E+00
.563E+00	.589E+00	0	−.214E−01	.75E+00
.640E+00	.650E+00	0	.111E−02	.80E+00
.723E+00	.768E+00	0	−.302E−01	.85E+00
.810E+00	.804E+00	0	.393E−01	.90E+00
.902E+00	.105E+01	0	−.308E−01	.95E+00
.100E+01	.888E+00	0	.888E+00	.10E+01

TABLE B.1d

$$\mathbf{f}_4 = \begin{pmatrix} 0 \\ Z^2 \end{pmatrix}$$

$f_4(Z)$	$\frac{1}{N}\sum_{M=1}^{N} S_M^{(1)}$	$g_4(Z)$	$\frac{1}{N}\sum_{M=1}^{N} S_M^{(2)}$	Z
0	.235E−02	0	−.401E−02	0
0	.446E−02	.250E−02	−.360E−02	.50E−01
0	.223E−02	.100E−01	.642E−02	.10E+00
0	.433E−02	.225E−01	.172E−01	.15E+00
0	.186E−02	.400E−01	.377E−01	.20E+00
0	.401E−02	.625E−01	.587E−01	.25E+00
0	.126E−02	.900E−01	.897E−01	.30E+00
0	.333E−02	.122E+00	.121E+00	.35E+00
0	.433E−03	.160E+00	.162E+00	.40E+00
0	.197E−02	.203E+00	.204E+00	.45E+00
0	−.588E−03	.250E+00	.255E+00	.50E+00
0	−.657E−03	.303E+00	.309E+00	.55E+00
0	−.176E−02	.360E+00	.367E+00	.60E+00
0	−.579E−02	.422E+00	.435E+00	.65E+00
0	−.300E−02	.490E+00	.497E+00	.70E+00
0	−.165E−01	.563E+00	.582E+00	.75E+00
0	−.419E−02	.640E+00	.640E+00	.80E+00
0	−.433E−01	.723E+00	.752E+00	.85E+00
0	−.504E−02	.810E+00	.771E+00	.90E+00
0	−.178E+00	.902E+00	.932E+00	.95E+00
0	.226E−02	.100E+01	.226E−02	.10E+01

TABLE B.1e

$$\mathbf{f}_5 = \begin{pmatrix} Z \\ 0 \end{pmatrix}$$

$f_5(Z)$	$\frac{1}{N}\sum_{M=1}^{N} S_M^{(1)}$	$g_5(Z)$	$\frac{1}{N}\sum_{M=1}^{N} S_M^{(2)}$	Z
0	0	0	0	0
.500E−01	.513E−01	0	−.122E−02	.50E−01
.100E+00	.102E+00	0	−.221E−02	.10E+00
.150E+00	.154E+00	0	−.364E−02	.15E+00
.200E+00	.205E+00	0	−.427E−02	.20E+00
.250E+00	.256E+00	0	−.601E−02	.25E+00
.300E+00	.307E+00	0	−.601E−02	.30E+00
.350E+00	.359E+00	0	−.829E−02	.35E+00
.400E+00	.408E+00	0	−.720E−02	.40E+00
.450E+00	.462E+00	0	−.105E−01	.45E+00
.500E+00	.510E+00	0	−.743E−02	.50E+00
.550E+00	.565E+00	0	−.125E−01	.55E+00
.600E+00	.610E+00	0	−.600E−02	.60E+00
.650E+00	.669E+00	0	−.146E−01	.65E+00
.700E+00	.708E+00	0	−.132E−02	.70E+00
.750E+00	.775E+00	0	−.168E−01	.75E+00
.800E+00	.803E+00	0	.111E−01	.80E+00
.850E+00	.892E+00	0	−.199E−01	.85E+00
.900E+00	.890E+00	0	.523E−01	.90E+00
.950E+00	.110E+01	0	−.119E−01	.95E+00
.100E+01	.919E+00	0	.919E+00	.10E+01

TABLE B.1f

$$\mathbf{f}_6 = \begin{pmatrix} 0 \\ Z \end{pmatrix}$$

$f_6(Z)$	$\frac{1}{N}\sum_{M=1}^{N} S_M^{(1)}$	$g_6(Z)$	$\frac{1}{N}\sum_{M=1}^{N} S_M^{(2)}$	Z
0	0	0	0	0
0	−.418E−03	.500E−01	.510E−01	.50E−01
0	−.710E−03	.100E+00	.102E+00	.10E+00
0	−.130E−02	.150E+00	.153E+00	.15E+00
0	−.143E−02	.200E+00	.204E+00	.20E+00
0	−.235E−02	.250E+00	.255E+00	.25E+00
0	−.215E−02	.300E+00	.305E+00	.30E+00
0	−.371E−02	.350E+00	.357E+00	.35E+00
0	−.288E−02	.400E+00	.406E+00	.40E+00
0	−.585E−02	.450E+00	.459E+00	.45E+00
0	−.359E−02	.500E+00	.507E+00	.50E+00
0	−.863E−02	.550E+00	.562E+00	.55E+00
0	−.425E−02	.600E+00	.606E+00	.60E+00
0	−.137E−01	.650E+00	.664E+00	.65E+00
0	−.473E−02	.700E+00	.701E+00	.70E+00
0	−.234E−01	.750E+00	.766E+00	.75E+00
0	−.475E−02	.800E+00	.789E+00	.80E+00
0	−.476E−01	.850E+00	.869E+00	.85E+00
0	−.319E−02	.900E+00	.846E+00	.90E+00
0	−.175E+00	.950E+00	.958E+00	.95E+00
0	.165E−01	.100E+01	.165E−01	.10E+01

TABLE B.2a

Convergence of the partial sums of the biorthogonal series (B.1) *corresponding to the functions which satisfy the same end conditions as the eigenfunctions.* $N = 10$.

$$\mathbf{f}_7 = \begin{pmatrix} 4 - Z^4 \\ 3Z^4 \end{pmatrix}$$

$f_7(Z)$	$S_N^{(1)}$	$g_7(Z)$	$S_N^{(2)}$	Z
.400E+01	.400E+01	0	.969E−04	0
.400E+01	.400E+01	.187E−04	−.719E−05	.50E−01
.400E+01	.400E+01	.300E−03	.212E−03	.10E+00
.400E+01	.400E+01	.152E−02	.159E−02	.15E+00
.400E+01	.400E+01	.480E−02	.486E−02	.20E+00
.400E+01	.400E+01	.117E−01	.116E−01	.25E+00
.399E+01	.399E+01	.243E−01	.243E−01	.30E+00
.398E+01	.398E+01	.450E−01	.452E−01	.35E+00
.397E+01	.397E+01	.768E−01	.768E−01	.40E+00
.396E+01	.396E+01	.123E+00	.123E+00	.45E+00
.394E+01	.394E+01	.188E+00	.188E+00	.50E+00
.391E+01	.391E+01	.275E+00	.275E+00	.55E+00
.387E+01	.387E+01	.389E+00	.389E+00	.60E+00
.382E+01	.382E+01	.536E+00	.535E+00	.65E+00
.376E+01	.376E+01	.720E+00	.721E+00	.70E+00
.368E+01	.368E+01	.949E+00	.949E+00	.75E+00
.359E+01	.359E+01	.123E+01	.123E+01	.80E+00
.348E+01	.348E+01	.157E+01	.157E+01	.85E+00
.334E+01	.334E+01	.197E+01	.197E+01	.90E+00
.319E+01	.319E+01	.244E+01	.244E+01	.95E+00
.300E+01	.300E+01	.300E+01	.300E+01	.10E+01

TABLE B.2b

$$\mathbf{f}_8 = \begin{pmatrix} 2Z - Z^3 \\ Z^3 \end{pmatrix}$$

$f_8(Z)$	$S_N^{(1)}$	$g_8(Z)$	$S_N^{(2)}$	Z
0	0	0	0	0
.999E−01	.990E−01	.125E−03	.994E−03	.50E−01
.199E+00	.199E+00	.100E−02	.985E−03	.10E+00
.297E+00	.298E+00	.338E−02	.247E−02	.15E+00
.392E+00	.392E+00	.800E−02	.804E−02	.20E+00
.484E+00	.483E+00	.156E−01	.166E−01	.25E+00
.573E+00	.573E+00	.270E−01	.269E−01	.30E+00
.657E+00	.658E+00	.429E−01	.418E−01	.35E+00
.736E+00	.736E+00	.640E−01	.641E−01	.40E+00
.809E+00	.808E+00	.911E−01	.924E−01	.45E+00
.875E+00	.875E+00	.125E+00	.125E+00	.50E+00
.934E+00	.935E+00	.166E+00	.165E+00	.55E+00
.984E+00	.984E+00	.216E+00	.216E+00	.60E+00
.103E+01	.102E+01	.275E+00	.276E+00	.65E+00
.106E+01	.106E+01	.343E+00	.342E+00	.70E+00
.108E+01	.108E+01	.422E+00	.420E+00	.75E+00
.109E+01	.109E+01	.512E+00	.513E+00	.80E+00
.109E+01	.108E+01	.614E+00	.616E+00	.85E+00
.107E+01	.107E+01	.729E+00	.727E+00	.90E+00
.104E+01	.104E+01	.857E+00	.858E+00	.95E+00
.100E+01	.100E+01	.100E+01	.100E+01	.10E+01

the convergence is more rapid. The conditions (B.4) are of the same form as the boundary conditions (A.13b) for the eigenfunctions. They may be referred to as *corner or end regularity conditions.* In Table B.2 we examine the numerical convergence of the expansions of

$$\text{(B.5)} \qquad \mathbf{f}_7=\begin{pmatrix}4-Z^4\\ 3Z^4\end{pmatrix}, \qquad \mathbf{f}_8=\begin{pmatrix}2Z-Z^3\\ Z^3\end{pmatrix}.$$

Both $\mathbf{f}^{(7)}$ and $\mathbf{f}^{(8)}$ satisfy the conditions (B.4).

Appendix C. The displacement and stress field in the sheared block near the corner points $x=\pm L$, $z=\pm 1$. Since the block is traction free on the faces, $z=\pm 1$ and is subject to prescribed displacements on the sides $x=\pm L$, it follows that there is a change of boundary condition at each of the four corner points $x=\pm L$, $z=\pm 1$. As a consequence, the stress field in the block is unbounded at each corner, a fact which has been recognized since the work of Williams (1952). A rigorous statement of the form of the elastic field in such a corner (see Gregory and Gladwell (1980, Thm. 1)) is that if r, θ are local polar coordinates in the corner $x=L$, $z=1$ (say), then $\boldsymbol{\sigma}(r,\theta)$ may be expanded in the form

$$\text{(C.1)} \qquad \boldsymbol{\sigma}(r,\theta)=\sum_{\rho\in R} A_\rho r^\rho \mathbf{F}_\rho(\theta),$$

where $r^\rho \mathbf{F}_\rho(\theta)$ are the eigenfunctions of an infinite right-angled wedge with zero displacements on one face and zero tractions on the other; R is the set of roots of[2]

$$\text{(C.2)} \qquad 2(3-4\nu)\cos\pi\rho+4(1+\rho)^2=(3-4\nu)^2+1$$

which lie in the half-plane

$$\text{(C.3)} \qquad \operatorname{Re}(\rho)>-1.$$

The member of R with least real part ($-\alpha$, say) is real and negative and lies in the range $-\frac{1}{2}<-\alpha\leqq 0$; for an incompressible material $\alpha \doteqdot 0.4054$.

As $r\to 0$, the expansion (C.1) is dominated by the term $\rho=-\alpha$ and it will be observed that $\boldsymbol{\sigma}$ is singular like $r^{-\alpha}$, but the displacement field is continuous; the displacement *gradients,* however, are singular like $r^{-\alpha}$. In particular, this means that $u(x,1)$ and $W(x,1)$ (as depicted in Figs. 9.1, 9.2) should have unbounded derivatives at $x=L$, i.e., the graphs should have vertical tangents at these points.

The numerical method of the present paper cannot possibly yield such a result since any *truncated* expansion of the form (8.1) must lead to bounded displacement gradients at $x=\pm L$, $z=\pm 1$. However, these errors are small and localized near the four corner points; away from these points, we expect the eigenfunction expansion to have its usual accuracy.

This corner point behavior does, however, serve to explain the curious change of curvature in the graph of $w(x,1)$, Fig. 9.1. By comparison with classical plate bending theory we would expect the curvature of $w(x,1)$ to be negative for $x>0$ except near the ends of the block; if we now make the reasonable assumption that at $x=L$ the tangent to $w(x,1)$ is vertically *upwards,* then a change of curvature is inevitable. This is clearly revealed in the present calculations.

Acknowledgment. We are indebted to R. D. Gregory for calling our attention to results mentioned in Appendix C.

[2] Here ν is Poisson's ratio, which in the present problem is $\frac{1}{2}$.

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