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Free Surface Problems induced by Motions Perturbing the Natural State of Simple Solids

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Chapter 1. Introduction

We develop a perturbation theory for solids along the lines which have been used to treat the motions of fluids which perturb states of rest or rigid motion. The perturbation theory for fluids does not assume special rheological models; it defines its own experimental rheology and finds application in characterizing materials by specifying the parameters which govern the perturbed motions. In fluids it is possible to measure some of these parameters by comparing the theoretical predictions of the shape of free surfaces with experimental observations (see JOSEPH & BEAVERS, 1977). In this paper we develop similar algorithms for computing the shape of free surfaces which arise from perturbing the natural state of incompressible viscoelastic solids.

In the theory of fluids and solids the simplest results are obtained for deformations which perturb those giving rise to zero stress. It is, therefore, natural to use a constitutive equation which is a functional expansion of the stress around the class of deformations giving rise to zero stress. We follow GREEN & RIVLIN (1957), COLEMAN & NOLL (1961), PIPKIN & RIVLIN (1961), and PIPKIN (1964) in this approach to constitutive theory and, like them, we assume that the functional derivatives can be represented by integrals. These earlier studies are contributions to the asymptotic theory of simple materials for small deformations which perturb zero. The perturbed stresses appear naturally in something akin to Taylor series arranged in powers of the small deformation. The stress is then expressed in terms of multilinear integrals with kernels which are simplified to the degree required by material symmetry.

The purpose of the asymptotic theory is to specify the simple forms of the stress when the motion is small in the appropriate norm. We try to go one step further in treating the cases in which the small norm arises from small prescribed data, like small external forcing, filtered through the equations of motion, so the heart of our theory is an asymptotic sequence of boundary value problems in which redundant terms are purged from the canonical forms of the stress. The usual questions of existence, stability, and uniqueness which are associated with such boundary value problems naturally cannot come up in studies which are abandoned at the point where the stress is arranged into suitably invariant series of powers of the deformation.

In Chapters 2 and 3 of this paper we derive the perturbed equations of motion. Our derivation requires us to specify the loads in the deformed configuration. In the end we find equations valid in the undeformed configuration and on its boundary. In other words we *derive* the boundary conditions which must be applied on the boundary of the region occupied by the undeformed body. In Chapter 4, we show that the solutions of our perturbation problems are unique. In many cases this demonstration is equivalent to a demonstration that the natural state of a simple solid is stable in the sense of linearized theory. The demonstration in Chapter 4 is formal; rigourously all follows if a conditional stability theorem relating the nonlinear problem to the spectral problem is assumed. In Chapter 5 we give equations for determining the shape of traction free surfaces for axisymmetric problems. In Chapter 6 we solve the problem of rod climbing in a simple solid. A rigid rod is fitted snugly in a hole on the axis of a cylindrical specimen. The ends of the cylinder are initially plane and perpendicular to the rod. We imagine that the rod must turn the solid as if bonded, but the rod can slide freely along its axis relative to the contacting solid, as is the case with lubricated gears in mesh. We derive simple formulas for the shape of the initially plane traction free surfaces. In Chapter 7 we solve the problem of the free surface on the traction free cylindrical surface of a simple solid undergoing torsional oscillations driven from rigid end plates. Unlike rod climbing, there can be no distortion of the traction free cylindrical surface under static twists. But the free surface will distort under dynamic twists. To solve this problem we have to do some new things with biorthogonal series arising in problems of incompressible solids using a stream function. Some elementary mathematical properties of these biorthogonal series are derived in Appendix A. In Appendix B we show that the perturbation stresses, including the linearized ones, are all frame indifferent in a strict exact sense and not just in the sense of approximations. In Appendix C we report some preliminary experimental results on the problem of rod climbing.

The results presented in all Chapters of this paper except Chapter 6 and Appendices B and C are joint results of DIXIT and JOSEPH. DIXIT worked out nearly all the details of the theory as a portion of his doctoral dissertation. A. NARAIN worked out the details of the solutions given in Chapter 6. The results presented in Appendix C were obtained in the Rheology Laboratory of the Department

of Aerospace Engineering and Mechanics at the University of Minnesota by NARAIN and JOSEPH. Some of the materials given here and other results can be found in the paper by DIXIT & JOSEPH (1979), in the paper by JOSEPH (1979) and in the thesis of DIXIT (1979).

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Chapter 2. Unperturbed Equations of Motion

2.1 The Constitutive Equation

We may describe the motion of particles in a solid by the relation

$$x = \chi(X, t) \tag{2.1}$$

where x is the position of a particle X at the present time t. If ξ is the position of the same particle at an earlier time, $-\infty < \tau \leqq t$, then the relative motion is defined by the relation

$$\xi = \chi_t(x, \tau). \tag{2.2}$$

$$F(X, t) = \nabla \chi(X, t), \quad (F)_{ij} = \partial \chi_i / \partial X_j \tag{2.3}$$

is the deformation gradient,

$$F_t(x, \tau) = \operatorname{grad} \chi_t(x, \tau), \quad (F_t)_{ij} = \partial(\chi_t)_i / \partial x_j \tag{2.4}$$

is the relative deformation gradient and

$$F(\tau) = F_t(\tau)\, F(t). \tag{2.5}$$

$$B = FF^T \tag{2.6}$$

is the left Cauchy-Green strain tensor and

$$C_t = F_t^T F_t \tag{2.7}$$

is the right relative Cauchy-Green strain tensor.

For a homogeneous, isotropic, simple solid, the Cauchy stress T at a particle is given by TRUESDELL & NOLL (1965):

$$T = \mathfrak{F}\left(B, \overset{\infty}{\underset{s=0}{G}}(s)\right) \equiv \mathfrak{F}(B, G(s)) \tag{2.8}$$

where

$$G(s) = C_t(\tau) - 1, \quad s = t - \tau \tag{2.9}$$

and $\mathfrak{F}$ satisfies

$$Q\mathfrak{F}(B, G(s))\, Q^T = \mathfrak{F}(QBQ^T, QG(s)\, Q^T) \tag{2.10}$$

for all orthogonal tensors Q and

$$T = \mathfrak{F}(1, 0) = 0. \tag{2.11}$$

Equation (2.11) says that there is no stress in the undeformed state of the solid. The stress-free, undeformed state of a solid is called the *natural state* of the solid. Equation (2.8) can be rewritten as

$$T = f(B) + \mathfrak{X}\Big(B, \underset{s=0}{\overset{\infty}{G(s)}}\Big) \equiv f(B) + \mathfrak{X}(B, G(s)) \tag{2.12}$$

where

$$\mathfrak{X}(B, 0) = 0 \tag{2.13}$$

and f and $\mathfrak{X}$ satisfy a relation similar to (2.10).

Nonlinear elasticity arises from (2.12) when T is independent of $G(s)$:

$$T(t) = f(B(t)). \tag{2.14}$$

It is probable that real materials never satisfy (2.14). However, the constitutive equation for viscoelastic solids reduces to (2.14) when the deformations are restricted to time-independent ones. The class of time-independent deformations is defined as the *rest state* of a viscoelastic solid. When x is independent of t, $F_t(\tau) = 1$, $G(s) \equiv 0$ and B is independent of t. Equation (2.12) then reduces to

$$T = f(B). \tag{2.15}$$

Thus from the point of view of material science, nonlinear dynamic viscoelasticity and nonlinear static elasticity are not different subjects but just different realizations of the same governing equations corresponding to dynamic or static data.

Dynamic elasticity defined by (2.14) does not appear to be a viable subject except in an asymptotic sense. For example, it is often asserted, and it is not hard to show, that certain fast deformations of viscoelastic materials satisfy a linearized form of (2.14), but even in this case, the "elastic moduli" depend not only on the parameters of $f(B)$ but also on viscoelastic parameters. (In the notation of (3.23) the "elastic modulus" depends not only on β but also on a certain moment of $\zeta(s)$.)

The constitutive equation (2.12) is general enough to describe many deformations of many different solids. But it is too general to be of much practical use. To simplify the constitutive equation, we restrict the class of deformations to those in which $G(s)$ is small and follow ideas introduced by GREEN & RIVLIN (1957) and COLEMAN & NOLL (1961). In the work of COLEMAN & NOLL, integral forms of stress are derived under the assumption that the stresses are Fréchet differentiable in the weighted $L^2[0, \infty)$ spaces of fading memory. Like them we assume that

a) $\mathfrak{X}$ is N-times Fréchet-differentiable at $G(s) \equiv 0$. The Fréchet-differentiability is uniform in B.

b) For each n, $1 \leqq n \leqq N$, the n^{th} Fréchet-differential can be represented as an integral of the form

$$\int_0^\infty \int_0^\infty \cdots \int_0^\infty L(B, s_1, s_2, \ldots, s_n)\, G(s_1)\, G(s_2) \ldots G(s_n)\, ds_1\, ds_2 \ldots ds_n{}^{*} \quad n\text{-times}$$

where the integral is finite and the kernel $L(B, s_1, s_2, \ldots, s_n)$ is a tensor of order $2n + 2$ whose components are monotonically decreasing functions of

$$s_i, \quad i = 1, 2, \ldots, n.$$

For $N = 2$,

$$T = f(B) + \int_0^\infty K(B, s)\, G(s)\, ds + \int_0^\infty \int_0^\infty L(B, s_1, s_2)\, G(s_1)\, G(s_2)\, ds_1\, ds_2 + O(\|G(s)\|^3). \tag{2.16}$$

where, because of (2.10), f, K and L are isotropic tensor-valued functions, *i.e.*

$$Qf(B)\, Q^T = f(QBQ^T), \tag{2.17}$$

$$Q[K(B, s)\, G(s)]\, Q^T = K(QBQ^T, s)\, QG(s)\, Q^T, \tag{2.18}$$

$$Q[L(B, s_1, s_2)\, G(s_1)\, G(s_2)]\, Q^T = L(QBQ^T, s_1, s_2)\, QG(s_1)\, Q^T QG(s_2)\, Q^T \tag{2.19}$$

for all orthogonal tensors Q. The forms of f and K satisfying (2.17) and (2.18) were given by COLEMAN & NOLL (1961) and the form of L satisfying (2.19) was given by DIXIT (1979):

$$f(B) = f_0\, 1 + f_1 B + f_2 B^2, \tag{2.20}$$

$$\begin{aligned} K(B, s)\, G(s) = {} & \operatorname{tr}[(\phi_{00} 1 + \phi_{01} B + \phi_{02} B^2)\, G(s)]\, 1 \\ & + \operatorname{tr}[(\phi_{10}\, 1 + \phi_{11} B + \phi_{12} B^2)\, G(s)]\, B \\ & + \operatorname{tr}[(\phi_{20}\, 1 + \phi_{21} B + \phi_{22} B^2)\, G(s)]\, B^2 \\ & + (\phi_{30}\, 1 + \phi_{31} B + \phi_{32} B^2)\, G(s) \\ & + G(s)\, (\phi_{30}\, 1 + \phi_{31} B + \phi_{32} B^2), \end{aligned} \tag{2.21}$$

$$\begin{aligned} L(B, s_1, s_2)\, G(s_1)\, G(s_2) = {} & \{\operatorname{tr}[G(s_1)\, G(s_2)\, (\psi_1 1 + \psi_2 B + \psi_3 B^2)] \\ & + \operatorname{tr}[G(s_1)\, (\psi_4 1 + \psi_5 B + \psi_6 B^2)] \operatorname{tr} G(s_2) \\ & + \operatorname{tr}[G(s_1)\, (\psi_7 1 + \psi_8 B + \psi_9 B^2)] \operatorname{tr} BG(s_2) \\ & + \operatorname{tr}[G(s_1)\, (\psi_{10} 1 + \psi_{11} B + \psi_{12} B^2)] \operatorname{tr} B^2 G(s_2)\}\, 1 \\ & + \{\text{similar expression}\}\, B \\ & + \{\text{similar expression}\}\, B^2 \end{aligned} \tag{2.22}$$

* The representation of the first integral follows from the Riesz Representation Theorem where it is assumed that $G(s)$ lies in the Hilbert space with the inner product which gives rise to the weighted $L^2[0, \infty)$ norm (COLEMAN & NOLL, 1961).

$$+ \operatorname{tr}[G(s_2)(\psi_{37}1 + \psi_{38}B + \psi_{39}B^2)]\,G(s_1)$$
$$+ \operatorname{tr}[G(s_2)(\psi_{40}1 + \psi_{41}B + \psi_{42}B^2)]\,(BG(s_1) + G(s_1)\,B)$$
$$+ \operatorname{tr}[G(s_2)(\psi_{43}1 + \psi_{44}B + \psi_{45}B^2)](B^2G(s_1) + G(s_1)\,B^2)$$
$$+ \{\text{similar expression}\}\,G(s_2)$$
$$+ \{\text{similar expression}\}\,(BG(s_2) + G(s_2)\,B)$$
$$+ \{\text{similar expression}\}\,(B^2G(s_2) + G(s_2)\,B^2)$$
$$+ (\psi_{55}1 + \psi_{56}B + \psi_{57}B^2)\,G(s_1)\,G(s_2)$$
$$+ G(s_1)\,G(s_2)\,(\psi_{55}1 + \psi_{56}B + \psi_{57}B^2)$$
$$+ (\psi_{55}1 + \psi_{58}B + \psi_{59}B^2)\,G(s_2)\,G(s_1)$$
$$+ G(s_2)\,G(s_1)\,(\psi_{55}1 + \psi_{58}B + \psi_{59}B^2)$$
$$+ \psi_{60}\{B^2G(s_1)\,BG(s_2) + G(s_2)\,BG(s_1)\,B^2\}$$

where f_i, ϕ_{ij} and ψ_j are the material parameters; f_i are functions of the three principal invariants of B:

$$I = \operatorname{tr} B,$$
$$II = [(\operatorname{tr} B)^2 - \operatorname{tr} B^2]/2,$$
$$III = \det B;$$

ϕ_{ij} are functions of the same three invariants and s, while ψ_j are functions of the same three invariants and s_1 and s_2.

For an incompressible solid, (2.16) reduces to

$$T = -p\,1 + \hat{f}(B) + \int_0^\infty \hat{K}(B, s)\,G(s)\,ds \tag{2.23}$$
$$+ \int_0^\infty\int_0^\infty \hat{L}(B, s_1, s_2)\,G(s_1)\,G(s_2)\,ds_1\,ds_2 + O(\|G(s)\|)^3)$$

where p, called the pressure, is the constitutively indeterminate isotropic part of T and $\hat{f}$, $\hat{K}$ and $\hat{L}$ are respectively equal to f, K and L modulo terms proportional to $\mathbf{1}$. The material parameters f_i, ϕ_{ij} and ψ_j now depend only on I and II because $III = 1$ for incompressible solids.

2.2 Lagrangian Equations of Motion

The displacement $u(X, t)$ of a particle X at the present time t is defined as

$$u = x - X. \tag{2.24}$$

The velocity $\dot{u}(x, t)$ and the acceleration $\ddot{u}(X, t)$ of the particle are defined by

$$\dot{u}(X, t) = \left.\frac{\partial u(X, t)}{\partial t}\right|_{X\text{-fixed}} \tag{2.25}$$

and

$$\ddot{u}(X, t) = \left.\frac{\partial^2 u(X, t)}{\partial t^2}\right|_{X\text{-fixed}}. \tag{2.26}$$

Conservation of mass is expressed as

$$\varrho_0(X) = \varrho(X, t) \det F(X, t) \quad \text{in} \quad \mathscr{V}_0 \quad \text{for} \quad t > 0 \tag{2.27}$$

and the balance of linear momentum as

$$\varrho_0(X)\,\ddot{u}(X, t) = \varrho_0(X)\,b(X, t) + \operatorname{div} S^T(X, t) \quad \text{in} \quad \mathscr{V}_0 \quad \text{for} \quad t > 0 \tag{2.28}$$

where $\mathscr{V}_0$ is the reference domain, $\varrho_0(X)$ and $\varrho(X, t)$ are the densities in the reference and the present configurations respectively, $b(X, t)$ is the prescribed body force per unit mass and S^T is the Piola-Kirchhoff stress tensor which is related to the Cauchy stress tensor T and the deformation gradient F through

$$S^T = T^T(F^T)^{-1} \det F = T(F^T)^{-1} \det F. \tag{2.29}$$

For an incompressible solid, (2.27) reduces to

$$\det F(X, t) = 1 \quad \text{in} \quad \mathscr{V}_0 \quad \text{for} \quad t > 0. \tag{2.30}$$

2.3 Boundary and Initial Conditions

Let $\partial\mathscr{V}$ be the boundary of the deformed domain. In the free surface problems which are solved in this paper, deformation and stresses are forced by data prescribed on different parts $\partial\mathscr{V}$ of the boundary of $\mathscr{V}$. If $A(x, t)$ is prescribed for $x \in \partial\mathscr{V}$, then $\hat{A}(X, t) = A(\chi(X, t), t)$ is prescribed for $X \in \partial\mathscr{V}_0$ where $\partial\mathscr{V}_0$ is the image of $\partial\mathscr{V}$ in the reference domain. Our data is always prescribed on parts of the boundary of the deformed domain but the perturbation equations and the perturbed boundary conditions are necessarily posed on the boundary of the undeformed domain. Three types of boundary conditions will be considered.

Type 1: Displacements prescribed

$$u(x, t), \quad x \in \partial\mathscr{V} \text{ is specified for } t > 0. \tag{2.31}$$

Type 2: Tractions prescribed

$$T(x, t)\,n(x, t), \quad x \in \partial\mathscr{V} \text{ is specified for } t > 0. \tag{2.32}$$

$n(x, t)$ is the unit outward normal on the boundary of the deformed domain at a point x where x is related to X and t through the relation (2.1).

Type 3: Mixed data prescribed

Let $t_1 = t_1(x, t)$ and $t_2 = t_2(x, t)$ be mutually orthogonal unit tangents on the boundary of the deformed domain at a point $x = \chi(X, t)$. Then either two components of the displacement and one traction component are specified, *i.e.*

$$\left.\begin{array}{l} u \cdot n,\ u \cdot t_1 \ \text{ and } \ (Tn) \cdot t_2 \\ \text{or} \\ (Tn) \cdot n,\ u \cdot t_1 \ \text{ and } \ u \cdot t_2 \end{array}\right\} \quad \text{specified on } \partial\mathscr{V} \text{ for } t > 0, \tag{2.33}$$

or one component of the displacement and two traction components are specified, *i.e.*

$$\left.\begin{array}{l}(Tn)\cdot n,\ (Tn)\cdot t_1 \quad \text{and} \quad u\cdot t_2 \\ \text{or} \\ u\cdot n,\ (Tn)\cdot t_1 \quad \text{and} \quad (Tn)\cdot t_2\end{array}\right\} \text{specified on } \partial\mathscr{V} \text{ for } t>0. \tag{2.34}$$

To complete the prescription of the data, the initial history is prescribed, *i.e.*

$$u(\xi,\tau) \text{ is prescribed in } \xi\in\mathscr{V}(\tau) \text{ for } \tau\leq 0. \tag{2.35}$$

Chapter 3. Perturbed Equations of Motion

In our perturbation we develop a sequence of equations which may be systematically associated with a perturbation of data giving rise to the natural state. The data is all important and when we perturb it, we induce a perturbation of the kinematics as well as of the constitutive equations. Expressions for kinematic variables which perturb elastostatic states of deformation are given, along with the forms of stress and equations of motion by DIXIT & JOSEPH (1979). Here only the perturbations of the natural state are considered.

3.1 Kinematics for Perturbations of the Natural State

We start by expanding the displacement,

$$u(\chi(X,t),\tau,\varepsilon) = \varepsilon u^{\langle 1\rangle}(X,\tau) + \varepsilon^2 u^{\langle 2\rangle}(X,\tau) + O(\varepsilon^3). \tag{3.1}$$

This induces a perturbation of related kinematic variables. The expressions for the perturbed variables listed below are easy to derive:

$$F(t,\varepsilon) = 1 + \varepsilon F^{\langle 1\rangle}(t) + \varepsilon^2 F^{\langle 2\rangle}(t) + O(\varepsilon^3) \tag{3.2}$$

where

$$F^{\langle n\rangle}(t) = \nabla u^{\langle n\rangle}(t), \tag{3.3}$$

$$F^{-1} = 1 - \varepsilon F^{\langle 1\rangle} + \varepsilon^2(-F^{\langle 2\rangle} + F^{\langle 1\rangle}F^{\langle 1\rangle}) + O(\varepsilon^3). \tag{3.4}$$

The expression for F_t can be obtained with the help of (2.5), (3.2) and (3.4).

$$G(s,\varepsilon) = F_t^T(\tau,\varepsilon)\,F_t(\tau,\varepsilon) - 1 = \varepsilon G^{\langle 1\rangle}(s) + \varepsilon^2 G^{\langle 2\rangle}(s) + O(\varepsilon^3) \tag{3.5}$$

where

$$G^{\langle 1\rangle}(s) = 2[E^{\langle 1\rangle}(t-s) - E^{\langle 1\rangle}(t)],$$

$$G^{\langle 2\rangle}(s) = 2[E^{\langle 2\rangle}(t-s) - E^{\langle 2\rangle}(t)] + \xi^{\langle 2\rangle}(t,s),$$

$$E^{\langle n\rangle} = [F^{\langle n\rangle} + F^{\langle n\rangle T}]/2,$$

$$\xi^{\langle 2\rangle}(t,s) = F^{\langle 1\rangle T}(t-s)\,F^{\langle 1\rangle}(t-s) - F^{\langle 1\rangle T}(t)\,F^{\langle 1\rangle}(t) - 2F^{\langle 1\rangle T}[E^{\langle 1\rangle}(t-s) - E^{\langle 1\rangle}(t)] - 2[E^{\langle 1\rangle}(t-s) - E^{\langle 1\rangle}(t)]\,F^{\langle 1\rangle}(t).$$

$$\begin{aligned} B(t,\varepsilon) &= F(t,\varepsilon)\,F^T(t,\varepsilon) = 1 + \varepsilon B^{\langle 1\rangle}(t) + \varepsilon^2 B^{\langle 2\rangle}(t) + O(\varepsilon^3) \\ &= 1 + \varepsilon[2E^{\langle 1\rangle}(t)] + \varepsilon^2[2E^{\langle 2\rangle}(t) + F^{\langle 1\rangle}(t)\,F^{\langle 1\rangle T}(t)] + O(\varepsilon^3), \end{aligned} \tag{3.6}$$

$$\det F = 1 + \varepsilon[\operatorname{tr} F^{\langle 1\rangle}] + \varepsilon^2\{\operatorname{tr} F^{\langle 2\rangle} + [\operatorname{tr} F^{\langle 1\rangle}]^2/2 - [\operatorname{tr}(F^{\langle 1\rangle}F^{\langle 1\rangle})]/2\} + O(\varepsilon^3). \tag{3.7}$$

For a compressible solid, the expression for $\varrho(X,t,\varepsilon)$ is needed:

$$\varrho(X,t,\varepsilon) = \varrho_0(X) + \varepsilon\varrho^{\langle 1\rangle}(X,t) + \varepsilon^2\varrho^{\langle 2\rangle}(X,t) + O(\varepsilon^3). \tag{3.8}$$

Expanding equation (2.27) in powers of ε and identifying independent powers of ε, we find that

$$\varrho^{\langle 1\rangle} + \varrho_0 \operatorname{tr} F^{\langle 1\rangle} = 0, \tag{3.9}$$

$$\varrho^{\langle 2\rangle} + \varrho^{\langle 1\rangle}\operatorname{tr} F^{\langle 1\rangle} + \varrho_0\{\operatorname{tr} F^{\langle 2\rangle} + [\operatorname{tr} F^{\langle 1\rangle}]^2/2 - [\operatorname{tr}(F^{\langle 1\rangle}F^{\langle 1\rangle})]/2\} = 0. \tag{3.10}$$

The expression for the normal

$$n = N + \varepsilon n^{\langle 1\rangle} + O(\varepsilon^2) \tag{3.11}$$

can be obtained with the help of

$$n\tilde{J} = \det F(F^T)^{-1} N \tag{3.12}$$

where

$$\tilde{J} = \frac{da}{dA},$$

N is the unit outward normal at the particle X on the boundary of the reference domain, dA is the boundary element at X with normal N and da is its image under the mapping (2.1). Combining (3.11), (3.12), (3.7), (3.4) and

$$\tilde{J} = 1 + \varepsilon\tilde{J}^{\langle 1\rangle} + O(\varepsilon^2), \tag{3.13}$$

we find that

$$n^{\langle 1\rangle} + N\tilde{J}^{\langle 1\rangle} = (\operatorname{tr} F^{\langle 1\rangle})\,N - F^{\langle 1\rangle T}N. \tag{3.14}$$

n is a unit vector; hence

$$N\cdot n^{\langle 1\rangle} = 0. \tag{3.15}$$

The equations (3.14) and (3.15) then imply

$$\tilde{J}^{\langle 1\rangle} = (\operatorname{tr} F^{\langle 1\rangle}) - (F^{\langle 1\rangle T}N)\cdot N \tag{3.16}$$

and

$$n^{\langle 1\rangle} = [(F^{\langle 1\rangle T}N)\cdot N]\,N - F^{\langle 1\rangle T}N. \tag{3.17}$$

3.2 Perturbation of the Stress and the Traction Vector

We may determine the perturbed forms of f, K and L by expanding f_i, ϕ_{ij} and ψ_j in powers of ε, using (3.5) and (3.6). As a practical matter, the full second order theory of compressible solids has too many material parameters (DIXIT &

JOSEPH, 1979). For this reason, here and henceforth, we shall consider only incompressible solids. In this case (2.30) and (3.7) give

$$\text{tr}\, \boldsymbol{F}^{\langle 1\rangle} \equiv \text{div}\, \boldsymbol{u}^{\langle 1\rangle} = 0 \tag{3.18}$$

and

$$\text{tr}\, \boldsymbol{F}^{\langle 2\rangle} \equiv \text{div}\, \boldsymbol{u}^{\langle 2\rangle} = [\text{tr}\,(\boldsymbol{F}^{\langle 1\rangle}\boldsymbol{F}^{\langle 1\rangle})]/2. \tag{3.19}$$

It follows from (3.5) and (3.6) that

$$\text{tr}\, \boldsymbol{G}^{\langle 1\rangle}(s) = \text{tr}\, \boldsymbol{B}^{\langle 1\rangle} = 0. \tag{3.20}$$

The expression for ϕ_{ij} can be written as

$$\phi_{ij}(I(\varepsilon), II(\varepsilon), s) = \phi_{ij}{}^{\langle 0\rangle}(s) + \varepsilon\phi_{ij}{}^{\langle 1\rangle}(s) + O(\varepsilon^2) \tag{3.21}$$

where

$$\phi_{ij}{}^{\langle 0\rangle} = \phi_{ij}(3, 3, s),$$

$$\phi_{ij}{}^{\langle 1\rangle} = \left.\frac{\partial\phi_{ij}}{\partial I}\right|_{\varepsilon=0}\{\text{tr}\, \boldsymbol{B}^{\langle 1\rangle}\} + \left.\frac{\partial\phi_{ij}}{\partial II}\right|_{\varepsilon=0}\cdot\{[6\, \text{tr}\, \boldsymbol{B}^{\langle 1\rangle} - 2\, \text{tr}\, \boldsymbol{B}^{\langle 1\rangle}]/2\} = 0.$$

The expressions for f_i and ψ_j are similar. Substitution of (3.5)–(3.6), (3.20)–(3.21) and the similar expressions for f_i and ψ_j into (2.20)–(2.22), after very tedious manipulations, gives rise to the expressions for the kernels $\hat{\boldsymbol{f}}$, $\hat{\boldsymbol{K}}$ and $\hat{\boldsymbol{L}}$ in powers of ε. Every coefficient of ε in the three expressions contains terms proportional to $\mathbf{1}$. These terms are grouped with the pressure

$$p = \varepsilon p^{\langle 1\rangle} + \varepsilon^2 p^{\langle 2\rangle} + O(\varepsilon^3) \tag{3.22}$$

and the resultant is called the modified pressure. $p^{\langle n\rangle}$ denotes the modified pressure at order n. Finally,

$$\boldsymbol{T} = \varepsilon\boldsymbol{T}^{\langle 1\rangle} + \varepsilon^2\boldsymbol{T}^{\langle 2\rangle} + O(\varepsilon^2) \tag{3.23}$$

where

$$\boldsymbol{T}^{\langle 1\rangle} = -p^{\langle 1\rangle}\mathbf{1} + 2\beta\boldsymbol{E}^{\langle 1\rangle} + 2\int_0^\infty \zeta(s)\,[\boldsymbol{E}^{\langle 1\rangle}(t-s) - \boldsymbol{E}^{\langle 1\rangle}(t)]\,ds,$$

$$\begin{aligned}\boldsymbol{T}^{\langle 2\rangle} = {}& -p^{\langle 2\rangle}\mathbf{1} + 2\beta\boldsymbol{E}^{\langle 2\rangle} + 2\int_0^\infty \zeta(s)\,[\boldsymbol{E}^{\langle 2\rangle}(t-s) - \boldsymbol{E}^{\langle 2\rangle}(t)]\,ds \\ & + \beta\boldsymbol{F}^{\langle 1\rangle}\boldsymbol{F}^{\langle 1\rangle T} + \beta^{[2]}\boldsymbol{B}^{\langle 1\rangle}\boldsymbol{B}^{\langle 1\rangle} + \int_0^\infty \zeta(s)\,\boldsymbol{\xi}^{\langle 2\rangle}(t, s)\,ds \\ & + \int_0^\infty \zeta^{[2]}(s)\,[\boldsymbol{B}^{\langle 1\rangle}\boldsymbol{G}^{\langle 1\rangle}(s) + \boldsymbol{G}^{\langle 1\rangle}(s)\,\boldsymbol{B}^{\langle 1\rangle}]\,ds \\ & + \int_0^\infty\int_0^\infty \alpha(s_1, s_2)\,\boldsymbol{G}^{\langle 1\rangle}(s_1)\,\boldsymbol{G}^{\langle 1\rangle}(s_2)\,ds_1\,ds_2.\text{*}\end{aligned}$$

* An alternative way of obtaining these expressions is discussed in DIXIT & JOSEPH (1979).

The material constants β and $\beta^{[2]}$ and the material functions $\zeta(s)$, $\zeta^{[2]}(s)$ and $\alpha(s_1, s_2)$ are the combinations of $f_i{}^{\langle 0\rangle}$, $\phi_{ij}{}^{\langle 0\rangle}$ and $\psi_j{}^{\langle 0\rangle}$ shown below:

$$\begin{aligned}\beta &= f_1^{\langle 0\rangle} + 2f_2^{\langle 0\rangle}, \\ \beta^{[2]} &= f_2^{\langle 0\rangle}, \\ \zeta(s) &= 2(\phi_{30}^{\langle 0\rangle} + \phi_{31}^{\langle 0\rangle} + \phi_{32}^{\langle 0\rangle}), \\ \zeta^{[2]}(s) &= \phi_{31}^{\langle 0\rangle} + 2\phi_{32}^{\langle 0\rangle}, \\ \tilde{\alpha}(s_1, s_2) &= 2\psi_{55}^{\langle 0\rangle} + \psi_{56}^{\langle 0\rangle} + \psi_{57}^{\langle 0\rangle} + \psi_{58}^{\langle 0\rangle} + \psi_{59}^{\langle 0\rangle} + \psi_{60}^{\langle 0\rangle}, \\ \alpha(s_1, s_2) &= \tilde{\alpha}(s_1, s_2) + \tilde{\alpha}(s_2, s_1).\end{aligned} \tag{3.24}$$

From (3.24) it is clear that $\alpha(s_1, s_2)$ is symmetric in s_1 and s_2.

The Piola-Kirchhoff stress tensor $\boldsymbol{S}^T$ is now given by (2.29), (2.30), (3.4) and (3.23) as

$$\boldsymbol{S}^T = \varepsilon\boldsymbol{S}^{\langle 1\rangle T} + \varepsilon^2\boldsymbol{S}^{\langle 2\rangle T} + O(\varepsilon^3) \tag{3.25}$$

where

$$\boldsymbol{S}^{\langle 1\rangle T} = \boldsymbol{T}^{\langle 1\rangle},$$

$$\boldsymbol{S}^{\langle 2\rangle T} = \boldsymbol{T}^{\langle 2\rangle} - \boldsymbol{T}^{\langle 1\rangle}\boldsymbol{F}^{\langle 1\rangle T}.$$

To characterize the motion of a particular incompressible viscoelastic solid at first order, we need values for one elastic constant β and one viscoelastic function $\zeta(s)$. At second order, besides β and $\zeta(s)$ we need a second elastic constant $\beta^{[2]}$ and two more viscoelastic functions, $\zeta^{[2]}$ and $\alpha(s_1, s_2)$.

The expression for the traction vector $\boldsymbol{t}_n$ can be obtained by expanding the relation

$$\boldsymbol{t}_n = \boldsymbol{T}\boldsymbol{n}.$$

The equations (3.11), (3.17), and (3.23) lead to

$$\boldsymbol{t}_n = \varepsilon\boldsymbol{t}_n{}^{\langle 1\rangle} + \varepsilon^2\boldsymbol{t}_n{}^{\langle 2\rangle} + O(\varepsilon^3) \tag{3.26}$$

where

$$\boldsymbol{t}_n{}^{\langle 1\rangle} = \boldsymbol{T}^{\langle 1\rangle}\boldsymbol{N} \tag{3.27}$$

and

$$\boldsymbol{t}_n{}^{\langle 2\rangle} = \boldsymbol{T}^{\langle 2\rangle}\boldsymbol{N} + [(\boldsymbol{F}^{\langle 1\rangle T}\boldsymbol{N})\cdot\boldsymbol{N}]\,\boldsymbol{T}^{\langle 1\rangle}\boldsymbol{N} - \boldsymbol{T}^{\langle 1\rangle}\boldsymbol{F}^{\langle 1\rangle T}\boldsymbol{N}. \tag{3.28}$$

Equations (3.26)–(3.28) express the values of the traction on the boundary of the deformed domain in terms of quantities defined on the boundary of the reference domain.

3.3 Equations of Motion

With the help of the expressions in powers of ε of the variables $\boldsymbol{u}$, $\boldsymbol{t}_n$ and $\boldsymbol{S}^T$, the problem governing deformations can now be arranged in a set of equations to be solved sequentially. To simplify the writing, we set the body force equal to zero and obtain the problem governing the first order perturbation as

$$\varrho_0 \ddot{u}^{\langle 1\rangle} = -\nabla p^{\langle 1\rangle} + \gamma \nabla^2 u^{\langle 1\rangle} + \int_0^\infty \zeta(s)\, \nabla^2 u^{\langle 1\rangle}(t-s)\, ds,$$

$$\operatorname{div} u^{\langle 1\rangle} = 0 \quad \text{in} \quad \mathscr{V}_0 \quad \text{for} \quad t > 0;$$

(3.29) $$u^{\langle 1\rangle} \text{ is prescribed on } \partial\mathscr{V}_{01} \text{ for } t > 0;$$

$$T^{\langle 1\rangle} N = \left[-p^{\langle 1\rangle} 1 + 2\gamma E^{\langle 1\rangle} + 2\int_0^\infty \zeta(s)\, E^{\langle 1\rangle}(t-s)\, ds\right] N$$

$$\text{is prescribed on } \partial\mathscr{V}_{02} \text{ for } t > 0,$$

where $\partial\mathscr{V}_0 = \partial\mathscr{V}_{01} \cup \partial\mathscr{V}_{02}$ and

$$u^{\langle 1\rangle} \text{ is prescribed in } \mathscr{V}_0 \text{ for } t \leqq 0.$$

At second order

$$\varrho_0 \ddot{u}^{\langle 2\rangle} = -\nabla p^{\langle 2\rangle} + \gamma \nabla^2 u^{\langle 2\rangle} + \int_0^\infty \zeta(s)\, \nabla^2 u^{\langle 2\rangle}(t-s)\, ds + M_2,$$

$$\operatorname{div} u^{\langle 2\rangle} = \theta_2 \quad \text{in} \quad \mathscr{V}_0 \quad \text{for} \quad t > 0;$$

$$u^{\langle 2\rangle} \text{ is prescribed on } \partial\mathscr{V}_{01} \text{ for } t > 0;$$

(3.30) $$T^{\langle 2\rangle} N + [(F^{\langle 1\rangle T} N) \cdot N]\, T^{\langle 1\rangle} N - T^{\langle 1\rangle} F^{\langle 1\rangle T} N$$

$$\text{is prescribed on } \partial\mathscr{V}_{02} \text{ for } t > 0;$$

$$u^{\langle 2\rangle} \text{ is prescribed in } \mathscr{V}_0 \text{ for } t \leqq 0.$$

In (3.29) and (3.30),

(3.31) $$\gamma = \beta - \int_0^\infty \zeta(s)\, ds,$$

(3.32) $$\theta_2 = [\operatorname{tr}(F^{\langle 1\rangle} F^{\langle 1\rangle})]/2$$

and

(3.33) $$M_2 = F^{\langle 1\rangle T} \nabla \hat{p}^{\langle 1\rangle} + \operatorname{div}\{\beta^{[2]} B^{\langle 1\rangle} B^{\langle 1\rangle}$$
$$+ \int_0^\infty \zeta(s)\, [-G^{\langle 1\rangle}(s)\, B^{\langle 1\rangle} + 2(F^{\langle 1\rangle T}(t-s) - F^{\langle 1\rangle T})\, E^{\langle 1\rangle}(t-s)]\, ds$$
$$+ \int_0^\infty \zeta^{[2]}(s)\, [B^{\langle 1\rangle} G^{\langle 1\rangle}(s) + G^{\langle 1\rangle}(s)\, B^{\langle 1\rangle}]\, ds$$
$$+ \int_0^\infty \int_0^\infty \alpha(s_1, s_2)\, G^{\langle 1\rangle}(s_1)\, G^{\langle 1\rangle}(s_2)\, ds_1\, ds_2\}.$$

In deriving the expression for M_2, we have used the identity

$$\nabla[\operatorname{tr}(F^{\langle 1\rangle} F^{\langle 1\rangle})] = 2 \operatorname{div} [F^{\langle 1\rangle T} F^{\langle 1\rangle T}].$$

Thus, at every order $n \geqq 1$, $u^{\langle n\rangle} = v$ and $\hat{p}^{\langle n\rangle} = \pi$ satisfy

$$\left.\begin{aligned} \varrho_0 \ddot{v} &= -\nabla\pi + \gamma\, \nabla^2 v + \int_0^\infty \zeta(s)\, \nabla^2 v(t-s)\, ds + f_1(X,t), \\ \operatorname{div} v &= f_2(X,t) \end{aligned}\right\} \text{ in } \mathscr{V}_0 \text{ for } t > 0;$$

$$v = f_3(X,t) \quad \text{on} \quad \partial\mathscr{V}_{01} \quad \text{for} \quad t > 0;$$

(3.34) $$\left[-\pi 1 + 2\gamma E(v) + 2\int_0^\infty \zeta(s)\, E(v(t-s))\, ds\right] N = f_4(X,t)$$

$$\text{on} \quad \partial\mathscr{V}_{02} \quad \text{for} \quad t > 0;$$

$$v = f_5(X,t) \quad \text{in} \quad \mathscr{V}_0 \quad \text{for} \quad t \leqq 0$$

where

(3.35) $$E(v) = [\nabla v + (\nabla v)^T]/2.$$

The inhomogeneous terms in (3.34) are known from the prescribed data and lower order solutions. At each order the set of equations (3.34) is to be solved for four unknown fields $v(X,t)$ and $\pi(X,t)$ in $\mathscr{V}_0 \times [0,\infty)$.

3.4 The Second Order Elastic Parameter $\beta^{[2]}$

We have already noted that it is not possible to distinguish elastic from viscoelastic materials under static loadings. In the elastic case various authors (GREEN & SHIELD (1951), GREEN & ADKINS (1960), CHAN & CARLSON (1970)) have derived second order theories of elasticity framed in terms of hyperelasticity using Mooney-type constitutive expressions.

To relate our first and second order elastic parameter β and $\beta^{[2]}$ to the Mooney parameters $\frac{\partial W}{\partial I}$ and $\frac{\partial W}{\partial II}$ evaluated at the undeformed natural state we note the following:

The Cauchy stress T is given by (2.23) as:

(3.36) $$T = f(B) = -p\, 1 + f_1(I, II)\, B + f_2(I, II)\, B^2.$$

In the hyper-elastic case

(3.37) $$T = -p1 + \left(2\frac{\partial W}{\partial I} + 2I\frac{\partial W}{\partial II}\right) B - 2\frac{\partial W}{\partial II} B^2$$

where W is the strain-energy per unit reference volume and

(3.38) $$f_1(I, II) = 2\frac{\partial W}{\partial I} + 2I\frac{\partial W}{\partial II}, \qquad f_2(I, II) = -2\frac{\partial W}{\partial II}.$$

From (3.24) and (3.38) we find that:

$$\beta = f_1^{\langle 0\rangle} + 2f_2^{\langle 0\rangle} = f_1(3,3) + 2f_2(3,3) = 2\left\{\left.\frac{\partial W}{\partial I}\right|_{I,II=3,3} + \left.\frac{\partial W}{\partial II}\right|_{I,II=3,3}\right\}$$

and

$$\beta^{[2]} = f_2^{(0)} = f_2(3, 3) = -2\left(\frac{\partial W}{\partial II}\bigg|_{I,II=3,3}\right).$$

The derivatives of W with respect to I and II evaluated at the rest state have been determined for the rubber-like materials used in the experiments of JONES & TRELOAR (1975) and HAINES & WILSON (1979).

(1) For the material used by HAINES & WILSON (1979), we have

$$\frac{\partial W}{\partial I}\bigg|_{I,II=3,3} = C_{10} = 0.179,$$

$$\frac{\partial W}{\partial II}\bigg|_{I,II=3,3} = C_{01} = 0.009$$

where C_{10} and C_{01} are given in MegaNewtons per square meter. Hence

$$\beta = 2(C_{10} + C_{01}) = 0.376,$$

$$\beta^{[2]} = -2C_{01} = -0.018.$$

(2) HAINES & WILSON (1975) have evaluated C_{10} and C_{01} for the material used in the experiments of JONES & TRELOAR (1975):

$$\frac{\partial W}{\partial I}\bigg|_{I,II=3,3} = C_{10} = 0.207,$$

$$\frac{\partial W}{\partial II}\bigg|_{I,II=3,3} = C_{01} = 0.0273.$$

Hence

$$\beta = 2(C_{10} + C_{01}) = 0.4686,$$

$$\beta^{[2]} = -2C_{01} = -0.0546.$$

(3) For the natural gum rubber used in the celebrated experiments of RIVLIN (1947),

$$\frac{C_{10}}{C_{01}} = 7.1 = -\frac{\beta}{\beta^{[2]}} - 1.$$

For all of these materials, $\beta^{[2]}$ is small and negative.

The free surface measurements reported in Appendix C of this paper may also be used to determine β and $\beta^{[2]}$ (or C_{10} and C_{01}). But the urethane rubber used in our experiment is *definitely not* a Mooney material in any set of deformations not close to static ones.

Chapter 4. Uniqueness and Stability

The main aim of this chapter is to show that the solution of the perturbation equations (3.34) are uniquely determined when the body force field f_1, the mass source field f_2 and the prescribed boundary conditions f_3, f_4 are given. The

difference between two solutions of (3.34) with the same prescribed conditions but different initial conditions f_5, satisfies the following initial history boundary-value problem:

$$\left.\begin{aligned}\mathbb{J}\boldsymbol{v} + \nabla\pi &= \mathbf{0},\\ \operatorname{div}\boldsymbol{v} &= 0\end{aligned}\right\} \quad \text{in} \quad \mathscr{V}_0 \quad \text{for} \quad t > 0;$$

(4.1)
$$\boldsymbol{v} = \mathbf{0} \quad \text{on} \quad \partial\mathscr{V}_{01} \quad \text{for} \quad t > 0;$$

$$\left\{-\pi\boldsymbol{I} + 2\gamma\boldsymbol{E}(\boldsymbol{v}) + 2\int_0^\infty \zeta(s)\,\boldsymbol{E}(\boldsymbol{v}(t-s))\,ds\right\}\boldsymbol{N} = \mathbf{0} \quad \text{on} \quad \partial\mathscr{V}_{02} \quad \text{for} \quad t > 0;$$

$$\boldsymbol{v} = \boldsymbol{v}_0 \text{ is prescribed in } \mathscr{V}_0 \text{ for } t \leqq 0$$

where

$$\mathbb{J}(\cdot) \equiv \varrho_0\ddot{\overline{(\cdot)}} - \gamma\,\nabla^2(\cdot) - \int_0^\infty \zeta(s)\,\nabla^2(\cdot)\,(t-s)\,ds.$$

We wish to establish the conditions on the (material) parameters of $\mathbb{J}$ which guarantee that $\boldsymbol{v}(t) \to \mathbf{0}$ as $t \to \infty$ independent of $\boldsymbol{v}_0 \neq 0$.* In this case, two different solutions with the same prescribed data will ultimately be the same; that is, the solution of (3.34) is determined ultimately by prescribed conditions independent of the initial history f_5. Phrasing this statement in another way, we may say that the conditions which lead to asymptotic stability of the solution $\boldsymbol{v} = \mathbf{0}$ of (4.1) imply that the natural state of the body is stable whenever the principle of linearized stability is valid. Phrasing it in yet another way, we say that we are seeking the conditions under which $\mathbb{J}$ is uniquely invertible and new solutions with $\boldsymbol{v} \neq \mathbf{0}$ cannot bifurcate.

The importance of this type of uniqueness can be appreciated by analogy with the theory of slow flow of a Navier-Stokes fluid. That theory is relatively uncomplicated because when the flow is slow (or the Reynolds number is small), there is just one solution in the long run and it is uniquely determined by the boundary conditions and body forces, independent of initial conditions. The unique solution is globally stable in the sense that all disturbances, small or large, of this solution ultimately decay. Thus we expect to observe in nature what we calculate from the equations when the flow is slow. And we do. But this simplicity is lost when the flow is not slow because there are many solutions for prescribed boundary conditions and body forces and many of these are unstable.

The situation of viscoelastic fluids and solids is not so different. In any event, when the forcing data is small, the natural state is stable in the linearized approximation provided only that material parameters and their derivatives have the expected sign. For larger forcing data, the problem of stability is probably at least as complicated as in the Navier-Stokes theory. In fact, shocks may occur in viscoelastic bodies, and smooth solutions may cease to exist, both effects being without parallel in the Navier-Stokes theory.

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* DIXIT (1979) has shown that $\boldsymbol{v} = \mathbf{0}$ is the unique solution of (4.1) corresponding to the initial rest history $\boldsymbol{v}_0 = \mathbf{0}$. FICHERA (1979) has given an existence and uniqueness theory for a general set of linear viscoelastic problems which contains our problem as a special case.

It is probable that a theory for existence, uniqueness and stability of the type recently given by SLEMROD (1977, 1978) for JOSEPH'S (1976) theory of motions which perturb the state of rest of simple fluids can be adapted to the present problem (4.1). But here a different approach is followed.

4.1 Linearized Stability of the "Displacement" Problem

Existence, uniqueness and asymptotic stability of *generalized* solutions of equations similar to (4.1) have been given by DAFERMOS (1970) for the displacement problem, *i.e.* for the problem in which displacement is prescribed on the entire boundary $\partial\mathscr{V}_0$. In the equations treated by DAFERMOS, the velocity field $\boldsymbol{v}$ is not necessarily solenoidal and $\pi = 0$. The problem (4.1) can be reduced to the one considered by DAFERMOS by a method of projection used in mathematical studies of the Navier-Stokes equations (FUJITA & KATO, 1964; LADYZHENSKAYA, 1963). In this method one introduces a Hilbert Space H by completing the space of $C^\infty(\mathscr{V}_0)$ vectors with compact support under the norm generated by the usual inner product in H. The compact support is natural for the displacement problem. For such problems it is possible to decompose H into the orthogonal subspaces of solenoidal vectors (H_1) and gradients (H_2) where $H = H_1 \oplus H_2$. There is then the unique orthogonal projection P which commutes with $\mathbb{J}$ and annihilates the gradients. Then

$$\begin{aligned} &\mathbb{J}\mathrm{P}\boldsymbol{v} = \boldsymbol{0} \quad \text{in} \quad \mathscr{V}_0 \quad \text{for} \quad t > 0, \\ (4.2)\qquad &\mathrm{P}\boldsymbol{v} = \boldsymbol{0} \quad \text{on} \quad \partial\mathscr{V}_0 \quad \text{for} \quad t > 0, \\ &\mathrm{P}\boldsymbol{v} = \mathrm{P}\boldsymbol{v}_0 \text{ is prescribed in } \mathscr{V}_0 \text{ for } t \leqq 0. \end{aligned}$$

Problem (4.2) falls in the frame of the study of DAFERMOS (1970), who has shown that $\|\dot{\boldsymbol{v}}\| \to 0$ and $\int_{\mathscr{V}_0} |\nabla \boldsymbol{v}|^2 \, dX \to 0$ as $t \to \infty$ provided that

$$\begin{aligned} &1. \quad \beta > 0, \varrho_0 > 0, \\ &2\text{a}. \quad \zeta, \dot{\zeta} \in C^0[0,\infty) \cup L^1[0,\infty), \\ (4.3)\qquad &2\text{b}. \quad \zeta(s) \leqq 0 \quad \text{on} \quad [0,\infty), \\ &2\text{c}. \quad \dot{\zeta}(s) \geqq 0 \quad \text{on} \quad [0,\infty), \\ &2\text{d}. \quad \zeta \text{ does not vanish identically.} \end{aligned}$$

These conditions do not disagree with conditions which rheologists would require on physical grounds using experience and intuition.

4.2 Linearized Stability

The asymptotic behavior of the solution $\boldsymbol{v}$ for problem (4.1) when tractions are prescribed on a part $\partial\mathscr{V}_{02}$ of the boundary has not been studied. The main difficulty seems to involve the boundary condition on the part $\partial\mathscr{V}_{02}$. The following formal argument, which is based on the theory of the Laplace transform, may be helpful in further studies of this problem.

Problem (4.1) may be rewritten as

$$(4.4)\qquad \begin{aligned} &\varrho_0 \ddot{\boldsymbol{v}} - \gamma \nabla^2 \boldsymbol{v} - \int_0^t \zeta(s) \nabla^2 \boldsymbol{v}(t-s)\, ds + \nabla \pi = \boldsymbol{f}_1, \\ &\operatorname{div} \boldsymbol{v} = \boldsymbol{0} \quad \text{in} \quad \mathscr{V}_0 \quad \text{for} \quad t > 0; \\ &\boldsymbol{v} = \boldsymbol{0} \qquad \text{on } \partial\mathscr{V}_{01} \text{ for } t > 0; \\ &\left\{-\pi \boldsymbol{1} + 2\gamma \boldsymbol{E}(\boldsymbol{x}) + 2 \int_0^t \zeta(s) \boldsymbol{E}(\boldsymbol{x}(t-s))\, ds\right\} \boldsymbol{N} = \boldsymbol{f}_2 \quad \text{on } \partial\mathscr{V}_{02} \text{ for } t > 0 \end{aligned}$$

where

$$(4.5)\qquad \begin{aligned} \boldsymbol{f}_1 &= \int_t^\infty \zeta(s) \nabla^2 \boldsymbol{v}(t-s)\, ds, \\ \boldsymbol{f}_2 &= -2 \int_t^\infty \zeta(s) \boldsymbol{E}(\boldsymbol{x}(t-s)) \boldsymbol{N}\, ds, \end{aligned}$$

depend on $\boldsymbol{v}_0$ and are known.

A function $f(\boldsymbol{X}, t)$ is said to possess a Laplace transform in t uniformly with respect to $\boldsymbol{X}$ if and only if the integral

$$\hat{f}(\boldsymbol{X}, \xi) = \int_0^\infty f(\boldsymbol{X}, t)\, e^{-\xi t}\, dt$$

converges absolutely and uniformly in $\boldsymbol{X}$ for all real ξ greater than some real number ξ_0. The following results from the theory of the Laplace transform are needed:

a) $\hat{\ddot{f}} = \xi^2 \hat{f} - \xi f|_{t=0} - \dot{f}|_{t=0}$.

b) For a convolution

$$(f * g)(t) = \int_0^t f(s)\, g(t-s)\, ds = \int_0^t f(t-s)\, g(s)\, ds,$$

$$\widehat{f * g} = \hat{f}\hat{g}.$$

We now assume that ζ, $\boldsymbol{v}$, $\nabla\boldsymbol{v}$, $\operatorname{div}\boldsymbol{v}$, $\nabla^2\boldsymbol{v}$, π, $\nabla\pi$, $\boldsymbol{f}_1$ and $\boldsymbol{f}_2$ possess Laplace transforms in t uniformly with respect to $\boldsymbol{X}$. Then, using (4.4), we find that

$$\begin{aligned} &\lambda\hat{\boldsymbol{v}} + \nabla^2\hat{\boldsymbol{v}} - \nabla\hat{p} = \boldsymbol{f}_3, \quad \operatorname{div}\boldsymbol{v} = 0 \quad \text{in} \quad \mathscr{V}_0 \quad \text{for} \quad \xi > \xi_0; \\ &\hat{\boldsymbol{v}} = \boldsymbol{0}, \quad \text{on} \quad \partial\mathscr{V}_{01} \quad \text{for} \quad \xi > \xi_0; \\ &\{-\hat{p}\boldsymbol{1} + 2\boldsymbol{E}(\hat{\boldsymbol{v}})\}\, \boldsymbol{N} = \boldsymbol{f}_4 \quad \text{on} \quad \partial\mathscr{V}_{02} \quad \text{for} \quad \xi > \xi_0 \end{aligned}$$

where

$$\begin{aligned} \boldsymbol{f}_3(\boldsymbol{X}, \xi) &= \{\hat{\boldsymbol{f}}_1(\boldsymbol{X}, \xi) + \varrho_0[\xi \boldsymbol{v}(\boldsymbol{X}, 0) + \dot{\boldsymbol{v}}(\boldsymbol{X}, 0)]\}/\{-\chi(\xi)\}, \\ \boldsymbol{f}_4(\boldsymbol{X}, \xi) &= \hat{\boldsymbol{f}}_2(\boldsymbol{X}, \xi)/\chi(\xi), \\ \hat{p}(\boldsymbol{X}, \xi) &= \hat{\pi}(\boldsymbol{X}, \xi)/\chi(\xi), \end{aligned}$$

and

$$\lambda(\xi) = -\varrho_0 \xi^2/\chi(\xi)$$

where

$$\chi(\xi) = \beta + \int_0^\infty \zeta(s)\,[e^{-\xi s} - 1]\,ds.$$

To invert the Laplace transform, the analytic extension

$$\hat{V}(X, \sigma) \equiv \int_0^\infty v(X, t)\, e^{-\sigma t}\, dt$$

is introduced where $\sigma = \xi + i\eta$. Similar analytic extensions are introduced for π, f_1, f_2 and ζ. Then

$$\left.\begin{aligned} \lambda\hat{V} + \nabla^2\hat{V} - \nabla P &= F_3,\\ \operatorname{div}\hat{V} &= 0 \end{aligned}\right] \text{ in } \mathscr{V}_0;$$

(4.6) $$\hat{V} = \mathbf{0} \quad \text{on} \quad \partial\mathscr{V}_{01};$$

$$\{-P1 + 2E(\hat{V})\}\, N = F_4 \quad \text{on} \quad \partial\mathscr{V}_{02}$$

where F_3, F_4 and P are obtained by replacing ξ with σ and $\lambda = \lambda(\sigma)$. The spectrum of the linear operator defined by (4.6) is the collection of all complex values σ for which (4.6) is not uniquely invertible with continuous inverse. For other values of σ, not in the spectrum, (4.6) is uniquely invertible and

$$\hat{V} = R_\sigma F_3 + S_\sigma F_4$$

depends continuously on the initial history $v_0(X, \tau)$, $\tau \leqq 0$ through F_3, F_4 and the tensor-valued resolvent operators R_σ and S_σ. The values σ not in the spectrum are said to be in the resolvent set. One may use the Laplace inversion integral to compute

(4.7) $$v(t) = \frac{1}{2\pi i}\int_{\xi - i\infty}^{\xi + i\infty} e^{\sigma t}\hat{V}\, d\sigma = \frac{1}{2\pi i}\int_{\xi - i\infty}^{\xi + i\infty} e^{\sigma t}[R_\sigma F_3 + S_\sigma F_4]\, d\sigma$$

where $\xi > \xi_0$ so that $R_\sigma F_3 + S_\sigma F_4$ is analytic for $\operatorname{Re}\sigma > \xi_0$. The singularities of $R_\sigma F_3 + S_\sigma F_4$ coincide with the spectral values σ associated with the problem (4.6).

Suppose it can be shown that all the spectral values have negative real part ($\xi < 0$). Let δ be the spectral value with the largest real part. Then $R_\sigma F_3 + S_\sigma F_4$ is analytic for $\xi > \xi_1 = R_e\,\delta_1$ and in (4.7) one can choose ξ to be negative ($\xi_1 < \xi < 0$). Hence $v(t) \to \mathbf{0}$ as $t \to \infty$.

4.3 Linearized Stability of the Mixed Problem: Spectral Method

Substitution of exponential solutions $(v(X, t), \pi(X, t)) = e^{\sigma t}(v(X), \pi(X))$, $\sigma = \xi + i\eta$, into (4.1) leads us to the spectral problem

$$\left.\begin{aligned} \lambda v + \nabla^2 v - \nabla\tilde{\pi} &= \mathbf{0}\\ \operatorname{div} v &= 0 \end{aligned}\right] \text{ in } \mathscr{V}_0,$$

(4.8) $$v = \mathbf{0} \quad \text{on} \quad \partial\mathscr{V}_{01},$$

$$\{-\tilde{\pi}1 + 2E(v)\}\, N = \mathbf{0} \quad \text{on} \quad \partial\mathscr{V}_{02}$$

where $\tilde{\pi} = \pi/\chi(\sigma)$ and $\lambda = \lambda(\sigma)$. In the context of the spectral problem, the natural state is stable if $\operatorname{re}\sigma = \xi < 0$ for all spectral values σ and is unstable if there are values $\xi > 0$.

The determination of the sign of ξ in the spectrum of (4.8) is simplified by the fact that the spectral values of (4.8) may be characterized as critical points of a positive symmetric functional

(4.9) $$\lambda[v] = 2\int_{\mathscr{V}_0} |E(v)|^2\, dX \Big/ \int_{\mathscr{V}_0} |v|^2\, dX$$

on a Hilbert space

$$H = [v|_{\partial\mathscr{V}_0} = \mathbf{0},\quad \operatorname{div} v = 0,\quad \int_{\mathscr{V}_0} |E(v)|^2\, dX < \infty].$$

Equations (4.8) are Euler's variational equations for the functional (4.9). The last condition of (4.8) arises as a natural boundary condition. Standard theory for such variational problems implies that

(1) *The spectrum of* (4.8) *is real valued and discrete; that is the values λ for which* (4.8) *has non-trivial solutions are eigenvalues.*

(2) *The number of eigenvalues is countably infinite. They are of finite multiplicity, all semi-simple and may be arranged as an increasing sequence clustering at infinity*

$$0 \leqq \lambda_1 \leqq \lambda_2 \leqq \ldots < \infty.$$

(3) *If the measure of the set of points $\partial\mathscr{V}_{01}$ on which $v = \mathbf{0}$ is greater than zero, then $\lambda_1 > 0$* (CAMPANATO, 1959, 1962).

Condition (3) eliminates solutions $v \neq \mathbf{0}$, $E[v] = \mathbf{0}$ in $\mathscr{V}_0$ and guarantees the existence of a positive greatest lower bound for $\lambda[v]$.

Now we prove a theorem giving conditions on the material parameters β and $\zeta(s)$ which guarantee that $\xi < 0$ for all eigenvalues.

Theorem. *Let*

a) $\varrho_0 > 0$, $\beta > 0$,

b) $\zeta(s) \leqq 0$ *on* $[0, \infty)$ *and* $\zeta(s) \to 0$ *as* $s \to \infty$ *monotonically. Then all the eigenvalues* σ *of* (4.8) *have negative real parts* $\xi < 0$.

To prove this theorem, we note that the eigenvalues $\lambda = -\varrho_0\sigma^2/\chi(\sigma)$ are positive, finite and strictly greater than zero. Recalling now the definition of $\chi(\sigma)$, we may rewrite the defining equation for $\lambda(\sigma)$ as

(4.10) $$m\left\{\beta + \int_0^\infty \zeta(s)\,(e^{-\sigma s} - 1)\, ds\right\} = -\sigma^2$$

where

$$m = \lambda/\varrho_0 > 0.$$

Case 1: $\eta = 0$.

For $\eta = 0$, (4.10) reduces to

(4.11) $$m\left\{\beta + \int_0^\infty \zeta(s)\,(e^{-\xi s} - 1)\, ds\right\} = -\xi^2.$$

If $\xi \geqq 0$, the left hand side of (4.11) is positive while the right hand side is always non-positive. Hence, $\xi < 0$.

Case 2: $\eta \neq 0$.
In this case equation (4.10) may be decomposed into real and imaginary parts:

$$(4.12)\qquad \begin{aligned} &\text{a)}\quad m\left\{\beta + \int_0^\infty \zeta(s)\,(e^{-\xi s}\cos\eta s - 1)\,ds\right\} = -(\xi^2 - \eta^2),\\ &\text{b)}\quad (m/2)\left\{\int_0^\infty \zeta(s)\,e^{-\xi s}\,(\sin\eta s/\eta)\,ds\right\} = \xi. \end{aligned}$$

Let $\xi \geqq 0$. Then the integrand in (4.12b) is negative for small values of s, it changes sign at each zero of $\sin\eta s$ and the contribution to the integral on each interval is of decreasing magnitude. The negative contributions are therefore larger than the positive ones. Hence, the left side of (4.12b) is negative and ξ cannot be $\geqq 0$; thus $\xi < 0$.

Chapter 5. The Shape of Traction Free Surfaces for Axisymmetric Problems

The problems studied in the next two chapters involve deformation of visco-elastic solids which are right circular cylinders in the natural state. The normal stress effects which we compute are analogous to Weissenberg effects in fluids and, as in problems concerning fluids, these effects first appear at second order. Since the material parameters of the second order theory are not yet known, even for one solid, we may hope to determine something about them by comparing the shapes of predicted and observed free surfaces.

5.1 Kinematics for Axisymmetric Problems

In axisymmetric problems the stresses and the displacements are independent of the angular coordinate Θ. Let $(\boldsymbol{e}_r, \boldsymbol{e}_\theta, \boldsymbol{e}_z)$ and $(\boldsymbol{e}_R, \boldsymbol{e}_\Theta, \boldsymbol{e}_Z)$ denote the ortho-normal cylindrical bases in the distorted and the natural states, respectively. To compute the components of displacement,

$$(5.1)\qquad \boldsymbol{u} = \boldsymbol{x} - \boldsymbol{X}$$

where

$$(5.2)\qquad \begin{aligned} \boldsymbol{x} &= r\boldsymbol{e}_r(\theta) + z\boldsymbol{e}_z,\\ \boldsymbol{X} &= R\boldsymbol{e}_R(\Theta) + Z\boldsymbol{e}_Z. \end{aligned}$$

We need to expand the coordinates of $\boldsymbol{x}$ in powers of ε:

$$(5.3)\qquad \begin{aligned} r &= R + \varepsilon r^{\langle 1\rangle}(R, Z, t) + \varepsilon^2 r^{\langle 2\rangle}(R, Z, t) + O(\varepsilon^3),\\ \theta &= \Theta + \varepsilon\theta^{\langle 1\rangle}(R, Z, t) + \varepsilon^2\theta^{\langle 2\rangle}(R, Z, t) + O(\varepsilon^3),\\ z &= Z + \varepsilon z^{\langle 1\rangle}(R, Z, t) + \varepsilon^2 z^{\langle 2\rangle}(R, Z, t) + O(\varepsilon^3). \end{aligned}$$

The expansion of $\boldsymbol{e}_r(\theta)$ is

$$(5.4)\qquad \boldsymbol{e}_r(\theta) = \boldsymbol{e}_R + \varepsilon\theta^{\langle 1\rangle}\boldsymbol{e}_\Theta + \varepsilon^2[\theta^{\langle 2\rangle}\boldsymbol{e}_\Theta - (\theta^{\langle 1\rangle 2}/2)\,\boldsymbol{e}_R] + O(\varepsilon^3).$$

Combining (5.1)–(5.4), we get

$$(5.5)\qquad \boldsymbol{u} = \varepsilon\boldsymbol{u}^{\langle 1\rangle} + \varepsilon^2\boldsymbol{u}^{\langle 2\rangle} + O(\varepsilon^3)$$

where

$$(5.6)\qquad \boldsymbol{u}^{\langle 1\rangle} = r^{\langle 1\rangle}\boldsymbol{e}_R + R\theta^{\langle 1\rangle}\boldsymbol{e}_\Theta + z^{\langle 1\rangle}\boldsymbol{e}_Z$$

and

$$(5.7)\qquad \boldsymbol{u}^{\langle 2\rangle} = (r^{\langle 2\rangle} - R\theta^{\langle 1\rangle 2}/2)\,\boldsymbol{e}_R + (R\theta^{\langle 2\rangle} + r^{\langle 1\rangle}\theta^{\langle 1\rangle})\,\boldsymbol{e}_\Theta + z^{\langle 2\rangle}\boldsymbol{e}_Z.$$

The components of $\boldsymbol{F}^{\langle 1\rangle}$ in the basis $(\boldsymbol{e}_R, \boldsymbol{e}_\Theta, \boldsymbol{e}_Z)$ are

$$(5.8)\qquad [\boldsymbol{F}^{\langle 1\rangle}] = \begin{bmatrix} r_R^{\langle 1\rangle} & -\theta^{\langle 1\rangle} & r_Z^{\langle 1\rangle}\\ (R\theta^{\langle 1\rangle})_R & r^{\langle 1\rangle}/R & R\theta_Z^{\langle 1\rangle}\\ z_R^{\langle 1\rangle} & 0 & z_Z^{\langle 1\rangle} \end{bmatrix}$$

and $\boldsymbol{E}^{\langle 1\rangle}$ is the symmetric part of $\boldsymbol{F}^{\langle 1\rangle}$. Here and elsewhere $(\cdot)_R$ and $(\cdot)_Z$ denote partial derivatives.

5.2 Simplification due to Symmetry

The data to be prescribed in the two problems treated in Chapters 5 and 6 is some type of assignment of a twist proportional to ε. In all cases, the sign of the azimuthal deformation and shear stress changes with ε whereas the displace-ments of points in radial planes through the origin, the pressure, and the other components of the stress are unchanged. It therefore follows from uniqueness that

$$(5.9)\qquad \theta^{\langle 2n\rangle}, T_{RR}^{\langle 2n\rangle}, T_{Z\theta}^{\langle 2n\rangle}, r^{\langle 2n-1\rangle}, z^{\langle 2n-1\rangle}, T_{RR}^{\langle 2n-1\rangle}, T_{ZZ}^{\langle 2n-1\rangle}, T_{\Theta\Theta}^{\langle 2n-1\rangle}, T_{RZ}^{\langle 2n-1\rangle}, p^{\langle 2n-1\rangle}$$

all vanish for $n = 1, 2, 3, \ldots$

To simplify notations, we define

$$\phi \overset{\text{def}}{=} \theta^{\langle 1\rangle},$$
$$(\hat r, \hat z) \overset{\text{def}}{=} (r^{\langle 2\rangle}, z^{\langle 2\rangle}),$$
$$\hat p \overset{\text{def}}{=} p^{\langle 2\rangle}.$$

Then

$$(5.10)\qquad \begin{aligned} r &= R + \varepsilon^2\hat r(R, Z, t) + O(\varepsilon^4),\\ \theta &= \Theta + \varepsilon\phi(R, Z, t) + O(\varepsilon^3),\\ z &= Z + \varepsilon^2\hat z(R, Z, t) + O(\varepsilon^4),\\ \boldsymbol{e}_r(\theta) &= \boldsymbol{e}_R + \varepsilon\phi\boldsymbol{e}_\Theta - \varepsilon^2\phi^2\boldsymbol{e}_R/2 + O(\varepsilon^3),\\ \boldsymbol{u}^{\langle 1\rangle} &= R\phi\boldsymbol{e}_\Theta,\\ \boldsymbol{u}^{\langle 2\rangle} &= (\hat r - R\phi^2/2)\,\boldsymbol{e}_R + \hat z\boldsymbol{e}_Z, \end{aligned}$$

$$[F^{(1)}]=\begin{bmatrix}0 & -\phi & 0\\ (R\phi)_R & 0 & R\phi_Z\\ 0 & 0 & 0\end{bmatrix},$$

$$\operatorname{div} \boldsymbol{u}^{(2)}-\frac{1}{2}\operatorname{tr} F^{(1)}F^{(1)}=\frac{1}{R}\frac{\partial}{\partial R}(R\hat{r})+\frac{\partial \hat{z}}{\partial Z}=0.$$

The last of equations (5.10) shows that it is possible to identify $\hat{r}$ and $\hat{z}$ with a stream function

$$(\hat{r},\hat{z})=\frac{1}{R}(\psi_Z,-\psi_R).$$

5.3 Parametric Representation of the Shape of Free Surfaces

The traction free surface

(5.11) $$F(r,z,t,\varepsilon)=0$$

in the deformed state is the image of the surface

(5.12) $$R=\hat{R}(v),\quad Z=\hat{Z}(v)$$

given parametrically, with parameter v, in the natural state. The parameter ε is an amplitude given by prescribed data, t is the time and r, z are the displacement functions defined by $(5.10)_{1,3}$. It follows that (5.11) is also given parametrically by $(5.10)_{1,3}$ and (5.12):

(5.13) $$\begin{aligned} r&=\hat{R}(v)+\varepsilon^2\hat{r}(\hat{R}(v),\hat{Z}(v),t)+O(\varepsilon^4),\\ z&=\hat{Z}(v)+\varepsilon^2\hat{z}(\hat{R}(v),\hat{Z}(v),t)+O(\varepsilon^4).\end{aligned}$$

In the rod climbing problem studied in Chapter 6, the data is assumed to be time independent, so the material has a purely elastic response. The traction free surfaces are warped images of the end planes $Z=\pm L$ of the natural state (see Fig. 6.1). Each traction free warped plane, $F(r,z,\varepsilon)=0$, is defined parametrically with parameter R by

(5.14) $$\begin{aligned} r&=R+\varepsilon^2\hat{r}(R,\pm L)+O(\varepsilon^4),\\ z&=\pm L+\varepsilon^2\hat{z}(R,\pm L)+O(\varepsilon^4).\end{aligned}$$

Conservation of volume, through terms of order two implies that

$$\int_a^b \{\hat{z}(R,L)-\hat{z}(R,-L)\}\,R\,dR=0.$$

This condition is automatically satisfied when $\hat{z}=-\psi_{R/R}$ is expressed in terms of a stream function ψ.

In the problem of torsional oscillations of a viscoelastic cylinder treated in Chapter 7, the traction free surface is the warped image of the round surface of the cylinder of radius a of the natural state (see Figure 7.1). The traction free surface is defined parametrically with parameter Z by

(5.15) $$\begin{aligned} r&=a+\varepsilon^2\hat{r}(a,Z,t)+O(\varepsilon^4),\\ z&=Z+\varepsilon^2\hat{z}(a,Z,t)+O(\varepsilon^4).\end{aligned}$$

To compute the first significant nonlinear contribution to the distortion of the free surface, we must compute the second order displacement function $\hat{r}$ and $\hat{z}$. The volume of the cylinder shown on the left of Figure 7.1 will be conserved to second order if

$$2La^2=\int_{-L}^{L}\imath^2(a,z,t,\varepsilon)\,dz$$

where $F=0$ has been expressed by $r=\imath(a,z,t,\varepsilon)$. After changing variables,

$$2La^2=\int_{-L}^{L} r^2(a,Z,t,\varepsilon)\left.\frac{\partial z}{\partial Z}\right|_{R=a}dZ$$

and, using (5.15) and $\hat{z}(a,\pm L)=0$, we find that

$$\int_{-L}^{L}\hat{r}(a,Z,t)\,dZ=0$$

With $\hat{r}=\dfrac{1}{R}\partial\psi/\partial Z$ and $\psi(a,L)=-\psi(a,-L)$, this condition is automatically satisfied.

Chapter 6. Rod Climbing

If a small rod is rotated in a nonlinear viscoelastic fluid, the fluid will climb up the rod. This well known phenomenon is one of the principal manifestations of normal stress effects in fluids. These effects are sometimes called Weissenberg effects, without historical justification, and the problem giving rise to them is called rod climbing. A mathematical solution of the fluids problem was constructed by Joseph & Fosdick (1973) using perturbation methods in the spirit of this paper. The lowest order solution, at order ε^2, is elementary, bordering on trivial, but the physical consequences of the elementary solution are of definite interest (see Joseph & Beavers, 1977).

Since solids also exhibit normal stress effects, a theory of free surfaces invites definition of the problem for solids that seems nearest to the problem of rod climbing for fluids. We define that problem forthwith.

6.1 Prescribed Data for Rod Climbing

A viscoelastic solid cylindrical annulus is bounded internally by a rigid rod of radius a and externally by a rigid cylinder of radius b. The rigid rod and cylinder are designed so that the viscoelastic solid can be twisted by the rigid rod and held stationary by the rigid cylinder. But in contrast with the problem for fluids, here the rigid members are not bonded to the viscoelastic solid; instead the solid may slip along the straight generators of their round surfaces. Such bonding might be achieved by grooving the contacting boundaries along the generators of the cylinders. The solid will rotate with the rigid rod but may slide in the axial direction. The configuration of the solid in the natural and the deformed states is sketched in Figure 6.1.

(a) Natural state

$\mathscr{V}_0 = \{R, \Theta, Z: a \leqq R \leqq b,$
$0 \leqq \Theta < 2\pi, -L \leqq Z \leqq L\}$

(b) Deformed state

$\mathscr{V} = \{r, \theta, z: a \leqq r \leqq b, 0 \leqq \theta < 2\pi,$
$z(R, -L, \varepsilon) \leqq z \leqq z(R, L, \varepsilon)\}$

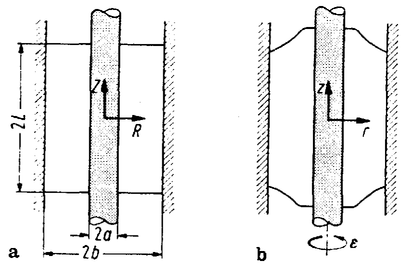


Fig. 6.1. Rod climbing of an elastic solid. The elastic solid twists with the rod but shear tractions along cylinder generators vanish.

The conditions which we have prescribed may be summarized as follows.
(i) Points on $R = a$ are rotated through an angle ε. Points on $R = b$ are prevented from rotating. Equation $(5.10)_2$ then implies that

(6.1) $$\phi(a, Z) = 1, \quad \phi(b, Z) = 0.$$

(ii) There is no radial displacement of points originally on $R = a$ and $R = b$. Equation $(5.10)_1$ then implies

(6.2) $$\hat{r}(a, Z) = \hat{r}(b, Z) = 0.$$

(iii) The shear stress along cylinder generators vanishes at $R = a$ and $R = b$,

(6.3) $$T_{rz}(a, Z) = T_{rz}(b, Z) = 0.$$

(iv) The traction vector $\boldsymbol{Tn}$ vanishes on the warped planes at the ends. Then (3.27) and (3.28) with $\boldsymbol{N} = \boldsymbol{e}_Z$ becomes

$(6.4)_1$ $$\boldsymbol{T}^{\langle 1\rangle}\boldsymbol{e}_Z = \boldsymbol{0},$$

$(6.4)_2$ $$\boldsymbol{T}^{\langle 2\rangle}\boldsymbol{e}_Z + [(\boldsymbol{F}^{\langle 1\rangle T}\boldsymbol{e}_Z)\cdot\boldsymbol{e}_Z]\,\boldsymbol{T}^{\langle 1\rangle}\boldsymbol{e}_Z - \boldsymbol{T}^{\langle 1\rangle}\boldsymbol{F}^{\langle 1\rangle T}\boldsymbol{e}_Z = \boldsymbol{0}.$$

Since the prescribed data is steady, the response of the material is purely elastic. In the remaining parts of this chapter we will speak of an elastic solid with the understanding that we mean the elastic response of a viscoelastic solid.

The static response of an incompressible viscoelastic solid is elastic and at second order it depends only on the two elastic constants β and $\beta^{[2]}$. The shear modulus β is known for many materials but $\beta^{[2]}$ is known only for a few rubber-

like solids (see § 3.4). The situation is vastly more complicated when data is not steady because then, besides the two elastic constants, we must know three viscoelastic functions. In the elastic case the equations under (3.23) giving the stress reduce to

(6.5) $$\boldsymbol{T}^{\langle 1\rangle} = 2\beta\boldsymbol{E}^{\langle 1\rangle},$$

(6.6) $$\boldsymbol{T}^{\langle 2\rangle} = -p^{\langle 2\rangle}\boldsymbol{1} + 2\beta\boldsymbol{E}^{\langle 2\rangle} + \beta\boldsymbol{F}^{\langle 1\rangle}\boldsymbol{F}^{\langle 1\rangle T} + \beta^{[2]}\boldsymbol{B}^{\langle 1\rangle}\boldsymbol{B}^{\langle 1\rangle}$$

where $\boldsymbol{B}^{\langle 1\rangle}(t) = 2\boldsymbol{E}^{\langle 1\rangle}(t) = \boldsymbol{F}^{\langle 1\rangle} + \boldsymbol{F}^{\langle 1\rangle T}$. In fact, we shall show that if

$$1 + \frac{4\beta^{[2]}}{\beta} > 0,$$

the stress free surface will climb the rigid rod at $r = a$ and sink at $r = b$. If $\beta^{[2]} < -\beta/4$, then the stress free surface will climb at $r = b$ and sink at $r = a$.

6.2 The First Order Solution

At first order (5.9) implies that

(6.7) $$r^{\langle 1\rangle} = z^{\langle 1\rangle} = T_{RR}{}^{\langle 1\rangle} = T_{ZZ}{}^{\langle 1\rangle} = T_{\Theta\Theta}{}^{\langle 1\rangle} = T_{RZ}{}^{\langle 1\rangle} = p^{\langle 1\rangle} = 0$$

and (3.29) may be reduced to an identity if $\phi = \phi(R)$, independent of Z, where

(6.8) $$(R\phi)'' + \frac{1}{R}(R\phi)' - \frac{\phi}{R} = 0, \quad \phi(a) = 1, \quad \phi(b) = 0.$$

Hence

(6.9) $$\phi = \frac{a^2}{b^2 - a^2}\left[\left(\frac{b}{R}\right)^2 - 1\right].$$

It follows from $(5.10)_7$ that

(6.10) $$\boldsymbol{F}^{\langle 1\rangle} = -\boldsymbol{e}_R\boldsymbol{e}_\Theta\phi + \boldsymbol{e}_\Theta\boldsymbol{e}_R(R\phi)',$$

and from (6.5) that

$$\boldsymbol{T}^{\langle 1\rangle} = \beta\, 2\boldsymbol{E}^{\langle 1\rangle} = \beta(\boldsymbol{e}_R\boldsymbol{e}_\Theta + \boldsymbol{e}_\Theta\boldsymbol{e}_R)\, R\phi'.$$

The other conditions of (3.29) are now automatically satisfied; in particular, $\boldsymbol{T}^{\langle 1\rangle}\boldsymbol{N} = \boldsymbol{T}^{\langle 1\rangle}\boldsymbol{e}_Z = \boldsymbol{0}$.

6.3 The Second Order Equations

To compute the solution at second order, we must evaluate the inhomogeneous terms in (3.30). We find, using $\boldsymbol{B}^{\langle 1\rangle} = 2\boldsymbol{E}^{\langle 1\rangle}$, that

(6.11) $$\begin{aligned} \boldsymbol{M}_2 &= \operatorname{div}\beta^{[2]}\boldsymbol{B}^{\langle 1\rangle}\boldsymbol{B}^{\langle 1\rangle} = \beta^{[2]}\operatorname{div}4\boldsymbol{E}^{\langle 1\rangle}\boldsymbol{E}^{\langle 1\rangle} \\ &= \beta^{[2]}\operatorname{div}(\boldsymbol{e}_R\boldsymbol{e}_R + \boldsymbol{e}_\Theta\boldsymbol{e}_\Theta)(R\phi')^2 = \beta^{[2]}\{R^2\phi'^2\}'\boldsymbol{e}_R \\ &= -\frac{16\beta^{[2]}a^4b^4}{(b^2 - a^2)^2R^5}\,\boldsymbol{e}_R. \end{aligned}$$

The only other inhomogeneous term appearing in the boundary value problem for $\boldsymbol{u}^{\langle 2\rangle}$ and $p^{\langle 2\rangle}=\hat{p}$ arises from

$$\nabla^2\boldsymbol{u}^{\langle 2\rangle}=\nabla^2(\hat{r}\boldsymbol{e}_R+\hat{z}\boldsymbol{e}_Z-\boldsymbol{e}_R R\phi^2/2). \tag{6.12}$$

Using (6.8) and (6.9), we compute

$$\nabla^2\boldsymbol{e}_R R\phi^2/2=R\phi'^2\boldsymbol{e}_R=\frac{4a^4b^4}{(b^2-a^2)^2R^5}\boldsymbol{e}_R.$$

We now show that no other inhomogeneous terms arise in the boundary value problem for $\hat{r}$, $\hat{z}$, and $\hat{p}$. The only place where such terms could arise are in the traction conditions (6.3) and (6.4). From (6.3) we have

$$T_{RZ}^{\langle 2\rangle}=0\quad\text{at}\quad R=a,b, \tag{6.13}$$

and from (6.4) we find, using (6.10) and (6.5), that

$$\boldsymbol{T}^{\langle 2\rangle}\boldsymbol{e}_Z=\boldsymbol{0}\quad\text{on}\quad Z=\pm L. \tag{6.14}$$

It suffices to show that $\boldsymbol{T}^{\langle 2\rangle}\boldsymbol{e}_Z$ does not depend explicitly on ϕ. We have

$$\boldsymbol{T}^{\langle 2\rangle}=-\hat{p}\boldsymbol{1}+2\beta\boldsymbol{E}^{\langle 2\rangle}+\beta\boldsymbol{F}^{\langle 1\rangle}\boldsymbol{F}^{\langle 1\rangle T}+\beta^{[2]}\boldsymbol{B}^{\langle 1\rangle}\boldsymbol{B}^{\langle 1\rangle}$$

where $\boldsymbol{E}^{\langle 2\rangle}=[\boldsymbol{F}^{\langle 2\rangle}+\boldsymbol{F}^{\langle 2\rangle T}]/2$ and, using $(5.10)_6$,

$$\begin{aligned}\boldsymbol{F}^{\langle 2\rangle}&=\nabla\boldsymbol{u}^{\langle 2\rangle}=\nabla[\hat{r}\boldsymbol{e}_R+\hat{z}\boldsymbol{e}_Z\}-\nabla(R\phi^2/2)\,\boldsymbol{e}_R\\&=[\hat{r}-R\phi^2/2]_R\,\boldsymbol{e}_R\boldsymbol{e}_R+\hat{z}_R\boldsymbol{e}_Z\boldsymbol{e}_R+\left(\frac{\hat{r}}{R}-\phi^2/2\right)\boldsymbol{e}_\Theta\boldsymbol{e}_\Theta+\hat{r}_Z\boldsymbol{e}_R\boldsymbol{e}_Z+\hat{z}_Z\boldsymbol{e}_Z\boldsymbol{e}_Z.\end{aligned}$$

Hence

$$[2\boldsymbol{E}^{\langle 2\rangle}]=\begin{bmatrix}2\hat{r}_R-(R\phi^2)' & 0 & \hat{r}_Z+\hat{z}_R\\ 0 & 2\dfrac{\hat{r}}{R}-\phi^2 & 0\\ \hat{r}_Z+\hat{z}_R & 0 & 2\hat{z}_Z\end{bmatrix}$$

and

$$2\boldsymbol{E}^{\langle 2\rangle}\boldsymbol{e}_Z=(\hat{r}_Z+\hat{z}_R)\,\boldsymbol{e}_R+2\hat{z}_Z\boldsymbol{e}_Z.$$

Since $\boldsymbol{F}^{\langle 1\rangle}\boldsymbol{F}^{\langle 1\rangle T}\boldsymbol{e}_Z=\boldsymbol{0}$ and $\boldsymbol{B}^{\langle 1\rangle}\boldsymbol{B}^{\langle 1\rangle}\boldsymbol{e}_Z=\boldsymbol{0}$, it follows that

$$T_{RZ}^{\langle 2\rangle}=\beta[\hat{r}_Z+\hat{z}_R] \tag{6.15}$$

and

$$T_{ZZ}^{\langle 2\rangle}=-\hat{p}+2\beta\hat{z}_Z. \tag{6.16}$$

The differential equation $(3.30)_1$ may now be written as

$$0=-\nabla\hat{p}+\beta\,\nabla^2(\hat{r}\boldsymbol{e}_R+\hat{z}\boldsymbol{e}_Z)-\beta k\boldsymbol{e}_R/R^5 \tag{6.17}$$

where

$$k=4\left(1+\frac{4\beta^{[2]}}{\beta}\right)a^4b^4/(a^2-b^2)^2.$$

We may now write (6.17) in component form and collect all the other conditions satisfied by $\hat{r}$, $\hat{z}$ and $\hat{p}$. The equations which govern in $\mathscr{V}_0$ (see Fig. 6.1) are

$$\left\{\begin{aligned}&-\frac{\partial p}{\partial R}+\beta\left[\nabla_{II}{}^2\hat{r}-\frac{\hat{r}}{R^2}\right]=\beta k/R^5,\\&-\frac{\partial p}{\partial Z}+\beta\,\nabla_{II}{}^2\hat{z}=0,\\&\nabla_{II}{}^2=\frac{1}{R}\frac{\partial}{\partial R}R\frac{\partial}{\partial R}+\frac{\partial^2}{\partial Z^2},\\&\frac{\partial}{\partial R}(\hat{r}R)+\frac{\partial}{\partial Z}(\hat{z}R)=0.\end{aligned}\right. \tag{6.18}$$

The boundary conditions (6.2), (6.3) and (6.4) are

$$\hat{r}=0\quad\text{and}\quad\hat{r}_Z+\hat{z}_R=0\quad\text{on}\quad R=a,b \tag{6.19}$$

and

$$\hat{r}_Z+\hat{z}_R=0\quad\text{and}\quad-\hat{p}+2\beta\hat{z}_Z=0\quad\text{on}\quad Z=\pm L. \tag{6.20}$$

6.4 The Stream Function

Equation $(6.18)_4$ may be reduced to an identity by a stream function, where

$$\begin{aligned}R\hat{z}&=-\psi_R,\\R\hat{r}&=\psi_Z.\end{aligned} \tag{6.21}$$

We introduce the stream function into (6.18), (6.19) and (6.20), eliminate pressure from $(6.18)_{1,2}$ by cross differentiation, and eliminate the pressure from $(6.20)_2$ by first differentiating with respect to R and then replacing $\partial\hat{p}/\partial R$ using $(6.18)_1$. In this way we find that

$$\begin{aligned}&\mathscr{L}^2\psi=0\quad\text{in}\quad\mathscr{V}_0,\\&\psi_Z=\hat{\mathscr{L}}\psi=0\quad\text{on}\quad R=a,b,\\&\hat{\mathscr{L}}\psi=0\quad\text{and}\quad\left\{3\left[\psi_{RR}-\frac{\psi_R}{R}\right]+\psi_{ZZ}\right\}_Z=k/R^4\quad\text{on}\quad Z=\pm L\end{aligned} \tag{6.22}$$

where

$$\mathscr{L}=\frac{\partial^2}{\partial R^2}-\frac{1}{R}\frac{\partial}{\partial R}+\frac{\partial^2}{\partial Z^2}$$

and

$$\hat{\mathscr{L}}=\frac{\partial^2}{\partial R^2}-\frac{1}{R}\frac{\partial}{\partial R}-\frac{\partial^2}{\partial Z^2}.$$

To reveal the dependence of the solution on parameters, we introduce dimensionless variables

$$(t,x,\lambda,\alpha,\Psi)=\left(\frac{R}{a},\frac{Z}{a},\frac{b}{a},\frac{L}{a},\frac{\psi a}{k}\right) \tag{6.23}$$

and dimensionless operators

$$L = a^2\mathscr{L} = \frac{\partial^2}{\partial t^2} - \frac{1}{t}\frac{\partial}{\partial t} + \frac{\partial^2}{\partial x^2}$$

and

$$\hat{L} = \frac{\partial^2}{\partial t^2} - \frac{1}{t}\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}$$

all related by

$$L^2\Psi = 0 \quad \text{in} \quad 1 \leqq t \leqq \lambda, \quad -\alpha \leqq x \leqq \alpha,$$

(6.24)
$$\Psi_x = \hat{L}\Psi = 0 \quad \text{on} \quad t = 1, \lambda,$$

$$\hat{L}\Psi = \left\{3\left[\Psi_{tt} - \frac{\Psi_t}{t}\right] + \Psi_{xx}\right\}_x - \frac{1}{t^4} = 0 \quad \text{on} \quad x = \pm\alpha.$$

$\Psi(t, x, \lambda, \alpha)$ depends only on the geometric parameters $\lambda = b/a$ and $\alpha = L/a$. We may consider the semi-infinite problem which is also governed by (6.24) except that the condition $(6.24)_3$ is applied only at $x = 0$ and $\Psi \to 0$ as $x \to -\infty$. This problem is completely independent of parameters. In general, we may expect that if λ is not too close to one, the deflections will be proportional to a; for example,

(6.25)
$$\frac{\hat{z}}{a} = -4\left(1 + \frac{4\beta^{[2]}}{\beta}\right)\frac{1}{t}\Psi_t(t, x)\Big/\left(\frac{1}{\lambda^2} - 1\right)^2.$$

The case $\lambda \to 1$ deserves special attention as a limiting problem with a bounded displacement.

To study this problem, we again change variables

$$t = 1 + (\lambda - 1)\tau,$$
$$x = (\lambda - 1)\chi,$$
$$\Psi = (\lambda - 1)^3\tilde{\Psi}$$

and in the limit $\lambda \to 1$, we find that

$$\nabla^4\tilde{\Psi} = 0 \quad \text{in} \quad 0 \leqq \tau \leqq 1, \quad 0 \geqq \chi > -\infty,$$

(6.26)
$$\tilde{\Psi}_\chi = \tilde{\Psi}_{\tau\tau} - \tilde{\Psi}_{\chi\chi} = 0 \quad \text{on} \quad \tau = 0, 1,$$
$$\tilde{\Psi}_{\tau\tau} - \tilde{\Psi}_{\chi\chi} = \{3\tilde{\Psi}_{\tau\tau} + \tilde{\Psi}_{\chi\chi}\}_\chi - 1 = 0 \quad \text{on} \quad \chi = 0,$$
$$\tilde{\Psi} \to 0 \quad \text{as} \quad \chi \to -\infty.$$

The solution of (6.26) is

$$\tilde{\Psi}(\tau, \chi) = 2\sum_{n=1}^{\infty}\left[\chi - \frac{1}{\gamma_n}\right]\frac{e^{\gamma_n\chi}}{\gamma_n^3}\sin\gamma_n\tau, \quad \gamma_n = (2n - 1)\pi.$$

The vertical deflection as $\lambda \to 1$ is given by

(6.27)
$$\frac{\hat{z}}{a} = -4\left(1 + \frac{4\beta^{[2]}}{\beta}\right)\tilde{\Psi}_\tau(\tau, \chi)\frac{(\lambda - 1)^3}{(\lambda - 1)\left(1 - \dfrac{1}{\lambda^2}\right)^2}$$
$$= -\left(1 + \frac{4\beta^{[2]}}{\beta}\right)\tilde{\Psi}_\tau(\tau, \chi).$$

The vertical deflections of a point on the free surface next to the rod $(R, Z) = (a, 0)$ is given by (6.27) and (6.26) with $(\tau, \chi) = (0, 0)$ as

$$\hat{z} = \frac{2a}{\pi^3}\left(1 + \frac{4\beta^{[2]}}{\beta}\right)\sum_{n=1}^{\infty}\frac{1}{(2n - 1)^3}$$

where

$$\sum_{n=1}^{\infty}\frac{1}{(2n - 1)^3} \doteq 1.05.$$

Hence

(6.28)
$$\hat{z}(a, 0) = 0.067\, a\left(1 + \frac{4\beta^{[2]}}{\beta}\right).$$

In the limit $\lambda \to 1$ the stresses are singular with a singularity proportional to $1/(\lambda - 1)$.

Returning now to the general problem (6.24), we find that

(6.29)
$$\Psi(t, x) = \sum_{n=1}^{\infty} C_n f_n(t)\, g_n(x),$$

where

$$f_n(t) = t\{Y_1(p_n t)\, J_1(p_n) - J_1(p_n t)\, Y_1(p_n)\} \equiv t\mathscr{C}_1(p_n t),$$

Y_1 and J_1 are two kinds of Bessel functions,

p_n are the roots of $\mathscr{C}_1(\lambda p_n) = 0$,

$$g_n(\chi) = (\sinh\alpha p_n + \alpha p_n\cosh p_n\alpha)\sinh p_n x$$
$$- p_n x\cosh p_n x\sinh\alpha p_n$$

and C_n is determined by the Fourier-Bessel expansion induced by the last equation in the group (6.24); that is,

(6.30)
$$\frac{1}{t^4} = -2\sum_{n=1}^{\infty} C_n p_n^3(\cosh\alpha p_n\sinh\alpha p_n + \alpha p_n) f_n(t) = \sum_{n=1}^{\infty} D_n f_n(t).$$

To compute $C_n = -D_n/2p_n^3(\cosh\alpha p_n\sinh x p_n + \alpha p_n)$, we introduce the scalar product

$$\langle a, b\rangle = \int_1^\lambda \frac{1}{t}\, a(t)\, b(t)\, dt$$

for any $a, b \in L^2(1, \lambda)$. It is easy to verify that

(6.31)
$$\langle f_n, f_m\rangle = K_n\,\delta_{nm}, \quad K_n = \langle f_n, f_n\rangle$$

for any $f_n(t)$ satisfying

$$f_n'' - \frac{1}{t}f_n' + p_n^2 f_n = 0, \quad f_n(1) = f_n(\lambda) = 0.$$

It follows from (6.30) and (6.31) that

(6.32)
$$D_n = \frac{\left\langle \dfrac{1}{t^4}, f_n\right\rangle}{\langle f_n, f_n\rangle}$$

and, collecting all these results, we may write (6.29) as

$$(6.33)\qquad \Psi(t,x) = -\sum_{n=1}^{\infty}\Big\{D_n f_n(t) \times \frac{(\sinh\alpha p_n + \alpha p_n\cosh\alpha p_n)\sinh p_n x - p_n x\cosh p_n x\sinh\alpha p_n}{2p_n^3(\cosh\alpha p_n\sinh\alpha p_n + \alpha p_n)}\Big\}.$$

The limits of large and small $\alpha = L/a$, $-\alpha \leqq x \leqq \alpha$, are of interest. In the limiting case of a thin disk, $\alpha \to 0$ $(L \to 0)$,

$$(6.34)\qquad \Psi(t,x) = -\sum_{n=1}^{\infty} D_n f_n(t)\, x/4p_n^2$$

is linear in x.

To obtain Ψ for the semi-infinite problem in which $\Psi \to 0$ as $x \to -\infty$, we set

$$(6.35)\qquad y = x - \alpha$$

and let α tend to ∞. In this limit

$$(6.36)\qquad \Psi(t,y) = -\sum_{n=1}^{\infty} D_n f_n(t)\,(1 - p_n y)\, e^{p_n y}/2p_n^3.$$

Displacements are given by

$$(6.37)\qquad (\hat{z},\hat{r}) = \frac{1}{R}(-\psi_R, \psi_Z) = \frac{4a\lambda^4}{(\lambda^2-1)^2}\left(1 + \frac{4\beta^{[2]}}{\beta}\right)\frac{1}{t}(-\Psi_t, \Psi_x).$$

In the limit $\alpha \to 0$, we find that

$$\Psi_t(t,x) \to 0 \quad \text{with} \quad \alpha \quad [\text{i.e. } \hat{z}(R,Z) \to 0],$$

but

$$\Psi_x(t,x) = -\sum_{n=1}^{\infty} D_n f_n(t)/4p_n^2$$

reaches a finite limit independent of x and

$$[\hat{r}(R,Z) \to \hat{r}(R) \text{ independent of } Z].$$

In the semi-infinite case, we replace Ψ_x with Ψ_y and use (6.36). The semi-infinite case has the interesting property that

$$\Psi_y(t,0) = 0, \quad 1 \leqq t \leqq \lambda.$$

Hence the radial displacement $\hat{r}(R,Z)$ vanishes on the free surface

$$\hat{r}(R,0) = 0$$

and in the limit $Z \to -\infty$,

$$\hat{r}(R,Z) = 0.$$

Numerical evaluation of solutions was carried out by standard methods. We computed one hundred roots p_n of $\mathscr{C}_1(p_n\lambda)$ using the method suggested by ABRAMOWITZ & STEGUN (1964, p. 374). Values for $\langle 1/t^4, f_n\rangle$ were computed by

numerical integration. Convergence of the partial sums

$$\sum_{n=1}^{N} D_n f_n(t)$$

of the series (6.30) to the prescribed data $1/t^4$ is slow. In fact, we do not achieve convergence with $N = 100$, but the Cesaro sum

$$\frac{1}{M}\sum_{N=1}^{M}\sum_{n=1}^{N}\{D_n f_n(t)\} \quad \text{with} \quad M = 100,$$

does converge to the prescribed data at all interior points slightly away from $t = 1$ and $t = \lambda$. The data and characteristic functions $f_n(t)$ are incompatible at $t = 1$ and $t = \lambda$ and the series for $1/t^4$ exhibits a Gibbs phenomenon. The series for the displacements $(\hat{r}, \hat{z})$ is much more rapidly convergent, also at $t = 1$ and $t = \lambda$, because it involves two integrations of the poorly convergent one. All these features are usual and well understood.

Representative graphs of the displacement of the free surface are shown in Figures (6.2)–(6.6). The principal features of these graphs are summarized in the captions.

The unknown second order elastic parameter $\beta^{[2]}$ may be determined from measurements of $\hat{z}(R,0)$ in the semi-infinite case. To optimize this deflection with respect to large deflections for small cylinders, it would appear to be sufficient to take $L = 2a$, $b/a = 3$ (see Appendix C).

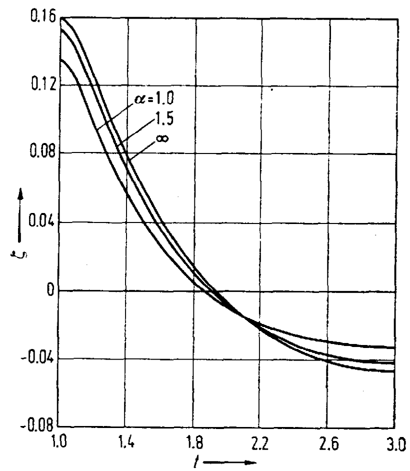


Fig. 6.2. Vertical displacement of the top free surface

$$\zeta = \hat{z}\Big/\left[a\left(1 + \frac{4\beta^{[2]}}{\beta}\right)\right]$$

for a wide gap $\lambda = b/a = 3$ and three different values of $\alpha = L/a$. The shape of the free surface on a cylinder whose height is greater than twice the radius of the inner cylinder $(\alpha > 2)$ is essentially independent of α (see Fig. 6.3). $\zeta(t,\alpha)$ goes to zero with α as α tends to zero.

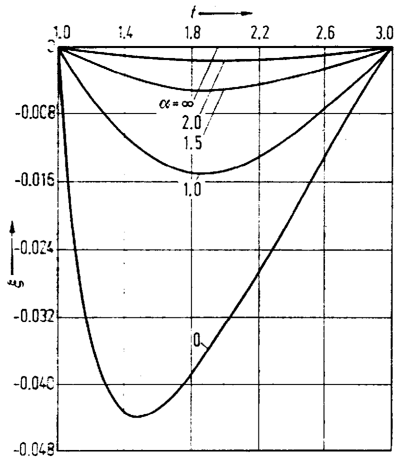


Fig. 6.3. Radial displacement of the top free surface

$$\xi = \hat{r}/a\left(1 + \frac{4\beta^{[2]}}{\beta}\right)$$

for a wide gap $\lambda = b/a = 3$ and five different values of $\alpha = L/a$. ξ tends to zero as $\alpha \to \infty$ and is essentially zero for $\alpha > 2$. ξ tends to a finite limit as $\alpha \to 0$. When α is small a circle drawn on the face of the free surface in the natural state will change its radius in the deformed state.

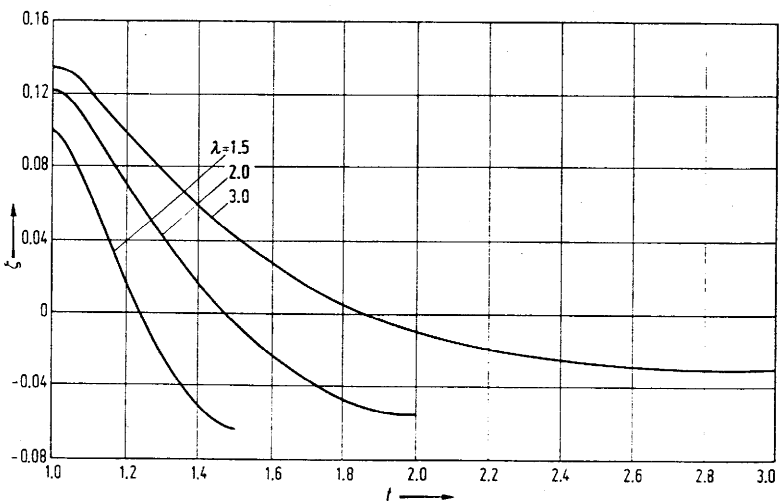


Fig. 6.4. Vertical displacement ζ of the top free surface in an annulus for which $L = a$ ($\alpha = 1$) with the outer radius $\lambda = b/a$ as a parameter.

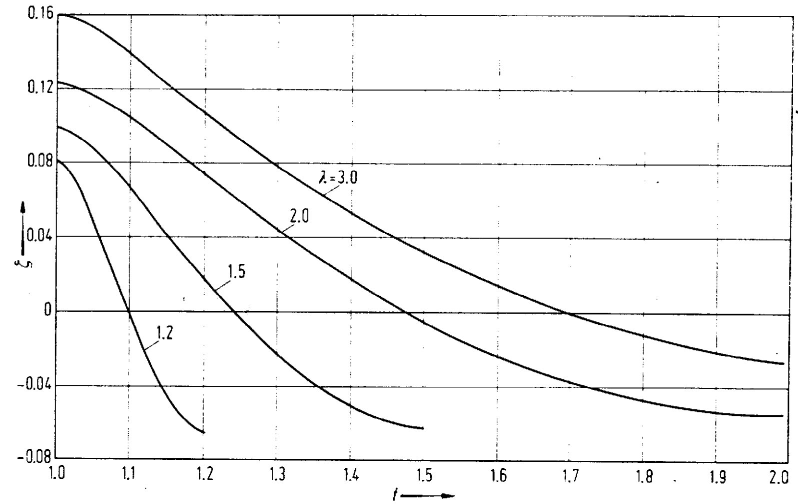


Fig. 6.5. Vertical displacement ζ of the free surface in the semi-infinite case ($\alpha = \infty$) with outer radius $\lambda = b/a$ as a parameter.

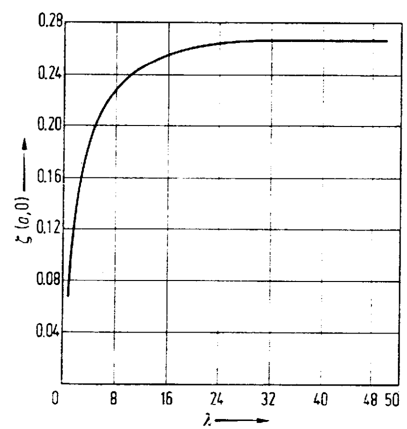


Fig. 6.6. Vertical displacement ζ of the point $(R, Z) = (a, 0)$ on the free surface next to the rod as a function of the outer radius $\lambda = b/a$ in the semi-infinite case. This displacement increases rapidly from $\zeta = 0.067$ at $\lambda = 1$ to $\zeta = 0.2642$.

Chapter 7. Torsion of a Viscoelastic Cylinder

Now instead of twisting the viscoelastic cylinder at its round side while monitoring the deformation of the stress-free end planes as in Figure 6.1, we shall grip the cylinder at the end planes and monitor the deformation of the cylindrical stress-free surface as in Figure 7.1.

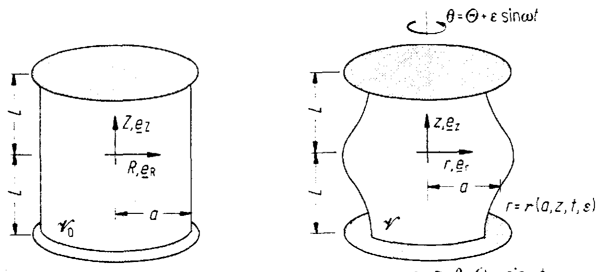


Fig. 7.1. A viscoelastic cylinder is sheared by the rotation of two end plates.
(a) $\mathscr{V}_0 = \{R, \Theta, Z: 0 \leqq R < a, 0 \leqq \Theta \leqq 2\pi, -L < Z < L\}$, natural state.
(b) $\mathscr{V} = \{r, \theta, z: 0 \leqq r < r(a, z, t, \varepsilon), 0 \leqq \theta \leqq 2\pi, -L < z < L\}$, deformed state.

The torsion problem (Fig. 7.1) is like the rod climbing problem (Fig. 6.1) in that the deformation of stress free surfaces due to normal stresses appears first at second order in ε. But unlike rod climbing, *the free surface at $R = a$ will not deform under a small static twist*. The reason is that under a static twist (replace $\varepsilon \sin \omega t$ with ε in Fig. 7.1), the viscoelastic cylinder responds elastically. The elastic problem falls into the class of universal deformations in nonlinear elastostatics found by RIVLIN (1949). These deformations are independent of the constitutive equation provided that the material is undergoing elastic deformations; then, globally, without perturbations we get

$$r = R, z = Z, \theta = \Theta + \frac{\varepsilon Z}{L}.$$

Thus the free surface on RIVLIN's solution will not change shape under a static twist. Since, under the conditions laid down in Chapter 4 and under other, yet weaker conditions, we get uniqueness for small prescribed displacements, RIVLIN's solution is unique when ε is small. Actually, the sides of the cylinder on RIVLIN's solution will never deform even when ε is large, but other static solutions, with more complicated symmetry breaking the uniformity of the free surfaces, bifurcate (see PENN & KEARSLEY, 1976) when the twist is large.

We are going to show that the stress-free cylindrical surface will distort under dynamic loading. We need to know the values of two material constants and three material functions to make a definite prediction. No one knows anything close to this even for one solid, so we do not have anything like the rheological data we need. Mathematically, using perturbations, we can use linearity to get around this problem by computing components of the solution which are independent of the 2nd order rheological parameters. You will see this for yourself later. But even here we cannot get around the rheological problem entirely; we need to know or guess β and $\zeta(s)$ to make rheological predictions.

7.1 Mathematical Formulation of the Torsion Problem

Consider a right circular cylinder of viscoelastic solid material of radius a and height $2L$ (Fig. 7.1). The cylindrical surface is stress-free while the top and bottom surfaces are connected to oscillating rigid plates. The distance between the plates is kept fixed. We shall consider two problems associated with this configuration. The two problems are exactly the same except that in

(I) *we prescribe zero radial shear stress and zero axial displacement on the end plates at $Z = \pm L$ whilst in*

(II) *we prescribe zero radial displacement and zero axial displacement on the end plates at $Z = \pm L$.*

Prescription (II) is achieved when the material is bonded to the end plates. In (I) we may imagine that the end plates have lubricated grooves or slots which lie in a fan along radii. The material is then free to slide radially along the slots but it must rotate with the slots.

The boundary conditions on the cylinder shown in Figure 7.1 are

$$\begin{aligned} &\boldsymbol{T}\boldsymbol{e}_r = 0 \quad \text{at} \quad R = a; \\ (7.1)\qquad &\theta = \Theta \pm \varepsilon \sin \omega t, \quad z = \pm L \quad \text{at} \quad Z = \pm L; \\ &\text{either (I) } T_{rz} = 0 \quad \text{or} \quad \text{(II) } r = R \quad \text{at} \quad Z = \pm L. \end{aligned}$$

The problems governing the first and second derivatives of the solution with respect to ε at $\varepsilon = 0$ are given by (3.29)–(3.30) with $\mathscr{V}_0$ as in Fig. 7.1. These equations are recast for axisymmetric problems as in Chapter 5 and reduced using symmetry (5.9). We make all lengths $(R, Z, L, r^{\langle n \rangle}, z^{\langle n \rangle})$ dimensionless with the radius a and continue to use the same symbols in the dimensionless version of the problem. All of the equations retain their original form in the new variables except that $\varrho_0 a^2$ replaces ϱ in the equations of motion and the radius of the undeformed cylinder is one.

At first order the only nonzero displacement is $\theta^{\langle 1 \rangle} = \phi$ where ϕ satisfies

$$(7.2)\qquad \begin{cases} \varrho_0 a^2 R \ddot{\phi} = \hat{L}[\nabla^2(R\phi) - \phi/R] \quad \text{in} \quad 0 \leqq R < 1, \quad -L < Z < L, \\ \hat{L}[R\phi_R] = 0 \quad \text{at} \quad R = 1, \end{cases}$$

$$\phi = \pm(e^{i\omega t} - e^{-i\omega t})/2i \quad \text{at} \quad Z = \pm L$$

where for any $a(t)$

$$\hat{L}[a(t)] \stackrel{\text{def}}{=} \gamma a(t) + \int_0^\infty \zeta(s)\, a(t-s)\, dt.$$

The solution of (7.2) is

$$(7.3)\qquad \phi(Z, t) = V(Z)\, e^{i\omega t} + \bar{V}(Z)\, e^{-\omega t}$$

independent of R where

$$V(Z) = \sinh(\Lambda Z)/[2i \sinh(\Lambda L)]$$

and

$$\Lambda = \left\{-\varrho_0 \omega^2 a^2 \Big/ \left[\gamma + \int_0^\infty \zeta(s)\, e^{-i\omega s}\, ds\right]\right\}^{\frac{1}{2}} = \Lambda_r + i\Lambda_i$$

is a dimensionless complex number.

To obtain the equations at second order satisfied by $(r^{\langle 2\rangle}, z^{\langle 2\rangle}, p^{\langle 2\rangle}) \overset{\text{def}}{=} (\hat{r}, \hat{z}, \hat{p})$ we must compute the inhomogeneous terms in the equations of motion $(3.30)_1$–$(3.30)_2$. All of these may be computed with the help of

$$F^{\langle 1\rangle} = \phi(-e_R e_\Theta + e_\Theta e_R) + R\phi' e_\Theta e_Z,$$

$$T^{\langle 1\rangle} = \hat{L}[R\phi'](e_\Theta e_Z + e_Z e_\Theta), \tag{7.4}$$

$$u^{\langle 2\rangle} = (\hat{r} - R\phi^2/2)\, e_R + \hat{z} e_Z.$$

Using $(3.30)_1$, (3.33), (7.3) and (7.4), we find that

$$\begin{aligned}
\varrho_0 a^2 \ddot{\hat{r}} + \hat{p}_R - \hat{L}[\nabla^2 \hat{r} - \hat{r}/R^2]
&= -\hat{L}[\nabla^2(R\phi^2/2) - \phi^2/(2R)] + \varrho_0 a^2 \ddot{\overline{R\phi^2/2}} + M_{2R} \\
&= R\{[n_1 \sinh^2(\Lambda Z)/\Lambda + n_2 \cosh^2(\Lambda Z)/\Lambda]\, e^{2i\omega t} \\
&\quad + \text{complex conjugate} \\
&\quad + (m_1 + m_2) \cosh(2\Lambda_r Z) + (-m_1 + m_2) \cos(2\Lambda_i Z)\}
\end{aligned} \tag{7.5}$$

where

$$\begin{cases}
n_1 = \varrho_0 \omega^2 a^2 \Lambda/[4 \sinh^2(\Lambda L)], \\
n_2 = \Lambda^3 \Big\{\beta + \beta^{[2]} + \int_0^\infty 2\zeta^{[2]}(s)\,(e^{-i\omega s} - 1)\, ds \\
\qquad + \int_0^\infty \int_0^\infty \alpha(s_1, s_2)\,(e^{-i\omega s_1} - 1)(e^{-i\omega s_2} - 1)\, ds_1\, ds_2\Big\} \Big/ [4 \sinh^2(\Lambda L)], \\
m_1 = \varrho_0 \omega^2 a^2 / |2 \sinh(\Lambda L)|^2, \\
m_2 = -|\Lambda|^2 \Big\{2\beta + 2\beta^{[2]} + \int_0^\infty 2\zeta^{[2]}(s)\,[(e^{-i\omega s} - 1) + (e^{i\omega s} - 1)]\, ds \\
\qquad + \int_0^\infty \int_0^\infty \alpha(s_1, s_2)\,[(e^{-i\omega s_1} - 1)(e^{i\omega s_2} - 1) \\
\qquad + (e^{i\omega s_1} - 1)(e^{-i\omega s_2} - 1)]\, ds_1\, ds_2\Big\} \Big/ \{2\, |2 \sinh(\Lambda L)|^2\}
\end{cases} \tag{7.6}$$

and

$$\begin{aligned}
\varrho_0 a^2 \ddot{\hat{z}} + \hat{p}_Z - \hat{L}[\nabla^2 \hat{z}] &= M_{2Z} \\
&= R^2\{n_3 \sinh(2\Lambda Z)\, e^{2i\omega t} + \text{complex conjugate} \\
&\quad + m_3[\Lambda_r \sinh(2\Lambda_r Z) - \Lambda_i \sin(2\Lambda_i Z)]\}
\end{aligned} \tag{7.7}$$

where

$$\begin{cases}
n_3 = -(n_1 + n_2 + n_4), \\
n_4 = \Lambda^3 \left[\int_0^\infty \zeta(s)\,(e^{-2i\omega s} - e^{-i\omega s})\, ds\right] \Big/ [4 \sinh^2(\Lambda L)], \\
m_3 = -2m_2 + m_4, \\
m_4 = |\Lambda|^2\, [2\varrho_0 \omega^2 a^2/\Lambda^2 + 2\varrho_0 \omega^2 a^2/\bar{\Lambda}^2 + 2\beta] / |2 \sinh(\Lambda L)|^2.
\end{cases} \tag{7.8}$$

The last equation of (5.10) expresses the conservation of volume.

The boundary conditions at second order are

$$T^{\langle 2\rangle} e_R + \phi T^{\langle 1\rangle} e_\Theta = \mathbf{0} \quad \text{at} \quad R = 1,$$

$$\hat{z} = 0 \quad \text{at} \quad Z = \pm L$$

and either

$$\left.\begin{array}{ll} \text{(I)} & T^{\langle 2\rangle}_{RZ} + \phi T^{\langle 1\rangle}_{\Theta Z} = 0 \\ \text{or} \quad \text{(II)} & \hat{r} = 0 \end{array}\right\} \quad \text{at} \quad Z = \pm L.$$

The expressions giving $T^{\langle 2\rangle}$ follow from (3.23) and (7.4). We find that at $R = 1$

$$\begin{aligned} -\hat{p} + L[2\hat{r}_R] &= 0, \\ \hat{L}[\hat{r}_Z + \hat{z}_R] &= 0, \end{aligned} \tag{7.9}$$

and at the top and bottom surfaces $Z = \pm L$,

$$\hat{z} = 0 \tag{7.10}$$

and either (I)

$$\hat{L}[\hat{r}_Z + \hat{z}_R] = 0 \tag{7.11}$$

or (II)

$$\hat{r} = 0. \tag{7.12}$$

The equation (5.10) shows that it is possible to obtain $\hat{r}$ and $\hat{z}$ from a stream function

$$\begin{aligned} \hat{r} &= \tilde{\bar{\Psi}}_Z/R, \\ \hat{z} &= -\tilde{\bar{\Psi}}_R/R. \end{aligned} \tag{7.13}$$

We obtain the equation governing $\tilde{\bar{\Psi}}$ in $\mathscr{V}_0$ by eliminating the pressure p from (7.5) and (7.7). The pressure in the boundary condition $(7.9)_1$ is eliminated by using the equation of motion (7.7) in the Z-direction.

The nature of the inhomogeneous terms is such that the stream function $\tilde{\bar{\Psi}}$ can be decomposed into a mean part and a part with a zero mean oscillating with a frequency 2ω. Thus

$$\tilde{\bar{\Psi}} = \psi + [\Psi e^{2i\omega t} + \bar{\Psi} e^{-2i\omega t}]. \tag{7.14}$$

The *mean problem* satisfied by ψ is given by

a) $\mathscr{L}^2\psi = R^2[m_5 \sinh(2\Lambda_r Z) + m_6 \sin(2\Lambda_i Z)]$ in $\mathscr{V}_0$;

b) at $R=1$, $\dfrac{1}{R}(\mathscr{L}\psi)_R + 2\left(\dfrac{1}{R}\psi_{ZZ}\right)_R = m_8[\Lambda_r \sinh(2\Lambda_r Z) - \Lambda_i \sin(2\Lambda_i Z)]$,

(7.15) $$\frac{1}{R}\left(\psi_{RR} - \frac{1}{R}\psi_R - \psi_{ZZ}\right) = 0;$$

c) at $Z = \pm L$, $\dfrac{1}{R}\psi_R = 0$

and (I)

$$\frac{1}{R}\left(\psi_{RR} - \frac{1}{R}\psi_R - \psi_{ZZ}\right) = 0$$

or (II)

$$\frac{1}{R}\psi_Z = 0$$

where

(7.16) $$\mathscr{L}(\cdot) = R\frac{\partial}{\partial R}\left[\frac{1}{R}\frac{\partial(\cdot)}{\partial R}\right] + \frac{\partial^2(\cdot)}{\partial Z^2}$$

and

(7.17) $$\begin{aligned} m_5 &= -2\Lambda_r[m_1 + m_2 - m_3]/\beta, \\ m_6 &= -2\Lambda_i[m_1 - m_2 + m_3]/\beta, \\ m_8 &= m_3/\beta \end{aligned}$$

are dimensionless real quantities depending on the density ϱ_0, the frequency ω, the radius a, the dimensionless height L and the first and second order material parameters. Similarly, we obtain the *time-periodic problem* satisfied by Ψ:

(7.18) $$\begin{cases} \text{a)}\ \ \mathscr{L}^2\Psi - \Gamma^2\mathscr{L}\Psi = -n_5 R^2 \sinh(2\Lambda Z) \ \text{ in } \mathscr{V}_0; \\ \text{b)}\ \text{ at } R=1,\ \dfrac{1}{R}(\mathscr{L}\Psi - \Gamma^2\Psi)_R + 2\left(\dfrac{1}{R}\Psi_{ZZ}\right)_R = -n_6 \sinh(2\Lambda Z), \\ \qquad \dfrac{1}{R}\left(\Psi_{RR} - \dfrac{1}{R}\Psi_R - \Psi_{ZZ}\right) = 0; \\ \text{c)}\ \text{ at } Z = \pm L,\ \dfrac{1}{R}\Psi_R = 0 \end{cases}$$

and (I)

$$\frac{1}{R}\left(\Psi_{RR} - \frac{1}{R}\Psi_R - \Psi_{ZZ}\right) = 0$$

or (II)

$$\frac{1}{R}\Psi_Z = 0$$

where

(7.19) $$\begin{cases} \Gamma^2 = -4\varrho_0\omega^2 a^2/\eta \equiv -4\varrho_0\omega^2 a^2\Big/\left[\gamma + \int\limits_0^\infty \zeta(s)\, e^{-2i\omega s}\, ds\right], \\ n_5 = (n_1 + n_2 - 2n_3)/\eta, \\ n_6 = -n_3/\eta \end{cases}$$

are dimensionless complex quantities depending on ϱ_0, ω, a, L and the first and second order material parameters.

Now we shall show how to obtain $\bar{\bar{\Psi}}$ by solving the problems (7.15) and (7.18) for various cases. Once $\bar{\bar{\Psi}}$ is obtained, the dimensionless second order displacements $\hat{r}$ and $\hat{z}$ can be found from the relation (7.13). The second order pressure $\hat{p}$ is determined (up to an additive constant) from the equations of motion (7.5) and (7.7). The constant is found from the boundary condition $(7.9)_1$. The shape of the free surface can then be plotted with the help of the parametric representation (5.15).

7.2 Solution of the Time-Periodic Problem (I)

We seek the solution as a linear combination of the particular solution $\hat{\Psi}$ and the homogeneous solution $\breve{\Psi}$; *i.e.*,

(7.20) $$\Psi = \hat{\Psi} - \breve{\Psi}$$

where $\hat{\Psi}$ is given by

(7.21) $$\hat{\Psi} = R^2[A \sinh(2\Lambda Z) + BZ + C\sinh(\Gamma Z)]$$

and

(7.22) $$\begin{aligned} A = -n_5/(4\Lambda^2\Sigma^2),\quad \Sigma^2 &= 4\Lambda^2 - \Gamma^2, \\ B &= A\Sigma^2 \sinh(2\Lambda L)/(\Gamma^2 L), \\ C &= -A(4\Lambda^2)\sinh(2\Lambda L)/[\Gamma^2 \sinh(\Gamma L)]. \end{aligned}$$

Then $\breve{\Psi}$ satisfies

(7.23) $$\begin{cases} \text{a)}\ \ \mathscr{L}^2\breve{\Psi} - \Gamma^2\mathscr{L}\breve{\Psi} = 0 \ \text{ in } \mathscr{V}_0; \\ \text{b)}\ \text{ at } R=1,\ \dfrac{1}{R}(\mathscr{L}\breve{\Psi} - \Gamma^2\breve{\Psi})_R + 2\left(\dfrac{1}{R}\breve{\Psi}_{ZZ}\right)_R \\ \qquad = [n_6 + 2A(4\Lambda^2 + \Sigma^2)]\sinh(2\Lambda Z) - 2\Gamma^2 BZ + 2\Gamma^2 C\sinh(\Gamma Z) \\ \qquad \equiv \mathscr{F}(Z), \\ \dfrac{1}{R}\left(\breve{\Psi}_{RR} - \dfrac{1}{R}\breve{\Psi}_R - \breve{\Psi}_{ZZ}\right) = -4\Lambda^2 A \sinh(2\Lambda Z) - \Gamma^2 C\sinh(\Gamma Z) \\ \qquad \equiv \mathscr{G}(Z); \\ \text{c)}\ \text{ at } Z = \pm L,\ \dfrac{1}{R}\left(\breve{\Psi}_{RR} - \dfrac{1}{R}\breve{\Psi}_R - \breve{\Psi}_{ZZ}\right) = 0,\ \ \dfrac{1}{R}\breve{\Psi}_R = 0. \end{cases}$$

The solution of (7.23a) and (7.23c) is

(7.24) $$\tilde{\Psi} = \sum_{n=1}^{\infty} [c_n R I_1(\alpha_n R) + d_n R I_1(\beta_n R)] \sin(\alpha_n Z)$$

where $\alpha_n = n\pi/L$ and

(7.25) $$\beta_n = (\alpha_n^2 + \Gamma^2)^{\frac{1}{2}}.$$

The coefficients c_n and d_n are determined by application of the orthogonality relation

(7.26) $$\int_{-L}^{L} \sin \alpha_n Z \sin \alpha_m Z \, dZ = L\delta_{nm},$$

to the boundary condition (7.23b). This leads to

(7.27) $$\begin{aligned} c_n a_{11}(n) + d_n a_{12}(n) &= -\gamma_n/\alpha_n, \\ c_n a_{21}(n) + d_n a_{22}(n) &= \delta_n \end{aligned}$$

where

$$\begin{aligned} a_{11}(n) &= (\alpha_n^2 + \beta_n^2) I_0(\alpha_n) - 2\alpha_n I_1(\alpha_n), \\ a_{12}(n) &= 2\alpha_n[\beta_n I_0(\beta_n) - I_1(\beta_n)], \\ a_{21}(n) &= 2\alpha_n^2 I_1(\alpha_n), \\ a_{22}(n) &= (\alpha_n^2 + \beta_n^2) I_1(\beta_n) \end{aligned}$$

and

(7.28) $$\begin{aligned} \gamma_m L &= \int_{-L}^{L} \mathscr{F}(Z) \sin(\alpha_m Z) \, dZ, \\ \delta_m L &= \int_{-L}^{L} \mathscr{G}(Z) \sin(\alpha_m Z) \, dZ. \end{aligned}$$

At each n, it is possible to invert (7.27) uniquely as

(7.29) $$\det \begin{bmatrix} a_{11}(n) & a_{12}(n) \\ a_{21}(n) & a_{22}(n) \end{bmatrix} = (\alpha_n^2 + \beta_n^2)^2 I_0(\alpha_n) I_1(\beta_n) - 2\alpha_n \Gamma^2 I_1(\beta_n) I_1(\alpha_n) - 4\beta_n \alpha_n^3 I_1(\alpha_n) I_0(\beta_n) \neq 0.$$

The boundary conditions (7.23b) at $R = 1$ are satisfied by the Fourier series (7.24).

7.3 Fourier Series Solution of the Mean Problem (I)

Let

(7.30) $$\psi = \hat{\hat{\psi}} - \bar{\bar{\psi}}$$

where the particular solution $\hat{\hat{\psi}}$ is given by

(7.31) $$\hat{\hat{\psi}} = R^2[A_1 \sinh(2\Lambda_r Z) + B_1 \sin(2\Lambda_i Z) + C_1 Z + D_1 Z^3]$$

and

(7.32) $$\begin{aligned} A_1 &= m_5/(16\Lambda_r^4), \\ B_1 &= m_6/(16\Lambda_i^4), \\ C_1 &= [2\Lambda_r^2 L/3 - 1/L] A_1 \sinh(2\Lambda_r L) - [2\Lambda_i^2 L/3 + 1/L] B_1 \sin(2\Lambda_i L), \\ D_1 &= -(2\Lambda_r^2/3L) A_1 \sinh(2\Lambda_r L) + (2\Lambda_i^2/3L) B_1 \sin(2\Lambda_i L). \end{aligned}$$

Then $\bar{\bar{\psi}}$ satisfies

(7.33) $$\begin{cases} \text{a)} \quad \mathscr{L}^2 \bar{\bar{\psi}} = 0 \quad \text{in} \quad \mathscr{V}_0; \\ \text{b)} \quad \text{at} \quad R = 1, \quad \dfrac{1}{R}(\mathscr{L}\bar{\bar{\psi}})_R + 2\left(\dfrac{1}{R}\bar{\bar{\psi}}_{ZZ}\right)_R \\ \qquad = -(m_8 - 16A_1\Lambda_r)\Lambda_r \sinh(2\Lambda_r Z) \\ \qquad\quad + (m_8 - 16B_1\Lambda_i)\Lambda_i \sin(2\Lambda_i Z) + 24D_1 Z \equiv F(Z), \\ \qquad \dfrac{1}{R}\left(\bar{\bar{\psi}}_{RR} - \dfrac{1}{R}\bar{\bar{\psi}}_R - \bar{\bar{\psi}}_{ZZ}\right) \\ \qquad = -4\Lambda_r^2 A_1 \sinh(2\Lambda_r Z) \\ \qquad\quad + 4\Lambda_i^2 B_1 \sin(2\Lambda_i Z) - 6D_1 Z \equiv G(Z); \\ \text{c)} \quad \text{at} \quad Z = \pm L, \quad \dfrac{1}{R}\left(\bar{\bar{\psi}}_{RR} - \dfrac{1}{R}\bar{\bar{\psi}}_R - \bar{\bar{\psi}}_{ZZ}\right) = 0, \\ \qquad \dfrac{1}{R}\bar{\bar{\psi}}_R = 0. \end{cases}$$

The solution of (7.33a) and (7.33c) is

(7.34) $$\bar{\bar{\psi}} = \sum_{n=1}^{\infty} [c_n R I_1(\alpha_n R) + d_n R^2 I_0(\alpha_n R)] \sin(\alpha_n Z)$$

where

(7.35) $$\alpha_n = n\pi/L.$$

To determine the constants c_n and d_n, we use the boundary conditions (7.33b). Then

(7.36) $$\begin{cases} \displaystyle\sum_{n=1}^{\infty} \gamma_n \sin(\alpha_n Z) = F(Z), \\ \displaystyle\sum_{n=1}^{\infty} \delta_n \sin(\alpha_n Z) = G(Z) \end{cases}$$

where

(7.37) $$\begin{cases} \gamma_n = (-2\alpha_n^2)\{c_n b_{11}(\alpha_n) + d_n b_{12}(\alpha_n)\}, \\ \delta_n = (2\alpha_n)\{c_n b_{21}(\alpha_n) + d_n b_{22}(\alpha_n)\} \end{cases}$$

and

$$\text{(7.38)} \qquad \begin{cases} b_{11}(\alpha_n) = \alpha_n I_0(\alpha_n) - I_1(\alpha_n), \\ b_{12}(\alpha_n) = b_{21}(\alpha_n) = \alpha_n I_1(\alpha_n), \\ b_{22}(\alpha_n) = \alpha_n I_0(\alpha_n) + I_1(\alpha_n). \end{cases}$$

Application of the orthogonality condition (7.26) to (7.36) leads to

$$\text{(7.39)} \qquad \begin{aligned} \gamma_m L &= \int_{-L}^{L} F(Z) \sin(\alpha_m Z)\, dZ, \\ \delta_m L &= \int_{-L}^{L} G(Z) \sin(\alpha_m Z)\, dZ. \end{aligned}$$

Since

$$\operatorname{Det}(\alpha_n) \equiv \det \begin{bmatrix} b_{11}(\alpha_n) & b_{12}(\alpha_n) \\ b_{21}(\alpha_n) & b_{22}(\alpha_n) \end{bmatrix}$$

$$\text{(7.40)} \qquad = \alpha_n^2 I_0^2(\alpha_n) - (\alpha_n^2 + 1)\, I_1^2(\alpha_n)^{*} \neq 0,$$

c_n and d_n can be determined uniquely from (7.37) and (7.39). The Fourier series (7.36) converge to the boundary data.

7.4 *Biorthogonal Series Solution of the Mean Problem* (I)

Biorthogonal series are more appropriate than Fourier series for problems which, like (II), prescribe both components of displacement at the rigid boundary. Problem (II) is solved in the next section. However, it is instructive to use biorthogonal series to solve the problem (I) which was solved with Fourier series in the last section. For this purpose it is useful to make the boundary conditions at $R = 1$ homogeneous with the help of the particular solution

$$\text{(7.41)} \qquad \begin{aligned} \hat{\psi} = {} & [A_1 R^2 + A_2 R J_1(2\Lambda_r R) + A_3 R^2 J_0(2\Lambda_r R)] \sinh(2\Lambda_r Z) \\ & + [A_4 R^2 + A_5 R I_1(2\Lambda_i R) + A_6 R^2 I_0(2\Lambda_i R)] \sin(2\Lambda_i Z) \end{aligned}$$

where

$$\text{(7.42a)} \qquad A_4 = B_1$$

and A_1 and B_1 are given by (7.32). The other quantities satisfy

$$\text{(7.42b)} \qquad \begin{aligned} A_2 b_{11}(i\,2\Lambda_r) + iA_3 b_{12}(i\,2\Lambda_r) &= i(m_8 - 16A_1\Lambda_r)/(8\Lambda_r), \\ -iA_2 b_{21}(i\,2\Lambda_r) + A_3 b_{22}(i\,2\Lambda_r) &= -iA_1\Lambda_r \end{aligned}$$

and

$$\text{(7.42c)} \qquad \begin{aligned} A_5 b_{11}(2\Lambda_i) + A_6 b_{12}(2\Lambda_i) &= (m_8 - 16A_4\Lambda_i)/(8\Lambda_i), \\ A_5 b_{21}(2\Lambda_i) + A_6 b_{22}(2\Lambda_i) &= -A_4\Lambda_i \end{aligned}$$

* Compare this expression with the eigenvalue equation (A.8) of Appendix A.

where $b_{11}(\cdot)$, $b_{12}(\cdot)$, $b_{21}(\cdot)$ and $b_{22}(\cdot)$ are defined by the equations (7.38). Since $\operatorname{Det}(i\,2\Lambda_r)$ and $\operatorname{Det}(2\Lambda_i)$ are nonzero,* A_2, A_3, A_5 and A_6 can be determined uniquely from (7.42b) and (7.42c). The difference

$$\text{(7.43)} \qquad \tilde{\psi} = \hat{\psi} - \psi$$

now satisfies

$$\text{(7.44)} \qquad \begin{cases} \text{a)} \quad \mathscr{L}^2 \tilde{\psi} = 0 \quad \text{in} \quad \mathscr{V}_0; \\ \text{b)} \quad \text{at} \quad R = 1, \; \dfrac{1}{R} (\mathscr{L}\tilde{\psi})_R + 2\left(\dfrac{1}{R}\tilde{\psi}_{ZZ}\right)_R = 0, \\ \qquad \dfrac{1}{R}\left(\tilde{\psi}_{RR} - \dfrac{1}{R}\tilde{\psi}_R - \tilde{\psi}_{ZZ}\right) = 0; \\ \text{c)} \quad \text{at} \quad Z = \pm L, \dfrac{1}{R}\left(\tilde{\psi}_{RR} - \dfrac{1}{R}\tilde{\psi}_R - \tilde{\psi}_{ZZ}\right) = \pm \tilde{f}_1, \\ \qquad \dfrac{1}{R}\tilde{\psi}_R = \pm \tilde{g} \end{cases}$$

where

$$\text{(7.45)} \qquad \begin{cases} \tilde{f}_1 = -4\Lambda_r^2 \sinh(2\Lambda_r L)\{A_1[R] + A_2[2J_1(2\Lambda_r R)] \\ \qquad + A_3[J_1(2\Lambda_r R)/\Lambda_r - 2RJ_0(2\Lambda_r R)]\} \\ \qquad + 4\Lambda_i^2 \sin(2\Lambda_i L)\{A_4[R] + A_5[2I_1(2\Lambda_i R)] \\ \qquad + A_6[I_1(2\Lambda_i R)/\Lambda_i + 2RI_0(2\Lambda_i R)]\}, \\ \tilde{g} = 4\Lambda_r^2 \sinh(2\Lambda_r L)\{A_1[1/(2\Lambda_r^2)] + A_2[J_0(2\Lambda_r R)/(2\Lambda_r)] \\ \qquad + A_3[J_0(2\Lambda_r R)/(2\Lambda_r^2) - RJ_1(2\Lambda_r R)/(2\Lambda_r)]\} \\ \qquad + 4\Lambda_i^2 \sin(2\Lambda_i L)\{A_4[1/(2\Lambda_i^2)] + A_5[I_0(2\Lambda_i R)/(2\Lambda_i)] \\ \qquad + A_6[I_0(2\Lambda_i R)/(2\Lambda_i^2) + RI_1(2\Lambda_i R)(2\Lambda_i)]\}. \end{cases}$$

The solution of (8.44a)–(8.44b) is

$$\text{(7.46)} \qquad \tilde{\psi} = \frac{c_0 R^2 Z}{2L} + \lim_{\substack{N \to \infty \\ n \neq 0}} \sum_{-N}^{N} \frac{c_n \phi_1^n(R) \sinh(p_n z)}{p_n^2 \sinh(p_n L)},$$

where the eigenfunction ϕ_1^n and the eigenvalues p_n are given by the equations (A.7)–(A.8). For $n \geqq 1$, p_n are the first quadrant eigenvalues which are numbered according to the size of their real part. To make $\tilde{\psi}$ real-valued, we must choose

$$c_{-n} = \bar{c}_n, \quad p_{-n} = \bar{p}_n.$$

It is not necessary to include the second and third quadrant eigenvalues as

$$\phi_1^n(-p_n, R) = -\phi_1^n(p_n, R).$$

* See the eigenvalue equation (A.8) of Appendix A.

The coefficients c_0 and c_n are to be chosen to satisfy the boundary conditions at $Z = \pm L$ (7.44c). The procedure for determining them is as follows: First we modify the conditions (7.44c):

$$\left.\begin{aligned}\tilde{\psi}_{ZZ} &= \pm f,\\ \tilde{\psi}_{RR} - \frac{1}{R}\tilde{\psi}_R &= \pm g\end{aligned}\right\} \quad \text{at} \quad Z = \pm L \tag{7.47}$$

where

$$\begin{aligned} f &= R(\tilde{g}' - \tilde{f}_1),\\ g &= R\tilde{g}'. \end{aligned} \tag{7.48}$$

Substitution of (7.46) into (7.47) gives

$$\operatorname*{Lim}_{N\to\infty} \sum_{\substack{-N\\ n\neq 0}}^{N} c_n \boldsymbol{\Phi}^n = \boldsymbol{f}, \tag{7.49}$$

where

$$\boldsymbol{f} = \begin{bmatrix} f \\ g \end{bmatrix}$$

and

$$\boldsymbol{\Phi}^n = \begin{bmatrix} \phi_1^n \\ \phi_2^n \end{bmatrix}$$

is given by the equations (A.15)–(A.16). Now we apply the biorthogonality condition (A.24) to (7.49) and find that

$$c_m = \langle \boldsymbol{f}, \boldsymbol{\Phi}^{*m} \rangle_A / k_m \tag{7.50}$$

where k_m, the inner product $\langle , \rangle_A$ and the adjoint eigenfunctions $\boldsymbol{\Phi}^{*m}$ are defined in Appendix A. To determine c_0, we integrate the second component of (7.49) and find that

$$\operatorname*{Lim}_{N\to\infty} T_N = \tilde{g} + K \tag{7.51}$$

where

$$T_N = \sum_{\substack{-N\\ n\neq 0}}^{N} c_n \phi_1^{n\prime}/(R p_n^2) \tag{7.52}$$

and K, the constant of integration, may be found by evaluating (7.51) at some point on the interval $0 \leqq R \leqq 1$. Substitution of the series solution (7.46) into the equation $(7.44\text{c})_2$ gives

$$c_0 + \operatorname*{Lim}_{N\to\infty} T_N = \tilde{g}. \tag{7.53}$$

Comparing (7.51) and (7.53), we find that by choosing $c_0 = -K$ it is possible to satisfy the equation (7.53) and, hence, the boundary condition $(7.44\text{c})_2$. Evaluating (7.51) at $R = 0$, we get

$$c_0 = -K = \left[\tilde{g} - \operatorname*{Lim}_{N\to\infty} T_N\right]\Big|_{R=0}. \tag{7.54}$$

7.5 Biorthogonal Series for Solution of the Mean Problem (II)

This problem cannot be conveniently solved with Fourier series. To solve it we use biorthogonal series. Let

$$\psi = \hat{\psi} - \tilde{\psi} \tag{7.55}$$

where the "particular" solution $\hat{\psi}$ is given by (7.41). $\tilde{\psi}$ then satisfies

$$\left\{\begin{aligned} &\text{a)} \quad \mathcal{L}^2\tilde{\psi} = 0 \quad \text{in} \quad \mathcal{V}_0;\\ &\text{b)} \quad \text{at} \quad R = 1, \frac{1}{R}\mathcal{L}(\tilde{\psi})_R + 2\left(\frac{1}{R}\tilde{\psi}_{ZZ}\right)_R = 0,\\ &\qquad \frac{1}{R}\left(\tilde{\psi}_{RR} - \frac{1}{R}\tilde{\psi}_R - \tilde{\psi}_{ZZ}\right) = 0;\\ &\text{c)} \quad \text{at} \quad Z = \pm L, \frac{1}{R}\tilde{\psi}_Z = \tilde{f}_2,\\ &\qquad \frac{1}{R}\tilde{\psi}_R = \tilde{g} \end{aligned}\right. \tag{7.56}$$

where $\tilde{g}$ is given by $(7.45)_2$ and

$$\begin{aligned}\tilde{f}_2 = {} & 4\Lambda_r^2 \sinh(2\Lambda_r L)\{\coth(2\Lambda_r L)/(2\Lambda_r)\,[A_1 R + A_2 J_1(2\Lambda_r R)\\ & + A_3 R J_0(2\Lambda_r R)]\}\\ & + 4\Lambda_i^2 \sin(2\Lambda_i L)\{\cot(2\Lambda_i L)/(2\Lambda_i)\,[A_4 R + A_5 I_1(2\Lambda_i R) + A_6 R I_0(2\Lambda_i R)]\}.\end{aligned} \tag{7.57}$$

The solution of (7.56a) and (7.56b) is given by (7.46) where the coefficients c_0 and c_n are now determined from (7.56c). Substitution of (7.46) into (7.56c) gives

$$c_0 \begin{bmatrix} R/(2L) \\ 1 \end{bmatrix} + \operatorname*{Lim}_{N\to\infty} \sum_{\substack{-N\\ n\neq 0}}^{N} c_n \begin{bmatrix} \phi_1^n \coth(p_n L)/(R p_n) \\ \phi_1^{n\prime}/(R p_n^2) \end{bmatrix} = \begin{bmatrix} \tilde{f}_2 \\ \tilde{g} \end{bmatrix}. \tag{7.58}$$

We rewrite the expression (7.58) in a form convenient for biorthogonal projections:

$$\operatorname*{Lim}_{N\to\infty} \sum_{\substack{-N\\ n\neq 0}}^{N} c_n \left\{\boldsymbol{\Phi}^n + [\coth(p_n L)/p_n - 1]\begin{bmatrix} \phi_1^n \\ 0 \end{bmatrix}\right\} = \boldsymbol{f}^1 - [c_0/(2L)]\,\boldsymbol{f}^2 \equiv \boldsymbol{f} \tag{7.59}$$

where

$$\boldsymbol{f}^1 = \begin{bmatrix} R\tilde{f}_2 \\ R\tilde{g}' \end{bmatrix}, \quad \boldsymbol{f}^2 = \begin{bmatrix} R^2 \\ 0 \end{bmatrix}. \tag{7.60}$$

The function f now contains the unknown coefficient c_0. To overcome this difficulty we decompose the coefficients

$$c_n = c_n^1 + d_0 c_n^2 \tag{7.61}$$

and expand each part of the function f,

$$\lim_{N\to\infty} \sum_{\substack{-N\\ n\neq 0}}^{N} c_n^i \left\{ \boldsymbol{\Phi}^n + [\coth(p_n L)/p_n - 1] \begin{bmatrix} \phi_1^n \\ 0 \end{bmatrix} \right\} = f^i. \tag{7.62}$$

The constants c_0 and d_0 will be determined later. Applying the biorthogonality condition (A.24) to (7.62), we get

$$\lim_{N\to\infty} \sum_{\substack{-N\\ n\neq 0}}^{N} B_{mn}[\coth(p_n L)/p_n - 1]\, c_n^i + k_m c_m^i = f_m^i, \quad m = \pm 1, \pm 2, \dots \tag{7.63a}$$

where

$$B_{mn} = \int_0^1 \phi_1^m \phi_1^n \frac{1}{R}\, d_R, \qquad f_m^i = \langle f^i, \boldsymbol{\Phi}^{*m} \rangle_A \tag{7.63b}$$

and k_m, the inner product $\langle , \rangle_A$ and the adjoint eigenfunctions $\boldsymbol{\Phi}^{*m}$ are defined in Appendix A. The system (7.63) must be solved by truncation. There are no theorems about completeness of the eigenfunctions $\{\boldsymbol{\Phi}^n\}$ so we must verify that these series do actually converge to prescribed values (see tables of convergence in the thesis of DIXIT, 1979).

7.6 Numerical Results

To compute numerical values from our solutions, we need values of the first and second order material parameters since the data depends on them. Functional forms and values for all these parameters are not known for any material. The restrictions (4.3) on the form of the relaxation modulus are not inconsistent with the assumption that

$$\zeta(s) = -\mu e^{-\nu s}, \quad \mu > 0, \quad \nu > 0.$$

The elastic modulus β is known for many materials. (For rubber and polymeric material β lies in the range of 10^4 to 10^{10} dynes/cm^2.) But we have no basis for assumptions about the form of the second order material parameters. To deal with this difficulty we use superposition and present results for each component of the second order solution. In this way we can avoid assumptions about the second order material parameters.

The numerical computation of solutions require that we make some assumptions about first order parameters. In choosing a representative value of the frequency ω we take into consideration the fact that for viscoelastic fluids good agreement was reported between this kind of theory and experiments for small values of ω (see JOSEPH & BEAVERS, 1977).

After consulting with some rheologists about representative values for the first order parameters, we guessed that

$$\begin{aligned} a &= 2 \text{ cm (Radius of the cylinder)}, \\ \varrho_0 &= 1 \text{ gm/cm}^3, \\ \omega &= 35 \text{ rad/sec}, \\ \beta &= 10^4 \text{ dynes/cm}^2, \\ \mu &= 3.5\times 10^6 \text{ dynes/cm}^2\text{/sec}, \\ \nu &= 70 \text{ sec}^{-1}. \end{aligned} \tag{7.64}$$

Then the values of Λ, Γ^2 and Σ^2 are:

$$\begin{aligned} \Lambda &= 0.1593 + i0.3845, \\ \Gamma^2 &= -0.3708 + i0.2649, \\ \Sigma^2 &= -0.1192 + i0.2251. \end{aligned} \tag{7.65}$$

See the equations $(7.22)_2$, $(7.19)_1$ and $(7.3)_3$ for definitions of the symbols on the left of (7.65).

(i) *Time-periodic displacements of the free surface of problem (I).* The values of the real part of the dimensionless time-periodic displacements

$$U_2 = \Psi_Z/R, \quad W_2 = -\Psi_R/R \tag{7.66}$$

of the surface $R = 1$ are shown in Figure 7.2. They are computed for the following values of the parameters;

$$n_5 = -4\Lambda^2\Sigma^2, \quad n_6 = 1 - 2(4\Lambda^2 + \Sigma^2). \tag{7.67}$$

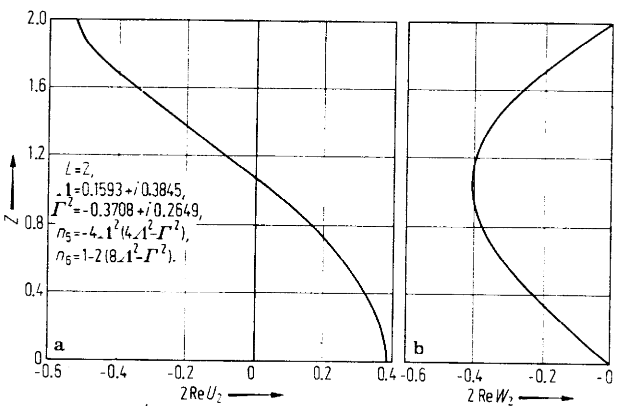


Fig. 7.2. The real part of the dimensionless time-periodic displacements of the free surface at $R = 1$ in problem (I): (a) Radial displacement $U_2(1, Z) = U_2(1, -Z)$, (b) axial displacement $W_2(1, Z) = -W_2(1, -Z)$.

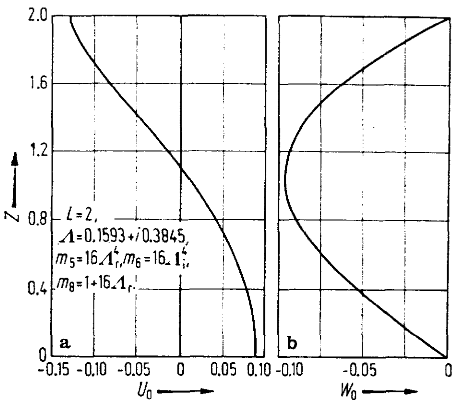


Fig. 7.3. Mean displacements of the free surface at $R=1$ in problem (I): (a) Radial displacement $U_0(1, Z) = U_0(1, -Z)$, (b) axial displacement $W_0(1, Z) = -W_0(1, -Z)$.

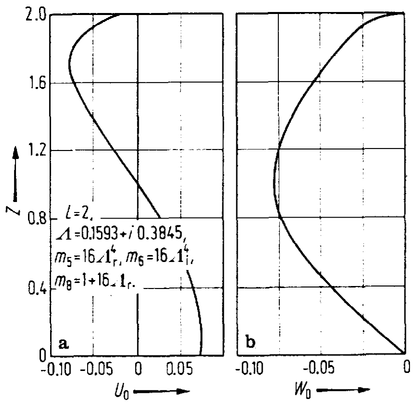


Fig. 7.4. Mean displacements of the free surface at $R=1$ in problem (II): (a) Radial displacement $U_0(1, Z) = U_0(1, -Z)$, (b) axial displacement $W_0(1, Z) = -W_0(1, -Z)$.

(ii) *Mean displacement of the free surface of problem (I).* We computed the solutions of the mean problem (I) with Fourier series in § 7.3 and with biorthogonal series in § 7.4. To compare the two solutions we choose

(7.68) $$m_5 = 16\Lambda_r^4, \quad m_6 = 16\Lambda_i^4, \quad m_8 = 1 + 16\Lambda_r.$$

The two series are different representations of the same solution and they give the same displacements (see Table 7.1). But the biorthogonal series converge

more rapidly than the Fourier series. Displacements $U_0 = \psi_Z/R$ and $W_0 = -\psi_R/R$ of the free surface at $R = 1$ are shown in Figure 7.3.

(iii) *Mean displacement of the free surface of problem (II).* These displacements are shown in Fig. 7.4.

(iv) *Convergence of the biorthogonal series to the prescribed data of problem (I).* The coefficients c_n, $n = 0, 1, \ldots$, defined in § 7.4 were computed for the values (7.65) of the material parameters by methods described in Appendix A. The solution by series obviously satisfies (7.44a) and (7.44b) but it is not obvious that the series represent the prescribed functions f_1 and g appearing in (7.44c). Equations (7.47) show that if the series (7.49) converge to f, then (7.44c) and, hence, the solution by series will satisfy the entire problem (7.44). We have verified numerically that the Cesaro sum

(7.69)$_1$ $$\frac{1}{N}\sum_{M=1}^{N} S_M, \quad S_M = \sum_{\substack{-M \\ n\neq 0}}^{M} c_n(\phi_2^n - \phi_1^n)/R$$

for

(7.69)$_2$ $$\frac{1}{R}\left(\tilde{\psi}_{RR} - \frac{1}{R}\tilde{\psi}_R - \tilde{\psi}_{ZZ}\right)\Bigg|_{Z=L}$$

converges pointwise to $\tilde{f}_1$, and we verified numerically that the Cesaro sum

(7.70) $$c_0 + \frac{1}{N}\sum_{M=1}^{N} T_M \cong \frac{1}{R}\tilde{\psi}_R\Bigg|_{Z=L}$$

converges to $\tilde{g}$. Pointwise convergence to three figures is achieved when $N = 50$ except that there is a sort of "Gibbs phenomenon" in the expansion (7.69) of $\tilde{f}_1$; that is, convergence very near $R = 1$ where $\tilde{f}_1(1) \neq 0$ and $S_M(1) = 0$ is not obtained. The convergence of partial sums is much less rapid than the convergence of Cesaro sums. Summability theorems and theorems about Gibbs phenomena have not yet been formulated for biorthogonal series.

(v) *Convergence of the biorthogonal series, under truncation to the prescribed data of problem (II).* To compute from the solution given in § 7.5, we solve the system (7.63) for the coefficients c_n^i by truncation. In our sample computation we use the values (7.68) in (7.32) and (7.42). With $N = 50$, the series

(7.72) $$\mathcal{S}^i_N \equiv \sum_N c_n^{i(N)} \phi_1^n \coth(p_n L)/(R p_n)$$

converges to $\tilde{f}_2$ ($i = 1$) and R ($i = 2$) while the series

(7.73) $$\mathcal{T}^i_N \equiv \sum_N c_n^{i(N)} \phi_1^{n\prime}/(R p_n^2)$$

converges to $\tilde{g} - \tilde{g}(0) + K^{1(N)}$ $(i = 1)$ and $K^{2(N)}$ $(i = 2)$ where

(7.74) $$K^{i(N)} = \mathscr{T}^{i}_{N}\big|_{R=0}$$

(see Table 6.4 of Dixit, 1979). The series (7.73) is obtained by integrating the second component of the series in (7.62).

Our approximate solution is the expression (7.46) with the coefficients c_0 and c_n replaced by the truncated coefficients $c_0^{(N)}$ and $c_n^{(N)}$. $c_0^{(N)}$ and $d_0^{(N)}$ are determined so that equation (7.58) and, hence, the boundary conditions (7.56c) are satisfied. Equations (7.58) and (7.61) and the convergence results (7.72)–(7.73) imply that

(7.75) $$\begin{gathered} c_0^{(N)}(1/2L) + d_0^{(N)} = 0, \\ c_0^{(N)} + d_0^{(N)}(K^{2(N)}) = \tilde{g}(0) - K^{1(N)}. \end{gathered}$$

Since $K^{2(N)} \neq 2L$, $c_0^{(N)}$ and $d_0^{(N)}$ can be determined uniquely.

Table 7.1. Comparison of the values of the displacements of the free surface at $R = 1$ given by biorthogonal and by Fourier series for $\Lambda = 0.1593 + i0.3845$, $m_5 = 16.1_r^4$, $m_6 = 16.1_i^4$, $m_8 = 1 + 16.1_r$ and $L = 2$. Both series have been truncated after N terms (a) $N = 5$, (b) $N = 10$. The biorthogonal series converges more rapidly.

Biorthogonal U_0	W_0	Fourier U_0	W_0	Z
.893E−01	0	.897E−01	0	0
.885E−01	−.140E−01	.887E−01	−.140E−01	.10E+00
.861E−01	−.277E−01	.861E−01	−.277E−01	.20E+00
.822E−01	−.408E−01	.818E−01	−.409E−01	.30E+00
.767E−01	−.532E−01	.763E−01	−.532E−01	.40E+00
.696E−01	−.644E−01	.695E−01	−.644E−01	.50E+00
.612E−01	−.743E−01	.614E−01	−.743E−01	.60E+00
.513E−01	−.826E−01	.517E−01	−.826E−01	.70E+00
.400E−01	−.891E−01	.403E−01	−.891E−01	.80E+00
.275E−01	−.936E−01	.275E−01	−.937E−01	.90E+00
.139E−01	−.960E−01	.135E−01	−.961E−01	.10E+01
−.756E−03	−.962E−01	−.123E−02	−.962E−01	.11E+01
−.163E−01	−.940E−01	−.165E−01	−.940E−01	.12E+01
−.325E−01	−.893E−01	−.321E−01	−.894E−01	.13E+01
−.491E−01	−.823E−01	−.484E−01	−.824E−01	.14E+01
−.659E−01	−.730E−01	−.654E−01	−.731E−01	.15E+01
−.823E−01	−.616E−01	−.825E−01	−.616E−01	.16E+01
−.978E−01	−.482E−01	−.988E−01	−.482E−01	.17E+01
−.112E+00	−.331E−01	−.112E+00	−.331E−01	.18E+01
−.122E+00	−.169E−01	−.122E+00	−.169E−01	.19E+01
−.128E+00	.164E−03	−.125E+00	−.817E−13	.20E+01

(a) $N = 5$

Table 7.1. (Continuation)

Biorthogonal U_0	W_0	Fourier U_0	W_0	Z
.893E−01	0	.893E−01	0	0
.885E−01	−.140E−01	.885E−01	−.140E−01	.10E+00
.861E−01	−.277E−01	.862E−01	−.277E−01	.20E+00
.822E−01	−.408E−01	.822E−01	−.408E−01	.30E+00
.767E−01	−.532E−01	.766E−01	−.532E−01	.40E+00
.696E−01	−.644E−01	.697E−01	−.644E−01	.50E+00
.612E−01	−.743E−01	.612E−01	−.743E−01	.60E+00
.513E−01	−.826E−01	.512E−01	−.826E−01	.70E+00
.400E−01	−.891E−01	.400E−01	−.891E−01	.80E+00
.275E−01	−.936E−01	.276E−01	−.937E−01	.90E+00
.139E−01	−.960E−01	.140E−01	−.961E−01	.10E+01
−.756E−03	−.962E−01	−.815E−03	−.962E−01	.11E+01
−.163E−01	−.940E−01	−.163E−01	−.940E−01	.12E+01
−.325E−01	−.893E−01	−.324E−01	−.894E−01	.13E+01
−.491E−01	−.823E−01	−.491E−01	−.824E−01	.14E+01
−.659E−01	−.730E−01	−.660E−01	−.731E−01	.15E+01
−.823E−01	−.616E−01	−.823E−01	−.616E−01	.16E+01
−.978E−01	−.482E−01	−.976E−01	−.482E−01	.17E+01
−.111E+00	−.331E−01	−.112E+00	−.332E−01	.18E+01
−.122E+00	−.169E−01	−.122E+00	−.169E−01	.19E+01
−.127E+00	.578E−04	−.127E+00	−.817E−13	.20E+01

(b) $N = 10$

Appendix A. Biorthogonality of Traction Free Eigenvectors of Stokes' Stream Function

All cases of separable solutions

(A.1) $$\tilde{\psi}(R, Z) = \phi_1(R)\mathscr{Z}(Z)$$

of the equation

(A.2) $$\mathscr{L}^2\tilde{\psi} = 0$$

have been discussed by Yoo & Joseph (1978). Here it is enough to note that the case

(A.3) $$\mathscr{Z}''/\mathscr{Z} = p^2 \quad (p^2 \text{ complex})$$

satisfies the boundary conditions (7.44b) in a nontrivial way. Then ϕ_1 satisfies

(A.4) $$\begin{aligned} &\text{a)}\quad L^2\phi_1 + 2p^2L\phi_1 + p^4\phi_1 = 0: \\ &\text{b)}\quad \left.\begin{aligned} &(L\phi_1 + p^2\phi_1)'/R + 2(p^2\phi_1/R)' = 0, \\ &(L\phi_1 - p^2\phi_1)/R = 0 \end{aligned}\right\} \quad \text{at} \quad R = 1 \end{aligned}$$

where

(A.5) $$L(\cdot) = R\frac{d}{dR}\left[\frac{1}{R}\frac{d(\cdot)}{dR}\right].$$

Solutions of the eigenvalue problem (A.4), which satisfy the conditions

(A.6) $$\phi_1/R,\ \phi_1'/R,\ \phi_1/R^2 \text{ and } L\phi_1 R \text{ bounded at } R = 0^*,$$

are

(A.7) $$\phi_1^n = -[p_n J_0(p_n) + J_1(p_n)]\, RJ_1(p_n R) + p_n J_1(p_n)\, R^2 J_0(p_n R).$$

The eigenvalues p_n satisfy the equation

(A.8) $$p^2 J_0^2(p) + (p^2 - 1)\, J_1^2(p) = 0.$$

There are countably infinite eigenvalues which are symmetrically located in the complex plane. They can be determined by an iterative scheme using Muller's method with deflation. To find the initial guesses, we use Hankel's asymptotic expansion for the Bessel functions. The first term in the asymptotic expansion of (A.8) is

(A.9) $$p^2 J_0^2(p) + (p^2 - 1)\, J_1^2(p) = (1/\pi)\{2p - \cos 2p + O(1/p)\}.$$

For large n,

(A.10) $$p_n = n\pi + i[\ln(4n\pi)]/2 + O(\alpha_n)$$
$$(\alpha_n \to 0 \quad \text{as} \quad n \to \infty)$$

is the asymptotic form of the roots of the equation

$$\cos 2p = 2p.$$

The expressions (A.10) serve as good initial guesses. The first 25 eigenvalues of the first quadrant are shown in the Table A.1.

The zero^th eigenfunction of (A.4) corresponding to the eigenvalue $p = 0$ and satisfying the conditions (A.6) is

(A.11) $$\phi_1^0 = R^2.$$

The scalar eigenvalue problem (A.4) for $p \neq 0$ may be put into the vector form by defining

(A.12) $$\boldsymbol{\Phi} = \begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix}$$

where

$$\phi_2 = L\phi_1/p^2.$$

---

* The first two conditions are necessary to make the displacements bounded at $R = 0$. The last two conditions ensure that stresses are bounded at $R = 0$.

Table A.1. The first 25 first quadrant roots (eigenvalues) of the equation
$$p^2 J_0^2(p) + (p^2 - 1)\, J_1^2(p) = 0.$$

Re p_n	Im p_n	n
2.81056171828106	1.33993315859693	1
6.09472909409848	1.63429584646030	2
9.28885015543096	1.82659055752229	3
12.45876664276079	1.96618159333002	4
15.61833174130800	2.07553463135092	5
18.77243716214105	2.16536957862459	6
21.92328592041747	2.24158088523701	7
25.07202594981356	2.30774812397932	8
28.21931754299874	2.36620589609645	9
31.36556883321293	2.41856053381630	10
34.51104634067383	2.46596420591960	11
37.65593183601800	2.50927125152231	12
40.80035373830697	2.54913261824994	13
43.94440546420128	2.58605563821421	14
47.08815666246687	2.62044335753963	15
50.23166036564976	2.65262127124818	16
53.37495769592556	2.68285601032618	17
56.51808105107148	2.71136872204357	18
59.66105631561845	2.73834485420629	19
62.80390442926796	2.76394144360164	20
65.94664252108532	2.78829263532425	21
69.08928474393406	2.81151392436163	22
72.23184289796336	2.83370545843036	23
75.37432690300511	2.85495464097259	24
78.51674516108369	2.87533820484295	25

The problem satisfied by $\boldsymbol{\Phi}$ is

(A.13)
$$\text{a)}\quad \mathbb{L}\boldsymbol{\Phi} + p^2\boldsymbol{\Phi} = \mathbf{0};$$
$$\text{b)}\quad \left.\begin{aligned} &3\phi_1' + \phi_2' - 2\phi_1/R = 0,\\ &\phi_1 - \phi_2 = 0 \end{aligned}\right\} \quad \text{at} \quad R = 1$$

where

(A.14) $$\mathbb{L}(\cdot) = A^{-1}L(\cdot), \quad A = \begin{bmatrix}0 & -1\\ 1 & 2\end{bmatrix}.$$

The problem (A.13) has no nontrivial solution corresponding to $p = 0$ which also satisfies the conditions (A.6). Eigenvalues of (A.13) are given by the equation (A.8). The eigenfunctions are

(A.15) $$\boldsymbol{\Phi}^n = \begin{bmatrix}\phi_1^n\\ \phi_2^n\end{bmatrix}$$

where ϕ_1^n is given by (A.7) and

$$\phi_2^n = -\phi_1^n - 2J_1(p_n)\,RJ_1(p_nR). \tag{A.16}$$

It is easy to verify that all the eigenfunctions satisfy

$$\left.\begin{array}{l}\phi_1 = \phi_2 = 0\\ \phi_1'/R \text{ and } \phi_2'/R \text{ bounded}\end{array}\right\} \quad \text{at} \quad R = 0. \tag{A.17}$$

The adjoint operator $\mathbb{L}^*$ and the adjoint boundary conditions are found from the relation

$$\langle \mathbb{L}\boldsymbol{\Phi}, \boldsymbol{\Phi}^*\rangle_A = \langle \boldsymbol{\Phi}, \mathbb{L}^*\boldsymbol{\Phi}^*\rangle_A \tag{A.18}$$

where the inner product is defined by

$$\langle \boldsymbol{a}, \boldsymbol{b}\rangle_A = \int_0^1 (A\boldsymbol{a})\cdot\bar{\boldsymbol{b}}\,\frac{1}{R}\,dR. \tag{A.19}$$

$\boldsymbol{\Phi}$, in (A.18), satisfies the conditions (A.13b) and (A.17) while $\boldsymbol{\Phi}^*$ is required to satisfy the adjoint boundary conditions.

The adjoint vector eigenvalue problem is

$$\begin{array}{ll} \text{a)} & \mathbb{L}^*\boldsymbol{\Phi}^* + \bar{p}^2\boldsymbol{\Phi}^* = \boldsymbol{0};\\ \text{b)} & \left.\begin{array}{l}\phi_1^{*\prime} + \phi_2^{*\prime} - 2\phi_2^*/R = 0,\\ \phi_1^* - 3\phi_2^* = 0\end{array}\right\} \text{ at } R = 1;\\ \text{c)} & \left.\begin{array}{l}\phi_1^* = \phi_2^* = 0,\\ \phi_1^{*\prime}/R \text{ and } \phi_2^{*\prime}/R \text{ bounded}\end{array}\right\} \text{ at } R = 0\end{array} \tag{A.20}$$

where

$$\mathbb{L}^*(\cdot) = (A^T)^{-1}L(\cdot). \tag{A.21}$$

The adjoint eigenvalues satisfy the equation (A.8). The adjoint eigenfunctions are

$$\boldsymbol{\Phi}^{*m} = \begin{bmatrix}\phi_1^{*m}\\ \phi_2^{*m}\end{bmatrix} \tag{A.22}$$

where

$$\begin{aligned}\phi_1^{*m} &= \overline{\phi_1^m} - 2J_1(\bar{p}_m)\,RJ_1(\bar{p}_mR),\\ \phi_2^{*m} &= \overline{\phi_1^m}.\end{aligned} \tag{A.23}$$

$p = 0$ is not an eigenvalue of the problem (A.20).

The biorthogonal condition

$$\langle \boldsymbol{\Phi}^n, \boldsymbol{\Phi}^{*m}\rangle_A = k_n\,\delta_{nm} \tag{A.24}$$

where

$$k_m = 2J_1^2(p_m)\,[p_mJ_0(p_m)\,J_1(p_m) - p_m^2J_0^2(p_m) - J_1^2(p_m)]/p_m^2 \tag{A.25}$$

follows from (A.13a), (A.18) and (A.20a).

In deriving the expressions for k_m (A.25), the coefficients c_m (7.50), the matrix elements B_{mn} $(7.63)_1$ and the inhomogeneous terms f_m^i $(7.63)_2$, we used certain integrals of Bessel functions which are derived in Appendix B of the paper by Yoo & Joseph (1978) and listed in the thesis of Dixit (1979).

Appendix B. Frame Indifference and Perturbation Theory

It is well known that isotropic functionals of the type (2.8) are frame indifferent. We wish to show that the stresses we deduce from them under perturbation are also indifferent. The stress functionals of $\boldsymbol{B}(t, \varepsilon)$ and $\boldsymbol{G}(s, \varepsilon)$ which are induced under perturbation of the natural state ($\boldsymbol{B}^{\langle 0\rangle}(t) = 1$, $\boldsymbol{G}^{\langle 0\rangle}(s) = \boldsymbol{0}$) are isotropic. They are also indifferent if their arguments, the derivatives $\boldsymbol{B}^{\langle n\rangle}$ with respect to ε at $\varepsilon = 0$, are indifferent. We must show how such derivatives transform under a change of frame. We have

$$\boldsymbol{B}^*(\boldsymbol{X}, t, \varepsilon) = \boldsymbol{F}^*(\boldsymbol{X}, t, \varepsilon)\,\boldsymbol{F}^{*T}(\boldsymbol{X}, t, \varepsilon) \tag{B.1}$$

where $\boldsymbol{X}$ is independent of ε and

$$\boldsymbol{F}^*(\boldsymbol{X}, t, \varepsilon) = \nabla_{\boldsymbol{X}}\boldsymbol{x}^*(\boldsymbol{X}, t, \varepsilon).$$

Moreover, under the change of frame

$$\boldsymbol{x}^*(\boldsymbol{X}, t, \varepsilon) = \boldsymbol{C}(t) + \boldsymbol{Q}(t)\,\boldsymbol{x}(\boldsymbol{X}, t, \varepsilon) \tag{B.2}$$

where $\boldsymbol{Q}(t)$ is an orthogonal tensor and

$$\boldsymbol{F}^*(\boldsymbol{X}, t, \varepsilon) = \boldsymbol{Q}(t)\,\boldsymbol{F}(\boldsymbol{X}, t, \varepsilon). \tag{B.3}$$

Since $\boldsymbol{x}(\boldsymbol{X}, t, 0) = \boldsymbol{X}$ we have $\boldsymbol{F}^{\langle 0\rangle}(\boldsymbol{X}, t) = \boldsymbol{1}$

and

$$\boldsymbol{F}^{*\langle 0\rangle}(\boldsymbol{X}, t) = \boldsymbol{Q}(t), \tag{B.4}$$

whilst for $n > 1$

$$\boldsymbol{F}^{*\langle n\rangle}(\boldsymbol{X}, t) = \boldsymbol{Q}(t)\,\boldsymbol{F}^{\langle n\rangle}(\boldsymbol{X}, t). \tag{B.5}$$

We find from (B.4) that

$$\boldsymbol{B}^{*\langle 0\rangle}(\boldsymbol{X}, t) = \boldsymbol{Q}^T(t)\,\boldsymbol{Q}(t) = 1 \tag{B.6}$$

and from (B.5) with $n = 1$ that

$$\begin{aligned}\boldsymbol{B}^{*\langle 1\rangle}(\boldsymbol{X}, t) &= \boldsymbol{F}^{*\langle 0\rangle}(\boldsymbol{X}, t)\,\boldsymbol{F}^{*\langle 1\rangle T} + \boldsymbol{F}^{*\langle 1\rangle}\boldsymbol{F}^{*\langle 0\rangle T}\\ &= \boldsymbol{Q}(t)\,2\boldsymbol{E}^{\langle 1\rangle}(\boldsymbol{X}, t)\,\boldsymbol{Q}^T(t)\\ &= \boldsymbol{Q}(t)\,\boldsymbol{B}^{\langle 1\rangle}(\boldsymbol{X}, t)\,\boldsymbol{Q}^T(t).\end{aligned} \tag{B.7}$$

Hence $\boldsymbol{B}$ is indifferent.

We note that $2E^{\langle 1\rangle} = F^{\langle 1\rangle} + F^{T\langle 1\rangle}$ is the linear strain but

$$B^{*\langle 1\rangle} \neq F^{*\langle 1\rangle} + F^{*\langle 1\rangle T}. \tag{B.8}$$

In studies which assume linearity from the start (TRUESDELL & TOUPIN, 1960; TRUESDELL, 1977; FOSDICK & SERRIN, 1979), it is natural to assume equality in (B.8); then we get

$$E^{*\langle 1\rangle} = F^{*\langle 1\rangle} + F^{*T\langle 1\rangle} \quad \text{and} \quad E^{\langle 1\rangle}(X, t)$$

is not indifferent. In general, since

$$B^*(X, t, \varepsilon) = Q(t)\, B(X, t, \varepsilon)\, Q^T(t),$$

we have

$$B^{\langle n\rangle *}(X, t) = Q(t) \mathrm{x}\, B^{\langle n\rangle}(X, t)\, Q^T(t). \tag{B.9}$$

We calculate $B^{\langle n\rangle}(X, t)$ by perturbations and (B.9) then defines $B^{\langle n\rangle *}(X, t)$.

The same reasoning works when the present place x is the reference. Then

$$G(s, \varepsilon) = F_t^T(x, \tau, \varepsilon)\, F_t(x, \tau, \varepsilon) - 1,$$

$s = t - \tau$, where

$$F_t(x, \tau, \varepsilon) = \nabla_x \chi_t(x, \tau, \varepsilon).$$

In this case we have x independent of ε and

$$F_t(x, \tau, 0) = F_t^{\langle 0\rangle}(x) = 1, \quad \tau \leqq t.$$

Under a change of frame we get

$$F_t^*(x^*, \tau, \varepsilon) = Q(\tau)\, F_t(x, \tau, \varepsilon)\, Q^T(t),$$

$$F_t^{*\langle 0\rangle}(x^*, \tau) = Q(\tau)\, Q^T(t)$$

and

$$F_t^{*\langle n\rangle}(x, \tau) = Q(\tau)\, F_t^{\langle n\rangle}(x, \tau)\, Q^T(t).$$

Then

$$\begin{aligned} G^{*\langle 1\rangle}(s, \varepsilon) &= F_t^{*\langle 0\rangle T} F_t^{*\langle 1\rangle} + F_t^{*\langle 1\rangle T} F_t^{*\langle 1\rangle} \\ &= Q(t)\, 2E^{\langle 1\rangle}(x, \tau)\, Q^T(t) \\ &= Q(t)\, G^{\langle 1\rangle} Q^T(t) \end{aligned}$$

is indifferent. We find that

$$2E^{\langle 1\rangle} \stackrel{\text{def}}{=} F_t^{\langle 1\rangle} + F_t^{\langle 1\rangle T} = G^{\langle 1\rangle}$$

but

$$2E^{*\langle 1\rangle} \stackrel{\text{def}}{=} F_t^{*\langle 1\rangle} + F_t^{*\langle 1\rangle T} \neq G^{*\langle 1\rangle}.$$

In general,

$$G^{*\langle n\rangle}(s) = Q(t)\, G^{\langle n\rangle} Q^T(t). \tag{B.10}$$

We compute $G^{\langle n\rangle}(s)$ by perturbations and (B.10) then defines $G^{*\langle n\rangle}(s)$. The same formula (B.10) holds when, as in the solids theory given here, $x = \chi(X, t, \varepsilon)$ depends on ε.

Appendix C. Preliminary Experiments on Rod Climbing

We have initiated experiments to determine if and how our theory can be used.* In fluids, free surface deformations can be used to determine material properties. But in fluids the deformations are large even when the stresses are small while in solids the deformations can be small when the stresses are large. Thus it might be better to determine material properties of solids through measurements of stresses. We could use our theory to design experiments for measuring stresses, but here we are interested in seeing if we get measurable deformations of stress-free surfaces in solids. In fact, our first experiments on rod climbing (Chap. 6) show that we do get large deformations.

In our experiment, the steel rod is rigidly bonded to the polyurethane ** (urethane rubber) sample. We were not succesful in achieving the partial bonding required in Chapter 6. (Partial bonding means that the sample must turn with the rod but can slide along the rod.) A theoretical study of the fully bonded case is presently under preparation†.

We tried casting many rubber-like materials for experiments but the bond between the rod and the sample would break before we could get a twist large enough to deform the sample noticeably. The experiment reported below is a description of what we saw and measured on the very first sample in which the bond did not break. We have not yet perfected our bonding procedures, our measurements or our interpretation of the measurements. Thus our report is preliminary. We have no reason to believe that nonlinear effects in our randomly chosen sample are relatively large. It seems likely that we can find solids in which the climb is orders of magnitude larger than in the polyurethane.

The dimensions of polyurethane sample shown in photographs of Fig. C.1 were chosen to maximize the climb under constraints of size and weight. We simulated the semi-infinite case by chosing $\alpha = L/a = 5/2$ (see Fig. 6.2):

$$2L = \frac{5}{4}\,\text{in.}, \quad a = \frac{1}{4}\,\text{in.}$$

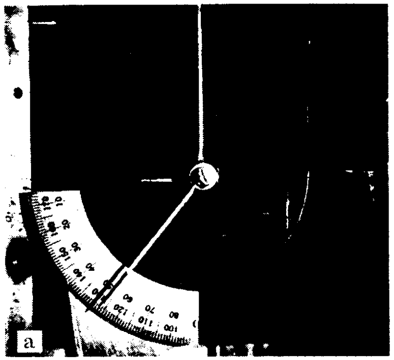
In the theory, the outer boundary is a circle of radius b and already the bulge of the free surface is nearly independent of b when $\lambda = b/a > 20$. In the experi-

* We wish to thank to A. CERS for his help with all aspects of the experimental work. More experiments, based on the theory of this paper, are presently in progress in the Rheology laboratory of the Department of Aerospace Engineering and Mechanics at the University of Minnesota with the collaboration of G. S. BEAVERS.

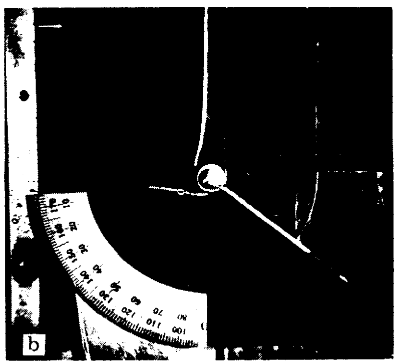
** Manufactured by "Devcon" under the brand name Devcon 60.

† See note added in proof.

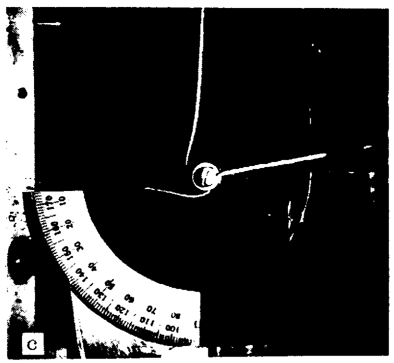
ment, the outer boundary is square and is fitted snuggly into a square steel casing with 6 inch sides. The effect of the far removed outer boundary on the bulge at the rod is negligible. The 1/2 in diameter steel rod was rigidly bonded at the



(a) The natural state of urethane rubber. The twist angle $\varepsilon = 0$ when the protractor reads $\theta = 53°$, $\varepsilon = \theta - 53°$.



(b) Torque = 40 lbs × 3 in.
$\varepsilon = 138° - 53° = 85°$.



(c) Torque = 60 lb × 3 in.
$\varepsilon = 188° - 53° = 135°$.

Fig. C.1. *Rod climbing of urethane rubber.* A protractor is attached to a 1/2 inch steel rod which is bonded at the center of a urethane rubber sample. The undeformed state is shown in (a); a zero angle of twist ε corresponds to the 53° mark on the protractor scale. The size of the "climb" or "bulge" of the traction-free surface near the rod is made visual by the Moiré fringing of vertical lines on a diffraction grating. The bulge is axisymmetric and the curvature of the fringe lines in (a)–(c) is proportional to the product of the rise and the angle between a radial and horizontal line (the projection perpendicular to the etched lines). The angle of twist ε seems to be very nearly a linear function of the torque applied to the rod (see Fig. C.2) and the "bulge" increases approximately like the square of ε (see Fig. C.3).

geometric center of the polyurethane sample. The whole sample was fixed rigidly on the bed of a lathe and the rod was gripped rigidly by the chuck head (radius 3″) which was free to rotate under torques induced by the tangential pull of a steel wire loaded under gravity through pulleys and weights.

To determine the bulge induced by normal stresses we used a diffraction grating which produced Moiré fringes. Pictures of these fringes are shown in the photographs of Fig. C.1. The fringes are visual projections of the bulge (actually, the slope) of the traction free bulge perpendicular to the etched lined of the grating. In the loaded samples the increasing density and curvature of the fringes is caused by the bulge of the free surface. The maximum bulge observed in our experiment was about 1/8 of inch.

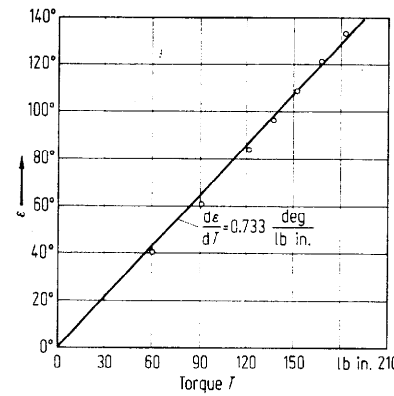


Fig. C.2. Twist angle ε versus torque T for polyurethane rubber, Devcon 60-A.

In Fig. C.2 we have plotted the applied torque T versus the angle of twist ε. The applied torque is given by product of the chuck radius (3 inches) and the applied weight. The twist angle ε was read off the protractor shown in Fig. C.1; ε = protractor reading $\theta - 53°$, where $\varepsilon = 0$ corresponds to the unloaded, undeformed natural state of the urethane rubber. In Fig. C.3 we have plotted the bulge measured by a dial indicator at $R = 0.437''$ versus the angle of twist. These two figures show that the torque is nearly a linear function and the bulge nearly a quadratic function of the angle of twist. A table of values of experimental points used in constructing Figs. C.2 and C.3 shown in Table C.1 and photographs corresponding to some of the points are shown in Fig. C.1. The urethane rubber is very viscous and transients decay slowly. Each data point in Table 1 was taken three minutes after changing loads. These deflections may be less than true deflections after transients have completely decayed.

If the observed deflections satisfy the equations of our perturbation theory truncated after terms of order two, the torque would be exactly linear and the bulge exactly quadratic in the angle ε of twist.

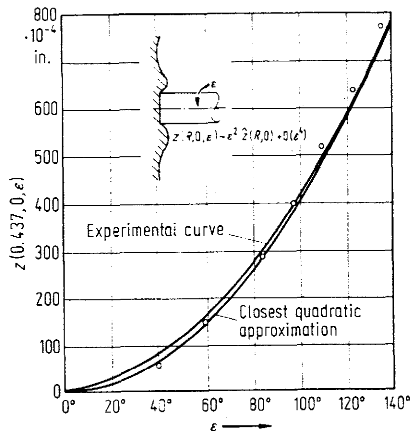


Fig. C.3. Bulge z at $R = 0.437''$ versus twist angle ε for polyurethane rubber, Devcon 60-A.

Table C.1. Experimental data for polyurethane rubber Devcon 60-A. All readings were taken three minutes after changing loads.

Torque T (lb.-in.)	Twist ε (deg.)	Deflection z at $R = 0.437''$ (in 10^{-4} inches)	z/ε^2
60	41	52	3.09
90	62	150	3.90
120	85	292	4.04
135	98	402	4.18
150	110	520	4.29
165	123	643	4.25
180	135	775	4.25

An exact comparison of the second order theory given in Chapter 6 with the experiment is not possible because the problem solved in Chapter 6 assumes partial bonding and in the experiment the rod is fully bonded to the urethane rubber. The solution at first order is the same for partial and full bonding. The second order theory in the fully bonded case also satisfies (6.24) except that $(6.24)_2$ is replaced with the full bonding condition $\Psi_x = \Psi_t = 0$ on $t = 1, \lambda$. The remarks following (6.24) also hold in the fully bonded case. Hence when L and b are both large, the shear stress on the rod is given under (6.10) as

$$\tau = -T_{r\theta} = -\beta R\phi\varepsilon + O(\varepsilon^3) \tag{C.1}$$

where $\varepsilon = 0$ and $\phi = a^2/R^2$ and

$$z(R, Z, \varepsilon) = Z + \hat{z}(R, Z)\,\varepsilon^2 + O(\varepsilon^4) \tag{C.2}$$

where

$$\hat{z}(R, 0) = C(R) \quad a \quad \left(1 + \frac{4\beta^{[2]}}{\beta}\right). \tag{C.3}$$

The function $C(R)$ is to be determined from the solution of $(6.24)_1$, $(6.24)_3$ applied to $x = 0$ and $\psi_x = \psi_t = 0$ at $t = 1, \infty$. In the partially bonded case, $C(a) = 0.2645$.

In the urethane rubber at $R = a$, $\tau = 2\beta\varepsilon + O(\varepsilon^3)$. The torque T at $r = a$ is $T = a\tau \times$ area of contacting surface $= 4\pi a^2 L\tau$ and

$$\beta = \frac{1}{2}\frac{d\tau}{d\varepsilon} = \frac{1}{2}\left\{\frac{1}{4\pi a^2 L}\right\}\frac{dT}{d\varepsilon},$$

$dT/d\varepsilon = (1/0.733)\,\dfrac{\text{lb.} - \text{in.}}{\text{deg}}$, $a = 1/4''$, $L = 5/4''$. Hence

$$\beta = 0.549\,\frac{\text{MN}}{\text{m}^2} \quad (79.6 \text{ psi}).$$

To check the order of magnitude of $\beta^{[2]}$ we put $C = 0.2645$, $a = 0.437''$, $Z = 0$ in (C.2) and (C.3) and compute $\hat{z}(a, 0) = 4.0 \times 100^6$ in/deg.$^2 = 33.35 \times 100^6$ m/rad^2. It then follows that $\beta^{[2]} = -0.12\, MN/m^2$ in the urethane rubber. This value is of the same order and the same sign as experimentally determined values for other rubber-like materials given in § 3.4.

Our urethane rubber is very viscous and changing loads give rise to slow monotonic transients. The urethane rubber cannot be treated as an elastic material, much less a "Mooney" material, in any deformation which is not close to equilibrium.

Note added in proof: The fully bonded problem for rod climbing in the elastic case has been solved in the following two works:

(1) BÖHME, G.: Nichtlineare Theorie Rotationssymetrischen Elasticher Deformationen. THO Wissenschaft und Technik Schriftenreihe der TH – Darmstadt, 1980.

(2) PÖPPLAU, J.: Die Berechnung der Sekundärdeformationen eines Hohlzylinders aus Mooney-Rivlin Material mit der Finite Elemente Methode. Ing.-Archiv 50 (1981).

We learned of these works after our paper was sent to press. Only elastic materials are considered in these papers and the only problem treated is the fully bonded problem for rod climbing in the semi-infinite case. We are under the general impression that their calculations give the bulge that we have observed in the experiments reported here. Our analysis shows that in the partially bonded case normal stress effects produce radial displacements as well as a bulge, but only in finite cylinders and not infinite ones.

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