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DANIEL D. JOSEPH

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I am going to prove that the rest state of fluids of grade n, any $n \neq 1$, is unstable in the spectral sense of linearized theory when the ratio of the coefficients of A_n and A_{n-1} in the constitutive equation is negative. Negative ratios, and only negative ratios, are implied by integral expansions of the stress. Moreover, in the only case ($n=2$) which has been checked in experiments with polymeric liquids, the ratio is negative.

To explain my result I have to specify further what I mean by linearized theory and what I mean by the ratio of coefficients. I am following the analysis of my book on stability (1976) and carrying it further. I take the stress for an incompressible simple fluid in the form $\boldsymbol{T}=-p\mathbf{1}+\boldsymbol{S}$ where instead of normalizing the extra stress $\boldsymbol{S}$ in the usual way with $\operatorname{Tr}\boldsymbol{S}=0$ I normalize so that $\boldsymbol{S}=0$ on rigid motions and, of course, $\boldsymbol{S}=0$ when there is no motion. Thus $\boldsymbol{S}$ need not be traceless and in fact the usual approximations set down for $\boldsymbol{S}$ are not traceless.

There are two types of approximations which are used to express the stress on motions perturbing the rest state. I will follow COLEMAN & NOLL (1960, 1961) in setting my work in the frame of simple fluids with fading memory. I will assume that the reader is familiar with the common notations of continuum mechanics and I recall that the stress is given by a functional of the history of the Cauchy strain

(1) $$\text{(1)}\quad \boldsymbol{S}=\underset{0}{\overset{\infty}{\mathscr{F}}}[\boldsymbol{G}(s)], \qquad s=t-\tau,$$

where

$$\boldsymbol{G}(s)=\boldsymbol{F}_t^T(t-s)\,\boldsymbol{F}_t(t-s)-\mathbf{1}.$$

COLEMAN & NOLL (1960) derived asymptotic forms for retarded motions, which are slow motions in slow time, and they got a sequence of approximations $\boldsymbol{S}^{(N)}\sim\boldsymbol{S}$ perturbing the rest state

$$
\begin{aligned}
&S^{(1)} = \mu A_1,\\
&S^{(2)} = S^{(1)} + \alpha_1 A_2 + \alpha_2 A_1^2,\\
&S^{(3)} = S^{(2)} + \beta_1 A_3 + \beta_2 (A_2 A_1 + A_1 A_2) + \beta_3 (\mathrm{tr}\, A_2) A_1,\\
&S^{(n)} = S^{(n-1)} + \hat{\phi}_n A_n + \text{nonlinear terms}
\end{aligned} \tag{2}
$$

where μ, α_1, α_2, β_1, β_2, β_3, $\hat{\phi}_n$ are constants. COLEMAN & NOLL (1961) also derived asymptotic forms $\mathscr{F}^{(N)} \sim S$ for small amplitude motions which need not be in slow time, and they justified an integral representation for the first one

$$
\mathscr{F}^{(1)} = \int_0^\infty \mathring{G}(s)\, G(s)\, ds = -\int_0^\infty G(s)\, \mathbb{J}(s)\, ds \tag{3}
$$

where $\mathbb{J}(s) = \mathring{G}(s)$ comes up after integrating by parts, using $G(s) \to 0$ as $s \to \infty$. I like $\mathbb{J}(s)$ better than $G(s)$ because it leads to a perturbation theory (JOSEPH, 1976; JOSEPH & BEAVERS, 1977) in which only velocities and not particle paths are active, but of course $\mathbb{J}$ and G are equivalent whenever the kernels in the integrals are such as to justify integration by parts for $G(s)$ in the weighted $L_2(0, \infty)$ spaces of fading memory. It may be assumed that

$$
\begin{aligned}
&\mathscr{F}^{(2)} = \mathscr{F}^{(1)} + \int_0^\infty \int_0^\infty \gamma(s_1, s_2)\, \mathbb{J}(s_1)\, \mathbb{J}(s_2)\, ds_1\, ds_2,\\
&\mathscr{F}^{(3)} = \mathscr{F}^{(2)} + \text{triple integrals trilinear in } \mathbb{J}(s_j),\\
&\mathscr{F}^{(n)} = \mathscr{F}^{(n+1)} + N \text{ fold integrals } n\text{-linear in } \mathbb{J}(s_j).
\end{aligned} \tag{4}
$$

GREEN & RIVLIN (1957) were the first to assume general expressions for the stress in terms of multiple integrals. COLEMAN & NOLL (1961) set such approximations in terms of NOLL's theory using COLEMAN's theory of fading memory. If integrals are assumed, isotropy determines the form of the kernels.

COLEMAN & MARKOVITZ (1964) noted that some of the coefficients in (2) could be obtained as moments of the kernels in (4). For example, since

$$
\mathbb{J}(s) = \sum_{n=1}^{\infty} \frac{(-1)^n}{(n-1)!} s^{n-1} A_n(t) \tag{5}
$$

is analytic in s and $A_n(t)$ are Rivlin-Ericksen tensors computed on the velocity field $U(x, t)$. Combining (5) and (4) we can compute the $S^{(N)}$. For example,

$$
\mathscr{F}^{(1)} = \sum_{n=1} \frac{(-1)^{n-1}}{(n-1)!} \phi_n A_n(t) \tag{6}
$$

where

$$
\phi_n = \int_0^\infty s^{n-1} G(s)\, ds.
$$

Now set $\hat{\tau} = \varepsilon t$ (slow time) and $U(x, t) = \varepsilon V(x, \hat{\tau})$ (slow motion) and compare the coefficient of successive powers of ε to the expressions (2). In this way we find that

(7) $$\mu=\int_0^\infty G(s)\,ds=\phi_1,$$

(8) $$-\alpha_1=\int_0^\infty sG(s)\,ds=\phi_2,$$

(9) $$\beta_1=\tfrac{1}{2}\int_0^\infty s^2 G(s)\,ds=\tfrac{1}{2}\phi_3,$$

$$\alpha_2=\int_0^\infty\int_0^\infty \gamma(s_1,s_2)\,ds_1\,ds_2$$

and so on. It is obvious that moments of the same kernels are not independent; if $G(s)$ is given, the ϕ_n are all known. Moreover, I observed (JOSEPH, 1980) that even without specifying the kernels, certain integral inequalities would hold among moments. For example, if $G(s)\geqq 0$ and if (3) is assumed when the motion is retarded, then

(10) $$-\alpha_1\leqq\mu^{\frac{1}{2}}(2\beta_1)^{\frac{1}{2}}$$

is implied by applying Schwarz's inequality to (7), using (8) and (9).

For my demonstration here it is enough that all the moments ϕ_n of $G(s)$ are positive. This condition holds when $G(s)>0$ for finite s and together with the assumptions leading to (7), (8) and (9) it implies that

(11) $$\mu>0,\quad -\alpha_1>0,\quad \beta_1>0\qquad\text{and so on.}$$

All the experimental determinations of material parameters of fluids which are known to me are consistent with (11).

Now I shall explain what I mean by linearized theory. I mean that we judge stability, as in Lyapounov's theory, by the sign of the real part of the principal spectral value associated with the linearized equations of dynamics. First I linearize the stress. From (2) I get

(12)
$$\begin{aligned}
\boldsymbol{S}^{(1)}&=\mu\boldsymbol{A}_1,\\
\boldsymbol{S}^{(2)}&=\boldsymbol{S}^{(1)}+\alpha_1\frac{\partial\boldsymbol{A}_1}{\partial t},\qquad \alpha_1=-\phi_2,\\
\boldsymbol{S}^{(3)}&=\boldsymbol{S}^{(2)}+\beta_1\frac{\partial^2\boldsymbol{A}_1}{\partial t^2},\qquad \beta_1=\frac{\phi^3}{2},\\
\boldsymbol{S}^{(n)}&=\boldsymbol{S}^{(n-1)}+\frac{(-1)^{n-1}}{(n-1)!}\phi_n\frac{\partial^{n-1}\boldsymbol{A}_1}{\partial t^{n-1}}.
\end{aligned}$$

Rivlin-Ericksen fluids of complexity n have the same linearized stresses. Thus, in my work here, there is no difference between complexity and grade. On the other hand, under linearization $\mathbb{J}(s)\rightarrow-\boldsymbol{A}_1(s)$ and the integral representations (4)

(13) $$\mathscr{F}^{(n)}\rightarrow\int_0^\infty G(s)\,\boldsymbol{A}_1(s)\,ds$$

have the *same linearization independent of n*. The only tensor that comes up in the linearized stresses is

$$A_1(s)=L(s)+L^T(s), \qquad L(s)=\nabla U(t-s).$$

For each and every linearization I get

(14) $$S=\mathscr{L}(A_1)$$

where $\mathscr{L}$ is linear and

$$\nabla\cdot S=\mathscr{L}(\nabla^2 U).$$

To study the stability of the rest state, we consider a container of fluid $\mathscr{V}$ with closed boundary $\partial\mathscr{V}$ on which zero velocity is prescribed. A state of rest with $U=0$ solves the equations of dynamics and we perturb this rest state keeping only linear terms:

(15) $$\rho\frac{\partial U}{\partial t}=-\nabla P+\mathscr{L}(\nabla^2 U), \quad \operatorname{div} U=0, \quad U|_{\partial\mathscr{V}}=0.$$

The spectral problem for (15) is obtained by putting

(16) $$(U, P)=e^{-\sigma t}(\zeta, \pi).$$

For ζ and π we get

(17) $$-\rho\sigma\zeta=-\nabla\pi+k(\sigma)\,\nabla^2\zeta, \quad \operatorname{div}\zeta=0, \quad \zeta|_{\partial\mathscr{V}}=0$$

where the $k(\sigma)$ associated with (12) are

(18) $$\begin{aligned} k(\sigma)&=\mu-\alpha_1\sigma=\phi_1+\sigma\phi_2,\\ k(\sigma)&=\mu-\alpha_1\sigma+\beta_1\sigma^2=\phi_1+\sigma\phi_2+\tfrac{1}{2}\sigma^2\phi_3,\\ k(\sigma)&=\sum_{l=1}^{n}\frac{\phi_l\sigma^{l-1}}{(l-1)!}, \qquad n>1, \end{aligned}$$

and the $k(\sigma)$ associated with (13) is given by

(19) $$k(\sigma)=\int_0^\infty G(s)\,e^{\sigma s}\,ds.$$

It is well known and easy to show that (17) are Euler equations for the Rayleigh quotient

(20) $$\Lambda[u]=\frac{\int_{\mathscr{V}}(\nabla u)^2\,d\mathscr{V}}{\int_{\mathscr{V}}|u|^2\,d\mathscr{V}}$$

where

$$u\in H=\left(u\colon \operatorname{div} u=0,\ u|_{\partial\mathscr{V}}=0,\ \int(\nabla u)^2\,d\mathscr{V}<\infty\right).$$

Moreover the critical values Λ_n of the functional (20) are positive, bounded below and cluster only at infinity:

(21) $$0<\Lambda, \quad \Lambda_n \geqq \Lambda_{n-1}, \quad \Lambda_n \to \infty \quad \text{as } n \to \infty.$$

These critical values are related to the spectral values σ by the equation

(22) $$k(\sigma) = \sigma/\Lambda_n.$$

Any σ solving (22) for any integer n is an acceptable spectral value for (17).

Consider the case $n \to \infty$; then if $k(\sigma)=0$ determines bounded spectral values σ, (22) will imply that $k(\sigma)=0$ as $n \to \infty$. Equation (18) shows that when $n>1$, $k(\sigma)=0$ is a polynomial of degree $n-1$ in σ with strictly positive coefficients. The coefficient of σ^{n-2} in this polynomial is minus the sum of all eigenvalues and it is positive because the coefficient of σ^{n-2} is positive; that is, the roots σ_l of $(n-1)!\, k(\sigma)/\phi_n = (\sigma-\sigma_1)(\sigma-\sigma_2)\dots(\sigma-\sigma_{n-1})=0$ satisfy

(23) $$-(\sigma_1+\sigma_2+\dots+\sigma_{n-1}) = (n-1)\frac{\phi_{n-1}}{\phi_n} > 0.$$

Hence at least one root has a negative real part, leading to *instability* (see (16)).

The instability of the rest state of a layer of "fluid" of grade two against two dimensional disturbances was demonstrated by CRAIK (1968). Putting the members of $(18)_1$ equal to zero, we get CRAIK's result:

(24) $$\sigma = \frac{\mu}{\alpha_1} = \frac{\phi_1}{\phi_2} < 0.$$

COLEMAN, DUFFIN & MIZEL (1965) and COLEMAN & MIZEL (1966) proved a result about the unboundness of certain derivatives of shear flow in a fluid of second grade when $\alpha_1 < 0$. Their result might be interpreted as implying some kind of instability were it not for the fact that their method of analysis does not lead to unbounded derivatives in fluids of grade $n>2$ (unpublished result of N. T. DUNWOODY & J. T. DUNWOODY, 1980).

When $n=3$, we put $(18)_2$ equal to zero and find that

(25) $$\sigma = \frac{1}{2}\frac{\alpha_1}{\beta_1} \pm \frac{1}{2\beta_1}\sqrt{\alpha_1^2 - 4\mu\beta_1} = -\frac{\phi_2}{\phi_3} \pm \frac{1}{\phi_3}\sqrt{\phi_2^2 - \phi_1^2\phi_3},$$

The inequality (10) shows that $\alpha_1^2 - 4\mu\beta_1 < 0$ so that

(26) $$\operatorname{re}\sigma = -\frac{\phi_2}{\phi_3} < 0,$$

and the rest state of the fluid of grade 3 is unstable. Equation (23) shows that we get this same instability for each and every n. Equation (23) does not require that one accept the view that the parameters of the fluid of grade n be expressible as moments of the kernel functions. The general result implied by (23) is stated in the following

Theorem. *The rest state of each and every "fluid" of grade $n+1$ is unstable in the sense of the spectral problem of linearized theory if the ratio of the coefficients of $\boldsymbol{A}_n$ to $\boldsymbol{A}_{n-1}$ is negative.*

In contrast, CRAIK (1968), JOSEPH (1976) and SLEMROD (1976) have shown that the rest state of fluids obeying (4) and (13) is stable in the spectral sense of linearized theory provided only that $G(s)$ is a positive, monotone, decreasing function.

I say that it is wrong to study stability using the constitutive expressions for "fluids" of grade n. These expressions arise in response to slow deformations. In fact, there is no such thing as a constitutive equation without prior specification of the domain of deformations in which the constitutive equation lives. However good rigid body mechanics is for some problems, it is obviously no good for studying the deformation of strained bodies. It is equally no good to study stability with constitutive expressions which do not allow you to consider to small disturbances with large frequencies. Such disturbances can never be classed as retarded and they do not lead to fluids of grade n.

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Department of Aerospace
Engineering & Mechanics
University of Minnesota
Minneapolis

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