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The stick-slip problem for a round jet II. Small surface tension

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With 18 figures

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Summary

The stick-slip problem for a round jet studied in Part I gives a good approximation for the swell of a low speed jet when the surface tension is large but it fails when the surface tension is small. In this paper a new stick-slip problem (II) is defined and solved using matched eigenfunction expansions. The new problem reduces to that solved in Part I when the surface tension is large and gives good results in the case of zero and small surface tension.

Zusammenfassung

Das in Teil I untersuchte Haft-Gleit-Problem für einen runden Strahl liefert eine gute Näherung für die Strahlaufweitung bei langsamer Austrittsgeschwindigkeit, wenn die Oberflächenspannung groß ist, versagt dagegen bei kleiner Oberflächenspannung. In der vorliegenden Veröffentlichung wird daher ein neues Haft-Gleit-Problem (II) definiert und mittels aneinander angeschlossener Eigenfunktionsentwicklungen gelöst. Dieses Problem geht in das schon früher in Teil I gelöste Problem über, wenn die Oberflächenspannung groß ist, liefert aber auch bei kleiner oder gar verschwindender Oberflächenspannung gute Ergebnisse.

Key words

Stick-slip, low speed jet, surface tension, matched eigenfunction expansion

1. Introduction

In Part I (1) of this paper we solved the stick-slip problem for a low speed round jet by expressing the solution for the flow in the pipe as a biorthogonal series and in the jet as a Fourier series. In the stick-slip problem the jet boundary is first assumed to be a straight cylinder of constant circular cross-section of radius $\hat{r} = a$ where a is the pipe radius. The series satisfy all requirements in the interior of the pipe and jet, the velocity and stresses are continuous at the exit plane $\hat{x} = 0$ where the pipe meets the jet, the velocity at the pipe wall ($\hat{x} < 0$) vanishes and the normal component of velocity and shear stress vanish on the cylinder $\hat{r} = a$, $\hat{x} > 0$. An approximation to the shape of the free surface $\hat{r} = \hat{h}(\hat{x})$ may then be obtained by requiring that the jump in the normal component of the stress on $\hat{r} = a$ balance surface tension times the mean curvature. This method for computing the free surface fails completely when the surface tension is zero. The finite element solution of Silliman and Scriven (2) for the plane jet shows that the analytic method for computing the free surface is accurate when the surface tension is large (perfect, when infinite) and is increasingly inaccurate when the surface tension is small. In this paper we define a new stick-slip problem which can be solved by eigenfunction methods for all values of surface tension including zero. The problem reduces to that treated in Part I when the surface tension is infinite and is in good agreement with direct numerical computations for zero and small surface tension.

2. Governing equations

A Newtonian fluid is extruded from a semininfinite ($\hat{x} < 0$) circular pipe of radius $\hat{r} = a$ by an applied uniform pressure gradient as $\hat{x} \to -\infty$ giving rise to Poiseuille flow there. A jet of fluid of radius $\hat{r} = \hat{h}(\hat{x})$, $\hat{h}(0) = a$, $\hat{x} \geqslant 0$ is extruded from the pipe. It is assumed that wind shears and body forces are negligible so that the shear stress on $\hat{r} = \hat{h}(\hat{x})$ vanishes and the normal stresses are balanced by interfacial tension. Under such assumptions the flow far upstream, $\hat{x} \to \infty$ is constant with final velocity $\hat{\boldsymbol{u}} \cdot \boldsymbol{e}_x = \hat{U}_f$ and jet radius $\hat{h} = \hat{h}_f$. The swelling ratio is defined as $\chi = \hat{h}_f/a$ (see fig. 2.1).

With inertial effects neglected the Stokes equations of motion are

$$\operatorname{div} \hat{\boldsymbol{T}} = \operatorname{div} \hat{\boldsymbol{u}} = 0, \qquad [2.1]$$

where

$$\hat{\boldsymbol{T}} = -\hat{\Phi}\boldsymbol{1} + \hat{\boldsymbol{S}}$$

is the total stress,

$$\hat{\boldsymbol{S}} = \mu[\operatorname{grad}\hat{\boldsymbol{u}} + (\operatorname{grad}\hat{\boldsymbol{u}})^{\mathrm{T}}]$$

is the deviatoric stress and

$$\hat{\Phi} = \hat{P} - P_a$$

is the reduced pressure for zero body force. Eq. $[2.1]_2$, along with the requirement that the flow be axisymmetric, implies the existence of a stream function $\hat{\Psi}(\hat{x}, \hat{r})$. We are interested in determining this stream function and the jet shape $\hat{h}(\hat{x})$ for different values of surface tension, σ. We seek these field quantities in terms of dimensionless variables

$$(x, r) = \frac{1}{a}(\hat{x}, \hat{r})$$

$$\Psi(x, r) = \hat{\Psi}(\hat{x}, \hat{r})/\varepsilon,$$

where

$$\varepsilon = \int_0^{\hat{h}(\hat{x})} \boldsymbol{e}_x \cdot \hat{\boldsymbol{u}}(\hat{x}, r) r\,dr = \frac{\text{volume flux}}{2\pi}$$

is independent of $\hat{x}$,

$$(u, w) = \frac{a^2}{\varepsilon}(\hat{u}, \hat{w}),$$

$$S = \frac{a^3}{\varepsilon\mu}\hat{S},$$

$$\Phi = \frac{a^3}{\varepsilon\mu}\hat{\Phi}$$

and

$$h(x) = 1 + \eta(x) = \hat{h}(\hat{x})/a.$$

The dimensionless variables are related by

$$\boldsymbol{u} = \operatorname{curl}\left(\boldsymbol{e}_\theta \frac{\Psi}{r}\right),$$

$$u = \frac{1}{r}\frac{\partial \Psi}{\partial r},$$

$$w = -\frac{1}{r}\frac{\partial \Psi}{\partial x},$$

$$S_{\langle rr \rangle} = 2\frac{\partial w}{\partial r},$$

$$S_{\langle xr \rangle} = \frac{\partial u}{\partial r} + \frac{\partial w}{\partial x},$$

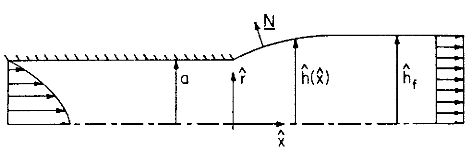


Fig. 2.1. The capillary jet

$$S_{\langle xx \rangle} = 2\frac{\partial u}{\partial x},$$

$$S_{\langle \theta\theta \rangle} = 2\frac{w}{r}.$$

Now we list the equations satisfied by $\Psi(x, r)$ in the pipe, $x < 0$:

$$\mathcal{L}^2\Psi = 0 \left[\mathcal{L}(\cdot) \stackrel{\text{def}}{=} r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(\cdot)\right) + \frac{\partial^2}{\partial x^2}(\cdot)\right],$$

$$\Psi(x, 1) = 1,$$

$$\frac{\partial \Psi}{\partial r}(x, 1) = 0, \qquad [2.2]$$

$$\Psi(x, r) \to r^2(2 - r^2) \quad \text{as} \quad x \to -\infty.$$

The equations satisfied by $\Psi(x, r)$ and $h(x)$ in the jet $x > 0$ are

$$\mathcal{L}^2\Psi = 0$$

$$\Psi(x, r) \to U_f r^2/2 \quad \text{as} \quad x \to \infty, \qquad [2.3]$$

$$h(x) \to \chi \quad \text{as} \quad x \to \infty.$$

On the free surface $r = h(x)$ we have (see Joseph (3))

$$F_1(x, h(x)) = w - h'u = 0, \; h' \stackrel{\text{def}}{=} dh/dx,$$

$$F_2(x, h(x)) = h'(S_{\langle rr \rangle} - S_{\langle xx \rangle}) + (1 - h'^2)S_{\langle xr \rangle} = 0, \qquad [2.4]$$

$$F_3(x, h(x)) = S_{\langle rr \rangle} - h'S_{\langle xr \rangle} - \Phi + \frac{2\zeta}{hh'}\left(\frac{h}{(1 + h'^2)^{1/2}}\right)' = 0,$$

where

$$\zeta = \frac{\sigma}{\mu\bar{U}}$$

is the dimensionless surface tension parameter and $\bar{U} = 2\varepsilon/a^2$ is the average velocity in the pipe. Eqs. [2.4] express the requirements that the normal velocity, the shear stress and the jump in normal stress minus interfacial tension vanish on $r = h(x)$. All the governing equations except [2.4] are linear.

The flow in the pipe and the flow in the jet are connected by the requirements that

the total stress and velocity are continuous across the plane $x = 0$. [2.5]

3. Linearization of the free surface conditions

Our linearization of [2.4] exploits the fact that the swell of the Newtonian jet is small. Numerical studies show that $\eta(x) < 0.14$, $(x \geqslant 0)$. At the same time a small Reynolds number need not imply that the final velocity U_f be particularly small. We then note that

$$F_j(x, 1 + \eta(x)) = F_j(x, 1) + \eta\frac{\partial F_j}{\partial r}(x, 1) + 0(\eta^2), \quad j = 1, 2, 3.$$

After dropping the second-order terms in the expansion of F_j we get the following conditions on $r = 1$

$$w - \eta' u + \eta\frac{\partial w}{\partial r} = 0,$$

$$\eta'(S_{\langle rr \rangle} - S_{\langle xx \rangle}) + S_{\langle xr \rangle} + \eta\frac{\partial S_{\langle xr \rangle}}{\partial r} = 0, \qquad [3.1]$$

$$S_{\langle rr \rangle} + \eta\frac{\partial S_{\langle rr \rangle}}{\partial r} - \eta' S_{\langle xr \rangle} - \Phi - \eta\frac{\partial \Phi}{\partial r} + 2\zeta\left[\frac{1}{1 + \eta} - \eta''\right] = 0,$$

where $(1 + \eta)^{-1} \sim 1 - \eta$. It is easy to ascertain from [2.3] and $[2.4]_3$ that the asymptotic values for $x \to \infty$ of Ψ, h and Φ are not zero and it is convenient to extract these asymptotic values. To do this we set

$$\Phi(x, r) = \frac{2\zeta}{1 + \eta_f} + \tilde{\Phi}(x, r),$$

$$\Psi(x, r) = \frac{U_f}{2}r^2 + \tilde{\Psi}(x, r), \qquad [3.2]$$

$$(\tilde{\Phi}, \tilde{\Psi}) \to (0, 0) \quad \text{as} \quad x \to \infty$$

and find that on $r = 1$

$$\tilde{w} - \eta' U_f - \eta'\tilde{u} + \eta\frac{\partial \tilde{w}}{\partial r} = 0,$$

$$\eta'(\tilde{S}_{\langle rr\rangle} - \tilde{S}_{\langle xx\rangle}) + \tilde{S}_{\langle xr\rangle} + \eta \frac{\partial \tilde{S}_{\langle xr\rangle}}{\partial r} = 0\,, \quad [3.3]$$

$$\tilde{S}_{\langle rr\rangle} + \eta \frac{\partial \tilde{S}_{\langle rr\rangle}}{\partial r} - \eta' \tilde{S}_{\langle xr\rangle} - \tilde{\Phi} - \eta \frac{\partial \tilde{\Phi}}{\partial r}$$
$$+ 2\zeta\eta_f - 2\zeta(\eta'' + \eta) = 0\,,$$

where $\tilde{S}_{\langle mn\rangle}$ is related to

$$(\tilde{u}, \tilde{w}) = \frac{1}{r}\left(\frac{\partial \tilde{\Psi}}{\partial r}, -\frac{\partial \tilde{\Psi}}{\partial x}\right)$$

in the same way as $S_{\langle mn\rangle}$ is related to (u, w). All the tilde overbar quantities vanish as $x \to \infty$ and $\mathcal{L}^2 \tilde{\Psi} = 0$.

The stick-slip problem treated in Part I arises from discarding the quadratic terms in [3.3]:

$$\tilde{w} = \tilde{S}_{\langle xr\rangle} = \tilde{S}_{\langle rr\rangle} - \tilde{\Phi} + 2\zeta\eta_f - 2\zeta(\eta'' + \eta)$$
$$= 0\,. \quad [3.4]$$

The stick-slip problem I is uniquely determined by [2.2], [2.3], [2.5] and the first two of the conditions of [3.4]. The continuity conditions [2.5] are satisfied for the stick-slip problem I if $\left(\Psi, \frac{\partial \Psi}{\partial x}, \frac{\partial^2 \Psi}{\partial x^2}, \frac{\partial^3 \Psi}{\partial x^3}\right)$ are all continuous across $x = 0$ (see Part I). It follows that all the conditions will be satisfied if we can satisfy the normal stress conditions $[3.4]_3$. Since Ψ, $\tilde{\Psi}$ and $\tilde{\Phi}$ are known without $[3.4]_3$ we can satisfy $[3.4]_3$ by determining $\eta(x)$; that is, by determining the shape of the free surface.

The method used in the stick-slip problem I fails when $\zeta = 0$. We get a flow, but the free surface cannot be determined.

Silliman and Scriven (2) computed the free surface on a plane jet by a finite element method. They showed much better convergence for small ζ when the shear and normal stress condition were treated as primary and the shape of the free surface was computed, iteratively from the condition of vanishing normal velocity. For larger ζ they showed it better to treat the normal velocity and shear stress as primary, iterating the normal stress, as in the stick-slip problem I. This observation together with the remark that U_f need not be small leads to the linearization

$$\tilde{w} - \eta' U_f = \tilde{S}_{\langle xr\rangle} = \tilde{S}_{\langle rr\rangle} - \tilde{\Phi} + 2\zeta\eta_f$$
$$-2\zeta(\eta'' + \eta) = 0\,. \quad [3.5]$$

Since the continuity requirements at $x = 0$ have $\tilde{w}(0, 1) = 0$ it follows from $[3.5]_1$ that $\eta'(0) = 0$. When $\zeta \to \infty$, $\eta(x) \to 0$. Then $\eta'(x) = 0$ and we recover stick-slip problem I in the limit $\zeta \to \infty$.

On the other hand, when $\zeta = 0$ we may determine Ψ, $\tilde{\Psi}$, Φ and $\tilde{\Phi}$ from the stated conditions and the second two of [3.5],

$$\tilde{S}_{\langle xr\rangle} = \tilde{S}_{\langle rr\rangle} - \tilde{\Phi} = 0\,.$$

Then $\eta'(x)$ and $\eta(x)$ are determined from $[3.5]_1$. So the linearization [3.5] leads to stick-slip problem II which can be solved for all ζ, $0 \leqslant \zeta < \infty$. Indeed since $\tilde{w} = -\frac{\partial \tilde{\Psi}}{\partial x}$ on $r = 1$ we have $[3.5]_1$ in the form

$$\frac{\partial \tilde{\Psi}}{\partial x} + \eta' U_f = 0\,. \quad [3.6]$$

Since $\Psi(x, r) = \frac{U_f}{2} r^2 + \tilde{\Psi}(x, r)$, $\Psi(x, 1) = 1$ for $x \leqslant 0$ and $\tilde{\Psi}(x, 1) \to 0$ for $x \to \infty$, we find that

$$\eta(x) U_f = 1 - \frac{U_f}{2} - \tilde{\Psi}(x, 1), \ \eta(0) = 0 \quad [3.7]$$

and

$$\eta_f = \frac{1}{U_f}\left(1 - \frac{U_f}{2}\right). \quad [3.8]$$

We next observe that the stick-slip problem II may be formulated in terms of the stream function alone; that is, the pressure $\tilde{\Phi}$ and the free surface function $\eta(x)$ may be eliminated entirely. To eliminate $\tilde{\Phi}$ we differentiate $[3.5]_3$ with respect to x,

$$\frac{\partial \tilde{S}_{\langle rr\rangle}}{\partial x} - \frac{\partial \tilde{\Phi}}{\partial x} = 2\zeta(\eta''' + \eta')$$

where, from the linearized equations of motion

$$\frac{\partial \tilde{\Phi}}{\partial x} = \frac{1}{r}\frac{\partial^3 \tilde{\Psi}}{\partial r^3} - \frac{1}{r^2}\frac{\partial^2 \tilde{\Psi}}{\partial r^2} + \frac{1}{r^3}\frac{\partial \tilde{\Psi}}{\partial r}$$
$$+ \frac{1}{r}\frac{\partial}{\partial r}\frac{\partial^2 \tilde{\Psi}}{\partial x^2}\,. \quad [3.9]$$

The same equation holds in $x > 0$ and $x < 0$ with $\tilde{\Phi}$ and $\tilde{\Psi}$ replaced by Φ and Ψ. Noting now that

$$\tilde{S}_{\langle rr\rangle} = -2\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \tilde{\Psi}}{\partial x}\right)$$

we find that on $r = 1$

$$2\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial^2 \tilde{\Psi}}{\partial x^2}\right) + \nabla^2\left(\frac{1}{r}\frac{\partial \tilde{\Psi}}{\partial r}\right) \quad [3.19]$$
$$= -2\zeta(\eta''' + \eta') = 2\xi\left(\frac{\partial^3 \tilde{\Psi}}{\partial x^3} + \frac{\partial \tilde{\Psi}}{\partial x}\right),$$

where η' has been eliminated with [3.6], $\xi = \frac{\zeta}{U_f}$ and

$$\nabla^2(\cdot) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}(\cdot)\right) + \frac{\partial^2}{\partial x^2}(\cdot)\,.$$

We may study the effects of quadratic terms neglected in going from [3.3] to [3.5] by successive approximations but we do not consider successive approximations here.

4. Biorthogonal eigenfunction solutions of the stick-slip problem II

The equations derived so far are listed below: In the pipe $x < 0$ we have

$$\mathcal{L}^2 \Psi = 0\,,$$
$$\Psi(x, r) \to r^2(2 - r^2) \quad \text{as} \quad x \to -\infty\,,$$
$$\Psi(x, 1) = 1\,, \quad [4.1]$$
$$\frac{\partial \Psi}{\partial r}(x, 1) = 0\,.$$

In the jet $x > 0$ we have

$$\Psi(x, r) = \frac{U_f}{2} r^2 + \tilde{\Psi}(x, r)\,,$$
$$\mathcal{L}^2 \Psi = \mathcal{L}^2 \tilde{\Psi} = 0\,,$$
$$\tilde{\Psi}(x, r) \to 0 \quad \text{as} \quad x \to \infty\,, \quad [4.2]$$
$$\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \tilde{\Psi}}{\partial r}\right) - \frac{\partial^2 \tilde{\Psi}}{\partial x^2}\right]_{r=1} = 0\,,$$
$$\left[2\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial^2 \tilde{\Psi}}{\partial x^2}\right) + \nabla^2\left(\frac{1}{r}\frac{\partial \tilde{\Psi}}{\partial r}\right)\right.$$
$$\left. -2\xi\left(\frac{\partial^3 \tilde{\Psi}}{\partial x^3} + \frac{\partial \tilde{\Psi}}{\partial x}\right)\right]_{r=1} = 0\,.$$

The continuity conditions at $x = 0$ (see Part I) are

$$\Psi(0^-, r) = \frac{U_f}{2} r^2 + \tilde{\Psi}(0^+, r)\,,$$
$$\frac{\partial \Psi}{\partial x}(0^-, r) = \frac{\partial \tilde{\Psi}}{\partial x}(0^+, r)\,, \quad [4.3]$$
$$\frac{\partial^2 \Psi}{\partial x^2}(0^-, r) = \frac{\partial^2 \tilde{\Psi}}{\partial x^2}(0^+, r)\,,$$
$$\frac{\partial^3 \Psi}{\partial x^3}(0^-, r) = \frac{\partial^3 \tilde{\Psi}}{\partial x^3}(0^+, r)\,.$$

Eqs. [4.1–4.3] determine $\Psi(x, r; \xi)$ uniquely. The solution depends on the surface tension parameter

$$\xi = \frac{\zeta}{U_f} = \frac{\sigma}{\mu \bar{U} U_f}\,.$$

The free surface $h(x) = 1 + \eta(x)$ is given by

$$\eta(x) = \frac{1}{U_f}\left(1 - \frac{U_f}{2} - \tilde{\Psi}(x, 1)\right). \quad [4.4]$$

The foregoing equations may be solved in terms of two different expansions of eigenfunctions, one set for the pipe, one set for the jet, matched by [4.3] at their common boundary. For $x < 0$

$$\Psi(x, r) = r^2(2 - r^2) + \sum_{-\infty}^{\infty} \frac{C_n}{p_n^2} \phi_n(r) e^{p_n x}\,, \quad [4.5]$$

$$\phi_n(r) = J_0(p_n) r J_1(p_n r) - J_1(p_n) r^2 J_0(p_n r)\,,$$

where p_n are selected so that $\phi_n'(1) = 0$. The p_n are the first quadrant roots of

$$p J_0^2(p) - 2J_0(p) J_1(p) + p J_1^2(p) = 0\,, \quad [4.6]$$
$$|p_1| < |p_2| < \cdots < |p_n|\,,$$
$$p_{-n} = \bar{p}_n\,.$$

Since $\Psi(x, r)$ is real $C_{-n} = \bar{C}_n$ where overbar designates complex conjugate.

For $x > 0$

$$\Psi(x, r) = \frac{U_f}{2} r^2 + \sum_{-\infty}^{\infty} \frac{D_n}{\alpha_n^2} \hat{\phi}_n(r) e^{-\alpha_n x}\,, \quad [4.7]$$
$$\hat{\phi}_n(r) = [J_1(\alpha_n) + \alpha_n J_0(\alpha_n)] r J_1(\alpha_n r)$$
$$- \alpha_n J_1(\alpha_n) r J_0(\alpha_n r)\,,$$

where $\alpha_n(\xi)$ are the first quadrant roots of

$$\alpha^2 J_0^2(\alpha) + \left[(\alpha^2 - 1) + \left(\frac{\alpha^2 + 1}{2\alpha}\right)\xi\right] J_1^2(\alpha) = 0, \quad [4.8]$$

$|\alpha_1| < |\alpha_2| < \cdots < |\alpha_n|$,

$\alpha_{-n} = \bar{\alpha}_n$,

$D_{-n} = \bar{D}_n$.

The eigenvalues $\alpha_n(\xi)$ are selected so that the boundary conditions [4.2] on $r = 1$ are satisfied. A point of novelty is that the eigenvalues α depend on the surface tension parameter ξ. The effects of surface tension in the stick-slip problem II are taken up fully in eigenvalues which are induced by separating variables.

To determine the coefficients U_f, C_n and D_n we reexpress the continuity conditions [4.3] in series form. The condition $[4.3]_1$ gives

$$\sum_{-\infty}^{\infty} \frac{D_n}{\alpha_n^2} \hat{\phi}_n(r) = r^2\left(2 - \frac{U_f}{2} - r^2\right) + \sum_{-\infty}^{\infty} \frac{C_n}{p_n^2} \phi_n(r). \quad [4.9]$$

We may reexpress U_f in series form by noting that $\phi_n(1) = 0$. Then

$$\sum_{-\infty}^{\infty} \frac{D_n}{\alpha_n^2} \hat{\phi}_n(1) = \left(1 - \frac{U_f}{2}\right). \quad [4.10]$$

This condition insures that $\Psi(0^+, 1) = 1$, that the volume flux is continuous at $x = 0$. Combining [4.9] and [4.10] and expressing the other conditions [4.3] in series form we get

$$\sum_{-\infty}^{\infty} \frac{D_n}{\alpha_n^2} [\hat{\phi}_n(r) - \hat{\phi}_n(1)] = r^2(1 - r^2) + \sum_{-\infty}^{\infty} \frac{C_n}{p_n^2} \phi_n(r),$$

$$\sum_{-\infty}^{\infty} \frac{D_n}{\alpha_n} \hat{\phi}_n(r) = -\sum_{-\infty}^{\infty} \frac{C_n}{p_n} \phi_n(r),$$

$$\sum_{-\infty}^{\infty} D_n \hat{\phi}_n(r) = \sum_{-\infty}^{\infty} C_n \phi_n(r),$$

$$\sum_{-\infty}^{\infty} \alpha_n D_n \hat{\phi}_n(r) = -\sum_{-\infty}^{\infty} p_n C_n \phi_n(r). \quad [4.11]$$

We next note that $\phi_n(r)$ and $\hat{\phi}_n(r)$ are r times various Bessel functions. This suggests that we may enforce [4.11] divided by r as infinite series independent of r by setting to zero the Fourier Bessel coefficients of expansion of [4.11] divided by r in a series of Bessel functions of the form

$$\sum_{1}^{\infty} k_l J_1(q_l r),$$

where q_l are the positive roots of $J_1(q) = 0$. To implement this idea we project by multiplying [4.11] by $J_1(q_l r)$ and integrating from 0 to 1. In this way the condition $k_l = 0$ can be written as

$$\sum_{-\infty}^{\infty} (A_{ln} - G_{ln}) \frac{D_n}{\alpha_n^2} = f_l + \sum_{-\infty}^{\infty} B_{ln} \frac{C_n}{p_n^2},$$

$$\sum_{-\infty}^{\infty} A_{ln} \frac{D_n}{\alpha_n} = -\sum_{-\infty}^{\infty} B_{ln} \frac{C_n}{p_n},$$

$$\sum_{-\infty}^{\infty} A_{ln} D_n = \sum_{-\infty}^{\infty} B_{ln} C_n,$$

$$\sum_{-\infty}^{\infty} \alpha_n A_{ln} D_n = -\sum_{-\infty}^{\infty} p_n B_{ln} C_n \quad [4.12]$$

where

$$A_{ln} = \int_0^1 \hat{\phi}_n(r) J_1(q_l r)\, dr = \frac{q_l(3\alpha_n^2 - q_l^2) J_0(q_l) J_1^2(\alpha_n)}{(\alpha_n^2 - q_l^2)^2},$$

$$B_{ln} = \int_0^1 \phi_n(r) J_1(q_l r)\, dr = \frac{2 q_l p_n J_0(q_l) J_1^2(p_n)}{(p_n^2 - q_l^2)^2},$$

$$G_{ln} = \hat{\phi}_n(1) \int_0^1 r^2 J_1(q_l r)\, dr = -\frac{J_0(q_l) J_1^2(\alpha_n)}{q_l},$$

$$f_l = \int_0^1 r^2(1 - r^2) J_1(q_l r)\, dr = -\frac{8}{q_l^3} J_0(q_l).$$

Eqs. [4.12] are an infinite system of equations for the coefficients C_n and D_n which we solve, as in Part I, by truncation. In a truncation of order N we replace the summation $\sum_{-\infty}^{\infty}$ with $\sum_{-N}^{N}$ and let $l = 1, 2, \ldots, N$. The truncated system may be written as follows: $[4.12]_2$ and $[4.12]_4$ are in the form

$$A^{(N)} \cdot D^{(N)} = B^{(N)} \cdot C^{(N)}, \quad [4.13]$$

where $A^{(N)}$ and $B^{(N)}$ are $2N \times 2N$ matrices and $D^{(N)}$ and $C^{(N)}$ are $2N$ component column vectors whose components $D_n^{(N)}$ and $C_n^{(N)}$ are approximations under truncation of the coefficients D_n and C_n. The entries on the first N rows of $A^{(N)}$ are A_{ln}/α_n and the entries on the second N rows of $A^{(N)}$ are $\alpha_n A_{ln}$, $-N \leqslant n \leqslant N$, $l = 1, 2, \ldots, N$. The first N rows of $B^{(N)}$ are $-B_{ln}/p_n$ and the second N rows are $-p_n B_{ln}$. Similarly eqs. $[4.12]_1$ and $[4.12]_2$ may be written as

$$\hat{A}^{(N)} \cdot D^{(N)} = \hat{B}^{(N)} \cdot C^{(N)} + f^{(N)}. \quad [4.14]$$

We find from [4.13] that

$$D^{(N)} = [A^{(N)}]^{-1} B^{(N)} \cdot C^{(N)} \quad [4.15]$$

and from [4.14] that

$$\{\hat{A}^{(N)}[A^{(N)}]^{-1} B^{(N)} - \hat{B}^{(N)}\} \cdot C^{(n)} = f^{(N)}.$$

Eq. [4.16] gives a system of equations for the coefficients $C^{(N)}$ which is solved by Gauss elimination. Once the $C^{(N)}$ are determined the $D^{(N)}$ may be determined from [4.15].

In Part I we examined convergence under truncation for increasing N and obtained good results with $N > 20$. Here we used $N = 50$, the limit for computer storage. The solution obtained this way reduces perfectly to that given in Part I when the surface tension is infinite.

5. Evaluation of the pressure

To find the pressure we integrate [3.9] in the pipe ($x < 0$) to obtain

$$\Phi(x, r) = -16x + 2 \sum_{-\infty}^{\infty} \frac{C_n}{p_n} J_1(p_n) J_0(p_n r) e^{p_n x} + \Phi_-, \quad [5.1]$$

and in the jet ($x > 0$) to obtain

$$\Phi(x, r) = \frac{2\zeta}{\chi} - 2 \sum_{-\infty}^{\infty} D_n J_1(\alpha_n) J_0(\alpha_n r) e^{-\alpha_n x},$$

where Φ_- is to be determined by requiring that $\Phi(x, r)$ is continuous at $x = 0$. Equating [5.1] and [5.2] at $x = 0$, we find that

$$2 \sum_{-\infty}^{\infty} \frac{C_n}{p_n} J_1(p_n) J_0(p_n r) + \Phi_- - \frac{2\zeta}{\chi} - 2 \sum_{-\infty}^{\infty} D_n J_1(\alpha_n) J_0(\alpha_n r) = 0. \quad [5.3]$$

The projection $\int_0^1 r(5.3)\, dr$ yields

$$\Phi_- - \frac{2\zeta}{\chi} = -4 \sum_{-\infty}^{\infty} \frac{D_n}{\alpha_n} J_1^2(\alpha_n) - 4 \sum_{-\infty}^{\infty} \frac{C_n}{p_n^2} J_1^2(p_n) \quad [5.4]$$

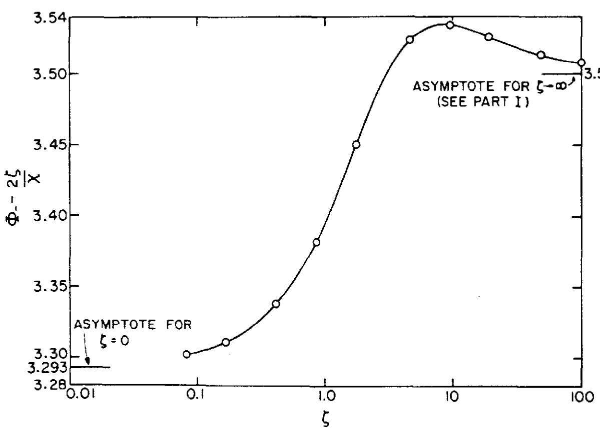


Fig. 5.1. Constant in the pressure for $x < 0$ as a function of the dimensionless surface tension parameter

In figure 5.1 we have shown how $\Phi_- - \dfrac{2\zeta}{\chi}$ varies with the surface tension parameter, ζ.

6. The singularity at the exit lip

We may study the properties of the solution of [4.1] and [4.2] near the exit lip using the method of Michael (4). We find that at the exit lip the solution is in the form

$$\tilde{\Psi} \sim r^{\lambda+1} f_\lambda(\theta),$$

and either

$$\cos \lambda\pi = 0, \quad \lambda = \lambda_n = \frac{2n-1}{2}, \quad n = 1, 2, \ldots$$

(leading to singular stresses proportioned to $r^{-1/2}$) or,

$$\cos \lambda\pi - \xi \sin \lambda\pi = 0 .$$

The second choice gives

$$0 \leqslant \lambda_1(\xi) \leqslant \frac{1}{2}, \quad \lambda_1(0) = \frac{1}{2},$$

where $\lambda_1(\xi)$ is a decreasing function of ξ;

$$\lambda_1(\xi) \to 0 \quad \text{as} \quad \xi \to \infty .$$

However we know from direct analysis of the case with $\xi = \infty$ that the characteristic exponent is $\lambda_1 = 1/2$. The choice $\lambda_1 = 1/2$ is the only one which is continuous for all values of ξ and also at $\xi = \infty$. This value of λ_1 does not appear to be inconsistent with our numerical results. The shear stress on the boundary of the pipe appears to be insensitive to changes in the surface tension parameter as the exit plane is approached. The analysis leading to the foregoing results is given in the Appendix.

We note that the stresses, though singular, are integrable. It was already noted by Richardson (5) that the inertial terms computed on the stick-slip solution are finite at the exit lip. The same is true of the inertial terms computed on the solution given in this paper. The square root singularity makes the computation of nonlinear stresses by perturbations difficult. To obtain the stress in a second-order non-Newtonian fluid by perturbations we must square the square root singularity. This leads to an unacceptable logarithmically infinite force. The perturbation method suggested by Joseph (3) for the problem of die swell in a non-Newtonian fluid will encounter these singular forces and is therefore unacceptable. It would appear that the nonlinear part of the stress at the exit cannot be treated as a perturbation of the linear part. In fact, Huilgol and Tanner (6), let the quadratic part of the stress in a fluid of grade two determine the singularity at the exit and they obtain $\lambda_1 = .711$ which leads to integrable quadratic stresses. Similar exponents for the problem of die swell of other non-linear (Maxwellian) fluids are suggested by the numerical results of Chang et al. (7).

7. Results and discussion

The results of our analysis are shown in graphical form in this section. The main results are given in figures 7.1 and 7.2. Figure 7.1 gives the shape of the free surface for different values of the surface tension parameter, ζ, while figure 7.2 gives the variation of the final diameter with ζ. The final diameter of the jet when $\zeta = 0$ in our analysis is $\chi = 1.111$. This compares with the value 1.13 computed by Nickell et al. (8), and Chang et al. (9).

In figure 7.2 we compare the final diameter by the method of Part I with the present method. The differences are small when $1/\zeta < .1$, but they become unacceptably large when the surface tension tends to zero. A similar graph comparing a finite element solution for the swell of a plane jet with Richardson's analytical solution (5) is given by Silliman and Scriven (2). They obtain qualitatively the same results as our Figure 7.2 and find that Richardson's stick-slip result approximates well the swell of a plane jet whenever $1/\zeta < .1$.

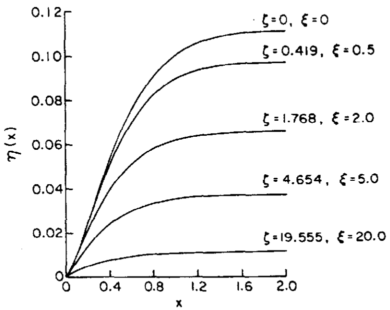


Fig. 7.1. The free surface correction for different values of the dimensionless surface tension parameter

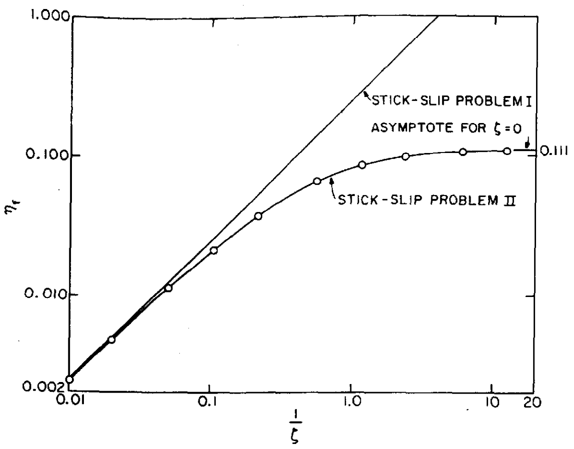


Fig. 7.2. The final free surface correction as a function of the dimensionless surface tension parameter

In figures 7.3 through 7.16 we have exhibited graphs of the same velocities and stresses for $\zeta = 0$ which were shown in Part I for $\zeta = \infty$. Here, as in Part I, the basic dynamics of the jet are reflected in the velocity adjustments shown in figures 7.3 and 7.4. Figure 7.5 shows the variation of the final velocity, U_f, with the surface tension parameter, ζ. As expected $U_f \to 2$, the value of Part I, as $\zeta \to \infty$. The remaining figures are plots of the stresses and the pressure for various axial coordinates, x. Again as in Part I, the singular nature of the stresses (except $S_{\langle\theta\theta\rangle}$) and the pressure is apparent at the exit lip, $x = 0$, $r = 1$.

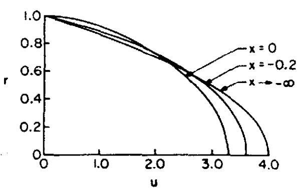


Fig. 7.3a. The axial component of velocity in the pipe ($\zeta = 0$)

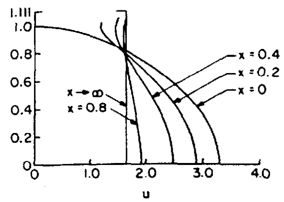


Fig. 7.3b. The axial component of velocity in the jet ($\zeta = 0$)

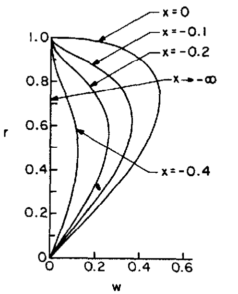


Fig. 7.4a. The radial component of velocity in the pipe ($\zeta = 0$)

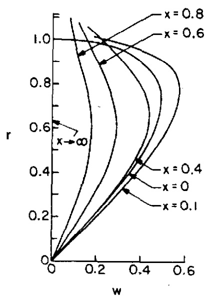


Fig. 7.4b. The radial component of velocity in the jet ($\zeta = 0$)

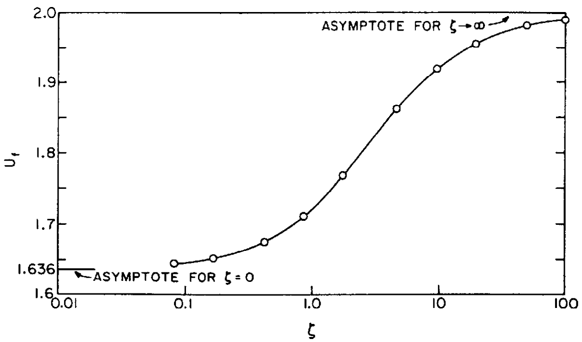


Fig. 7.5. Final jet velocity as a function of the dimensionless surface tension parameter. The final diameter decreases for increasing ζ so the final velocity must increase

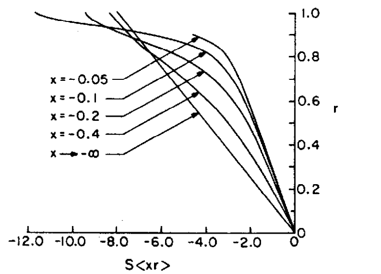
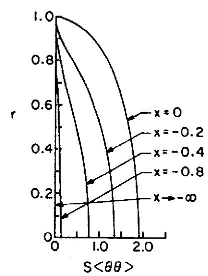


Fig. 7.6. The shear stress distribution in the pipe ($\zeta = 0$)



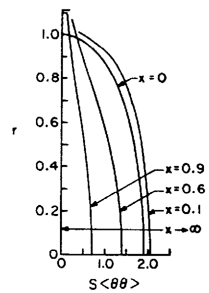


Fig. 7.8a. The hoop stress distribution in the pipe ($\zeta = 0$)

Fig. 7.8b. The hoop stress distribution in the jet ($\zeta = 0$)

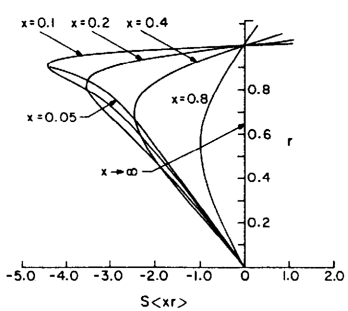


Fig. 7.7. The shear stress distribution in the jet ($\zeta = 0$)

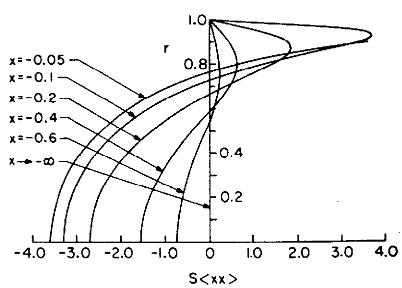


Fig. 7.9. The axial stress distribution in the pipe ($\zeta = 0$)

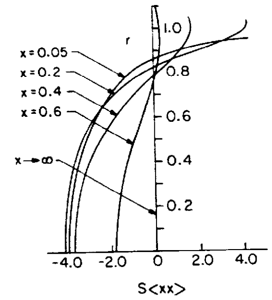


Fig. 7.10. The axial stress distribution in the jet ($\zeta = 0$)

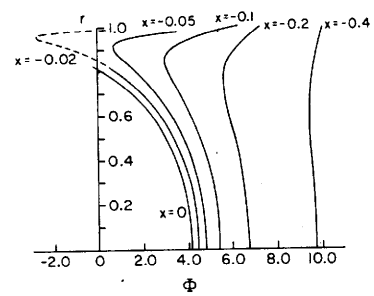


Fig. 7.13. The pressure distribution in the pipe ($\zeta = 0$)

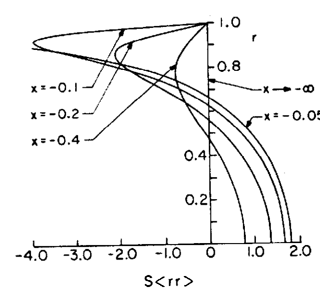


Fig. 7.11. The radial stress distribution in the pipe ($\zeta = 0$)

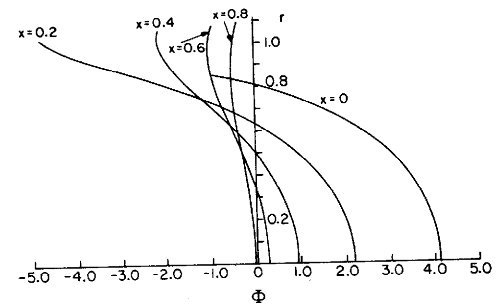


Fig. 7.14. The pressure distribution in the jet ($\zeta = 0$)

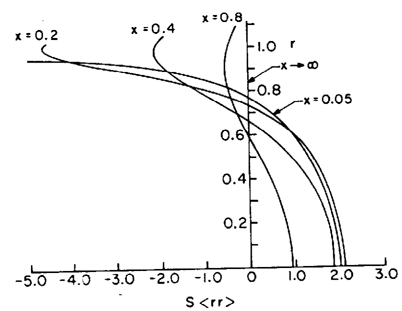


Fig. 7.12. The radial stress distribution in the jet ($\zeta = 0$)

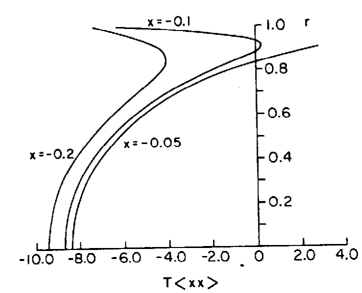


Fig. 7.15. The total axial stress distribution in the pipe ($\zeta = 0$)

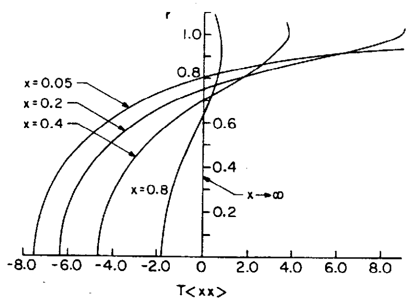


Fig. 7.16. The total axial stress distribution in the jet ($\zeta = 0$)

Appendix: Analysis at a point of separation of a viscous fluid with surface tension

We wish to examine the nature of the singularity at the exit lip, $x = 0$, $r = 1$ for the problem of Sect. 4. To do this we assume that locally at the exit lip second derivatives are more important than first derivatives, and thus the fluid jet can be regarded as plane. We follow Michael (4), Moffatt (10) and Lugt and Schwiderski (11) in examining the singularity for this plane case.

The analogous plane flow problem to [4.1] and [4.2] is summarized below:

$$\nabla^4 \bar{\Psi} = 0, \quad -1 < y < 1, \quad -\infty < x < \infty,$$

$$\left.\begin{aligned} \frac{\partial \bar{\Psi}}{\partial y} &= 0 \\ \frac{\partial \bar{\Psi}}{\partial x} &= 0 \end{aligned}\right\} \text{on } y = 1, x \leqslant 0,$$

$$\left.\begin{aligned} &\frac{\partial^2 \bar{\Psi}}{\partial y^2} - \frac{\partial^2 \bar{\Psi}}{\partial x^2} = 0 \\ &2\frac{\partial}{\partial y}\left(\frac{\partial \bar{\Psi}}{\partial x}\right) + \Phi = \xi \frac{\partial^2 \bar{\Psi}}{\partial x^2} \end{aligned}\right\} \text{on } y = 1, x \geqslant 0, \qquad \text{[A.1]}$$

$$\xi = \frac{\zeta}{2U_f},$$

$$\zeta = \frac{\sigma}{\mu \bar{U}},$$

where U_f is the final velocity in the jet and $\bar{U}$ the average velocity in the channel.

Under the transformation

$$x = -r\cos\theta,$$

$$y = -r\sin\theta + 1$$

[A.1] is transformed into the following problem in polar coordinates:

$$\nabla^4 \bar{\Psi} = 0, \quad r \geqslant 0, \quad 0 \leqslant \theta \leqslant \pi,$$

$$\left.\begin{aligned} \frac{1}{r}\frac{\partial \bar{\Psi}}{\partial \theta} &= 0 \\ \frac{\partial \bar{\Psi}}{\partial r} &= 0 \end{aligned}\right\} \text{on } \theta = 0,$$

$$\left.\begin{aligned} &\frac{\partial^2 \bar{\Psi}}{\partial r^2} - \frac{1}{r}\frac{\partial \bar{\Psi}}{\partial r} - \frac{1}{r^2}\frac{\partial^2 \bar{\Psi}}{\partial \theta^2} = 0 \\ &2\left[\frac{1}{r}\frac{\partial^2 \bar{\Psi}}{\partial r \partial \theta} - \frac{1}{r^2}\frac{\partial \bar{\Psi}}{\partial \theta}\right] + \Phi = \xi \frac{\partial^2 \bar{\Psi}}{\partial r^2} \end{aligned}\right\} \text{on } \theta = \pi. \qquad \text{[A.2]}$$

Separable solutions of $[A.2]_1$ of the form

$$\bar{\Psi} \sim r^{\lambda+1} f_\lambda(\theta)$$

exist for $Re\lambda > 0$ (see Lugt and Schwiderski (11)) where,

$$\left.\begin{aligned} f_\lambda(\theta) &= A_\lambda \sin(\lambda+1)\theta + B_\lambda \cos(\lambda+1)\theta \\ &\quad + C_\lambda \sin(\lambda-1)\theta + D_\lambda \cos(\lambda-1)\theta, \\ \Phi_\lambda &= -4\lambda r^{\lambda-1}[D_\lambda \sin(\lambda-1)\theta \\ &\quad - C_\lambda \cos(\lambda-1)\theta] \end{aligned}\right\} \lambda \neq 1 \qquad \text{[A.3]}$$

and

$$f_1(\theta) = A_1 \sin 2\theta + B_1 \cos 2\theta + C_1\theta + D_1,$$

$$\Phi_1 = 4C_1 \log r.$$

It is easy to show that no non-trivial solution of [A.2] exists for $\lambda = 1$. Implementing the conditions $[A.2]_2$ through $[A.2]_5$, respectively, we find that

$$f'_\lambda(0) = 0,$$

$$f_\lambda(0) = 0,$$

$$(\lambda-1)(\lambda+1)f_\lambda(\pi) - f''_\lambda(\pi) = 0,$$

$$2f'_\lambda(\pi) - 4\left[B_\lambda \sin\lambda\pi - \frac{\lambda+1}{\lambda-1}A_\lambda \cos\lambda\pi\right]$$

$$= \xi(\lambda+1)f_\lambda(\pi).$$

Substituting [A.3] into the above yields

$$D_\lambda = -B_\lambda,$$

$$C_\lambda = -\frac{\lambda+1}{\lambda-1}A_\lambda,$$

$$B_\lambda \cos\lambda\pi = 0,$$

$$-3(\lambda-1)B_\lambda \sin\lambda\pi + (\lambda+1)A_\lambda$$

$$\cdot [2\cos\lambda\pi - \xi\sin\lambda\pi] = 0.$$

These equations show that if $Re\lambda > 0$ and $\lambda \neq 1$ either

$$\cos\lambda\pi = 0,$$

or

$$2\cos\lambda\pi - \xi\sin\lambda\pi = 0.$$

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