

Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

Series: Mathematics Institute, University of Warwick
Adviser: D.B.A. Epstein

898

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Lectures on Bifurcation from Periodic Orbits.

Lectures given by D.D. Joseph.
Notes by K. Burns.

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CONTENTS

- §1. Introduction.
- §2. Derivation of the Autonomous Equation.
- §3. Hopf Bifurcation.
- §4. Subharmonic Bifurcation.
- §5. Rotation Number and Lock-Ins.
- §6. Experiments.

These lectures are about bifurcations from a periodic orbit of an evolution equation with periodic forcing. The analysis applies to equations in an arbitrary Hilbert space, not just to finite dimensional problems. The results described here are joint work of G. Iooss and D.D. Joseph [3,4,5]. In the lectures I will outline the methods, proofs are given in [3,4,5].

1. Introduction.

We consider the equation

$$d\underline{V}/dt = F(t, \mu, \underline{V}) \quad (1)$$

Here $\underline{V}(t, \mu)$ lies in a real Hilbert space $(H, \langle \cdot, \cdot \rangle)$, μ is a real bifurcation parameter, and F is T -periodic i.e., $F(T, \mu, \underline{V}) = F(t+T, \mu, \underline{V})$. Assume that there is a T -periodic solution

$$\underline{V} = \underline{U}(t, \mu) = \underline{U}(t+T, \mu) . \quad (2)$$

We rewrite (1) in local form about \underline{U} . If $\underline{u} = \underline{V} - \underline{U}$, then

$$d\underline{u}/dt = \underline{f}(t, \mu, \underline{u}) \quad (3)$$

where

$$\underline{f}(t, \mu, \underline{u}) = F(t, \mu, \underline{U} + \underline{u}) - F(t, \mu, \underline{U})$$

We shall study (3) with

$$\underline{f}(t, \mu, \cdot) = \underline{f}_{\underline{u}}(t, \mu | \cdot) + \underline{N}(t, \mu, \cdot)$$

where $\underline{f}_{\underline{u}}(t, \mu | \cdot)$ is linear and $\underline{N}(t, \mu, \underline{v}) = O(\|\underline{v}\|^2)$.

We assume that the periodic orbit \underline{U} , that is the orbit $\underline{u} = 0$ of (3), is stable if

$\mu < 0$, and loses stability for $\mu > 0$. To express this precisely consider the linearisation of (3)

$$d\underline{v}/dt = \underline{f}_{\underline{u}}(t, \mu | \underline{v}) \quad (4)$$

This is to be thought of as a complex linear equation (with real coefficients) on $H^{\mathbb{C}}$, the complexification of H . Associated with (4) is a linear operator on the space $P_T^{\mathbb{C}}$ of T -periodic vector fields on $H^{\mathbb{C}}$,

$$J_{\mu} = -d/dt + \underline{f}_{\underline{u}}(t, \mu | \cdot) \quad (5)$$

Eigenvalues of J_{μ} are called Floquet exponents. The orbit $\underline{u} = 0$ is stable if all Floquet exponents have negative real part, and unstable if any has positive real part. The loss of stability at $\mu = 0$ is assumed to occur in the simplest way :

Bifurcation Assumptions :

There is a Floquet exponent $\sigma(\mu) = \xi(\mu) + i\eta(\mu)$ such that

- (i) $\sigma(0) = i\omega_0 = \frac{2\pi r}{T}$ $0 \leq r < 1$.
- (ii) $\sigma(\mu)$ and $\bar{\sigma}(\mu)$ are isolated algebraically simple eigenvalues of J_{μ} .
- (iii) $d\xi/d\mu(0) > 0$.
- (iv) All eigenvalues of J_0 other than $\sigma(0)$ and $\bar{\sigma}(0)$ have negative real part.

The type of bifurcation that occurs depends on the value of r .

- (i) Strong Resonance : if $r = m/n$ and $n = 1, 2, 3$, or $n = 4$ and a certain inequality holds then nT -periodic solutions bifurcate.
- (ii) Y.H. Wan [6] has shown that there is an invariant torus when $n = 4$ and the inequality does not hold.
- (iii) Weak resonance : if $r = m/n$, $n \geq 5$, and certain exceptional conditions hold then nT -periodic solutions bifurcate.
- (iv) If $r \neq m/n$, $n = 1, 2, 3, 4$ there is a Hopf bifurcation to an invariant torus.

The next section describes how to approximate the original problem (3) with an autonomous equation in \mathbb{R}^2 . Sections 3 and 4 outline how to solve the approximate equation for Hopf and subharmonic bifurcations. The final section touches on the question of "lock-ons".

It should be mentioned that the asymptotic representations can be constructed directly, without normal forms, by methods of applied analysis (see appendices to Chapter X in [4]).

2. Derivation of the Autonomous Equation.

We assume that $r \neq 0, \frac{1}{2}$ (see [3,4,5] for a study of these cases). This means that the periodic orbit $\underline{u} = 0$ loses stability in two real dimensions instead of just one. The first step is to decompose (3) into a part in this plane and a complementary part.

There is an inner product on $\mathbf{P}_T^{\mathbb{C}}$,

$$[\underline{\xi}_1, \underline{\xi}_2] = \frac{1}{T} \int_0^T \langle \underline{\xi}_1(t), \underline{\xi}_2(t) \rangle dt .$$

Let J_μ^* be the adjoint of J_μ with respect to $[\cdot, \cdot]$. It can be verified that

$$J_\mu^* = d/dt + \underline{f}_u^*(t, \mu | \cdot) , \quad (6)$$

where $\underline{f}^*(t, \mu | \cdot)$ is the adjoint of $\underline{f}(t, \mu | \cdot)$ with respect to $\langle \cdot, \cdot \rangle$. Now $\sigma(\mu), \bar{\sigma}(\mu)$ are eigenvalues of J_μ, J_μ^* respectively; let $\underline{\xi}_\mu, \bar{\xi}_\mu$ be corresponding eigenfunctions. Using (6) and the assumption that $r \neq 0, \frac{1}{2}$, one can show that

$$\langle \underline{\xi}_\mu(t), \bar{\xi}_\mu^*(t) \rangle \equiv \langle \underline{\xi}_\mu(0), \bar{\xi}_\mu^*(0) \rangle$$

$$\langle \bar{\xi}_\mu(t), \underline{\xi}_\mu^*(t) \rangle \equiv 0 .$$

Normalise $\underline{\xi}_\mu, \bar{\xi}_\mu$ so $\langle \underline{\xi}_\mu, \bar{\xi}_\mu^* \rangle = 1$. Now we can write

$$\underline{u} = z \underline{\xi}_\mu + \bar{z} \bar{\xi}_\mu + \underline{W}$$

where $z = \langle \underline{u}, \bar{\xi}_\mu^* \rangle$ and \underline{W} is real. Equation (3) becomes

$$dz/dt = \sigma(\mu)z + b \quad (7a)$$

$$d\underline{W}/dt = \underline{f}_u(t, \mu | \underline{W}) + \underline{B} \quad (7b)$$

where

$$b(t, \mu, z, \bar{z}, \underline{W}) = \langle \underline{N}(t, \mu, \underline{u}), \bar{\xi}_\mu^*(t) \rangle$$

$$\underline{B}(t, \mu, z, \bar{z}, \underline{W}) = \underline{N}(t, \mu, \underline{u}) - \langle \underline{N}(t, \mu, \underline{u}), \bar{\xi}_\mu^* \rangle \underline{\xi}_\mu - \langle \underline{N}(t, \mu, \underline{u}), \bar{\xi}_\mu^* \rangle \bar{\xi}_\mu .$$

We have $b = b_0 + b_1$, $\underline{B} = \underline{B}_0 + \underline{B}_1$, where $b_0 = b(t, \mu, z, \bar{z}, 0)$, $b_1 = O(|z| \|\underline{W}\| + \|\underline{W}\|^2)$
 $\underline{B}_0 = \underline{B}(t, \mu, z, \bar{z}, 0) \underline{B}_1 = O(|z| \|\underline{W}\| + \|\underline{W}\|^2)$.

Roughly speaking (7b) will be eliminated and (7a) made autonomous up to $O(|z|^{N+1})$.

To do this we change variables

$$y = z + \gamma(t, \mu, z, \bar{z}) = z + \sum_{p+q \geq 2}^N z^p \bar{z}^q \gamma_{pq}(t, \mu) \quad (8)$$

$$\underline{Y} = \underline{W} + \underline{\Gamma}(t, \mu, z, \bar{z}) = \underline{W} + \sum_{p+q \geq 2}^N z^p \bar{z}^q \underline{\Gamma}_{pq}(t, \mu)$$

where N is arbitrary, γ_{pq} and $\underline{\Gamma}_{pq}$ are T -periodic, and $\underline{\Gamma}_{pq} \perp \underline{\xi}_\mu, \bar{\xi}_\mu$. We chose $\gamma_{pq}, \underline{\Gamma}_{pq}$ later, after (7) has been rewritten in terms of y, \underline{Y} . Now

$$\frac{dy}{dt} = \sigma z + b + \frac{\partial \gamma}{\partial t} + \frac{\partial \gamma}{\partial z} (\sigma z + b) + \frac{\partial \gamma}{\partial \bar{z}} (\bar{\sigma} \bar{z} + \bar{b})$$

$$= \sigma y + \left\{ \frac{\partial \gamma}{\partial t} + \sigma z \frac{\partial \gamma}{\partial z} + \bar{\sigma} \bar{z} \frac{\partial \gamma}{\partial \bar{z}} - \sigma y + \bar{b} \right\} + b_1 \left(1 + \frac{\partial \gamma}{\partial z} \right) + \bar{b}_1 \frac{\partial \gamma}{\partial \bar{z}}$$

where $\bar{b}(t, \mu, z, \bar{z}) = b_0 \left(1 + \frac{\partial \gamma}{\partial z} \right) + \bar{b}_0 \frac{\partial \gamma}{\partial \bar{z}}$;

$$\frac{dY}{dt} = \underline{f}_u(t, \mu | Y) + \left\{ \frac{\partial \Gamma}{\partial t} + \sigma z \frac{\partial \Gamma}{\partial z} + \bar{\sigma} \bar{z} \frac{\partial \Gamma}{\partial \bar{z}} - \underline{f}_u(t, \mu | \Gamma) + \bar{B} \right\} + B_1 \left(1 + \frac{\partial \Gamma}{\partial z} \right) + \bar{B}_1 \frac{\partial \Gamma}{\partial \bar{z}}$$

where $\bar{B}(t, \mu, z, \bar{z}) = B_0 \left(1 + \frac{\partial \Gamma}{\partial z} \right) + \bar{B}_0 \frac{\partial \Gamma}{\partial \bar{z}}$.

Expand $\tilde{b} = \sum_{p+q \geq 2}^N \tilde{b}_{pq} (t, \mu) z^p \bar{z}^q + O(|z|^{N+1})$
 $\tilde{B} = \sum_{p+q \geq 2}^N \tilde{B}_{pq}$

where $\tilde{b}_{pq}, \tilde{B}_{pq}$ are T periodic and $\tilde{B}_{pq} \perp \xi^*, \bar{\xi}^*$. Then

$$\frac{dy}{dt} = \sigma y + \sum_{p+q \geq 2}^N \left\{ \frac{\partial \gamma_{pq}}{\partial t} + [\sigma(p-1) + \bar{\sigma}q] \gamma_{pq} + \tilde{b}_{pq} \right\} z^p \bar{z}^q + O(|z| \|w\| + \|w\|^2 + |z|^{N+1}),$$

$$\frac{dY}{dt} = \underline{f}_u(t, \mu | Y) + \sum_{p+q \geq 2}^N \left\{ \frac{\partial \Gamma_{pq}}{\partial t} - \underline{f}_u(t, \mu | \Gamma_{pq}) + [\sigma p + \bar{\sigma}q] \Gamma_{pq} + \tilde{B}_{pq} \right\} z^p \bar{z}^q + O(|z| \|w\| + \|w\|^2 + |z|^{N+1}).$$

Finally use (8) on the right hand side to get

$$\frac{dy}{dt} = \sigma y + \sum_{p+q \geq 2}^N \left\{ \frac{\partial \gamma_{pq}}{\partial t} + [\sigma(p-1) + \bar{\sigma}q] \gamma_{pq} + \bar{b}_{pq} \right\} y^p \bar{y}^q + O(|y| \|Y\| + \|Y\|^2 + |y|^{N+1}) \quad (9a)$$

$$\frac{dY}{dt} = \underline{f}_u(t, \mu | Y) + \sum_{p+q \geq 2}^N \left\{ -J_\mu(\Gamma_{pq}) + [\sigma p + \bar{\sigma}q] \Gamma_{pq} + \bar{B}_{pq} \right\} + O(|y| \|Y\| + \|Y\|^2 + |y|^{N+1}) \quad (9b)$$

where \bar{b}_{pq} and \bar{B}_{pq} are functions of γ_{ij}, Γ_{ij} with $i+j < p+q$ with T -periodic coefficients and such that all terms in 9b are orthogonal to $\xi_\mu^*, \bar{\xi}_\mu^*$.

Now γ_{pq}, Γ_{pq} are chosen successively for $p+q = 2, 3, \dots, N$ so as to simplify (9). This is the key step. We choose Γ_{pq} to make $-J_\mu(\Gamma_{pq}) + [\sigma p + \bar{\sigma}q] \Gamma_{pq} + \bar{B}_{pq} = 0$ for small μ . This is always possible since $\Gamma_{pq}, \bar{B}_{pq} \in \{\xi \in P_T : \xi \perp \xi_\mu^*, \bar{\xi}_\mu^*\}$ and the bifurcation assumptions mean that for small μ none of the eigenvalues of J_μ on this space has real part as small as $\text{Re}(\sigma p + \bar{\sigma}q)$. This reduces (9b) to

$$\frac{dY}{dt} = \underline{f}_u(t, \mu | Y) + O(\|Y\| |y| + \|Y\|^2 + |y|^{N+1}). \quad (10a)$$

In order to choose γ_{pq} , write

$$\bar{b}_{pq}(t, \mu) = \sum_{\ell \in \mathbb{R}} b_{pq\ell}(\mu) e^{\frac{2\pi i \ell t}{T}}$$

$$\gamma_{pq}(t, \mu) = \sum_{\ell \in \mathbb{R}} \gamma_{pq\ell}(\mu) e^{\frac{2\pi i \ell t}{T}}$$

$$\text{Then } \frac{\partial \gamma_{pq}}{\partial t} + [\sigma(p-1) + \bar{\sigma}q] \gamma_{pq} + \bar{b}_{pq} = \sum_{\ell \in \mathbb{Z}} \alpha_{pq\ell}(\mu) e^{\frac{2\pi i \ell t}{T}}$$

where $\alpha_{pq\ell}(\mu) = \left\{ \frac{2\pi i \ell}{T} + [\sigma(p-1) + \bar{\sigma}q] \right\} \gamma_{pq\ell}(\mu) + b_{pq\ell}(\mu)$

$$\alpha_{pq\ell}(0) = \frac{2\pi i}{T} \{ \ell + r[p-1-q] \} \gamma_{pq\ell}(0) + b_{pq\ell}(0) .$$

We see that we can always choose $\gamma_{pq\ell}$ to make $\alpha_{pq\ell}(\mu) \equiv 0$ for small μ unless $\ell + r[p-1-q] = 0$. We call $\{(p, q, \ell, r) : \ell + r[p-1-q] = 0\}$ the Exceptional Set. It is the union of two disjoint subsets.

I The mean set : $(p, q, \ell, r) = (q+1, q, 0, r) \quad 2 \leq 2q+1 \leq N$

II The resonant set : $(p, q, \ell, r) = (q+1+nk, q, -km, \frac{m}{n}) \quad 0 \leq m < n, k \geq 1 \quad 2 \leq 2q+1+nk \leq N .$

The mean set is present for any r , but the resonant set arises only when r is rational.

When (p, q, ℓ, r) is in the exceptional set choose $\gamma_{pq\ell}(\mu) \equiv 0$; otherwise choose $\gamma_{pq\ell}(\mu)$ to make $\alpha_{pq\ell}(\mu) \equiv 0$. This reduces (9a) to

$$\begin{aligned} \frac{dy}{dt} = \sigma(\mu)y + \sum_{q \geq 1}^{2q+1 \leq N} y^{q+1} \frac{1}{y^q} b_{q+1, q, 0}(\mu) + \sum_{k \geq 0} \sum_{q \geq 0}^{2q-1+nk \leq N} \{ y^{q+1+nk} \frac{1}{y^q} b_{q+1+nk, q, -mk}(\mu) \\ - 2\pi i m k t / T \} \\ + y^q \frac{1}{y^{q-1+nk}} b_{q, q-1+nk, mk}(\mu) e^{2\pi i m k t / T} \\ + O(|y| \| \underline{Y} \| + \| \underline{Y} \|^2 + |y|^{N+1}) \end{aligned} \quad (10b)$$

The asymptotic representation is obtained by neglecting the order terms in (10a, b).

The truncation number N in (10b) is arbitrary. The justification of this approximation will not be attempted here; see [3, 4, 5]. We proceed to study the approximate problem.

It is clear that (10a) gives $\underline{Y}(t, \mu) \equiv 0$. To study (10b) set

$$y = x e^{i\omega_0 t} \quad (11)$$

Substitution in (10b) gives an autonomous equation of the form

$$\begin{aligned} \frac{dx}{dt} = \mu \hat{\sigma}(\mu)x + \sum_{q \geq 1}^{2q+1 < N} + x |x|^{2q} a_q(\mu) \\ + \sum_{k \geq 0} \sum_{q \geq 0} |x|^{2q} \{ x^{1+nk} a_{qk} + x^{-nk-1} a_{q, -k} \} \end{aligned} \quad (12)$$

where $\mu \hat{\sigma}(\mu) = \sigma(\mu) - \sigma(0)$ and $a_{q, k}(\mu) \equiv 0$ if r is irrational.

We shall look for the equilibrium solutions of (12). We expect to find fixed points and closed curves. These will be cross sections of subharmonic trajectories and

invariant tori for the original problem. The type of solution will depend on which terms on the right hand side of (12) have lowest order in x after $\mu\hat{\sigma}(\mu)x$. If $n = 3$ the term from the resonant set $a_{0,-1}x^{-n-1}$ is the only term of order 2, and we shall find fixed points for (12). If $n = 4$, $a_{0,-1}x^{-n-1}$ from the resonant set and $a_1x|x|^2$ from the mean set both have order 3, and either fixed points or an invariant circle can occur. If $n \geq 5$ then terms from the mean set have lower order, and we expect a closed orbit of (12). Normally it is traversed at a speed $O(\epsilon^2)$, but if enough exceptional conditions hold this speed can be so low that the terms from the resonant set break up the closed orbit into fixed points. This is weak resonance.

All of the above remarks assume that various terms are $\neq 0$. The exceptional cases where this is not true are ignored here. Also it will be assumed for simplicity that $\hat{\sigma}, a_q, a_{q,k}, a_{q,-k} \dots$ are independent of μ . This does not change the essence of the arguments.

3. Hopf Bifurcation.

This section outlines how to compute the trajectories on the torus when $n \geq 5$. We introduce an amplitude which is the mean radius of the invariant circle,

$$\epsilon = \frac{1}{2\pi} \int_0^{2\pi} x(s) e^{-is} ds.$$

We assume the orbit can be written in the form

$$\begin{aligned} x(t, \mu) &= \epsilon e^{is} \chi(s, \epsilon) \\ \mu &= \epsilon \tilde{\mu}(\epsilon) \\ s &= \epsilon \Omega(\epsilon) t \end{aligned} \tag{13}$$

where χ is 2π periodic in θ . Note that $2\pi/\epsilon\Omega(\epsilon)$ is the period of the closed orbit of (12).

Substitution in (12) gives

$$\begin{aligned} (i\Omega - \tilde{\mu}\hat{\sigma})\chi + \Omega \frac{d\chi}{ds} &= \sum \chi |x|^{2q} a_q \epsilon^{2q-1} + \sum_{k>0} \sum_{q \geq 0} |x|^{2q} \{ a_{q,k} e^{ink\theta} \chi^{1+nk} \epsilon^{2q+nk-1} \\ &+ a_{q,-k} e^{-ink\theta} \chi^{nk-1} \epsilon^{2q+nk-3} \}. \end{aligned}$$

Expand in powers of ϵ :

$$\begin{aligned} \chi(s, \epsilon) &= \sum_{j=0}^{\infty} \chi_j(s) \epsilon^j, \\ \tilde{\mu}(\epsilon) &= \sum_{j=0}^{\infty} \tilde{\mu}_j \epsilon^j, \\ \Omega(\epsilon) &= \sum_{j=0}^{\infty} \Omega_j \epsilon^j. \end{aligned}$$

The functions $\chi_j(\cdot)$ are 2π -periodic; and

$$1 = \frac{1}{2\pi} \int_0^{2\pi} \chi(s, \varepsilon) ds ,$$

so

$$\frac{1}{2\pi} \int_0^{2\pi} \chi_j(s, \varepsilon) ds = \begin{cases} 1 & j = 0 \\ 0 & j \geq 1 \end{cases} \quad (14)$$

We now solve by evaluating coefficients of successive powers of ε . From the terms of order 0,

$$(i\Omega_0 - \tilde{\mu}_0 \hat{\sigma}) \chi_0 + \Omega_0 \frac{d\chi_0}{ds} = 0 .$$

Taking the mean over $(0, 2\pi)$ gives

$$i\Omega_0 = \tilde{\mu}_0 \hat{\sigma} .$$

Now it follows from the bifurcation assumptions that $\hat{\sigma}$ has positive real part. Hence, since Ω_0 and $\tilde{\mu}_0$ are both real,

$$\Omega_0 = \tilde{\mu}_0 = 0 .$$

The terms of order 1 in ε now give

$$(i\Omega_1 - \tilde{\mu}_1 \hat{\sigma}) \chi_0 + \Omega_1 \frac{d\chi_0}{ds} = |\chi_0|^2 \chi_0 a_1 . \quad (15)$$

Taking the mean over $(0, 2\pi)$ gives

$$i\Omega_1 - \tilde{\mu}_1 \hat{\sigma} = a_1 \frac{1}{2\pi} \int_0^{2\pi} |\chi_0|^2 \chi_0 ds . \quad (16)$$

It can be shown from (14), (15) and (16) that $\chi_0(s) \equiv 1$. From (13) we now obtain

$$i\Omega_1 - \tilde{\mu}_1 \hat{\sigma} = a_1 .$$

Taking real and imaginary parts gives

$$\begin{aligned} \tilde{\mu}_1 \xi + \alpha_1 &= 0 \\ \Omega_1 - \tilde{\mu}_1 \hat{\eta} &= \beta_1 . \end{aligned}$$

To continue we have to assume that $\Omega_1 \neq 0$. It will be seen in the next section that $\Omega_1 = 0$ is the first of the special conditions leading to weak resonance.

The terms in ε^2 give

$$\Omega_1 \frac{d\chi_1}{ds} - a_1 (\chi_1 + \bar{\chi}_1) = g_1(s) - (i\Omega_2 - \tilde{\mu}_2 \hat{\sigma})$$

where

$$g_1(s) = \begin{cases} a_{0,-1} e^{-5is} & n = 5 \\ 0 & n \geq 5 \end{cases} .$$

We see from (14) that we must have

$$\int_0^{2\pi} \{g_1(s) - (i\Omega_2 - \tilde{\mu}_2 \hat{\sigma})\} ds = 0 .$$

This is true if and only if

$$\Omega_2 = \tilde{\mu}_2 = 0 .$$

It is easily shown using Fourier series that the equation $\Omega_1 \frac{dy}{ds} - a_1(y+\bar{y}) = \hat{g}(s)$ where \hat{g} is 2π -periodic and $\int_0^{2\pi} \hat{g}(s) ds = 0$ has a unique 2π -periodic solution. We see that

$$\begin{aligned} \chi_1(s) &= Ae^{5is} + Be^{-5is} & n = 5 \\ &= 0 & n \geq 5 . \end{aligned}$$

The analysis continues along these lines. It is found that $\tilde{\mu}(\cdot)$ and $\Omega(\cdot)$ are both odd functions, and that $\chi(\cdot, \varepsilon)$ is $\frac{2\pi}{n}$ periodic (constant if r is irrational). This is to be expected since (12) is invariant under rotation through $\frac{2\pi}{n}$. By tracing back through the derivation in section 2, we see that our approximate solution is quasi-periodic with the two frequencies $\frac{2\pi}{T}$ and $\omega_0 + \varepsilon^2 \omega(\varepsilon^2) = \omega_0 + \Omega_1 + \varepsilon^2 \Omega_3 + \dots$.

4. Subharmonic Bifurcation.

Suppose $x = \delta e^{i\varphi(\delta)}$ is a steady solution of (12). Note that $\delta e^{i\varphi(\delta)} e^{2\pi ik/n}$, $0 \leq k \leq n-1$, are all steady solutions of (12). They are the n -piercing points of a single nT -periodic trajectory. We have

$$0 = \mu \hat{\sigma} + \delta^2 a_1 + \delta^4 a_2 + \dots + \delta^{n-2} e^{-in\varphi} a_{0,-1} + \dots .$$

Assume

$$\begin{aligned} \varphi(\delta) &= \varphi_0 + \varphi_1 \delta + \varphi_2 \delta^2 + \dots \\ \mu &= \mu^{(1)} \delta + \mu^{(2)} \delta^2 + \dots . \end{aligned}$$

We evaluate the coefficients of increasing powers of δ .

For $n = 3$: the terms in δ give

$$\mu^{(1)} \hat{\sigma} + a_{0,-1} e^{-3i\varphi_0} = 0 .$$

Hence

$$\begin{aligned} \mu^{(1)} &= |a_{0,-1} / \hat{\sigma}| , \\ \varphi_0 &= \frac{1}{3} \arg(a_{0,-1} / \hat{\sigma}) + \frac{2k-1}{3} \quad k = 0, 1, 2 \end{aligned}$$

(taking $\mu^{(1)} = -|a_{0,-1} / \hat{\sigma}|$ will give the same solution).

The higher order terms can now be calculated. We obtain a single $3T$ -periodic trajectory. The bifurcation is two sided since $\mu(\delta) = 0(\delta)$.

If $n \geq 4$: the terms in δ give

$$\mu^{(1)} = 0 .$$

For $n = 4$ the terms in δ^2 give

$$\mu^{(2)}\hat{\sigma} + a_1 + e^{-4ik\varphi_0} a_{0,-1} = 0 ,$$

so

$$|\mu^{(2)}\hat{\sigma} + a_1|^2 = |a_{0,-1}|^2 .$$

This gives a quadratic equation for $\mu^{(2)}$. If the discriminant is positive we have two different values of $\mu^{(2)}$ which lead to two different 4T-periodic trajectories.

If $n \geq 5$: the terms in δ^2 give

$$\mu^{(2)}\hat{\sigma} + a_1 = 0 .$$

This is the first special condition for weak resonance; the requirement that $\mu^{(2)}$ be real restricts $\hat{\sigma}$ and a_1 . It can be verified that this restriction is equivalent to the requirement that $\Omega_1 = 0$ which was used in section 3.

For $n = 5$: the terms in δ^3 give

$$\mu^{(3)}\hat{\sigma} + a_{0,-1} e^{-5ik\varphi_0} = 0 .$$

This determines $\mu^{(3)}$ and φ_0 . Higher order terms can then be calculated. Since $\mu(\delta) = O(\delta^2)$ the bifurcation is one sided. Since $\mu^{(3)} \neq 0$ $\mu(\delta)$ is not even we obtain two 5T-periodic trajectories.

If $n \geq 6$: the terms in δ^3 give

$$\mu^{(3)} = 0 .$$

For $n = 6$: the terms in δ^4 give

$$\mu^{(4)}\hat{\sigma} + a_2 + a_{0,-1} e^{-6ik\varphi_0} = 0 .$$

This gives a quadratic equation for $\mu^{(4)}$; if the discriminant is positive two 6T-periodic trajectories bifurcate.

If $n \geq 7$: the terms in δ^4 give

$$\mu^{(4)}\hat{\sigma}_0 + a_2 = 0 .$$

This is the second special condition for weak resonance.

The results continue along these lines. As n increases subharmonic trajectories are possible only if more and more special conditions hold. When they do hold (and for

even n , if an extra inequality holds) there is one sided bifurcation of two nT -periodic trajectories. In the supercritical case, one solution is unstable and one is stable. In the subcritical case the torus itself is repelling and the periodic solution which is stable to disturbances or the torus is unstable in the larger space.

5. Rotation Number and Lock-Ins.

We conclude with a few remarks about the phenomenon of frequency locking when there is an invariant torus. This occurs when all the trajectories on the torus are captured by a single (subharmonic) trajectory.

We introduce the Poincaré (first return map). This is the map from the invariant circle to itself, this map takes a point on the circle to where the trajectory passing through it meets the circle again after going round the torus once (i.e. after time T). Consider its rotation number, ρ (defined for example in [4]; the reader may think of ρ as a frequency ratio). If ρ is irrational there is a change of coordinates which makes the Poincaré map a rotation, and the flow on the torus is quasiperiodic. The Poincaré map has no periodic points. If $\rho = p/q$ is rational, the Poincaré map must have periodic points of order q , to which correspond subharmonic trajectories. Generally there will be two such trajectories one attracting, the other repelling.

It is important to distinguish between the rotation number $\hat{\rho}(\varepsilon)$ for the asymptotic representation computed in section 3, and the rotation number $\rho(\varepsilon)$ for the real flow. It is known that $\rho(\varepsilon)$ is continuous but it is generally not differentiable. What happens is that if $\rho(\varepsilon_0) = p/q$ then $\rho(\varepsilon) \equiv p/q$ on an interval about ε_0 . The rotation number locks on to each rational value. This happens because if θ_0 is a periodic point of order q of the Poincaré map, $f_{\varepsilon_0}^q$, then generically $\partial/\partial\theta(f_{\varepsilon}^q)|_{\varepsilon=\varepsilon_0, \theta=\theta_0} \neq 0$. This enables us to solve for a fixed point of f_{ε}^q when ε is near ε_0 , so $\rho(\varepsilon)$ cannot change near ε_0 .

In particular the set of values of ε for which $\rho(\varepsilon)$ is rational has positive measure. It is an important result of M. Herman [2] that the set on which $\rho(\varepsilon)$ is irrational also has positive measure.

The results from section 3 show that the approximate rotation number is of the form

$$\hat{\rho}(\varepsilon) = \omega_0 + \varepsilon^2 \omega(\varepsilon^2).$$

It can be concluded from this that the true rotation number lies between two polynomials

$$\rho(\varepsilon) = \hat{\rho}(\varepsilon) \pm K\varepsilon^N,$$

where N is arbitrary. It follows that the lengths of the flat line segments on which lock-ins occur must tend to zero faster than any power of N as $\varepsilon \rightarrow 0$.

6. Experiments.

The type of dynamics which I have discussed in these lectures is characteristic of the observed dynamics in some mechanical systems involving fluid motions. The fact that an analysis of the kind given here does seem to fit well the observations of motion in small boxes of liquid heated from below, and in flow systems like the Taylor problem may surprise some readers. The surprise is that an analysis in two dimensions, and low dimensions greater than 2 give results in agreement with observations of continuum systems with "infinitely" many dimensions. In fact, this kind of agreement is associated with the fact that the spectrum of eigenvalues in the small scale systems for which agreements is sought is widely separated and the dimension of active eigenvalues is actually small.

I do not want to give a too cryptic explanation of the relevance to real fluid mechanics of the kind of analysis sketched in these lectures. In fact this kind of analysis is recommended for actual computation of bifurcated objects in fluid mechanics near the point of bifurcation [4]. A not too cryptic explanation of relevance can be found in my two review papers (D.D. Joseph, Hydrodynamic Instability and Turbulence, Ed. H. Swinney and J. Gollub, Topics in Physics, Springer, 1980) or in "Bifurcation in Fluid Mechanics" in the translation of the XIIIth International Congress of Theoretical and Applied Mechanics, (IUTAM), Toronto 1980.

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D.D. Joseph, Department of Aerospace Engineering & Mechanics, 107 Aeronautical Engineering, University of Minnesota, 110, Union St. S.E., Minneapolis, Minnesota 55455, U.S.A.

K. Burns, Mathematics Institute, University of Warwick, Coventry, CV4 7AL, England.