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Hydrodynamic Instabilities and the Transition to Turbulence

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3. Hydrodynamic Stability and Bifurcation

D. D. Joseph

With 14 Figures

The goal of hydrodynamics is to describe and predict the motions of fluids under applied forces. For incompressible Navier-Stokes fluids, in many circumstances, these forces scale with the Reynolds number. When the Reynolds number is small, hydrodynamics is not so difficult because there is a unique correspondence between the given boundary and internal forcing data and the predicted motions. But when the Reynolds number is larger, hydrodynamics is complicated; there are many solutions; nonuniqueness is the rule; sets of solutions must be described, and stable and observable subsets must be separated from the others.

A mathematical basis for the study of these hard problems is the theory of stability and the theory of bifurcation. In this review I will discuss some basic features of these two theories, their relation to one another, and their hydrodynamic applications[1].

3.1 The Navier-Stokes Equations and the Prescribed Data

The starting point for the study of laminar and turbulent motion of a viscous, incompressible, Newtonian fluid is the initial-value problem for the Navier-Stokes equations. We assume that the fluid occupies a confined region $\mathscr{V}$ of space and that the fluid velocity $\boldsymbol{U}(\boldsymbol{x},t)$ is prescribed in $\mathscr{V}$ at instant $t=0$. The mass of the incompressible fluid with density ϱ will be conserved in each small part of $\mathscr{V}$ if $\operatorname{div}\boldsymbol{U}=0$, and its momentum will be balanced if

$$\frac{\partial \boldsymbol{U}}{\partial t}+\boldsymbol{U}\cdot\nabla\boldsymbol{U}=-\nabla P+\nu\nabla^2\boldsymbol{U}+\boldsymbol{G}(\boldsymbol{x},t)\,. \tag{3.1}$$

Here $\nu=\mu/\varrho$ is the kinematic viscosity and P is a scalar field, called the "reaction pressure". P may be regarded as one of four unknown fields $(\boldsymbol{U},P)$ governed by the four equations (3.1) and $\operatorname{div}\boldsymbol{U}=0$. $\boldsymbol{G}(\boldsymbol{x},t)$ is a prescribed body force field. We shall also assume that the motion of the boundary $\partial\mathscr{V}$ of $\mathscr{V}$ is

1 Readers who wish to pursue the study of stability and bifurcation will find a relatively complete list of books, monographs, and review papers in the references. In the course of this review I will make frequent reference to materials in my two-volume treatment *Stability of Fluid Motions* [3.1, 2].

prescribed and that the fluid adheres to this boundary,

$$U(x,t)=U_B(x,t) \quad \text{for} \quad x\in\partial\mathscr{V}. \tag{3.2}$$

When other fields, like temperature or magnetic fields, enter into the dynamics, it is necessary to modify (3.1) and to supplement (3.1) and (3.2) with additional equations.

We may write the Navier-Stokes equations in dimensionless form, with velocity scale $\tilde{U}$ and length scale l. The dimensionless and dimensional equations are identical except that the kinematic viscosity ν is replaced with the reciprocal of the Reynolds number $R=\tilde{U}l/\nu$. Then all conclusions about the Navier-Stokes equations hold for all flows with different velocities $\tilde{U}$, in different domains $\mathscr{V}$ (proportional to l^3) and for fluids with different ν provided only that they share a common R. It will be convenient to think of ν^{-1} in (3.1) as the Reynolds number.

I call the boundary velocities $U=U_B$ and the body forces G which drive the motion, the *data*. We can think of $\mathscr{V}$ as a black box of fluid which makes contact with the outside world through the data. To a limited extent the fluid does what the data do, but in most cases the fluid has a mind of its own; it does what it wants. The data could be steady and flow inside $\mathscr{V}$ time periodic or even turbulent.

The flow through a round pipe is a good example of this anthropomorphic character of fluid motions. Suppose that $\mathscr{V}$ is an infinitely long pipe with axis x and radius l and that the data are given by

$$U_B=0 \quad \text{on} \quad r=l\,, \quad G(x,t)=-e_x\hat{P}\,, \tag{3.3}$$

where $\hat{P}=$ constant is the pressure drop per unit length. Then (3.1) has a simple solution with one nonzero component of velocity

$$U(r)=\tilde{U}(1-r^2/l^2)\,, \tag{3.4}$$

where $\tilde{U}=\hat{P}l^2/4\nu$ is the maximum velocity. The motion (3.4) is called laminar for obvious reasons. When $R=\tilde{U}l/\nu$ is small, this flow is always realized in experiments. But when R is larger, the flow is time dependent and erratic even though the data are steady and do not vary from place to place. The erratic flow is called turbulent and for a fixed value of $\hat{P}$ it has a smaller mass flux than the laminar flow (3.4). (It would be good if it were possible to run oil through pipes in laminar flow at large values of R; we could then reduce the cost of moving oil from Alaska to Minnesota. But such is not nature's design.)

The first serious scientific investigation of transition to turbulence in pipes was that of *Reynolds* [3.3] in 1883. His experiments revealed that the breakdown of laminar flow to "sinuous" motion occurred at a critical R which is the same for different diameter pipes, different velocities, and fluids with different viscosities. In the same classic paper *Reynolds* posed the question of why and how transition from laminar flow to turbulent flow takes place and he proposed an answer, which he attributed to Stokes, in terms of the breakdown of stability of the laminar flow. "The general cause of the change from steady to eddying motion was in 1843 pointed out Professor Stokes, as being that under certain circumstances the steady motion becomes unstable, so that an indefinitely small disturbance may lead to a change to sinuous motion". *Reynolds* first postulated that the circumstances referred to are that R, gradually increased, reaches a critical value at which the laminar flow becomes unstable to infinitesimal disturbances. But investigation based on infinitesimal disturbances did not give critical values like those observed in experiments. In *Reynolds*' [3.3, 4] address "On two manners of the motion of water" he noted pessimistically that "it has long been a matter of very general regret to those who are interested in Natural Philosophy, that in spite of the most strenuous efforts of the ablest mathematicians, the theory of fluid motion fits very ill with the actual behaviour of fluids, and this for unexplained reasons. The theory itself appears to be tolerably complete, and affords the means of calculating the results to be expected in almost every case of fluid motion, but while in many cases the theoretical results agree with those actually obtained, in other cases they are altogether different".

Pipe flow is evidently stable to infinitesimal disturbances at all values of the Reynolds number [Ref. 3.1, p. 120]. The instability observed by *Reynolds* seems to depend on nonlinearity in an important way. *Reynolds* [3.5] himself noted that the abruptness of the transition from laminar to turbulent flow "at once suggested the idea that the condition might be one of instability for disturbances of a certain magnitude, and stability for smaller disturbances".

The hydrodynamics of flow through round pipes is not very different from other types of shearing flow. For most of these, linearized theories of stability give a critical R but it is much larger than experimentally observed values. In still other types of hydrodynamical situations linearized theories give results which are close to actual behavior. The superficially anomalous relation of linearized theories to experiments has been clarified to a considerable extent by concepts which arise from bifurcation theory.

Bifurcation theory, in its broadest sense, attempts to classify and characterize the properties of all of the solutions which the initial-value problem can support when the transients have died away, the initial values have been forgotten, and the interior motions are driven by the data. In its more usual, less ambitious form, bifurcation theory classifies and characterizes all solutions which can arise from the instability of a given solution. I do not want to give a too cryptic description of the application of bifurcation theory to hydrodynamical problems at this point. For now it will suffice to remark that the theory allows one to make useful statements about the behavior of classes of problems even when explicit computations are not possible. By treating the problem from a more general point of view we learn how to distinguish the forest from the trees.

3.2 Uniqueness and Stability of Solutions when the Reynolds Number is Small

We start our discussion of stability and bifurcation with the comforting observation that when the Reynolds number is small, all solutions of the Navier-Stokes equations tend to a single one, determined by the data after the initial conditions have died away. So if the data are steady, the solution is steady; if the data are time periodic, so is the solution. It is of interest, and not too hard to show this. A perturbed solution $(\boldsymbol{U}^*, P^*)$ satisfies (3.1) and (3.2) but differs from $(\boldsymbol{U}, P)$ initially. The difference $(\boldsymbol{U}-\boldsymbol{U}^*, P-P^*)=(\boldsymbol{u}, p)$ satisfies

$$\frac{\partial \boldsymbol{u}}{\partial t}+(\boldsymbol{U}\cdot\nabla)\boldsymbol{u}+(\boldsymbol{u}\cdot\nabla)\boldsymbol{U}+(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}=-\nabla \mathrm{p}+\nu\nabla^2\boldsymbol{u}\ , \tag{3.5}$$

where

$$\boldsymbol{u}\in H=\{\boldsymbol{u}:\operatorname{div}\boldsymbol{u}=0, \boldsymbol{u}|_{\partial\mathscr{V}}=0, \langle|\nabla\boldsymbol{u}|^2\rangle<\infty\}\ , \tag{3.6}$$

and

$$\langle|\nabla\boldsymbol{u}|^2\rangle=\int_{\mathscr{V}}|\nabla\boldsymbol{u}|^2 d\mathscr{V}\equiv\int_{\mathscr{V}}\frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j}d\mathscr{V}\ . \tag{3.7}$$

Of course, $\boldsymbol{u}(\boldsymbol{x},0)=\boldsymbol{u}_0$ is prescribed.

To get the stability and uniqueness result, we work with an energy identity derived from (3.5–7). In preparation for the derivation of this identity we note that

$$\boldsymbol{u}\cdot(\boldsymbol{U}\cdot\nabla\boldsymbol{u}+\boldsymbol{u}\cdot\nabla\boldsymbol{u}+\nabla \mathrm{p})=\nabla\cdot(\boldsymbol{U}|\boldsymbol{u}|^2/2+\boldsymbol{u}|\boldsymbol{u}|^2/2+\boldsymbol{u}p)\ . \tag{3.8}$$

Then multiplying (3.5–7) by $\boldsymbol{u}$ and integrating we get

$$\frac{1}{2}\frac{d\langle|\boldsymbol{u}|^2\rangle}{dt}=-\langle\boldsymbol{u}\cdot\nabla\boldsymbol{U}\cdot\boldsymbol{u}\rangle-\nu\langle|\nabla\boldsymbol{u}|^2\rangle\leqq\langle|\nabla\boldsymbol{u}|^2\rangle(\nu_{\mathrm{E}}-\nu)\ , \tag{3.9}$$

where the existence of

$$\nu_{\mathrm{E}}=\max_{H}\left(\frac{-\langle\boldsymbol{u}\cdot\nabla\boldsymbol{U}\cdot\boldsymbol{u}\rangle}{\langle|\nabla\boldsymbol{u}|^2\rangle}\right)>0 \tag{3.10}$$

is guaranteed by the calculus of variations (provided that $\mathscr{V}$ can be confined between two parallel planes). We get monotonic decay of $\langle|\boldsymbol{u}|^2\rangle$ when the dissipation $\nu\langle|\nabla\boldsymbol{u}|^2\rangle$ is larger than the "production" $|\langle\boldsymbol{u}\cdot\nabla\boldsymbol{U}\cdot\boldsymbol{u}\rangle|$ of energy. The Navier-Stokes equations are special in that the only nonlinear term $(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}$ is inertial and it vanishes after integration. This feature, which is not typical of nonlinear problems, allows one to get (3.9) in a form independent of the amplitude $\boldsymbol{u}$; we get (3.9) for $\boldsymbol{v}$ if $\boldsymbol{u}$ is replaced by $a\boldsymbol{v}$, even when $a \to 0$. So (3.9) applies to (3.5–7) equally when $\boldsymbol{u} \cdot \nabla \boldsymbol{u}$ is finite or, as in the linear theory, when $(\boldsymbol{u} \cdot \nabla)\boldsymbol{u}$ is set to zero. Since the ratio of quadratic forms in $\boldsymbol{u}$ mentioned in (3.10) is standard in the calculus of variations, the existence of ν_E when $\boldsymbol{u}$ is a member of the set H, which contains at least all solutions of (3.5–7), is guaranteed, and $\langle|\boldsymbol{u}|^2\rangle$ certainly decays when $\nu > \nu_E$, that is, for small Reynolds numbers. In fact, the same theory guarantees the existence of $\Lambda > 0$ such that

$$\langle|\nabla \boldsymbol{u}|^2\rangle > \frac{\Lambda}{2}\langle|\boldsymbol{u}|^2\rangle . \tag{3.11}$$

Combining (3.9) with (3.11) we find, after integration, that

$$\langle|\boldsymbol{u}(t)|^2\rangle \leqq \langle|\boldsymbol{u}_0|^2\rangle \exp\left[\Lambda \int_0^t (\nu - \nu_E)dt\right] , \tag{3.12}$$

where ν_E depends on ν and t because $\boldsymbol{U}(\boldsymbol{x}, t, \nu)$ does. If $\boldsymbol{u}(t)$ is steady, then $\boldsymbol{u}(t) = \boldsymbol{u}_0$ and (3.12) holds if and only if $\langle|\boldsymbol{u}_0|^2\rangle = \langle|\boldsymbol{u}|^2\rangle = 0$. So steady flows are unique when $\nu > \nu_E$ [3.6]. Similarly, we can show that almost periodic motions are unique [Ref. 3.1, Chap. 1]. More generally, (3.12) shows that $\langle|\boldsymbol{u}|^2\rangle = \langle|\boldsymbol{U} - \boldsymbol{U}^*|^2\rangle \to 0$ as $t \to \infty$ so that $\boldsymbol{U}^* \to \boldsymbol{U}$ in the mean when ν is large or $R = 1/\nu$ is small. So for small Reynolds numbers there is just one solution $\boldsymbol{U}$ of (3.1) and (3.2) determined uniquely by the data $\boldsymbol{U}_B$ and $\boldsymbol{G}(\boldsymbol{x}, t)$ independent of the initial values $\boldsymbol{U}(\boldsymbol{x}, 0)$.

There is another interesting way to state the theorem of global stability just proved: *Flows which pertrub rigid motions are globally stable.* We may define a rigid motion as a motion $\boldsymbol{U}(\boldsymbol{x}, t)$ for which the stretching (rate of strain) tensor $\boldsymbol{D} = (\nabla \boldsymbol{U} + \nabla \boldsymbol{U}^T)/2 = 0$ ($2D_{ij} = \partial U_i/\partial x_j + \partial U_j/\partial x_i$) in $\mathscr{V}$. The velocity gradient $\nabla \boldsymbol{U}$ may be decomposed into symmetric and antisymmetric parts, $\nabla \boldsymbol{U} = \boldsymbol{D} + \boldsymbol{\Omega}$, where $\boldsymbol{\Omega} = (\nabla \boldsymbol{U} - \nabla \boldsymbol{U}^T)/2$ and $2\Omega_{ij} = \partial U_i/\partial x_j - \partial U_j/\partial x_i$, is the vorticity tensor with components $\Omega_{ij} = -\Omega_{ji}$. Since $\boldsymbol{u} \cdot \boldsymbol{\Omega} \cdot \boldsymbol{u} = u_i \Omega_{ij} u_j \equiv 0$, we find that $\langle \boldsymbol{u} \cdot \nabla \boldsymbol{U} \cdot \boldsymbol{u} \rangle = \langle \boldsymbol{u} \cdot \boldsymbol{D} \cdot \boldsymbol{u} \rangle$ always, and $\langle \boldsymbol{u} \cdot \boldsymbol{D} \cdot \boldsymbol{u} \rangle = 0$ for rigid motions. The energy of any disturbance of motions with $\boldsymbol{D} = 0$ decays monotonically since, by (3.9),

$$\frac{1}{2}\frac{d\langle|\boldsymbol{u}|^2\rangle}{dt} = -\nu\langle|\nabla \boldsymbol{u}|^2\rangle . \tag{3.13}$$

If $\boldsymbol{D}$ is small, the second term on the right (3.9) will outweigh the first, leading again to the decay of the energy of arbitrary disturbances.

Further discussion of energy methods in the theory of hydrodynamic stability can be found in [3.1, 2]. The results proved here are due to *Orr* [3.7], *Thomas* [3.8], *Hopf* [3.9], and *Serrin* [3.6].

3.3 Instability and Transition into Turbulence

We have seen that when the Reynolds number is small, the flow which evolves after a time is uniquely determined by the data independent of initial conditions. This unique flow has the maximum symmetry consistent with the data. At higher values of the Reynolds number we lose uniqueness. Other flows with different, usually more complicated patterns of symmetry are then observed after transients have decayed away. For example, motions which are spatially uniform in certain directions can be replaced by motions which are spatially periodic, quasi-periodic, or aperiodic. And motions which are steady can be replaced by motions which are time periodic, quasi-periodic, or aperiodic. So at larger values of the Reynolds numbers we may observe flows which do not follow the symmetry of the data (boundary conditions).

This breakdown in the symmetry of solutions is especially dramatic in the flows we call turbulent. In such flows the connection between the data and the flow is very elusive; even with steady data we can observe flows whose behavior after a long time is aperiodic with no (as yet identifiable) regularity. The spatial structure of turbulent flow is also very complicated with many little eddies and fluctuations which are sometimes called random because nobody knows how to characterize them in a precise way. Turbulent flows are extremely sensitive to initial conditions. Two flows with the same data but slightly different initial conditions evolve into two very different flows (see the discussions in Chaps. 2 and 4). There is surely a sense in which the data make themselves known in turbulent flow, but the connections between the data and the flow are subtle and elusive.

The following casual observations about transition to turbulence may be of value. When the data are steady and the Reynolds number is increased beyond the point at which turbulence first appears, the structure of the apparently chaotic flow does not seem to exhibit further qualitative changes. The most noticeable changes are that the parts of the flow which are turbulent gradually consume the whole flow. And, of course, the intensity of the fluctuating motions increases. It may be useful to think of turbulence as the least symmetric state of motion consistent with the given data. This way of thinking may or may not have intrinsic merit, but I use it to say that when we get turbulence you should read another book because I have arrived at the end of this story.

It is the very good and important observation of *Ruelle* and *Takens* [3.10] that there is actually an end to this story. They have given good theoretical support to the idea that we arrive at turbulence after a few bifurcations. This idea seems to be verified in the experiments known to me but it contradicts the appealing ideas about transition to turbulence which are associated with the names of *Landau* [3.11; see also 3.12] and *Hopf* [3.13, 14].

Landau and *Hopf* base their conjectures on ideas which derive from successive loss of stability of solutions to small disturbances. (The Landau idea is also discussed in Chap. 2 and Sect. 4.3.) At small Reynolds number $R=1/\nu$ there is just one steady flow, $\boldsymbol{U}=\boldsymbol{U}(\boldsymbol{x},R)$, and the scalar field $p=p[\boldsymbol{U}]$

belonging to this U[2]. The evolution of a disturbance $\boldsymbol{u}$ of $\boldsymbol{U}$ is governed by (3.5–7)

$$\frac{\partial \boldsymbol{u}}{\partial t}=F(\boldsymbol{u},R)-\nabla p\,,\qquad \boldsymbol{u}\in H\,, \tag{3.14}$$

where

$$F(\boldsymbol{u},R)=F_u(0,R|\boldsymbol{u})-\boldsymbol{u}\cdot\nabla\boldsymbol{u}\,, \tag{3.15}$$

and

$$F_u(0,R|\boldsymbol{u})=\frac{1}{R}\nabla^2\boldsymbol{u}-\boldsymbol{U}(R)\cdot\nabla\boldsymbol{u}-\boldsymbol{u}\cdot\nabla\boldsymbol{U}(R) \tag{3.16}$$

is a linear operator, the derivative of the operator $F(\boldsymbol{u},R)$, evaluated on the function $\boldsymbol{u}=0$. Since $F(0,R)=0$, $\boldsymbol{u}=0$ and $p[0]=0$ solve (3.14–16) for all R. When $\boldsymbol{u}$ is sufficiently small, we may seek conditions for the stability of $\boldsymbol{U}$ from the linearized problem,

$$\frac{\partial \boldsymbol{u}}{\partial t}=F_u(0,R|\boldsymbol{u})-\nabla p\,,\qquad \boldsymbol{u}\in H\,, \tag{3.17}$$

which arises from (3.14–16) when $\boldsymbol{u}\cdot\nabla\boldsymbol{u}$ is set to zero. $\boldsymbol{u}=e^{\sigma t}\zeta$, and $p[\boldsymbol{u}]=\exp(\sigma t)p[\zeta]$, $\zeta\in H$ solve (3.17) if ζ and $\sigma(R)=\xi(R)+\eta(R)$ solve the spectral problem,

$$\sigma\zeta=F_u(0,R|\zeta)-\nabla p[\zeta]\,,\qquad \zeta\in H\,. \tag{3.18}$$

When $\mathscr{V}$ is a bounded domain[3] there are a countably infinite number of isolated eigenvalues $\sigma(R)$ and all of these eigenvalues lie inside a parabola opening out to the left in the complex $\sigma(R)$ plane [3.15], as in Fig. 3.1.

The stability of $\boldsymbol{U}$ may be determined from the spectral problem: $\boldsymbol{U}$ is conditionally stable if for all eigenvalues $\xi(R)<0$ and is unstable if $\xi(R)>0$ for some eigenvalue. Conditionally stable means that $\boldsymbol{U}$ is stable to small disturbances; a conditionally stable flow may be unstable to large disturbances whose evolution is not governed by the linearized theory. As we increase R, there is a first critical value R_0 for which $\xi(R_0)=0$ for some eigenvalues $\sigma(R_0)=i\eta(R_0)$. If $\eta(R_0)\equiv\omega_0\neq 0$, then $\pm i\omega_0$ are equally eigenvalues of F_u with

2 Velocities and pressures are fields which depend on the position $\boldsymbol{x}$ at which they are evaluated. This dependence on $\boldsymbol{x}$ is suppressed in our notations.

3 The spectrum of F_u on unbounded domains is not fully understood. It is believed that there is a point spectrum (eigenvalues) controlling stability and a continuous spectrum which is confined to the left-hand side of the complex plane [$\xi(R)<0$ for all R].

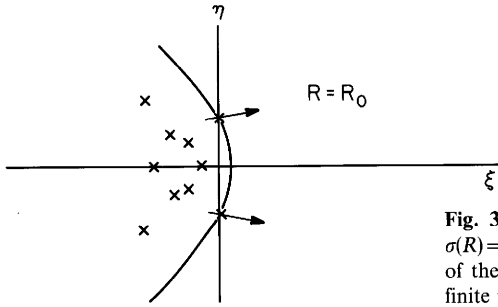


Fig. 3.1. When R is small, all the eigenvalues $\sigma(R)=\xi(R)+i\eta(R)$ of F_u are on the left-hand side of the complex plane. For larger values of R a finite number of eigenvalues have $\xi(R)>0$

eigenfunctions ζ and $\bar{\zeta}$, where $\bar{\zeta}$ is the complex conjugate of ζ. In general, a complex conjugate pair of eigenvalues crosses to the positive side of the complex σ plane as R is increased past R_0. If $\omega_0=0$, a single eigenvalue crosses at the origin.

In the usual situation the loss of stability of $\boldsymbol{U}$ implies the existence of a new solution $\boldsymbol{U}+\boldsymbol{u}$, which bifurcates from $\boldsymbol{U}$. The usual situation is as follows: ζ solving (3.18) is determined uniquely up to a multiplicative constant when $\sigma(R_0)=\mathrm{i}\omega_0$ [where $\sigma(R_0)$ is a simple eigenvalue of F_u], $\pm\mathrm{i}\omega_0$ are the only eigenvalues of F_u at criticality and the loss of stability of $\boldsymbol{U}$ is strict, $d\xi(R_0)/dR>0$. In this case, the solution which bifurcates is a steady solution if $\omega_0=0$ (see Sects. 3.5 and 3.8) and is a time-periodic solution if $\omega_0\neq 0$ (see Sects. 3.7, 8).

Returning now to the conjecture of *Landau-Hopf*, we suppose that $\omega_0\neq 0$, and a stable time-periodic flow with a characteristic frequency $\omega(\varepsilon)$, where ε is the amplitude of $\boldsymbol{u}$, exists for $R=R(\varepsilon)\geqq R_0$. [$\omega(\varepsilon)$ and $R(\varepsilon)$ are even functions (see Sect. 3.7).] So $\boldsymbol{u}(t,\varepsilon)=\boldsymbol{u}[t+2\pi/\omega(\varepsilon),\varepsilon]$ exchanges stability with $\boldsymbol{U}$ and is unique up to a choice of phase (a choice of the origin of t).

We need now to study the stability of the new periodic flow as R is increased. It is assumed that $R(\varepsilon)$ is an increasing function. We consider the evolution of a small disturbance $\boldsymbol{v}$ of $\boldsymbol{U}+\boldsymbol{u}$ and derive the linearized problem for $\boldsymbol{v}$,

$$\frac{\partial \boldsymbol{v}}{\partial t}=F_u(\boldsymbol{u}(t,\varepsilon),R(\varepsilon)|\boldsymbol{v})-\nabla p\,,\qquad \boldsymbol{v}\in H\,. \tag{3.19}$$

Equation (3.19) has time-periodic coefficients of period $2\pi/\omega(\varepsilon)$. A spectral problem for such equations may be obtained using the method of *Floquet* (see [Ref. 3.51, Chap. VII]). According to this method we set

$$\boldsymbol{v}=\mathrm{e}^{\tilde{\sigma}t}\tilde{\zeta}(t)\,,\qquad \tilde{\sigma}(\varepsilon)=\tilde{\xi}(\varepsilon)+\mathrm{i}\tilde{\eta}(\varepsilon)\,, \tag{3.20}$$

where $\tilde{\zeta}(t)=\tilde{\zeta}[t+2\pi/\omega(\varepsilon)]$ has the same period as the coefficients of (3.19) and is governed by the spectral problem

$$\tilde{\sigma}\tilde{\zeta}+\frac{\partial\tilde{\zeta}}{\partial t}=F_u(\boldsymbol{u}(t,\varepsilon),R(\varepsilon)|\tilde{\zeta})-\nabla p\,,\qquad \tilde{\zeta}\in H\,. \tag{3.21}$$

The eigenvalues $\tilde{\sigma}(\varepsilon)$ of (3.21) are called Floquet exponents. The solution $\boldsymbol{U}+\boldsymbol{u}$ is stable to small disturbances when $\tilde{\xi}(\varepsilon)<0$ and is unstable when $\tilde{\xi}(\varepsilon)>0$. The value ε_0 for which $\xi(\varepsilon_0)=0$ as ε is increased past ε_0 is the critical value for the loss of stability of $\boldsymbol{u}+\boldsymbol{U}$ and at criticality $\eta(\varepsilon_0)=\tilde{\omega}_0$.

If we assume that $\tilde{\sigma}(\varepsilon_0)=\mathrm{i}\tilde{\omega}_0$ is a simple eigenvalue of (3.21) and $\pm\mathrm{i}\tilde{\omega}_0$ are the only eigenvalues of (3.21) and $d\tilde{\xi}(\varepsilon_0)/d\varepsilon>0$, then a new solution bifurcates. The properties of this solution depend on the ratio of the frequencies $\tilde{\omega}_0/\omega(\varepsilon_0)$, $0\leqq\tilde{\omega}_0/\omega(\varepsilon_0)<1$. If $\tilde{\omega}_0/\omega(\varepsilon_0)=r/n$ is a fraction with $n=1$, 2, 3, or 4 the new solution has a period which is about n times the period of the bifurcating solution at criticality and the new periodic solution has a new frequency which depends on the amplitude. In the case of forced periodic motion the frequency $\omega(\varepsilon)$ is independent of amplitude and the bifurcating solution is strictly subharmonic with a fixed period $\tilde{T}=2\pi/\tilde{\omega}_0=2\pi n/\omega r=nT/r$ independent of amplitude. In both cases, supercritical solutions $[R>R(\varepsilon_0)]$ are stable and subcritical $[R<R(\varepsilon_0)]$ solutions are unstable when $n=1$ and 2. The $3T$-bifurcating solution is unstable on both sides of criticality when its amplitude is small. The stability properties of the $4T$-periodic bifurcating solution are slightly more complicated because there are several possibilities [3.16, 17]. If $n\neq 1$, 2, 3, or 4, the solutions which bifurcate are, in general, not periodic. They may be visualized as living on a torus which encircles the limit cycle representing the periodic solution. If the torus bifurcates subcritically $[R<R(\varepsilon_0)]$, all the bifurcating solutions are unstable; if it bifurcates supercritically $[R>R(\varepsilon_0)]$, the torus is stable but the solutions on the torus need not be stable. The analytical properties of the solution on the torus are not fully understood [3.16–19].

Landau [3.11, 12] conjectured that the solution which bifurcates when the periodic solution loses stability and ε [and $R(\varepsilon)$] is increased past ε_0 is a quasi-periodic solution with two frequencies. A quasi-periodic function of n variables $f(\omega_1 t,\omega_2 t,\ldots,\omega_n t)$ is a function containing a finite number of rationally independent frequencies $\omega_1,\omega_2,\ldots,\omega_n$ which is periodic with period 2π in each of its variables. For example, the function $f(\omega_1 t,\omega_2 t)=\cos t\cos\pi t$ is a quasi-periodic function with frequencies $\omega_1=2\pi$ and $\omega_2=2$. This function has the value $f(0,0)=1$ when $t=0$. $f(\omega_1 t,\omega_2 t)<1$ for any $t\neq 0$ but there is always a $t(\mu)>0$ such that $|f(\omega_1 t,\omega_2 t)-1|<\mu$ for preassigned $\mu>0$. The function

$$\mathrm{e}^{\mathrm{i}\tilde{\omega}_0 t}\tilde{\zeta}(t),\qquad \tilde{\zeta}(t)=\zeta[t+2\pi/\omega(\varepsilon_0)] \tag{3.22}$$

is a quasi-periodic function with two frequencies when $\omega(\varepsilon_0)/\tilde{\omega}_0$ is irrational.

So in the Landau-Hopf conjecture that transition to turbulence occurs by repeated quasi-periodic branching of solutions as R is increased, we get a two-frequency (ω_1,ω_2) solution

$$\boldsymbol{v}(t)=\tilde{\boldsymbol{v}}(\omega_1 t+\alpha_1,\omega_2 t+\alpha_2) \tag{3.23}$$

with two arbitrarily independent phases (α_1,α_2) (and an increasingly complicated spatial structure) when the periodic solution loses stability. A sequence of bifurcations like that just described does appear to occur in certain of the Bénard and Couette flow experiments described in Sect. 3.4 and elsewhere in this volume[4]. But this sequence is by no means universal because in many problems, like those described in the beginning of Sect. 3.4, we get a direct transition to turbulence even when $\boldsymbol{U}$ is stable according to the criteria of the linearized theory of stability. These cases, which defy *Landau* and *Hopf*, are examples of subcritical (or inverted) bifurcation which is a theoretical idea which I am going to explain in Sects. 3.5 and 3.7.

In the next step, *Landau* [3.11, 12] made a mistake and suggested a generalized Floquet theory which lacks a theoretical foundation and leads to results which contradict experience. He speculated that a spectral problem for the stability of the two-frequency solution $\boldsymbol{U}+\boldsymbol{u}+\boldsymbol{v}$ where $\boldsymbol{v}$ is given by (3.23) can be obtained by linearizing with disturbances of the form $\tilde{\tilde{\zeta}}(\omega_1 t, \omega_2 t)\exp(\tilde{\tilde{\sigma}}t)$, where $\tilde{\tilde{\zeta}}(\omega_1 t, \omega_2 t)$ has the same two periods as the coefficients of the linearized equation for the stability of $\boldsymbol{U}+\boldsymbol{u}+\boldsymbol{v}$. At criticality $\tilde{\tilde{\sigma}}=\mathrm{i}\tilde{\tilde{\omega}}$, introducing a third frequency, and a three-frequency solution with three arbitrary phases is said to bifurcate. So we get manifolds of quasi-periodic solutions of increasing dimension as R is increased and the process of transition to almost periodic turbulence with a discrete set of countably infinite frequencies occurs only in the limit $R\to\infty$.

The conjectures of *Landau* and *Hopf* can be criticized in several ways. *Bass* [3.21] called attention to the fact that, unlike turbulent solutions, quasi-periodic solutions do not phase mix. The values of quasi-periodic functions at two distant times are correlated as strongly as at close times. If $f(t)$ is defined on a turbulent field, then experiments show that the autocorrelation function

$$g(\tau)=\lim_{T\to\infty}\frac{1}{T}\int_0^T f(t+\tau)\bar{f}(t)dt\to 0 \quad \text{as} \quad \tau\to\infty\,. \tag{3.24}$$

Solutions with the property (3.24) are called *phase mixing.* If $f(t)$ is almost periodic (or quasi-periodic; see [3.1, p. 220]), then $f(t)$ has a Fourier series

$$f(t)=\sum_{-\infty}^{\infty} f_n \mathrm{e}^{-\mathrm{i}\lambda_n t}\,, \tag{3.25}$$

and

$$g(\tau)=\sum_{-\infty}^{\infty} |f_n|^2 \mathrm{e}^{-\mathrm{i}\lambda_n \tau} \tag{3.26}$$

does not tend to zero as $\tau\to\infty$.

---

4 Quasi-periodic flows with two frequencies usually lose stability to nonperiodic flow. An interesting experiment on forced periodic convection in bounded domains which undergoes transition from a flow with two frequencies to nonperiodic flow was reported by *Gollub* and *Benson* [3.20].

A second criticism of the Landau-Hopf conjecture raised by *Ruelle* and *Takens* [3.10] is that turbulent (phase mixing) solutions should appear after just a few bifurcations. In their early work they showed that even simple dynamical systems in four dimensions could be dominated by phase mixing solutions. A later analysis, leading to similar results in a simpler context, was given by *McLaughlin* and *Martin* [3.22]. Even earlier, *Lorenz* [3.23] and *Baker* et al. [3.24] had exhibited simple nonlinear ordinary differential equations in three dimensions with complicated turbulentlike dynamics. In all cases known to me turbulence appears after a finite number of bifurcations. In many flows, like the pipe flow discussed in the introduction, we get a direct transition to turbulence apparently without intervening bifurcations. In other flows, like the flow between rotating cylinders or rotating spheres, there are some symmetry-breaking bifurcations intervening between the unique basic flow which exists at small values of R and turbulence which exists at larger values.

3.4 Examples of Hydrodynamic Stability and Bifurcation

According to what I have already said, it is impossible to understand the hydrodynamics of flows which arise as the Reynolds number is increased without understanding instability and bifurcation of flows. To fix this important idea more firmly, it is useful to give a descriptive account of instability and bifurcation in some hydrodynamic examples. These examples are restricted to problems with steady data enjoying a high degree of spatial symmetry and are selected so as to represent certain general principles whose statement requires prior specification of some ideas from bifurcation theory. (Some of these ideas are developed in Sects. 3.5–7.)

Our first examples are the Poiseuille flows through the annulus between two long concentric cylinders. When there is no inner cylinder, we have the flow through a round pipe (Hagen-Poiseuille flow) whose most symmetric laminar state is given by (3.4). We have already discussed Reynolds experiments on transition to turbulence in pipe flow. The main features of the observations are also characteristic of flow through the annulus between concentric cylinders. The main features are: 1) There are no stable symmetry-breaking bifurcations of simple type; instead we go directly from laminar flow to turbulent flow. 2) In the low Reynolds turbulent regime only some parts of the flow are turbulent [3.25 26]; the flow is spatially segregated into distinct packets of laminar and turbulent flow (turbulent "puffs" when R is slightly above the critical R, and "slugs" at higher values of R). The transition from laminar to turbulent flow at a fixed place occurs suddenly as a puff or slug sweeps over the place, and the reverse transition occurs just as suddenly when it leaves the place. These observations suggest a sort of cycling in phase space between two weakly attracting solutions, one of which is laminar (see Fig. 3.2).

The second example is Couette flow between rotating cylinders when the inner cylinder is at rest and the outer one rotates with a steady angular velocity

◄ **Fig. 3.2a–c**

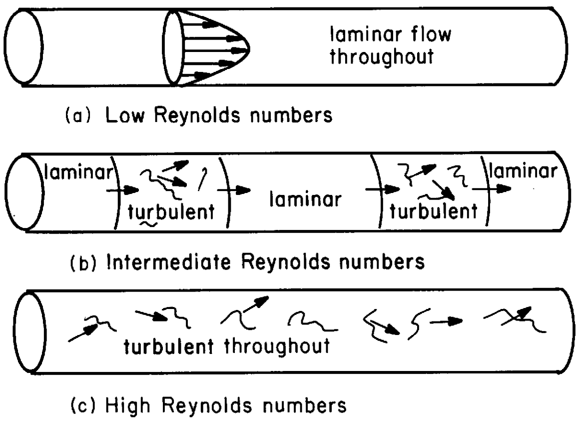


Fig. 3.3 ▼

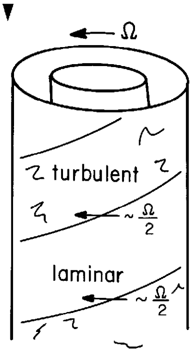


Fig. 3.2a–c. Sketch of the observational events in the transition to turbulence in pipes. The laminar flow which moves down the pipe at intermediate values of the Reynolds number is not the same as the Hagen-Poiseuille flow which exists at low Reynolds number. The alternate patches of laminar flow in (**b**) have a considerable amount of vorticity

Fig. 3.3. Spiral bands of turbulence which arise in the direct transition from Couette flow to turbulence. The outer cylinder rotates and the inner cylinder is stationary (or rotates slowly in the opposite direction). The spiral bands rotate at about one-half the angular speed of the outer cylinder. At higher Reynolds numbers the turbulence fills the whole annulus

Ω. The flows which evolve when the inner cylinder is at rest and the outer one rotates are much different than those which evolve when the outer cylinder is at rest and the inner one rotates. In the former case we get direct transition to turbulence, without intermediate bifurcations, as in Poiseuille flow. At low Reynolds numbers the flow is the laminar flow of maximum symmetry; in the idealized infinitely long annulus the flow is axisymmetric and does not vary along the cylinder axis, all components of velocity except the one in the azimuth vanish and it depends on the radius only. As in Poiseuille flow, Couette flow with the outer cylinder rotating undergoes direct transition to turbulence which first appears as alternate spiral bands of laminar and turbulent flow [3.27] (see Fig. 3.3), again as in Poiseuille flow.

The two examples just reviewed are examples of the direct transition to turbulence. Examples of transition to turbulence through a repeated finite number of symmetry-breaking bifurcations occurs in the Bénard problem, in the problem of Couette flow between cylinders when the outer cylinder is at rest, and in the problem of flow between concentric spheres when the outer sphere is at rest. The Bénard and Couette flow problems are reviewed in Chaps. 5 and 6, respectively. Here, I wish only to summarize features which appear to be common to all three problems and to draw attention to "abnormal" solutions which run against physical intuition but which are observed. These "abnormal" solutions appear much less abnormal when viewed in terms of general principles stemming from bifurcation theory (see remarks concluding Sect. 3.6).

The common features which appear in the three problems mentioned above are

1) The first bifurcation appears to break the spatial symmetry of the basic solution and to replace this solution of maximum symmetry with another steady flow having a different pattern of symmetry.

2) At higher values of the Reynolds number[5] new solutions, each with a different pattern of symmetry, appear to gain stability. Eventually there is a bifurcation of steady flow into a time-periodic motion with a single frequency. *Coles*' [3.27] study of stability and bifurcation of flow between rotating cylinders with the outer cylinder at rest draws attention to the marked degree of nonuniqueness and hysteresis which characterize these flows. For supercritical speed of a rotating inner cylinder up to about ten times critical, *Coles* finds that in one and the same apparatus the number of vortices and the number of waves travelling around these vortices are not uniquely determined by the speed. The number of Taylor cells in his apparatus ranges from 18 to 32 and the number of waves which travel around the axis of the cells ranges from 3 to 7. Moreover, "as many as 20 or 25 different states (each state being defined by the number of Taylor cells and the number of tangential waves) have been observed at a given speed".

3) At still higher values of Reynolds number the periodic solution with one frequency appears to give up its stability to a quasi-periodic solution with two frequencies both of which appear to vary smoothly with Reynolds number [3.28–30] (see Chaps. 5 and 6). As the Reynolds number is increased further, there is a continuous amplification of dynamical noise (turbulence) which eventually wipes out the sharp spectral peaks associated with discrete frequencies.

Now I will describe the "abnormal" solutions which are observed in the process of repeated bifurcation. In the problem of Couette flow between cylinders in which the outer cylinder is at rest, such abnormal solutions seem to have been observed first by *Benjamin* [3.31] and a possibly correct theoretical explanation of them has been given by him. Benjamin's apparatus is a short annulus which accommodates from two to four cells. (See the discussion of finite annulus height effects in Sect. 6.6.) In such experiments, cell cross sections never deviate very much from squares. He found that there are certain states, characterized by the number of cells and the sense of circulation in the cells, which can be reached only through sudden acceleration from rest. Such cells seem to be isolated from the basic cellular solution for most values of the height l of the rotating cylinders. In Benjamin's experiments and in the experiments on flow between spheres which are described below, there are an integral number of cells and an integral number of time-periodic undulations around the cells.

5 In the Bénard problem the relevant parameter is the Rayleigh number, measuring the temperature difference driving the motion, rather than the Reynolds number.

The different solutions correspond to different integers and the change from one to another set of integers appears always to be discontinuous. This suggests that we are dealing here with the problem of isolated solutions. (See Sect. 3.6 for a theoretical discussion of such possibilities.)

The abnormal cell observed by *Benjamin* appears to be one of the hard-to-reach solutions which can be reached only through special initial conditions like sudden acceleration from rest. To understand the abnormal cells we specify what is meant by a normal cell on the top or bottom of the finite cylinder. The top and bottom, like the outer cylinder, do not rotate. The cell is normal because the circulation in it is such that the fluid near the top or the bottom moves from the outer cylinder toward the inner cylinder. The reason is that in the center of the apparatus, away from the ends, the flow is like Couette flow (there are also weak secondary motions) in which an inward-pointing pressure gradient (pointing to the inner cylinder) is balanced against centripetal accelerations which in the absence of the opposing pressure gradient would throw the fluid outward. At the end plates this balance is broken because the centripetal accelerations are nullified by the boundary condition which requires that the fluid stick to the stationary end plates. So the fluid moves toward the inner cylinder driven by the unbalanced pressure gradient. In the "abnormal" cell observed by *Benjamin* the circulation is in the opposite sense.

Turning next to the allied problem of flow between rotating spheres we may cite experiments by *Sawatzki* and *Zierep* [3.32], *Munson* and *Menguturk* [3.33], *Wimmer* [3.34], *Belyaev* et al. [3.35], and *Yavorskaya* et al. [3.36]. The most symmetric laminar flow between spheres with the outer cylinder at rest is a steady axisymmetric spiral flow superposing motion in circles driven by shearing at the boundary and secondary motion between the equator and the poles driven by centripetal accelerations. This flow, unlike idealized Poiseuille and Couette flow between infinitely long cylinders, changes with the Reynolds number. The basic flow can be obtained by perturbation analysis or numerical analysis [3.37].

The characteristics of the flows which break the symmetries of the basic flow seem to depend strongly on the size of the gap between the spheres. When the gap is very small, the flow near the equator is very much like the flow between rotating cylinders and the sequence bifurcating flow is like that in the flow between cylinders. First, the basic flow loses stability and is replaced by a flow with vortices near the equator, but the flow near the poles is undisturbed. The vortices near the equator are like Taylor vortices; the cross section of each vortex is very nearly square, but the axis of the vortices spirals with a very slight pitch angle. For higher values of the Reynolds number the steady bifurcating flow with spiral vortices loses stability to a time-periodic motion in which the cell boundary becomes wavy and the waviness propagates around the sphere axis. This mode of instability is like the wavy vortex state in the Taylor problem (see Chap. 6). At higher values of the Reynolds number the wavy vortex mode gives up its stability to turbulence. The turbulent motion does not destroy the vortex structure when the gap is small; instead dynamical noise (fluctuations) of increasing (with the Reynolds number) amplitude is superposed on the equatorial vortices.

The situation is much more complicated when the gap between the spheres is large. The following observation may be helpful in interpreting the results. If we take the diameter of a vortex as the scale of length and then let the gap size tend to zero, the region near the equator looks locally like the infinitely long cylinders of the idealized Taylor problem. When the gap is large, we may expect that the vortices are influenced by the flow near the poles and the flow is more closely allied to a Taylor apparatus of finite height in which end effects are important.

Instability and bifurcation may be studied experimentally by measuring the torque the fluid exerts on the spheres, or by flow visualization. The two methods agree when the instability manifests itself as a break in the spatial symmetry of the solution. But *Munson* and *Menguturk* [3.33] found that when the gap is large (inner radius/outer radius = 0.44) there is no break in the spatial symmetry of the flow at the first value at which there is a break in the torque curve. They conjectured that the radial distribution of velocities is altered at this point of instability, but that the symmetry pattern of the new flow is basically the same as the old flow. *Munson* and *Menguturk* found three more breaks in the torque curve as the Reynolds number is increased, each of which is associated with some visual event, the last one leading to a bifurcation into turbulence.

The experimental results of *Munson* and *Menguturk* for stability of flow between spheres when the inner one rotates have been checked, but not reproduced, in the course of careful experiments of *Belyaev* et al. [3.35] and *Yavorskaya* et al. [3.36]. They did not observe the first three breaks in the torque curve, and the fourth break is observed as a three-dimensional wave, periodic in time. They said that this bifurcation leads to a new laminar flow and not to turbulence.

Similar experiments carried out by *Sawatzki* and *Zierep* [3.32] and *Wimmer* [3.34] revealed that five different types of flow can occur at the same Reynolds number when the gap between the cylinders is relatively wide. The five solutions observed in the experiments may be described as follows:

1) There is one mode which is visually like the basic laminar flow but has a different, nonlinear relation between the measured torque and the Reynolds number. This is possibly the same mode observed by *Munson* and *Menguturk* [3.33]. It is a very persistent solution which exists for a very large range of Reynolds numbers.

2, 3) There are paired steady solutions of normal and abnormal cellular motions in which there are a few vortices bordering the equator whose axes are parallel to the equator. In the normal cell the fluid is thrown out at the equator. This cell is normal because it is consistent with the physical fact that centripetal accelerations pushing the fluid out are greatest at the equator. In the abnormal flow the fluid at the equator moves inward, violating physical intuition (see Fig. 3.4). According to *Wimmer* [3.34] the abnormal flow has the lowest critical

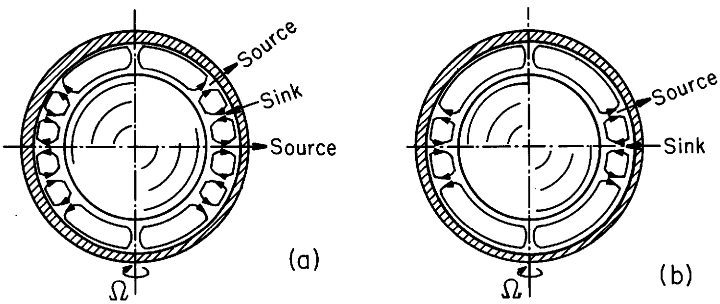


Fig. 3.4a, b. Two steady flows, (**a**) normal and (**b**) abnormal, which have been observed in the experiments of *Sawatzki* and *Zierep* [3.32] and *Wimmer* [3.34]

Reynolds number, which means that it is the first solution to replace the basic flow. It is also said to be the most stable solution, because it still occurs at very high Reynolds numbers close to the onset of turbulence.

4, 5) Corresponding to the normal and abnormal steady solutions is an analogous pair of unsteady solutions which may be time periodic. These solutions are like the ones described under 2) except that the axes of the vortices are no longer parallel but are inclined to the equator and the vortices penetrate this plane. Since the axes of the vortices now end in the midst of the flow field, single spots of vorticity are hurled off and travel to the poles. The whole process is said to be periodic.

The examples of instability, nonuniqueness and transition to turbulence which we have reviewed in this section will be discussed from a theoretical point of view in Sects. 3.6–8. The examples are merely a small sample of the types of problems which come up in trying to understand observed motions in fluids. A particularly interesting class of problems which is not well understood concerns the various common flows which occur in unbounded regions which cannot be contained between parallel planes. These include the instability of boundary layer, jets, shear layers, wakes, vortex streets, and thermal plumes. Some information about these problems can be found in Chaps. 7 and 8 and in the books, monographs, and review papers cited Sect. 3.10.

3.5 A Simplified Mathematical Discussion of some General Properties of Stability and Bifurcation

The difficulty I face in explaining the theory of hydrodynamic stability and bifurcation is that the equations are very complicated and admit very many different kinds of solutions. To my knowledge there are no good examples of realistic hydrodynamic situations which are simple enough to be understood at a glance. And in this subject physical intuition can betray you. For example, if you rely on physical intuition, you would dismiss as impossible the "abnormal" flows which appear between rotating cylinders and rotating spheres (see Sect. 3.4). I do not see any way to intuit the qualitative conditions under which a periodic flow will replace a steady one or a quasi-periodic flow a periodic one. What happens to flows depends mathematically on nonintuitive operations which in the simplest of cases involve roots of algebraic equations and the calculation of eigenvalues. It is dangerous to imagine that you can know such things without calculations, and the calculations in the hydrodynamic case are very involved.

To some extent the procedures of bifurcation theory simplify the problem by abstracting from specific problems the features which are essential in the description of the loss of stability and bifurcation. But in the process of abstraction we get to a more general problem in which the hydrodynamical equations are just one realization. The usual form for the abstracted problem is an evolution equation of the form

$$\frac{d\boldsymbol{u}}{dt}=F(\mu,\boldsymbol{u})\,, \tag{3.27}$$

where μ is a scalar parameter and $\boldsymbol{u}$ is a vector-valued field. In fact, the Navier-Stokes equations (3.14–16) can be written as in (3.27) when (3.14–16) are projected by the method explained in Sect. 3.8 with vectors ζ which have $\operatorname{div}\zeta=0$ in $\mathscr{V}$ and $\zeta\cdot\boldsymbol{n}=0$ on $\partial\mathscr{V}$. The method of projections is fundamental in theoretical studies of the Navier-Stokes equations (see, for example, [3.38–40]). I am not going to get into an involved discussion of projections here. For now it is enough to note that (3.27) is a general problem of which the Navier-Stokes equations is but one realization. There are a lot of properties of stability and bifurcation which are shared by all of the realizations of (3.27). And, in fact, we can learn a tremendous amount about the hydrodynamic case by studying the simplest realization of (3.27), the problem which arises when u is a scalar parameter and

$$\frac{du}{dt}=F(\mu,u) \tag{3.28}$$

governs its evolution. Of course (3.28) is not a hydrodynamical problem, but it is a good model and, in fact, it can be shown (for example, in [3.41, 42] that in important cases we can actually reduce the Navier-Stokes problem to the study of bifurcation and stability of steady solutions of (3.28). I do some of this showing in Sect. 3.8.

Some preliminary remarks about (3.28) are necessary for the analysis and some are useful for understanding the physical significance of the analysis. We first specify that both u and μ live on the real line, $-\infty<u<\infty$, $-\infty<\mu<\infty$. It is essential that $F(\mu,u)$ be a nonlinear function of two variables and it is necessary to assume that it and all of its first and second partial derivatives are continuous. It is good to regard (3.28) as an equation of motion. In most (nearly all) of the studies of bifurcation and stability it is conventional to imagine that $u=0$ satisfies (3.28) for all values of μ in an interval of special interest

$$F(\mu, 0)=0 . \tag{3.29}$$

I call the assumption (3.29) a reduction to local form. This reduction follows automatically when u is the difference between any solution and some special solution which exists for all values of μ (see [3.43]). In the hydrodynamic case the special solution is conventionally taken to be the unique one (the basic solution) which exists at small Reynolds numbers R and has the maximum symmetry consistent with the data [see discussion following (3.5–7) and (3.14–16)]. It is assumed that this solution continues to exist and is a sufficiently smooth function of R for all R, even for values $R>R_c$ for which the basic solution is unstable. In the hydrodynamic case we would put $\mu=(-1/R)+(1/R_c)$ so that the basic solution would be stable for $-\infty<\mu<0$ and unstable for positive $\mu\leqq 1/R_c$.

There is no great loss of generality in reducing bifurcation problems to local form. To see the small loss, suppose that there is a steady solution $u(\mu)$ of (3.28). Then a disturbance $x(t)$ of $u(\mu)$ satisfies

$$\frac{dx}{dt}=F[\mu, u(\mu)+x] . \tag{3.30}$$

So $\tilde{F}(\mu, x)=F[\mu, u(\mu)+x]$ obviously vanishes when $x=0$ for all μ such that $u(\mu)$ is defined. In the analysis of (3.28) it is neither necessary nor desirable to reduce $F(\mu, u)$ to local form. Instead we shall only introduce the notation $u=\varepsilon$, $d\varepsilon/dt=0$ for steady solutions of (3.28).

Now I am going to do some analysis, the simplest type of analysis I can do, which will still help in understanding broad features of stability and bifurcation in fluids. Until the end of this section, I suppress hydrodynamics and emphasize mathematics. We shall return to hydrodynamics, better prepared, in subsequent sections.

I start the mathematical discussion with a brief but thorough study of bifurcation and stability of steady solutions of (3.28). The bifurcation part of the study is just a review of the classical theory of singular points of plane curves. To this study we add results concerning the stability of bifurcating solutions (see [3.41–43]).

In our study of steady solutions (3.28) it is desirable to introduce the following classification of points:

i) A *regular point* of $F(\mu, \varepsilon)=0$ is one for which the implicit function theorem works,

$$F_\mu \neq 0 \quad \text{or} \quad F_\varepsilon \neq 0 . \tag{3.31}$$

If (3.31) holds, then we can find a unique curve $\mu=\mu(\varepsilon)$ or $\varepsilon=\varepsilon(\mu)$ through the point.

ii) A *regular turning point* is a point at which $\mu_\varepsilon(\varepsilon)$ changes sign and $F_\mu(\mu,\varepsilon)\neq 0$.

iii) A *singular point* of the curve $F(\mu,\varepsilon)=0$ is a point at which

$$F_\mu = F_\varepsilon = 0\ . \tag{3.32}$$

iv) A *double point* of the curve $F(\mu,\varepsilon)=0$ is a singular point through which pass two and only two branches of $F(\mu,\varepsilon)=0$ possessing distinct tangents.

v) A *singular turning (double) point* of the curve $F(\mu,\varepsilon)$ is a double point at which μ_ε changes sign.

vi) A *cusp point* of the curve $F(\mu,\varepsilon)=0$ is a point of second-order contact between two branches of the curve. The two branches of the curve have the same tangent at a cusp point.

vii) A *higher order singular point* of the curve $F(\mu,\varepsilon)=0$ is a singular point at which all three second derivatives of $F(\mu,\varepsilon)$ are null.

It is necessary to be precise about double points. Suppose that (μ_0,ε_0) is a singular point. The equilibrium curves passing through the singular point satisfy

$$2F(\mu,\varepsilon)=F_{\mu\mu}(\delta\mu)^2+2F_{\varepsilon\mu}\delta\varepsilon\delta\mu+F_{\varepsilon\varepsilon}(\delta\varepsilon)^2+O((\delta\mu)^2+\delta\varepsilon\delta\mu+(\delta\varepsilon)^2)=0\,, \tag{3.33}$$

where $\delta\mu=\mu-\mu_0$, $\delta\varepsilon=\varepsilon-\varepsilon_0$, and $F_{\mu\mu}=F_{\mu\mu}(\mu_0,\varepsilon_0)$, etc. In the limit as $(\mu,\varepsilon)\to(\mu_0,\varepsilon_0)$, the equation (3.33) for the curves $F(\mu,\varepsilon)=0$ reduces to the quadratic equation

$$F_{\mu\mu}(d\mu)^2+2F_{\varepsilon\mu}d\varepsilon d\mu+F_{\varepsilon\varepsilon}(d\varepsilon)^2=0 \tag{3.34}$$

for the tangents to the curve. We find two roots, designated with superscripts

$$\begin{pmatrix}\mu_\varepsilon^{(1)}(\varepsilon_0)\\ \mu_\varepsilon^{(2)}(\varepsilon_0)\end{pmatrix}=-\frac{F_{\varepsilon\mu}}{F_{\mu\mu}}\begin{pmatrix}1\\1\end{pmatrix}+\left(\frac{D}{F_{\mu\mu}^2}\right)^{1/2}\begin{pmatrix}1\\-1\end{pmatrix} \tag{3.35}$$

or

$$\begin{pmatrix}\varepsilon_\mu^{(1)}(\mu_0)\\ \varepsilon_\mu^{(2)}(\mu_0)\end{pmatrix}=-\frac{F_{\varepsilon\mu}}{F_{\varepsilon\varepsilon}}\begin{pmatrix}1\\1\end{pmatrix}-\left(\frac{D}{F_{\varepsilon\varepsilon}^2}\right)^{1/2}\begin{pmatrix}1\\-1\end{pmatrix}, \tag{3.36}$$

where

$$D=F_{\varepsilon\mu}^2-F_{\mu\mu}F_{\varepsilon\varepsilon}\ . \tag{3.37}$$

If $D<0$ there are no real tangents through (μ_0,ε_0) and the point (μ_0,ε_0) is an isolated point solution of $F(\mu,\varepsilon)=0$.

We shall consider the case when (μ_0,ε_0) is *not* a higher order singular point. Then (μ_0,ε_0) is a double point if and only if $D>0$. If $D=0$, then the slope at the singular point of higher order contact is given by (3.35) or (3.36). If $D>0$ and $F_{\mu\mu}\neq 0$, then there are two tangents with slopes $\mu_\varepsilon^{(1)}(\varepsilon_0)$ and $\mu_\varepsilon^{(2)}(\varepsilon_0)$ given by (3.35). If $D>0$ and $F_{\mu\mu}=0$, then $F_{\varepsilon\mu}\neq 0$ and

$$d\varepsilon(2d\mu F_{\varepsilon\mu}+d\varepsilon F_{\varepsilon\varepsilon})=0\ , \tag{3.38}$$

and there are two tangents $\varepsilon_\mu(\mu_0)=0$ and $\mu_\varepsilon(\varepsilon_0)=-F_{\varepsilon\varepsilon}/2F_{\varepsilon\mu}$. If $\varepsilon_\mu(\mu_0)=0$ then $F_{\mu\mu}(\mu_0,\varepsilon_0)=0$. So all possibilities are covered in the following two cases:

A) $D>0$, $F_{\mu\mu}\neq 0$ with tangents $\mu_\varepsilon^{(1)}(\varepsilon_0)$ and $\mu_\varepsilon^{(2)}(\varepsilon_0)$.

B) $D>0$, $F_{\mu\mu}=0$ with tangents $\varepsilon_\mu(\mu_0)=0$ and $\mu_\varepsilon(\varepsilon_0)=-F_{\varepsilon\varepsilon}/2F_{\varepsilon\mu}$.

Now I am going to connect stability and bifurcation. To study the stability of the solution $u=\varepsilon$ we study the linearized equation

$$Z_t=F_\varepsilon(\mu,\varepsilon)Z \tag{3.39}$$

by the spectral method

$$Z=e^{\gamma t}Z'\ , \tag{3.40}$$

where

$$\gamma=F_\varepsilon(\mu,\varepsilon)\ . \tag{3.41}$$

The solution $u=\varepsilon$ is stable when $\gamma<0$ and is unstable when $\gamma>0$.

Theorem 1 (Factorization Theorem): For every equilibrium solution $F(\mu,\varepsilon)=0$ for which $\mu=\mu(\varepsilon)$ we have

$$\gamma(\varepsilon)=F_\varepsilon[\mu(\varepsilon),\varepsilon]=-\mu_\varepsilon(\varepsilon)F_\mu[\mu(\varepsilon),\varepsilon]\equiv-\mu_\varepsilon\hat{\gamma}(\varepsilon)\ . \tag{3.42}$$

The proof of Theorem 1 follows from (3.41) and the equation

$$\frac{dF}{d\varepsilon}[\mu(\varepsilon),\varepsilon]=F_\varepsilon[\mu(\varepsilon),\varepsilon]+\mu_\varepsilon(\varepsilon)F_\mu[\mu(\varepsilon),\varepsilon]=0\ . \tag{3.43}$$

One of the main implications of the factorization theorem is that $\gamma(\varepsilon)$ *must change sign as ε is varied across a regular turning point.* This implies that the solution $u=\varepsilon$, $\mu=\mu(\varepsilon)$ is stable on one side of a regular turning point and is unstable on the other side (see Fig. 3.5).

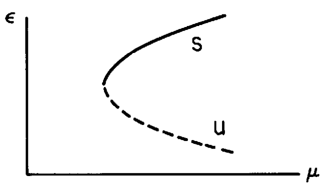


Fig. 3.5. Change of stability at a regular turning point

In the following theorem I make a connection between double-point bifurcation and the hypothesis that the loss of stability of a solution as a singular point is traversed is strict. This hypothesis seems to have been introduced by *Hopf* [3.44].

Theorem 2: Suppose that (μ_0, ε_0) *is a singular point and* (A) $\gamma_\varepsilon(\varepsilon_0) \neq 0$ *or* (B) $\gamma_\mu(\mu_0) \neq 0$. *Then* (μ_0, ε_0) *is a double point.* The proof of this theorem is elementary (see [3.41]) and will be omitted. The bifurcation picture is more complicated when the hypotheses of strict crossing in Theorem 2 are relaxed. If $\gamma_\varepsilon = 0$ when $\gamma = 0$, we may get cusp bifurcation; or if all three second derivatives vanish, then the cubic equation can give a triple point (three real roots for the slopes) or no bifurcation (two complex conjugate roots). If third derivatives also vanish, we face the problem of classifying the roots of a quartic. For example, we may get four bifurcating branches.

It is possible to make precise statements about the stability of solutions near double points of bifurcation. All of the possibilities for the stability of double-point bifurcation can be described by the cases A and B which were fully specified under (3.38). In case A, two curves $\mu^{(1)}(\varepsilon)$ and $\mu^{(2)}(\varepsilon)$ pass through the double point (μ_0, ε_0). In case B, two curves, $\varepsilon^{(1)}(\mu)$ [with $\varepsilon_\mu^{(1)}(\mu_0) = 0$] and $\mu_\varepsilon^{(2)}$, pass through the double point. The eigenvalue $\gamma^{(1)}$ belongs to the curve with superscript (1) and $\gamma^{(2)}$ to the curve with superscript (2).

Theorem 3: Suppose that (μ_0, ε_0) *is a double point. Then, in case* A,

$$\gamma^{(1)}(\varepsilon) = -\mu_\varepsilon^{(1)}(\varepsilon)[\hat{s}\sqrt{D}(\varepsilon - \varepsilon_0) + o(\varepsilon - \varepsilon_0)]\,. \tag{3.44}$$

and

$$\gamma^{(2)}(\varepsilon) = \mu_\varepsilon^{(2)}(\varepsilon)[\hat{s}\sqrt{D}(\varepsilon - \varepsilon_0) + o(\varepsilon - \varepsilon_0)]\,, \tag{3.45}$$

where $\hat{s} = F_{\mu\mu}/|F_{\mu\mu}|$ *and* D *and* $F_{\mu\mu}$ *are evaluated at* $\varepsilon = \varepsilon_0$. *And in case* B,

$$\gamma^{(1)}(\mu) = s\sqrt{D}(\mu - \mu_0) + o(\mu - \mu_0)\,, \tag{3.46}$$

and

$$\gamma^{(2)}(\varepsilon) = -s\mu_\varepsilon^{(2)}(\varepsilon)[\sqrt{D}(\varepsilon - \varepsilon_0) + o(\varepsilon - \varepsilon_0)]\,, \tag{3.47}$$

where $s = F_{\varepsilon\mu}/|F_{\varepsilon\mu}|$.

Proof: If $\mu = \mu(\varepsilon)$ we have (3.42) in the form,

$$\begin{aligned}\gamma(\varepsilon) &= -\mu_\varepsilon(\varepsilon)F_\mu[\mu(\varepsilon), \varepsilon] \\ &= -\mu_\varepsilon(\varepsilon)\{[F_{\mu\mu}(\mu_0, \varepsilon_0)\mu_\varepsilon(\varepsilon_0) + F_{\varepsilon\mu}(\mu_0, \varepsilon_0)](\varepsilon - \varepsilon_0) + o(\varepsilon - \varepsilon_0)\}\,.\end{aligned} \tag{3.48}$$

The formulas (3.44) and (3.45) arise from (3.48) when $\mu_\varepsilon(\varepsilon_0)$ is replaced with the values given by (3.35). If $\varepsilon=\varepsilon(\mu)$ with $\varepsilon_\mu(\mu_0)=0$, then $F_{\mu\mu}(\mu_0,\varepsilon_0)=0$, $F^2_{\varepsilon\mu}(\mu_0,\varepsilon_0)=D$ and

$$\begin{aligned}\gamma(\mu)&=F_\varepsilon[\mu,\varepsilon(\mu)]=F_{\varepsilon\mu}(\mu_0,\varepsilon_0)(\mu-\mu_0)+o(\mu-\mu_0)\\&=s\sqrt{D}(\mu-\mu_0)+o(\mu-\mu_0)\,.\end{aligned}\tag{3.49}$$

Theorem 3 gives an exhaustive classification relating the stability of solutions near a double point to the slope of the bifurcation curves near the point. The result may be summarized as follows. Suppose that $|\varepsilon-\varepsilon_0|>0$ is small. Then (3.44) and (3.45) show that $\gamma^{(1)}(\varepsilon)$ and $\gamma^{(2)}(\varepsilon)$ have the same (different) sign if $\mu_\varepsilon^{(1)}(\varepsilon)$ and $\mu_\varepsilon^{(2)}(\varepsilon)$ have different (the same) sign. The same conclusion can be drawn from (3.46) and (3.47). The possible distributions of stability of solutions is sketched in Fig. 3.6.

The analysis of double-point bifurcation is even easier when one first makes the reduction (3.29) to local form. It may be helpful to make a few remarks about the bifurcation diagrams which follow from analysis of (3.29). Nearly all the literature, not only the hydrodynamic literature, starts from a setup in which $u=0$ is a solution of the evolution problem. If $F(\mu,0)=0$, then $F_\mu(0,0)=F_{\mu\mu}(0,0)=0$ and the strict loss of stability of the solution $u=0$ as μ is increased past zero is

$$\gamma_\mu^{(1)}(0)=F_{\mu\varepsilon}(0,0)<0\,.\tag{3.50}$$

Then we get $D>0$ and

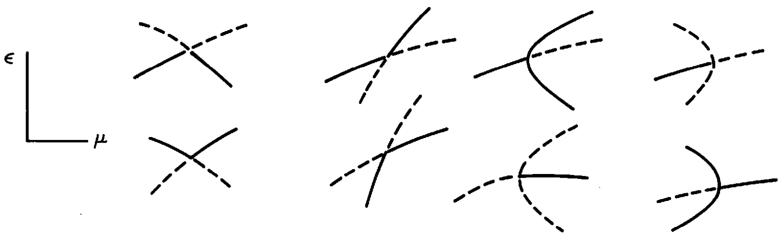
$$\gamma^{(2)}(\varepsilon)=-\mu_\varepsilon^{(2)}(\varepsilon)\gamma_\mu^{(1)}(0)[\varepsilon+o(\varepsilon)]\,.\tag{3.51}$$

The bifurcation diagrams which follow from these results and the conventional statements which we make about them are given by the diagrams and caption to Fig. 3.7.

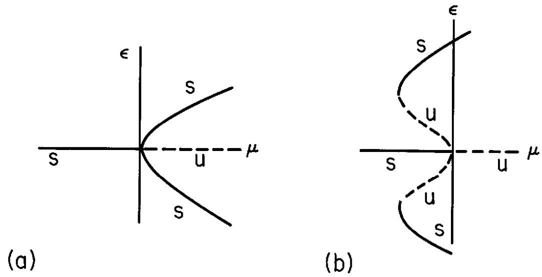
The results of our local analysis can be collected into a global theorem about double-point bifurcation.

Theorem 4: Assume that all singular points of solutions of $F(\mu,\varepsilon)=0$ are double points. The stability of such solutions must change at each regular turning point and at each singular point (which is not a turning point) and only at such points.

A marvelous demonstration which can help to fix the ideas embodied in Theorem 4 has been found by *Benjamin* (private communication). Benjamin's demonstration is an example of the buckling of a simple structure under the action of gravity. His apparatus is a board with two holes through which a viscoelastic wire is passed. The wire forms an arch above the board whose area length is l. The wire which is actually used in Benjamin's demonstration is like a



▲ **Fig. 3.6.** Stability of solutions in the neighborhood of double-point bifurcation. Dotted lines are unstable



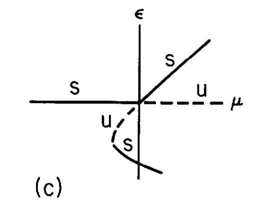


Fig. 3.7a–c. Stability of solutions bifurcating from $\varepsilon=0$. Supercritical solutions have $|\varepsilon|>0$ for values of μ ($\mu>0$ in the diagram) for which $\varepsilon=0$ is unstable. Subcritical solutions have $|\varepsilon|>0$ for values of μ for which $\varepsilon=0$ is stable

bicycle brake cable; it is wound like a tight coil spring and covered with a plastic sheaf. The demonstration apparatus is sketched in Fig. 3.8.

We imagine that the equation of motion for the wire arch is

$$\frac{d\theta}{dt}=F(l,\theta)\,. \tag{3.52}$$

The steady solutions of (3.52) are imagined to be in the form $F[l(\theta),\theta]=0$ shown in Fig. 3.9. Here $\theta=0$ is one solution (the upright one) and $l(\theta)$ is another solution (the bent arch). In fact there is a one-to-one correspondence between Benjamin's demonstration and the bifurcation diagram in Fig. 3.9; nothing is seen in the demonstration that does not appear in the diagram and there is nothing in the diagram that is not in the demonstration. The interpretation of events in the demonstration is given in the caption for Fig. 3.9.

Double-point bifurcation is perhaps the most common form of bifurcation which can occur at a singular point. Other types of bifurcation, cusp points, triple points, etc., are less common because they require that some higher order derivatives of $F(\mu,\varepsilon)$ vanish. Such situations are sometimes called

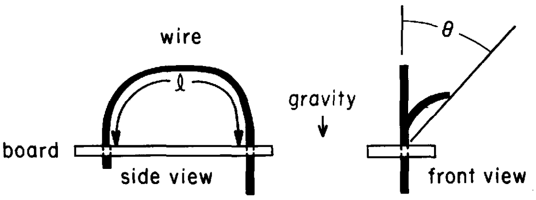


Fig. 3.8 ▲

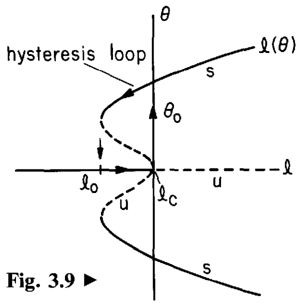


Fig. 3.9 ▶

Fig. 3.8. Benjamin's apparatus for demonstrating the buckling of a viscoelastic arch under gravity loading. The bifurcation diagram which fits this system is shown in Fig. 3.9. When l is small the only stable solution of (3.52) is the upright one ($\theta=0$). When $l>l_c$ is large the upright position is unstable and the arch falls to the left or to the right as shown in the front view. The bent position of the wire is also stable when $l<l_c$. When $l<l_c$ there are three stable steady solutions, the upright one ($\theta=0$) and the left or right bent one ($|\theta|\neq 0$)

Fig. 3.9. Bifurcation diagram for the buckling of the viscoelastic arch. When l is small the only equilibrium of (3.52) is the upright one ($\theta=0$). The solution $\theta=0$ loses stability when $u=l-l_c$ is increased past zero. A new solution $\mu(\theta)=l(\theta)-l_c$ corresponding to the bent arch then undergoes double-point bifurcation at a singular turning point $(l,\theta)=(l_c,0)$. The system is symmetric in θ. When $l<l_c$ only the left and right bent equilibrium configurations are stable. The points $(l,\theta)=(l_0,\pm\theta_0)$ are regular turning points. When $l_0\leqq l\leqq l_c$ there are three stable solutions $\theta=0$ and the symmetric left and right bent positions. In this region the system exhibits hysteresis. If the length l of the arch of the wire above the board is decreased while the wire is bent, the bent configuration will continue to be observed until $l=l_0$. When $l=l_0$ the bifurcating bent position is a regular turning point and for $l<l_0$ only $\theta=0$ is stable. So when l is reduced below l_0 the arch snaps through to the upright solution. Now if we increase l, the arch stays in the vertical position until $l=l_c$. When $l>l_c$, the upright solutions lose stability and the arch falls back into the left or right stable bent position

nongeneric. There is a technical mathematical sense for the word generic (having to do with dense open coverings). But most of the time the word is just a fancy alternative for the plain English word "typical". Analysis of typical problems does not help you if your problem is not typical. For example, at a cusp point where $D=0$ and all second derivatives are not null we get the bifurcation diagrams like those shown in Fig. 3.10 (see [3.42]).

All the results which we have asserted so far can be shown to apply to problems of partial differential equations, like the Navier-Stokes equations, under a condition, to be explained in Sect. 3.8, called bifurcation at simple eigenvalue.

It is very important that at this point we note with emphasis that it is not necessary for equilibrium solutions of evolution equations to be connected by bifurcations. There are isolated solutions, which are as common as rain, which are not connected to other solutions through bifurcation. Such isolated solutions of $F(\mu,\varepsilon)=0$ occur even in one-dimensional problems (see Fig. 3.11 for one typical example). In the one-dimensional case it is possible to prove

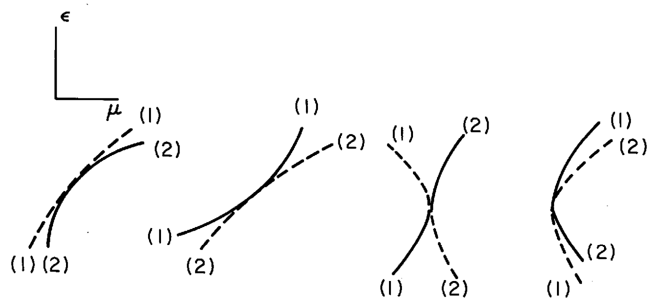


Fig. 3.10. Stability of solutions bifurcating at cusp point of second order

[3.42] that the stability of solutions which pierce the line $\mu =$ constant is of alternating sign as shown in Fig. 3.11. This result, however, is strictly one dimensional and does not apply to one-dimensional projections of higher dimensional problems in which curves of solutions which appear to intersect when projected onto the plane of the bifurcation diagram actually do not intersect in the higher dimensional space.

The possibilities for bifurcation which we have already indicated have important hydrodynamic applications. In particular we note that the supercritical form of bifurcation shown in Fig. 3.7 is typical of certain problems which have a high degree of spatial symmetry. The problem of bifurcation of Couette flow between rotating cylinders when the outer one is at rest is of this type when the cylinders are idealized to have an infinite length and the

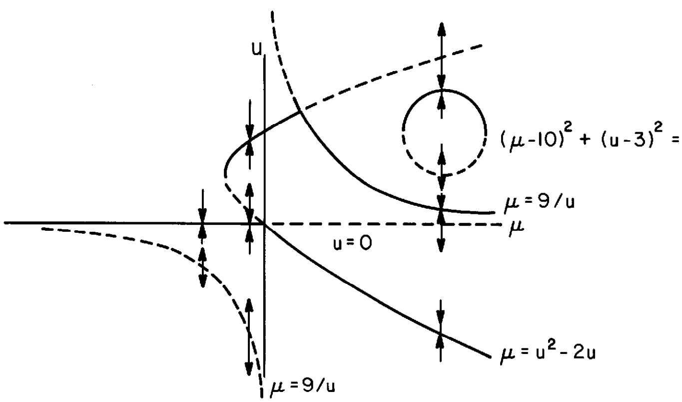


Fig. 3.11. Bifurcation, stability, and domains of attraction of equilibrium solutions of

$$du/dt = u(9-\mu u)(\mu+2u-u^2)\cdot[(\mu-10)^2+(u-3)^2-1].$$

The equilibrium solution $\mu = 9/u$ in the third quadrant and the circle are isolated solutions which cannot be obtained by bifurcation analysis disturbances may be assumed to be spatially periodic along the axis of the cylinders. In finite cylinders with closed ends the bifurcating flow is possibly of the transcritical type (see [3.31]). The first bifurcation of the idealized Bénard problem, in which the basic state is heat conduction without motion, is also of the supercritical type. Various modifications of this problem [Ref. 32, Chap. X; 3.45] lead to transcritical bifurcations. And the results of *Munson* and *Menguturk* [3.33] suggest that the first bifurcation of flow between rotating spheres may be of the transcritical type.

There are several important differences between bifurcations of the supercritical and transcritical type. Supercritical bifurcation is stable, and the stable flow is well approximated by eigenfunctions of linearized theory. In the transcritical case there is the possibility of hysteresis of the type shown in Fig. 3.9. In this case there are stable solutions which differ from the basic flow (here represented by the solution $u=0$) by a large amount ($|\varepsilon|$ in Fig. 3.7, θ in Fig. 3.9) at values of the Reynolds number (here $\mu<0$ in Fig. 3.7 and $l<l_c$ in Fig. 3.10) below those for which the basic flow is stable. So in the case of supercritical bifurcation we get agreement between what we find from studying the linear stability of the basic solution ($\varepsilon=0$) and what we observe. In the subcritical case and the transcritical case we may be observing the large amplitude solutions for which the linearized theory of stability of $\varepsilon=0$ is not relevant.

It is necessary to add that the conclusion which we have drawn from the study of problems in one dimension are only suggestive; they apply strictly in situations which may be described as bifurcation from a simple, real, isolated eigenvalue. We discuss these situations and draw closer to the actual complexities involved in the study of bifurcation and stability of solutions of the Navier-Stokes equation in Sect. 3.8.

3.6 Isolated Solutions Which Perturb Bifurcation

Isolated solutions are probably very common in hydrodynamical problems. One way to treat them is as a perturbation of problems which do bifurcate. This method of studying isolated solutions which are close to bifurcating solutions is known as imperfection theory. Some of the basic ideas involved in imperfection theory can be understood by comparing the bending of an initially imperfect, say bent, column (see Fig. 3.12). The first column will remain straight under end loadings P until a critical load P_c is reached. The column then undergoes supercritical, one-sided, double-point bifurcation (Euler buckling). In this perfect problem there is no way to decide if the column will buckle to the left or to the right. The situation is different for the initially bent column. The sidewise deflection starts as soon as the bent column is loaded and it deflects in the direction $x<0$ of the initial bending. If the initial bending is small, the deflection will resemble that of the perfect column. There will be small, nonzero

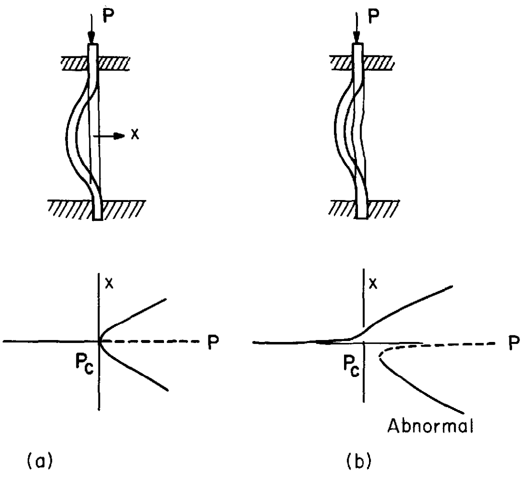


Fig. 3.12. (**a**) Buckling of a straight column: double-point supercritical bifurcation. (**b**) Bending of a bent column: isolated solutions which perturb double-point bifurcation

deflection with increasing load until a neighborhood of P_c is reached; then the deflection will increase rapidly with increasing load. When P is large it will be possible to push the deflected bent column into a stable "abnormal" position $(x>0)$ opposite to the direction of initial bending.

To understand the hydrodynamical implications of isolated solutions which perturb bifurcation, it is desirable to examine the possibilities with some generality. It is possible to do this simply, again by studying steady solutions of one-dimensional problems[6].

We first consider the problem

$$\frac{dx}{dt}=\tilde{F}(\mu,x,\delta)\,, \tag{3.53}$$

where δ and μ are parameters, and F has at least three derivatives with respect to each of its three variables at the point $(0,0,0)$. To simplify notations we drop the overbar on $\tilde{F}$ and the partial derivatives of $\tilde{F}$ at $(0,0,0)$. For example

$$\begin{aligned} F\ &=\tilde{F}(0,0,0)\,,\\ F_\mu&=\tilde{F}_\mu(0,0,0)\,,\quad \text{etc.} \end{aligned} \tag{3.54}$$

It is assumed that $(\mu,x)=(0,0)$ is a double point of $\tilde{F}(\mu,x,0)=0$ and that $\tilde{F}$ is in local form, that is, $x=0$ is a solution for all μ in a neighborhood of zero when $\delta=0$. Since $\tilde{F}(\mu,0)=0$ is an identity in μ, all the partial derivatives of $\tilde{F}(\mu,x,\delta)$

6 The results given here have much in common with the work of *Koiter* [3.46], *Thom* [3.47], *Benjamin* [3.48], *Matkowsky* and *Reiss* [3.49], and *Golubitsky* and *Schaeffer* [3.50]. The particular formulation of this section, the iterative procedures, and the stability results are taken from the forthcoming book on stability and bifurcation theory by *Iooss* and *Joseph* [3.51].

with respect to μ vanish at $(0,0,0)$. Since $(0,0,0)$ is a double point

$$F=F_x=0 \quad \text{and} \quad D=F^2_{\mu x}>0 . \tag{3.55}$$

We are interested in the steady solutions $x=\varepsilon$, $d\varepsilon/dt=0$ of

$$\tilde{F}(\mu,\varepsilon,\delta)=0 \tag{3.56}$$

which break the solutions which bifurcate at the double point into isolated solutions. For this, it is enough that

$$F_\delta \neq 0 . \tag{3.57}$$

Let us derive the form of the isolated solutions which break the bifurcation. The implicit function theorem and (3.57) guarantee that there is a function $\delta=\Delta(\mu,\varepsilon)$ with $\Delta(0,0)=0$ and

$$\tilde{F}[\mu,\varepsilon,\Delta(\mu,\varepsilon)]\equiv 0 . \tag{3.58}$$

It follows from (3.58) and the fact that $\tilde{F}(\mu,0,0)=0$ that we may take $\Delta(\mu,0)=0$ so that all the partial derivatives of $\Delta(\mu,\varepsilon)$ with respect to μ alone vanish when $(\mu,\varepsilon)=(0,0)$. Differentiating (3.58) with respect to ε and μ at $(\mu,\varepsilon)=(0,0)$, we find that

$$F_\varepsilon+F_\delta\Delta_\varepsilon=0 , \tag{3.59}$$

$$F_{\varepsilon\varepsilon}+F_\delta\Delta_{\varepsilon\varepsilon}=0 , \tag{3.60}$$

$$F_{\mu\varepsilon}+F_\delta\Delta_{\mu\varepsilon}=0 . \tag{3.61}$$

Now $F_\mu=0$ identically, and (3.59) shows that $\Delta_\varepsilon=0$ so that the surface $\delta=\Delta(\mu,\varepsilon)$ is tangent to the plane $\delta=0$ in the three-dimensional space with coordinates (μ,ε,δ) at the point $(0,0,0)$. From (3.61) we learn that

$$D[\Delta(0,0)]=\Delta^2_{\mu\varepsilon}=F^2_{\mu\varepsilon}/F_\delta>0 \tag{3.62}$$

so that the point $(0,0,0)$ is a saddle point.

We may find all of the derivatives of $\Delta(\mu,\varepsilon)$ at $(0,0)$ in terms of the derivatives of $\tilde{F}$ at $(0,0,0)$ by differentiating (3.58) repeatedly with respect to μ and ε. We find that

$$\begin{aligned}\Delta(\mu,\varepsilon)&=\frac{1}{2}(\Delta_{\varepsilon\varepsilon}\varepsilon^2+2\Delta_{\varepsilon\mu}\varepsilon\mu)\\&\quad+\frac{1}{3!}(\Delta_{\varepsilon\varepsilon\varepsilon}\varepsilon^3+3\Delta_{\mu\varepsilon\varepsilon}\mu\varepsilon^2+3\Delta_{\mu\mu\varepsilon}\mu^2\varepsilon)+\dots \\&=a\varepsilon^2+2b\varepsilon\mu+d\varepsilon^3+e\mu\varepsilon^2+f\mu^2\varepsilon+\dots ,\end{aligned} \tag{3.63}$$

where $\Delta_{\varepsilon\varepsilon}$ and $\Delta_{\mu\varepsilon}$ are given by (3.60) and (3.61) with similar equations for third derivatives.

Our problem now is to solve (3.63) for $\mu(\varepsilon, \delta)$ for a fixed value δ. The intersection of the surface $\delta = \Delta(\mu, \varepsilon)$ and the planes $\delta =$ constant determine these curves. The plane $\delta =$ constant may be written in parametric form with ε as a parameter.

$$\delta = \varepsilon\hat{\delta} = \varepsilon\hat{\Delta}(\mu, \varepsilon)\,, \tag{3.64}$$

$$\hat{\delta} = \hat{\Delta}(\mu, \varepsilon) = a\varepsilon + 2b\mu + d\varepsilon^2 + e\varepsilon\mu + f\mu^2 + \dots\,. \tag{3.65}$$

We can solve (3.64, 65) by successive approximations. First solve for

$$\mu = \frac{1}{2b}(\hat{\delta} - a\varepsilon - d\varepsilon^2 - e\mu\varepsilon - f\mu^2 + \dots)\,. \tag{3.66}$$

The first approximation is given by

$$\mu \sim \mu^{(1)} = \frac{1}{2b}(\hat{\delta} - a\varepsilon) = \frac{1}{2b}\left(\frac{\delta}{\varepsilon} - a\varepsilon\right). \tag{3.67}$$

Equation (3.67) gives two isolated solutions which break double-point bifurcation. For example, if $a = 0$, as in supercritical bifurcation, we get two bifurcating solutions when $\delta = 0$: $\varepsilon = 0$ and $\mu = 0$. The isolated solutions which perturb these bifurcating solutions when $\delta \neq 0$ are given by the pair of hyperbolas $\mu = \delta/2b\varepsilon$.

The second approximation is given by

$$\mu \sim \mu^{(2)} = \frac{1}{2b}(\hat{\delta} - a\varepsilon - d\varepsilon^2 - e\varepsilon\mu^{(1)} - f\mu^{(1)^2})$$

$$= \frac{1}{2b}\left[\frac{\delta}{\varepsilon} - a\varepsilon - d\varepsilon^2 - \frac{e}{2b}(\delta - a\varepsilon^2) - \frac{f}{4b^2}\left(\frac{\delta}{\varepsilon} - a\varepsilon\right)^2\right]. \tag{3.68}$$

For example, if $a = 0$ we get two bifurcating solutions when $\delta = 0$. These bifurcating solutions are given locally by

$$\varepsilon = 0 \quad \text{and} \quad \mu = -\frac{d}{2b}\varepsilon^2\,, \tag{3.69}$$

corresponding to one-sided supercritical bifurcation if $d/2b < 0$. The isolated solutions which break bifurcation when $\delta \neq 0$ are given by (3.68). In the supercritical case one possible pair of isolated solutions is that shown in Fig. 3.12b.

The stability of isolated solutions on the curve $\mu(\varepsilon, \delta)$ may be obtained from the factorization theorem. Perturbing the solutions $[\mu(\varepsilon, \delta), \varepsilon]$ of (3.53) with

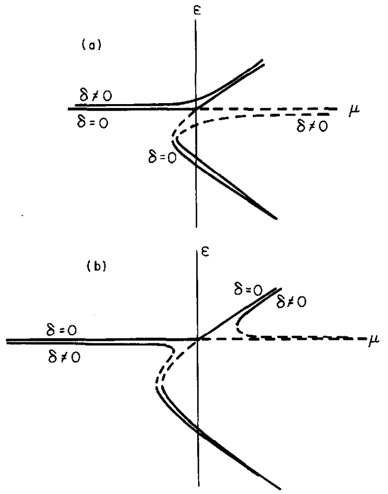


Fig. 3.13a, b. Isolated solutions which perturb transcritical bifurcation. The solution $\delta \neq 0$, $\mu<0$ in (**b**) exhibits hysteresis as shown

small disturbances proportional to $\exp \gamma t$, we find that

$$\gamma(\varepsilon)=F_{\varepsilon}[\mu(\delta,\varepsilon),\varepsilon,\delta]=-\mu_{\varepsilon}(\varepsilon,\delta)F_{\mu}[\mu(\delta,\varepsilon),\varepsilon,\delta] \ . \tag{3.70}$$

Then we have:

I) Stable branches in the bifurcation diagram perturb, by continuity, into stable branches of the perturbed solutions except at

II) regular turning points where the stability index of a solution changes sign.

Typical cases of the breaking of bifurcation with δ, in the second approximation, combined with the stability principles, are shown in Fig. 3.12 and in Fig. 3.13.

At a fixed value of μ there is one stable solution when μ is small. At larger values of μ there are two stable solutions, one for $\varepsilon>0$ and one for $\varepsilon<0$. In the hydrodynamic context $\varepsilon>0$ implies a cellular motion with one sense of circulation and $\varepsilon<0$ a circulation with the other sense. So the isolated solutions, like the bifurcating solutions, come in pairs. It follows then, even from the most elementary theoretical arguments, that stable cellular motions with different circulations are to be expected in flows which perturb bifurcation. In this sense the normal and abnormal cells which were discussed in Sect. 3.4 are to be expected, despite the fact that the "abnormal" cells appear to violate physical intuition.

3.7 Bifurcation of Steady Flow into Time-Periodic Flow

The problem of bifurcation of steady flow into time-periodic flow is basically two dimensional. It is not possible for a time-periodic solution to bifurcate from a steady one in one dimension. Time-periodic bifurcations are very important in hydrodynamics. In fact, a time-periodic bifurcation seems always to play a role in the transition to turbulence. It appears as one of the bifurcations in the finite sequence of supercritical bifurcations which lead to turbulence in the Bénard problem and in the Taylor problem. And a subcritical, unstable, time-periodic bifurcation may be characteristic of flows which undergo a direct, snap-through transition from steady laminar flow to turbulent flow.

The problem of bifurcation of steady solutions of the Navier-Stokes equations into time-periodic solutions can be reduced, after analysis involving projections (see Sect. 3.8), to a two-dimensional problem associated with the following pair of nonlinear ordinary differential equations:

$$\frac{dx_i}{dt}=F_i(\mu,x_1,x_2)=A_{ij}(\mu)x_j+k_{ijk}(\mu)x_jx_k$$
$$+\text{ higher order terms },\tag{3.71}$$

where the summation convention holds, $i=1,2$, $k_{ijk}=k_{ikj}$, and $A_{ij}(\mu)$ are components of

$$A(\mu)=\begin{pmatrix}a(\mu) & b(\mu)\\ c(\mu) & d(\mu)\end{pmatrix}.\tag{3.72}$$

We suppose that $(a-d)^2+4bc>0$ in a neighborhood of $\mu=0$. Then the eigenvalues $\sigma(\mu)=\xi(\mu)+\mathrm{i}\eta(\mu)$ and eigenvectors $\boldsymbol{\zeta}(\mu)$ of $\boldsymbol{A}(\mu)$ are complex conjugates, and

$$\sigma(\mu)\boldsymbol{\zeta}=\boldsymbol{A}\zeta \qquad (\sigma\zeta_i=A_{ij}\zeta_j)\ ,\tag{3.73}$$

and

$$\sigma(\mu)\bar{\boldsymbol{\zeta}}^*=\boldsymbol{A}^{\mathrm{T}}\bar{\boldsymbol{\zeta}}^*\ ,\tag{3.74}$$

where $\boldsymbol{\zeta}^*$ is the adjoint eigenvector with eigenvalue $\bar{\sigma}(\mu)$ in the scalar product, $\langle \boldsymbol{x},\boldsymbol{y}\rangle=\boldsymbol{x}\cdot\bar{\boldsymbol{y}}$. We may normalize so that

$$\begin{aligned}&\langle\zeta,\zeta^*\rangle=\zeta\cdot\bar{\zeta}^*=\zeta_k\bar{\zeta}_k^*=1\ ,\\ &\langle\zeta,\bar{\zeta}^*\rangle=\zeta_k\zeta_k^*=0\ .\end{aligned}\tag{3.75}$$

The eigenvalues $\sigma(\mu)=\xi(\mu)+\mathrm{i}\eta(\mu)$ arise in the spectral problem for the stability of the solution $x_i=0$ of (3.71). We suppose that this loss of stability occurs at $\mu=0$ so that $\xi(0)=0$. We will get bifurcation into periodic solutions if

$$\eta(0)=\omega_0\neq 0 \quad\text{and}\quad \xi_\mu(0)\neq 0\,[\text{say } \xi_\mu(0)>0]\ .\tag{3.76}$$

To prove bifurcation into periodic solutions under conditions (3.76), we note that ζ and $\bar{\zeta}$ are independent so that any real-valued two-dimensional vector $\boldsymbol{x}=(x_1, x_2)$ may be represented as

$$x_i = a(t)\zeta_i + \bar{a}(t)\bar{\zeta}_i \,. \tag{3.77}$$

Substitute (3.77) in (3.71) and use (3.73) to find

$$\begin{aligned} \mathring{a}\zeta_i + \mathring{\bar{a}}\bar{\zeta}_i &= \sigma(\mu)\zeta_i + \bar{\sigma}(\mu)\bar{\zeta}_i + a^2 k_{ijk}\zeta_j\zeta_k \\ &\quad + 2|a|^2 k_{ijk}\zeta_i\bar{\zeta}_k + \bar{a}^2 k_{ijk}\bar{\zeta}_i\bar{\zeta}_k \\ &\quad + O(|a|^3)\,. \end{aligned} \tag{3.78}$$

The orthogonality properties (3.75) are now employed to reduce (3.78) to a single, complex-valued, amplitude equation

$$\mathring{a} = f(\mu, a) = \sigma(\mu)a + \alpha(\mu)a^2 + 2\beta(\mu)|a|^2 + \gamma(\mu)\bar{a}^2 + O(|a|^3)\,, \tag{3.79}$$

where, for example, $\alpha(\mu) = k_{ijk}(\mu)\zeta_j\zeta_k\bar{\zeta}_i^*$. The linearized stability of the solution $a=0$ of (3.79) is determined by $\mathring{a}=\sigma(\mu)a$, $a=(\text{constant})\times\exp[\sigma(\mu)\mathrm{t}]$. At criticality ($\mu=0$), $a=(\text{constant})\times\exp(\mathrm{i}\omega_0 t)$ is 2π periodic in $s=\omega_0 t$.

A bifurcating time-periodic solution may be constructed from the solution of the linearized problem at criticality. This bifurcating solution is in the form

$$a(t) = b(s, \varepsilon)\,, \quad s = \omega(\varepsilon)t\,, \quad \omega(0) = \omega_0\,, \quad \mu = \mu(\varepsilon)\,, \tag{3.80}$$

where ε is the amplitude of a defined by

$$\varepsilon = \frac{1}{2\pi}\int_0^{2\pi} \exp(-\mathrm{i}s) b(s, \varepsilon) ds = [b]\,. \tag{3.81}$$

The solution (3.80) of (3.79) is unique to within an arbitrary translation of the origin of the time. This means that under translation $t \to t + c$ the solution $b(s, \varepsilon) \to b[s + c\omega(\varepsilon), \varepsilon]$ shifts its phase. This unique solution is analytic in ε when $f(\mu, a)$ is analytic in both variables and it may be expressed as a series

$$\begin{pmatrix} b(s, \varepsilon) \\ \omega(\varepsilon) - \omega_0 \\ \mu(\varepsilon) \end{pmatrix} = \sum_{n=1}^{\infty} \varepsilon^n \begin{pmatrix} b_n(s) \\ \omega_n \\ \mu_n \end{pmatrix}. \tag{3.82}$$

The perturbation problems which govern $b_n(s)$, ω_n, and μ_n can be obtained by identifying the coefficients of ε^n which arise when (3.82) is substituted into the two equations: $\omega\mathring{b}=f(\mu, b)$ and $\varepsilon=[b]$. We find that at order one

$$\omega_0\mathring{b}_1-\mathrm{i}\omega_0 b_1=0\,,\qquad [b_1]=1\,,\qquad b_1(s)=\mathrm{e}^{\mathrm{i}s}\,. \tag{3.83}$$

At order two, we find that $[b_2]=0$ and

$$\omega_0[\mathring{b}_2-\mathrm{i}b_2]+\omega_1\mathring{b}_1=\mu_1\sigma_\mu b_1+\alpha_0 b_1^2+2\beta_0|b_1|^2+\gamma_0\bar{b}_1^{\,2}\,, \tag{3.84}$$

where $\sigma_\mu=d\sigma(0)/d\mu$ and, for example, $\alpha_0=\alpha(0)$.

Equations of the form $\mathring{b}(s)-\mathrm{i}b(s)=f(s)=f(s+2\pi)$ are solvable for $b(s)=b(s+2\pi)$ if and only if $f(s)$ has no term proportional to $\exp(\mathrm{i}s)$. Hence

$$\mu_1=\omega_1=0 \quad \text{in (3.84)} \tag{3.85}$$

and

$$\mathring{b}_2-\mathrm{i}b_2=[\alpha_0\exp(2\mathrm{i}s)+2\beta_0+\gamma_0\exp(-2\mathrm{i}s)]/\omega_0\,. \tag{3.86}$$

We find that

$$b_2(s)=(\alpha_0\mathrm{e}^{2\mathrm{i}s}-2\beta_0-\gamma_0\mathrm{e}^{-2\mathrm{i}s}/3)/\mathrm{i}\omega_0\,. \tag{3.87}$$

The problem which governs at order 3 is $[b_3]=0$ and

$$\begin{aligned}\mathring{b}_3-\mathrm{i}b_3=[&-\omega_2\mathring{b}_1+\mu_2\sigma_\mu b_1+2\alpha_0 b_1 b_2\\&+2\beta_0(b_1\bar{b}_2+\bar{b}_1 b_2)+2\gamma_0\bar{b}_1\bar{b}_2]/\omega_0\,.\end{aligned} \tag{3.88}$$

To solve (3.88) we must eliminate terms proportional to $\exp(\mathrm{i}s)$ from the right side of (3.88). This is done if

$$\mathrm{i}\omega_2-\mu_2\sigma_\mu=(4\alpha_0\beta_0-4|\beta_0|^2-2\alpha_0\beta_0-2|\gamma_0|^2/3)/\mathrm{i}\omega_0\,. \tag{3.89}$$

The real part of (3.89) is solvable for μ_2 provided that $\xi'\neq 0$. The imaginary part of (3.89) is always solvable for ω_2.

Proceeding to higher orders, it is easy to verify that all of the perturbation problems are solvable when (3.76) holds and, in fact, $\omega(\varepsilon)=\omega(-\varepsilon)$, $\mu(\varepsilon)=\mu(-\varepsilon)$ are even functions. It follows that periodic solutions which bifurcate from steady solutions bifurcate to one or the other side of criticality and never to both sides; periodic bifurcating solutions cannot undergo two-sided or transcritical bifurcation (cf. Figs. 3.14 and 3.7c).

We now search for the conditions under which the bifurcating periodic solutions are stable. We consider a small disturbance $z(t)$ of $b(s,t)$. Setting $a(t)=b(s,\varepsilon)+z(t)$ in (3.79) we find the linearized equation $\mathring{z}(t)=f_a[\mu(\varepsilon), b(s,\varepsilon)]$ $z(t)$ where $f_a=\partial f/\partial a$ and $s=\omega(\varepsilon)t$. Then, using Floquet theory (see [3.51]), we

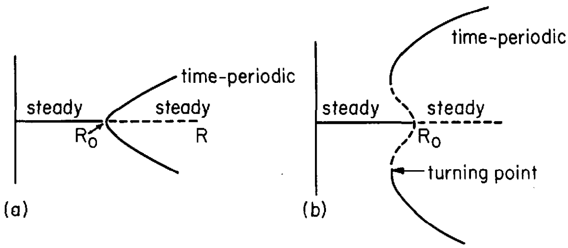


Fig. 3.14a, b. Supercritical (**a**) and subcritical (**b**) Hopf bifurcation. The part of the bifurcation diagram giving the values $R(\varepsilon)$ for which periodic solutions exist are symmetric, $R(\varepsilon)=R(-\varepsilon)$. For Navier-Stokes problems we always expect turning points at which the sign of the slope of $R(\varepsilon)$ changes, as in (**b**). It is not possible to have a periodic solution of a problem with steady data when R is small (see Sect. 3.2). One example of supercritical Hopf bifurcation (**a**) in hydrodynamics occurs in the problem of loss of stability of the steady Ekman layer [3.52]. The problem of loss of stability of Poiseuille flow is associated with subcritical Hopf bifurcation of type (**b**) (see Chap. 7)

set $z(t)=\exp(\gamma t)y(s)$ where $y(s)=y(s+2\pi)$ and find that

$$\gamma y=-\mathring{y}+f_a(\mu,b)y\equiv J[\mu(\varepsilon),b(s,\varepsilon)]y\ . \tag{3.90}$$

The stability result we need may be stated as a factorization theorem. To prove this theorem we use the fact that $\gamma=0$ is always an eigenvalue of J with eigenfunction $\mathring{b}(s,\varepsilon)$,

$$J\mathring{b}=0\ , \tag{3.91}$$

and the relation

$$\omega_\varepsilon(\varepsilon)\mathring{b}(s,\varepsilon)=\mu_\varepsilon(\varepsilon)f_\mu[\mu(\varepsilon),b(s,\varepsilon)]+Jb_\varepsilon\ , \tag{3.92}$$

which arises from differentiating $\omega\mathring{b}=f(\mu,b)$ with respect to ε at any ε.

Factorization Theorem:

$$\left.\begin{cases} y(s,\varepsilon)=c(\varepsilon)\left[\dfrac{\tau}{\gamma}\mathring{b}(s,\varepsilon)+b_\varepsilon(s,\varepsilon)+\mu_\varepsilon(\varepsilon)q(s,\varepsilon)\right]\ , \\ \tau(\varepsilon)=\omega_\varepsilon(\varepsilon)+\mu_\varepsilon(\varepsilon)\hat{\tau}(\varepsilon)\ , \\ \gamma(\varepsilon)=\mu_\varepsilon(\varepsilon)\hat{\gamma}(\varepsilon)\ , \end{cases}\right\} \tag{3.93}$$

where $c(\varepsilon)$ is an arbitrary constant and $q(s,\varepsilon)=q(s+2\pi,\varepsilon)$, $\hat{\tau}(\varepsilon)$, and $\hat{\gamma}(\varepsilon)$ satisfy the equation

$$\hat{\tau}\mathring{b}+\hat{\gamma}b_\varepsilon+f_\mu(\mu,b)+\varepsilon(\gamma q-Jq)=0 \tag{3.94}$$

and are real analytic functions in a neighborhood of $\varepsilon=0$. *Moreover,* $\hat{\tau}(\varepsilon)$ *and* $\hat{\gamma}(\varepsilon)/\varepsilon$ *are even functions and such that*

$$\hat{\gamma}_\varepsilon(0)=-\xi_\mu(0)\,, \qquad \hat{\tau}(0)=-\eta_\mu(0)\,. \tag{3.95}$$

Proof: Substitute the representations (3.93) into (3.90) utilizing (3.91) to eliminate $J\overset{\circ}{b}$ and (3.92) to eliminate Jb_ε. This leads to (3.94), and (3.94) may be solved by series

$$\begin{pmatrix} q(s,\varepsilon) \\ \hat{\gamma}(\varepsilon)/\varepsilon \\ \hat{\tau}(\varepsilon) \end{pmatrix} = \sum_{l=0} \begin{pmatrix} q_l(s) \\ \hat{\gamma}_l \\ \hat{\tau}_l \end{pmatrix} \varepsilon^l\,, \tag{3.96}$$

where $\hat{\gamma}_0=\hat{\gamma}_\varepsilon(0)$ and $\hat{\tau}_0=\hat{\tau}(0)$. Using the fact that to the lowest order, $b=\varepsilon\exp(\mathrm{i}s)$, $\gamma=0(\varepsilon^2)$ and [from (3.79)] $f_\mu(\mu,\mathrm{b})=\sigma_\mu(0)\exp(\mathrm{i}s)\varepsilon$, we find that

$$\exp(\mathrm{i}s)[\hat{\tau}(0)-\hat{\gamma}_\varepsilon(0)-\sigma_\mu]+J_0q_0=0\,. \tag{3.97}$$

Equation (3.97) is solvable for $q_0(s)=q_0(s+2\pi)$ if and only if the term in the bracket vanishes; that is, (3.95) holds. The remaining properties asserted in the theorem may be obtained by mathematical induction using the power series (3.96) (see [Ref. 3.1, Chap. 2]).

The linearized stability of the periodic solution for small values of ε can now be obtained from the spectral problem: $\boldsymbol{x}(s,\varepsilon)=\boldsymbol{x}(s+2\pi,\varepsilon)$ is stable when $\xi(\varepsilon)<0$ and is unstable when $\xi(\varepsilon)>0$ where

$$\begin{aligned}\gamma(\varepsilon)&=\xi(\varepsilon)+\mathrm{i}\eta(\varepsilon)=\mu_\varepsilon(\varepsilon)\hat{\gamma}(\varepsilon)\\ &=-\mu_\varepsilon(\varepsilon)\{[\xi_\mu(0)+\mathrm{i}\eta_\mu(0)]\varepsilon+O(\varepsilon^3)\}\,.\end{aligned} \tag{3.98}$$

We have already assumed that the basic flow loses stability strictly when μ is increased past zero, $\xi_\mu(0)>0$. So the branches for which $\mu_\varepsilon(\varepsilon)>0$ are stable and the ones for which $\mu_\varepsilon(\varepsilon)\varepsilon<0$ are unstable. There are two possibilities when ε is small: supercritical bifurcation (Fig. 3.14a) [3.52] or subcritical bifurcation (Fig. 3.14b). It is not possible to have transcritical periodic bifurcations as in Fig. 3.7c because $\mu(\varepsilon)=\mu(-\varepsilon)$.

3.8 Finite Dimensional Projections

You may think that the analysis of stability and bifurcation in one and two dimensions is merely suggestive of some typical hydrodynamical situations. In fact, the analysis applies strictly to general problems of bifurcation of solutions of the Navier-Stokes equations when certain typical conditions are satisfied. The typical conditions I have in mind are those associated with bifurcation at a simple eigenvalue; they are precisely described in the paragraph two up from (3.19) and in the paragraph one down from (3.21). The one- and two-dimensional problems arise as finite-dimensional projections of the evolution equation (3.14–16) by a procedure which I will now describe.

First we define a scalar product

$$\langle \boldsymbol{a}, \boldsymbol{b} \rangle \equiv \int \boldsymbol{a} \cdot \bar{\boldsymbol{b}} \, d\boldsymbol{r} = \langle \overline{\boldsymbol{b}, \boldsymbol{a}} \rangle \,. \tag{3.99}$$

Associated with this scalar product is a linear operator $F_u^*(0, R|\cdot)$ which is called the adjoint of $F_u(0, R|\cdot)$ and is defined by the relation

$$\langle F_u(0, R|\boldsymbol{a}), \boldsymbol{b} \rangle = \langle \boldsymbol{a}, F_u^*(0, R|\boldsymbol{b}) \rangle \tag{3.100}$$

for all fields $\boldsymbol{a}$ and $\boldsymbol{b}$ in H. It is readily verified that

$$\langle \nabla\phi, \boldsymbol{b} \rangle = \langle \nabla \cdot (\boldsymbol{b}\phi) - \phi \nabla \cdot \boldsymbol{b} \rangle = 0 \tag{3.101}$$

for all $\boldsymbol{b} \in H$ and any scalar field ϕ. It follows now from (3.101) and (3.18) that

$$\langle F_u(0, R|\zeta), \boldsymbol{b} \rangle = \sigma \langle \zeta, \boldsymbol{b} \rangle \tag{3.102}$$

for all $\boldsymbol{b} \in H$. An adjoint eigenvector ζ^* may be associated with the second member of (3.100)

$$\langle \boldsymbol{a}, F_u^*(0, R|\zeta^*) \rangle = \sigma \langle \boldsymbol{a}, \zeta^* \rangle = \langle \boldsymbol{a} \bar{\sigma} \zeta^* \rangle \tag{3.103}$$

for all $\boldsymbol{a} \in H$, so that

$$\bar{\sigma}\zeta^* = F_u^*(0, R|\zeta^*) - \nabla p[\zeta^*] \,, \qquad \zeta^* \in H \tag{3.104}$$

is the eigenvector problem adjoint to (3.18). It has exactly the same set of eigenvalues as (3.18) [cf. (3.74)].

There is an infinite number of eigenvalues $\sigma(R)$ which may be arranged in a sequence corresponding to the size of their real parts

$$\xi_1 \geqq \xi_2 \geqq \xi_3 \geqq \dots \geqq \xi_n \geqq \dots \tag{3.105}$$

clustering at $-\infty$ (see Fig. 3.1). To each eigenvalue there corresponds at most a finite number of eigenvectors ζ_n and adjoint eigenvectors ζ_n^*.

An eigenprojection is the scalar product of a field in H with one of adjoint eigenvectors. So we may form eigenprojections of the evolution equation (3.14–16)

$$\begin{aligned}\frac{d\langle \boldsymbol{u},\zeta_n^*\rangle}{dt} &= \langle F(u,R),\zeta_n^*\rangle \\ &= \langle F_u(0,R|\boldsymbol{u}),\zeta_n^*\rangle - \langle \boldsymbol{u}\cdot\nabla \boldsymbol{u},\zeta_n^*\rangle\,, \\ &= \langle \boldsymbol{u},F_u^*(0,R|\zeta_n^*)\rangle - \langle \boldsymbol{u}\cdot\nabla \boldsymbol{u},\zeta_n^*\rangle\,, \\ &= \sigma_n\langle \boldsymbol{u},\zeta_n^*\rangle - \langle \boldsymbol{u}\cdot\nabla \boldsymbol{u},\zeta_n^*\rangle\,. \end{aligned} \tag{3.106}$$

There are as many projections as there are adjoint eigenvectors. It is evident that when $\boldsymbol{u}$ is small

$$\langle \boldsymbol{u}(t),\zeta_n^*\rangle \simeq \langle \boldsymbol{u}(0),\zeta_n^*\rangle e^{\xi_n(R)t}e^{i\eta_n(R)t} \tag{3.107}$$

so that all the projections with $\xi(R)<0$ decay to zero. It follows that the important parts of the evolution problem (3.18) are the eigenprojections associated with eigenvalue $\sigma_n(R)$ for which $\xi_n(R)>0$.

In the problem of bifurcation at a simple eigenvalue we suppose that $\xi_n(R)<0$ when $n>1$. The important eigenprojection is then the one associated with $\zeta_1^*\equiv\zeta^*$ and $\sigma_1\equiv\sigma$,

$$\frac{d\langle \boldsymbol{u},\zeta^*\rangle}{dt} = \langle F(\boldsymbol{u},R),\zeta^*\rangle = \sigma(R)\langle \boldsymbol{u},\zeta^*\rangle - \langle \boldsymbol{u}\cdot\nabla \boldsymbol{u},\zeta^*\rangle\,. \tag{3.108}$$

If $\sigma(R)$ is real valued, (3.108) is a one-dimensional problem. If $\sigma(R)$ is complex values, there are two equations, the real and imaginary parts of (3.108), and the problem is essentially two-dimensional

I now want to delineate the sense in which the essentially two-dimensional problem is strictly two dimensional. We first decompose the bifurcating solution $\boldsymbol{u}$ into a real-valued orthogonal sum

$$\boldsymbol{u} = a(t)\zeta + \bar{a}(t)\bar{\zeta} + \boldsymbol{w}\,, \tag{3.109}$$

where

$$\langle \zeta,\zeta^*\rangle - 1 = \langle \zeta,\bar{\zeta}^*\rangle = \langle \boldsymbol{w},\zeta^*\rangle = 0\,. \tag{3.110}$$

Substitute (3.108) into (3.14–16) and use (3.18) to derive

$$[\mathring{a} - \sigma(R)a]\zeta + [\mathring{\bar{a}} - \bar{\sigma}(R)\bar{a}]\bar{\zeta} + \frac{\partial \boldsymbol{w}}{\partial t} = F_u(O,R|\boldsymbol{w}) - \boldsymbol{u}\cdot\nabla \boldsymbol{u} - \nabla p[\boldsymbol{w}]\,, \tag{3.111}$$

where $\nabla p[\boldsymbol{w}]=\nabla(p[\boldsymbol{u}]-ap[\zeta]-\bar{a}p[\bar{\zeta}])$. Now project (3.111) with ζ^*, use (3.109),

$$\left\langle\frac{\partial \boldsymbol{w}}{\partial t},\zeta^*\right\rangle=\frac{d\langle \boldsymbol{w},\zeta^*\rangle}{dt}=0\ ,$$
$$\langle F_u(0,R|\boldsymbol{w}),\zeta^*\rangle=\langle \boldsymbol{w},F_u^*(0,R|\zeta^*)\rangle=\sigma\langle \boldsymbol{w},\zeta^*\rangle=0\ , \tag{3.112}$$

and show that

$$\begin{aligned}\mathring{a}-\sigma(R)a=-\langle \boldsymbol{u}\cdot\nabla \boldsymbol{u},\zeta^*\rangle&=\alpha(R)a^2+2\beta(R)|a|^2+\gamma(R)\bar{a}^2\\&-[a\langle(\zeta\cdot\nabla \boldsymbol{w}+\boldsymbol{w}\cdot\nabla\zeta),\zeta^*\rangle+\bar{a}\langle(\bar{\zeta}\cdot\nabla \boldsymbol{w}+\boldsymbol{w}\cdot\nabla\bar{\zeta}),\zeta^*\rangle]\\&-\langle \boldsymbol{w}\cdot\nabla \boldsymbol{w},\zeta^*\rangle\ ,\end{aligned} \tag{3.113}$$

where

$$\begin{aligned}\alpha(R)&=-\langle\zeta\cdot\nabla\zeta,\zeta^*\rangle\ ,\\2\beta(R)&=-\langle(\bar{\zeta}\cdot\nabla\zeta+\bar{\zeta}\cdot\nabla\zeta),\zeta^*\rangle\ ,\\\gamma(R)&=-\langle\bar{\zeta}\cdot\nabla\bar{\zeta},\zeta^*\rangle\ .\end{aligned} \tag{3.114}$$

Returning now to (3.111) with (3.113), we find that

$$\frac{\partial \boldsymbol{w}}{\partial t}=F_u(0,R|\boldsymbol{w})-(\boldsymbol{u}\cdot\nabla \boldsymbol{u}-\langle \boldsymbol{u}\cdot\nabla \boldsymbol{u},\zeta^*\rangle\zeta-\langle \boldsymbol{u}\cdot\nabla \boldsymbol{u},\bar{\zeta}^*\rangle\bar{\zeta})-\nabla p[\boldsymbol{w}]\ , \tag{3.115}$$

and, using (3.108), we conclude from (3.115) that $\boldsymbol{w}=O(|a|^2)$. Equation (3.113) governs the evolution of the projection of the solution $\boldsymbol{u}$ into the eigensubspace belonging to the eigenvalue $\sigma(R)=\sigma(R)$, and (3.115) governs the evolution of the part of the solution orthogonal to ζ^*.

In bifurcation problems the complementary projection $\boldsymbol{w}$ plays a minor role; it arises only as a response generated by nonlinear coupling to the component of the solution spanned by ζ and $\bar{\zeta}$. Since $\boldsymbol{w}=O(|a|^2)$, we may dramatize the two-dimensional structure by comparing (3.113), written as

$$\mathring{a}=\sigma(R)a+\alpha(R)a^2+2\beta(R)|a|^2+\gamma(R)\bar{a}^2+O(|a|^3)\ , \tag{3.116}$$

with the equation (3.79) which governs in the strictly two-dimensional problem.

The nature of the relatively unimportant variation of the present problem with $\boldsymbol{w}\neq 0$ from the strictly two-dimensional problem studied in Sect. 3.7 can be best appreciated by carrying out an explicit computation of Hopf bifurcation starting from (3.113) and (3.115). We seek a 2π periodic solution $\boldsymbol{u}(s,\varepsilon)=\boldsymbol{u}(s+2\pi,\varepsilon)$, where $s=\omega(\varepsilon)t$ and $R=R(\varepsilon)$, $\boldsymbol{z}^*=\exp(\mathrm{i}s)\zeta^*$, and

$$\varepsilon=[\boldsymbol{u}(s,\varepsilon),\boldsymbol{z}^*(s)]\equiv\frac{1}{2\pi}\int_0^{2\pi}\langle \boldsymbol{u}(s,\varepsilon),\boldsymbol{z}^*\rangle ds\ . \tag{3.117}$$

Since the solution is to bifurcate from $\boldsymbol{u}(s,0)=0$ when $R(0)=R_0$ and $\omega(0)=\omega_0$, we may set $a(t)=b(s,\varepsilon)=\varepsilon B(s,\varepsilon)$, where $B(s,0)=b_1(s)$ is bounded. Equation (3.115) then shows that

$$\boldsymbol{w}(s,\varepsilon)\Rightarrow\varepsilon^2\boldsymbol{w}(s,\varepsilon)\ , \tag{3.118}$$

so that (3.108) becomes

$$\boldsymbol{u}=\varepsilon[B(s,\varepsilon)\zeta+\bar{B}(s,\varepsilon)\bar{\zeta}]+\varepsilon^2\boldsymbol{w}(s,\varepsilon)\ , \tag{3.119}$$

where $[\boldsymbol{w},\boldsymbol{z}^*]=0$. It follows from (3.119), (3.117), and (3.109) that

$$1=\frac{1}{2\pi}\int_0^{2\pi}\mathrm{e}^{-is}B(S,\varepsilon)ds,\equiv[B]\ , \tag{3.120}$$

which is exactly the normalizing condition (3.81). Moreover, (3.113) becomes

$$\begin{aligned}\omega(\varepsilon)\mathring{B}-\sigma[R(s)]B=\varepsilon[&\alpha(R)B^2+2\beta(R)|B|^2+\gamma(R)\bar{B}^2]\\ &-\varepsilon^3[B\langle(\zeta\cdot\nabla\boldsymbol{w}+\boldsymbol{w}\cdot\nabla\zeta),\zeta^*\rangle\\ &+\bar{B}\langle(\bar{\zeta}\cdot\nabla\boldsymbol{w}+\boldsymbol{w}\cdot\nabla\bar{\zeta}),\zeta^*\rangle]\\ &-\varepsilon^5\langle\boldsymbol{w}\cdot\nabla\boldsymbol{w},\zeta^*\rangle\end{aligned} \tag{3.121}$$

and (3.115) becomes

$$\begin{aligned}\omega(\varepsilon)\frac{\partial\boldsymbol{w}}{\partial s}&=F_u[0,R(\varepsilon)|\boldsymbol{w}]-M(B\zeta+\bar{B}\bar{\zeta}+\varepsilon\boldsymbol{w},B\zeta+\bar{B}\bar{\zeta}+\varepsilon\boldsymbol{w})\\ &\quad-\nabla p[\boldsymbol{w}]\ ,\end{aligned} \tag{3.122}$$

where

$$M(\boldsymbol{u},\boldsymbol{u})=\boldsymbol{u}\cdot\nabla\boldsymbol{u}-\langle\boldsymbol{u}\cdot\nabla\boldsymbol{u},\zeta^*\rangle\zeta-\langle\boldsymbol{u}\cdot\nabla\boldsymbol{u},\bar{\zeta}^*\rangle\bar{\zeta}\ . \tag{3.123}$$

We may construct the solution of (3.121) and (3.122) in a series equivalent to (3.82)

$$\begin{aligned}B(s,\varepsilon)&=b_1(s)+\varepsilon b_2(s)+\varepsilon^2 b_3(s)+\dots\\ \omega(\varepsilon)&=\omega_0+\omega_1\varepsilon+\omega_2\varepsilon^2+\dots\\ R(\varepsilon)&=R_0+R_1\varepsilon+R_2\varepsilon^2+\dots\\ \boldsymbol{w}(s,\varepsilon)&=\boldsymbol{w}_0(s)+\varepsilon\boldsymbol{w}_1(s)+\dots\ .\end{aligned} \tag{3.124}$$

Substituting these representations into (3.121) and (3.122), we find that $\omega_1=R_1=0$ and $b_1(s)$, $b_2(s)$, $b_3(s)$, ω_2, and R_2 are exactly as given in Sect. 3.7 (with μ replaced by R). The function $\boldsymbol{w}(s,\varepsilon)$ first enters into the analysis of $O(\varepsilon^3)$.

The equations governing the coefficients $\boldsymbol{w}_n(s)=\boldsymbol{w}_n(s+2\pi)$ are in the form

$$\omega_0 \frac{\partial \boldsymbol{w}_n}{\partial s} - \boldsymbol{F}_u(0, R_0|\boldsymbol{w}_n) = \boldsymbol{g}_n - \nabla p[\boldsymbol{w}_n] , \tag{3.125}$$

where $\boldsymbol{g}_n$ depends on lower order terms; for example,

$$\boldsymbol{g}_0 = -M(b_1\zeta + \bar{b}_1\bar{\zeta}, b_1\zeta + \bar{b}_1\bar{\zeta}) . \tag{3.126}$$

The necessary and sufficient condition for the solvability (3.125) is that $\boldsymbol{g}_n$ should be orthogonal to 2π periodic eigenvectors $\boldsymbol{z}^* = \exp(\mathrm{i}s)\zeta^*$ and $\bar{\boldsymbol{z}}^*$ of the operator

$$\omega_0 \frac{\partial}{\partial s} + \boldsymbol{F}_u^*(0, R_0|\,\cdot\,) , \tag{3.127}$$

which is adjoint to the operator on the left of (3.125) relative to the scalar product $[\,\cdot\,,\,\cdot\,]$ defined by (3.117). It is easy to verify that the problems (3.125) are uniquely solvable among vectors for which $[\boldsymbol{w}_n, \boldsymbol{z}^*]=0$.

All of the results of bifurcation theory, hydrodynamic or otherwise, are based on the method of projections. In mathematically strict discussions of projections it is necessary to introduce concepts from functional analysis. The end result of the application of functional analysis to partial differential equations is a reduction to problems of finite dimension. The existence and properties of bifurcation are associated with the finite dimensional projections and require classical analysis of functions and ordinary differential equations rather than functional analysis for their elucidation.

We have studied problems which could be reduced to one or two dimensions. It is also possible to consider multiple eigenvalue problems of multiplicity $N>2$. Such problems are N dimensional and in the case of symmetry-breaking bifurcations of steady uniform solutions into steady solutions they may be reduced to the study of nonlinear algebraic equations. Such breaking of symmetry is well known in the Bénard problem between infinite planes. *Sattinger* [3.53] has shown how to use group theory to solve such problems.

The problems which arise when $N>2$ in genuinely dynamic problems are much more difficult than in the symmetry-breaking bifurcations at eigenvalues of higher multiplicity. For example, the nonlinear problem of fluid convection which *Lorenz* [3.54] approximated with $N=3$ gives rise to very complicated dynamics, characterized by nonperiodic attractors with an exotic structure. It is possible to get interesting models of complicated dynamics, including turbulent-like behavior, with finite systems, of N nonlinear ordinary differential equations with $N \geqq 3$. For example, *Curry*'s [3.55] $N=14$ mode truncation of the system of convection equations which Lorenz truncated at $N=3$ gives a sequence of bifurcations which is in remarkable agreement with recent experiments of *Gollub* and *Benson* [3.20]. The same sequence of bifurcations is predicted by Curry's model and observed in the following experiments: bifurcation of a stationary solution into a time-periodic solution; bifurcation of the T periodic solution into another $2T$ periodic solution (a subharmonic solution); bifurcation of the $2T$ periodic solution into a quasi-periodic solution with two frequencies, and bifurcation of the quasi-periodic solution into a nonperiodic, turbulent like solution. The investigation of nonperiodic attractors generated by systems of nonlinear ordinary differential equations in $\mathbb{R}^N$, with N not too large, may have a big potential for understanding turbulence. (For further discussion see Chap 4 and [3.20].)

3.9 Bifurcation, Stability, and Transition in Poiseuille and Couette Flows

The bifurcation diagram for spatially periodic, time-periodic, axisymmetric solutions of the problem of Poiseuille flow through the annulus between two concentric cylinders is probably like those shown in Fig. 3.14b. This problem was studied and numerical calculations were carried out for small values of ε by *Joseph* and *Chen* [3.56] under the assumption that the bifurcating solution was axisymmetric and spatially periodic along the axis of the annulus. The Reynolds number $R(\varepsilon,\eta)=R(-\varepsilon,\eta)$ and the frequency $\omega(\varepsilon,\eta)=\omega(-\varepsilon,\eta)$ depend on the amplitude ε and radius ratio $\eta=a/b\leqq 1$, where a is the inner and b is the outer radius of the cylinders. The critical Reynolds numbers $R_0(\eta)=R(0,\eta)$ of the linearized theory of stability [3.57] depend strongly on η and $R_0(\eta)\to\infty$ as $\eta\to 0$. It is not known if the limiting flow $\eta=a/b\to 0$ is the same as Hagen-Poiseuille flow ($a=0$), but in both cases $R_0\to\infty$ and the time-periodic bifurcating solution is isolated from the laminar solution ($\varepsilon=0$).

When the gap between the cylinders is very small, $\eta\to 1$, the effects of curvature of the walls decreases to zero and the flow reduces to plane Poiseuille flow between two, infinitely extended, parallel plates. The problem of stability and bifurcation of plane Poiseuille flow has been studied by many authors (see [3.1, 58, 59] and Chap. 7). For plane flows a well-known theorem of Squires (see [3.60]) asserts that among all the critical spatially quasi-periodic disturbances a two-dimensional disturbance, in the plane of the flow, has the smallest critical value of R. This means that the most unstable disturbance in the annulus is axisymmetric when $\eta\to 1$. There is no special reason to believe that axisymmetric are most destabilizing when $\eta\neq 1$.

If we define $R=U_{\max}(b-a)/2\nu$, then $R(0,\eta)$ varies monotonically between $R(0,1)=5800$ and $R(0,0)=\infty$ when $R(0,\eta)$ is computed for axisymmetric disturbance. Axisymmetric disturbances are the most unstable when $\eta\to 1$ and $R(0,0)$ is probably infinite even when computed on nonaxisymmetric disturbances. Unlike the critical values of the linear theory of stability, the observed limit of stability to natural disturbances is not sensitive to η and seems to be fixed somewhere between 1000 and 1500 (see [Ref. 3.1, Fig. 35.1]).

It is of interest to discuss the two limiting cases $\eta \to 1$ and $\eta \to 0$ separately. We shall take the case $\eta \to 0$ as representative of the class of flows in which there is no finite critical value of $R(0, 0)$. In all such flows there is a direct transition to turbulence sharing common properties which I will describe later. The flows which appear to be in this class are the Poiseuille flows through an annular pipe in the limiting case $\eta \to 0$; Hagen-Poiseuille flow, $\eta = 0$; the flow between infinitely long rotating cylinders when the inner one is at rest (see Sect. 3.4); and plane Couette flow which is a limiting case of the shearing flow between cylinders when $\eta \to 1$ and the difference in the speed of the two cylinders Ω is reduced so that $\Omega/(1-\eta)$ is finite. In such flows the loss of stability is necessarily due to a disturbance of finite amplitude since infinitesimal disturbances may be presumed to decay at all finite Reynolds numbers. There do not appear to be strong differences between flows which have no finite linear stability limit and the flows close to them which have very large critical Reynolds numbers, for example, in pipe flow with small nonzero values of η or in the flow between rotating cylinders of finite height when the inner cylinder rotates and the outside, top, and bottom of the cylinders are stationary. The flow between rotating spheres when the inner one is at rest probably falls into the class of flows with large finite critical Reynolds numbers that undergo a direct transition to turbulence.

The subcritical finite-amplitude motions which occur in the cases when there is no finite critical Reynolds number seem to be characterized by the spatial segregation of the flow into laminar and turbulent patches as in the Poiseuille flow shown in Fig. 3.2 or the Couette flow shown in Fig. 3.3.

In the motions close to these, with finite large critical values of the Reynolds number, we also get subcritical direct transition to turbulence and, in addition, we may compute a subcritical, unstable, time-periodic bifurcation of the type described in Chaps. 7 and 8. We may regard Poiseuille flow between cylinders as representative of shearing flows which are linearly stable at high Reynolds numbers, and even the most studied case of plane Poiseuille flow with $\eta = 1$ is in this class. For such flows the unstable time-periodic solutions on the small ε part of the bifurcation curve shown in Fig. 3.14b)the dotted lines with $\varepsilon \neq 0$ near R_0) have been observed in the experiments of *Nishioka* et al. [3.61]. Their experiments were for the case $\eta \to 1$ and they found the unstable time-periodic bifurcating solution as a "metastable" or slowly changing transient state.

We could claim perfect agreement between bifurcation theory and experiments if the observed solutions on the stable large amplitude branch of the bifurcation curve in Fig. 3.14b were time periodic with the predicted values of $R(\varepsilon, \eta)$ and the frequencies $\delta(\varepsilon, \eta)$. In fact stable time-periodic solutions are not observed; instead, we see direct snap-through instability to turbulence. The reason for this discrepancy is that the stability analysis which leads to the conclusion that the upper branch of time-periodic bifurcating Poiseuille flow is stable is insufficiently general because it is carried out in a too-restricted set of disturbances in which three-dimensional disturbances and spatially aperiodic disturbances are arbitrarily excluded.

The difficulties involved in arriving at a correct mathematical interpretation of the mechanisms involved in the observed instability, bifurcation, and transition of shearing flows are enormous. Even the linear stability problem is difficult and proofs of various important points, like the linear stability of laminar flow in pipes at all finite values of the Reynolds number, have yet to be established in a mathematically secure way. Of course, the nonlinear problem is even more difficult and nearly all analytical results are restricted to small amplitudes. This type of restriction is especially serious for problems like the ones under discussion where, in the already cited words of Reynolds, "the condition might be one of instability for disturbances of a certain magnitude, and stable for smaller disturbances".

So it is nearly if not strictly true to say that the ordinary analytical methods of nonlinear stability theory and bifurcation theory have been unavailing for the problems of stability and transition in shear flows. Numerical methods have been only slightly more successful until recently. The recent numerical study of *Orszag* and *Kells* [3.59] of the fully nonlinear initial value problem for the stability of plane Poiseuille flow and plane Couette flow appears to represent a real breakthrough. The study of *Orszag* and *Kells* utilizes spectral, Galerkin-type methods of analysis combined with an efficient numerical method of computation of nonlinear terms. The methods used have been developed by *Orszag* and associates, and the mathematical foundations, justifications, and guides for use can be found in the recently published monograph of *Gottleib* and *Orszag* [3.62].

The results of *Orszag* and *Kells* [3.59] are summarized below.

1) The bifurcation picture for two-dimensional solutions of the equations governing disturbances of plane Poiseuille flow is like that shown in Fig. 3.14b. This confirms the qualitative picture given by the theory of bifurcation at a simple eigenvalue. The minimum value of $R(\varepsilon, \eta)$ for which a bifurcating solution of the classical Hopf type exists is about 2800. Earlier, *Zahn* et al. [3.63] found the minimum value of $R(\varepsilon, \eta)=2707$ the same type of disturbances, but in a severely truncated approximation. These maximum values are more than twice those observed in experiments. The numerical methods of *Orszag* and *Kells* enable one to compute solutions with large amplitudes and to separate the study of two- and three-dimensional disturbances. It is impossible to suppress three-dimensional disturbances in experiments and these evidently lead to the lower values of Reynolds number for which turbulent solutions are observed in experiments.

2) *Orszag* and *Kells* found that all spatially periodic two-dimensional disturbance of plane Couette flow decay to zero. It is generally agreed that plane Couette flow is stable against infinitesimal disturbances at all finite Reynolds numbers. But the result suggested by the computation of *Orszag* and *Kells* is new because it implies that Couette flow is also stable against finite amplitude two-dimensional disturbances. This suggests that a similar global

stability result holds for axisymmetric disturbances of Poiseuille flow through pipes when $\eta \to 0$ and when $\eta = 0$ and for axisymmetric disturbances of Couette flow between infinitely long cylinders when the inner cylinder is at rest and the outer one rotates.

3) *Orszag* and *Kells* found that small three-dimensional disturbances of plane Couette flow decay but that large amplitude disturbances persist and appear to evolve into turbulent flow at Reynolds numbers as low as 1000. And turbulent solutions will persist at slightly lower Reynolds numbers. This result is in rather good agreement with the experiments of *Reichardt* [3.64] and extends, but does not contradict, previous results on the linear stability of plane Couette flow. Three-dimensional disturbances of plane Poiseuille flow are also very destabilizing and lead to turbulentlike solutions at Reynolds numbers of about 1000, again in agreement with experiments. We may say that this shows that the stable two-dimensional solutions of the Hopf type, those in Fig. 3.14b, are unstable to three-dimensional disturbances and lead to turbulence. All disturbances of Couette and Poiseuille flow decay at Reynolds numbers of about 500.

3.10 Bibliographical Notes and Comments on Methods of Analysis

An extensive review of the analytical methods used in studying stability and bifurcation in fluids can be found in the Notes for Chap. II of my books [3.1, 2][7].

Linear stability theory is applied to interesting special problems in the following books and general reviews: *Chandrasekhar* [3.65] treated many kinds of special problems; *Lin* [3.60] also considered different problems but he emphasized the Orr-Sommerfeld equation; *Synge* [3.66], *Stuart* [3.67], *Shen*

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7 A corrigendum for errors and misprints which I have found in these books so far is as follows:

Vol. I [3.1]:

p.15 $v_g(\infty)$ is bounded.
p.29 Replace line 11 with "Then $y(t) = \boldsymbol{\Phi}(t) \cdot \psi$ and".
Replace (7.18) with "$y(t+T) = \boldsymbol{\Phi}(t+T) \cdot \psi = \boldsymbol{\Phi}(t)\boldsymbol{\Phi}(T) \cdot \psi = e^{-\gamma T}\boldsymbol{\Phi}(T) \cdot \psi = e^{-\gamma T} y(t)$".
Replace lines 17 through 27 with "and $w(t+T) = e^{\gamma t} e^{\gamma T} y(t+T) = e^{\gamma t} y(t) = w(t)$ is a T-periodic function".
p.167 Correct Eq. (44.20).
p.223 Change "$\phi \in C^3$" on line 11 to "$\phi \in C^2$"; change "$C^1(\mathscr{V})$" to "$C^3(\mathscr{V})$" on line 16.
p.224 Change lines 5 and 6 to "Since $\phi \in C^2(\mathscr{V})$ and $p^+ - p^-$ is continuous on S", we find, using Lemma 1, that $p^+ = p^-$ and p is single-valued.

Vol. II [3.2]:

p.59 Replace "$\mathscr{R}^{-1}$" with "$\mathscr{R}$" in (70.23–25).
p.60 Replace "$\boldsymbol{n} \cdot \boldsymbol{D} \cdot \boldsymbol{e}_z$" with "$2\boldsymbol{n} \cdot \boldsymbol{D} \cdot \boldsymbol{e}_z$" in (70.26) and in the first equality below (70.26).

[3.68], and *Monin* and *Yaglom* [3.69] gave general reviews; *Reid* [3.70] and the computer-oriented book of *Betchov* and *Criminale* [3.71] confined their analysis to the Orr-Sommerfeld theory; *Drazin* and *Howard* [3.72] considered the stability of inviscid flow; *Greenspan* [3.73] gave results for the stability of rotating flow; and *Yih* [3.74] dealt with the stability of stratified flows. *Davis* [3.75] is the only one on this list who treated the stability of time-dependent flows. Nonlinear problems of thermal convection are discussed in the review paper of *Segel* [3.76]. Other nonlinear problems of hydrodynamics were reviewed by *Stuart* [3.58]. A wider range of problems of interest can be found in the book of *Denn* [3.77]. The books of *Monin* and *Yaglom*, and *Denn* also treat nonlinear problems. The general problem of transition to turbulence has been reviewed in the paper of *Swinney* and *Gollub* [3.78].

The works listed in the last paragraph are concerned with linear stability theory and some of them with nonlinear stability theory. Nonlinear stability theory is different than bifurcation theory. The main differences are in the way you get hold of a problem and in what you aim to achieve. In nonlinear stability theory it is necessary to have explicit representations of some flow. Then one can do some explicit, if approximate, analysis of disturbances of this flow. Most of the time, studies of nonlinear stability theory are concerned with what happens when the unique flow which exists when the Reynolds number is small gives up its stability. Problems of secondary and higher bifurcation are hard to treat by the methods of nonlinear stability theory because explicit representations are known, usually in series form, only in asymptotic limits and the resulting analysis is, at the very best, difficult and approximate. When the explicit methods work they yield very good quantitative results suitable for comparison with experiments. But explicit methods lack generality because we cannot hope to have explicit representations for most flows and if we did, they would usually be too complicated to analyze by the ordinary methods of applied analysis.

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p.70 Replace "θ_0" in (73.4a) with "θ_0^*".
p.71 Replace "θ" in (73.9) with "$\bar{\bar{\theta}}^*$".
p.82 Replace "$M(a^2)\neq 0$" in Exercise 73.1 with "$M(a^2)=0$".
p.99 Replace "$-\sigma \boldsymbol{M}\cdot\boldsymbol{q}$" in (75.11) with "$-\sigma \boldsymbol{M}\cdot\boldsymbol{q}^*$".
p.196 In line 4, replace "...multilinear..." with "...multilinear and symmetric...".
p.197 Replace "$3\mathscr{F}_2$" in (93.6) with "$2\mathscr{F}_2$".
p.201 Replace "$d/d\tau^n$" on the last line with "$d^n/d\tau^n$".
p.203 Change "$A_m[\boldsymbol{u})x,t)]$" to "$A_m[\boldsymbol{u}(x,t)]$" at the end of (94.8)
p.211 Change "$\mathscr{F}^{\langle - \rangle}$" to "$\mathscr{F}^{\langle 2 \rangle}$" in line 12.
p.219 Replace "(94.52)" in the last line with "(94.51)".
p.225 Replace "...average of (94.67)..." with "...average of (94.67b)...".
p.226 On line 10, change "$\mu\langle \boldsymbol{u}^{\langle 1\rangle}\cdot(\nabla\cdot \boldsymbol{A}_1^{(2)})-\boldsymbol{u}^{\langle 2\rangle}\cdot \boldsymbol{A}_1^{(1)}\rangle+\dots$" to "$\mu\langle(\boldsymbol{u}^{\langle 1\rangle}\cdot\nabla)\boldsymbol{A}_1^{(2)} -(\boldsymbol{u}^{\langle 2\rangle}\cdot\nabla)\boldsymbol{A}_1^{(1)}\rangle+\dots$".
p.230 Replace the right-hand term with "$\varepsilon\partial U_j^{\langle 1\rangle}(\boldsymbol{X},\tau)/\partial X_i$".
p.256 Change the term on the right-hand side of (99.4b) from "$1-G\varepsilon^2/2+\dots$" to "$-1-G\varepsilon^2/2+\dots$".
Change the last term in the inequality below (99.4b) from "$\min_\Gamma$" to "$\min_{\tilde{\Gamma}}$".

Bifurcation theory does not require detailed knowledge of the flow which loses stability. Instead, analysis attaches itself to the problem through a classification of possibilities based on the spectrum, the eigenvalues of the linearized problem. For example, we already discussed the nature of bifurcation at an algebraically simple isolated eigenvalue. And in a longer work we would discuss bifurcation at multiple eigenvalues, bifurcation in the presence of continuous spectra, etc. So theory leads to a classification of the possible forms of bifurcation, and makes qualitative statements about their properties without computations based on explicit representations of flow. But, of course, many interesting details about the flow are lost in qualitative analysis. And to be certain, if a specific problem falls in one or the other classification, it is frequently necessary to verify the assumptions about the eigenvalues by explicit computations.

An introduction to the basic features of bifurcation theory, its principal applications, and an extensive bibliography was given by *Stakgold* [3.79]. The monographs of *Sattinger* [3.40] and *Iooss* [3.80] treat bifurcation problems in a Banach space. Excellent collections of papers on applications of bifurcation theory are found in the volumes edited by *Keller* and *Antman* [3.81], *Haken* [3.82], and *Rabinowitz* [3.83]. Many problems of bifurcation in hydrodynamical problems are studied in [3.1, 2]. Finally, an introductory treatment of bifurcation theory has been written as a textbook on "Elementary Bifurcation Theory" by *Iooss* and *Joseph* [3.51].

In my discussion of problems of stability and bifurcation in fluids I have tried to emphasize the main methods and I relied mostly on analytical methods. Most types of pure analysis are possible only when the amplitude of the disturbance is small. Explicit quantitative results can usually be obtained only when the amplitude is small and, in addition, when the domain occupied by the fluid and the data prescribed on the boundary of the domain have considerable symmetry. Many of the results we need in the theory of stability, bifurcation, and transition in fluids can be obtained most easily, and it is even possible that the only way to obtain them is by numerical methods.

The most successful numerical methods used so far are Galerkin-type methods, and the most far-reaching consequences of these methods have been reached by *Busse* and his associates (see his article in this volume, Chap. 5, and [Ref. 3.2, Sect. 81]) and, more recently, by the introduction of powerful spectral-numerical methods by *Orszag* and his associates (see, e.g., [3.62]). The various methods which I call Galerkin type involve expanding the solution in some set of eigenfunctions, truncating and examining the convergence of the method relative to some "residual" which gives the difference between the true solution and the approximating one. The application of this method to stability problems is explained in the book by *Finlayson* [3.84], and the newer and apparently more powerful spectral-numerical methods in the book by *Gottleib* and *Orszag* [3.62]. The explanation of numerical methods is beyond the scope of this chapter.

It seems likely to me that the theory of hydrodynamic stability, bifurcation, and transition will, in the future, come to rely increasingly on abstract methods for qualitative analysis and on numerical analysis for explicit results. I would expect an important role, but a decreasingly important one, to be played by the traditional methods of applied analysis.

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