BIFURCATION IN FLUID MECHANICS

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A broad discursive review of bifurcation theory in fluid mechanics is given. The review delineates the assumptions, methods and potential for application of bifurcation theory. The problem of sequential bifurcation of flows into other flows and finally into turbulence is considered and interpreted in terms of bifurcation theory.

I. INTRODUCTION

This lecture is a review of the applications of the theory of bifurcation to the problem of transition to turbulence. Most of the material in this lecture can be found in detail in my recent review [15], in other reviews in the same volume and in the monographs [16]. We shall discuss some new results having to do with frequency-locked solutions and bifurcation into higher dimensional tori in the transition to turbulence which were not discussed in [15] and [16]. Some of these results are derived in the new book on bifurcation theory by Iooss and Joseph [13]. To keep the lecture and this written report of it discursive, I am not going to do much citing and attributing of old results; more complete citations for the older work can be found in [15] and [16].

One traditional method of treating problems of stability in fluid mechanics is called non-linear stability theory. It involves explicit analysis of basic flows whose structure can be represented by explicit formulas. The goal of nonlinear stability theory is explicit calculation of details of the motion arising from instability. Bifurcation theory has a different goal and uses different methods to reach this goal. In bifurcation theory we classify the possibilities qualitatively. Explicit details of some basic flow are not required; instead we make assumptions about the eigenvalues of the linearized stability problem and classify the qualitative properties which can occur under each assumption. For example, we may discuss the problem of periodic solutions which arise from steady ones, or of quasi-periodic ones which arise from periodic ones under the assumption that the critical eigenvalue is simple.

The method of nonlinear stability theory cannot be used in most problems because it is usually not possible to give formulas for most of the flows we want to study.

Bifurcation theory has more generality but, in most applications, like nonlinear stability theory, it is limited to small amplitude motions.

Recent results, which I will discuss, suggest that hydrodynamical problems can be studied, without restricting the amplitude, by projections, using Galerkin approximations, into finite dimensions. The sets of ordinary differential which arise from these projections can then be treated by numerical methods.

II. UNIQUENESS AND GLOBAL ATTRACTIVITY AT SMALL R

I will confine my remarks to a discussion of the bifurcation of solutions of the Navier-Stokes equations for an incompressible fluid when the velocity $\underline{V}_B(\underline{x},t)$ of the boundary B of the region $\mathcal{V}$ occupied by the fluid is prescribed together with field forces $\underline{G}(\underline{x},t)$:

$$\left.\begin{array}{c} \dfrac{\partial \underline{V}}{\partial t} + \underline{V}\cdot\nabla\underline{V} = -\nabla p + \dfrac{1}{R}\nabla^2\underline{V} + \underline{G}(\underline{x},t), \\[2mm] \text{div } \underline{V} = 0 , \\[2mm] \underline{V} = \underline{V}_B(\underline{x},t) , \quad \underline{x} \in B . \end{array}\right\} \quad \underline{x} \in \mathcal{V}$$

We call the prescribed values $\underline{V}_B(\underline{x},t)$ and $\underline{G}(\underline{x},t)$, the data. R is the Reynolds number, a dimensionless parameter composed of the product of a velocity times a length divided by the kinematic viscosity. We can think of it as a dimensionless speed.

The motion of the fluid must ultimately be determined by the data. When the Reynolds number is small the motion is uniquely determined by the data. The meaning of this is as follows: given an initial condition

$$(1.2) \quad \underline{V}_0(\underline{x}) = \underline{V}(\underline{x},0) , \quad \underline{x} \in \mathcal{V}$$

we may suppose that the initial-boundary-value problem (1.1) and (1.2) have a unique solution. When R is small, each of these different solutions belonging to different $\underline{V}_0$, tend to a

single one determined ultimately by $\underline{G}$ and $\underline{V}_B$ and not by $\underline{V}_0$. So when R is small we ultimately get solutions which reproduce the symmetries of the data. Steady data gives rise to steady solutions, periodic data to periodic solutions.

When R is large solutions are not uniquely determined by the data. The relation between the data and solutions is subtle and elusive.

III. BIFURCATION OF STEADY SOLUTIONS

Let us consider what happens when the data is steady as we increase the Reynolds number. For technical reasons we suppose now and hereafter that $\mathcal{V}$ is a bounded domain, or it can be made bounded by devices such as restricting solutions to spatially periodic ones which can be confined to a period cell. We suppose $\underline{U}(R)$ is a steady solution which is the continuation of the unique steady solution which exists when R is small. When the equation satisfied by this solution is subtracted from Navier-Stokes equation, we get equations for the disturbance $\underline{u}$ of $\underline{U}(R)$ which has zero data and nonzero initial conditions

$$\frac{\partial \underline{u}}{\partial t} + \underline{U}\cdot\nabla\underline{u} + \underline{u}\cdot\nabla\underline{U} + \underline{u}\cdot\nabla\underline{u} = -\nabla p + \frac{1}{R}\nabla^2\underline{u}$$

(1.3) div $\underline{u}$ = 0   in $\mathcal{V}$

$$\underline{u} = 0 \ , \ \underline{x} \in B$$

(1.4) $\underline{u}(\underline{x},0) \neq 0, \ \underline{x} \in \mathcal{V}$

If the null solution $\underline{u} = 0$ of (1.3) is stable, then $\underline{U}(R)$ is stable. It is stable when R is small. We want to catalogue what can happen when $\underline{u} = 0$ loses stability as R is increased.

For simplicity we first write (1.3) as an evolution equation in some space, say a Banach space

(1.5) $\dfrac{d\underline{u}}{dt} = \underline{F}(R,\underline{u})$, $\underline{F}(R,0) = 0$.

It does no harm to think of (1.5) as a system of ordinary differential equations in $\mathbb{R}^n$.

To study the stability of $\underline{u} = 0$ we linearize (1.5) and introduce exponential solutions in order to derive the associated spectral problem:

$$\frac{d\underline{v}}{dt} = \underline{F}_u(R|\underline{v}) \ ,$$

$$\underline{v} = e^{\sigma t}\underline{\zeta} \ ,$$

$$\sigma = \xi(R) + i\eta(R) \in \Sigma \, \underline{F}_u(R|\cdot) \ ,$$

$$\alpha\underline{\zeta} = \underline{F}_u(R|\underline{\zeta})$$

where $\Sigma\underline{F}_u$ means the spectrum of $\underline{F}_u$. If σ is in the spectrum so is $\bar{\sigma}$. When is a bounded domain the spectrum of $\underline{F}_u$ is all of eigenvalues and when R is small all of the eigenvalues are bounded by a parabola on the left hand side of the complex σ plane. As R is increased past its first critical value some eigenvalues cross into the right side of the complex σ plane. In the usual case a single eigenvalue or a complex conjugate pair of eigenvalues cross over.

We state the foregoing conditions, which are sufficient for bifurcation, in a precise mathematical sense as follows. $R = R_c$ is the first critical value of R such that $\xi(R) < 0$ for all eigenvalues belonging to $\underline{F}_u(R|\cdot)$ when $R < R_c$, $\xi(R_c) = 0$, $\sigma(R_c) = i\omega_0$ (where $\omega_0 = \eta(R_c)$ is an algebraically simple eigenvalue of $\underline{F}_u(R_c|\cdot)$ and the loss of stability of $\underline{u} = 0$ at R_c is strict; that is,

$$\xi'(R_c) > 0 \ .$$

Given the assumptions made in the last paragraph there are two possibilities:

(I) $\omega_0 = 0$ and one real eigenvalue crosses at critically. A steady solution which breaks the spatial symmetry of the data, bifurcates.

It is usually enough to consider three possible types of bifurcation into steady solutions (see Figure 1). Transcritical bifurcation occurs when the projection of the quadratic part of the nonlinear terms into the null space of $F_u(R_c|\cdot)$ is nonvanishing. When this projection does vanish, bifurcation is controlled by cubic terms. When these terms do not vanish, here are two possibilities: bifurcation to the right (supercritical) and bifurcation to the left (subcritical). Solutions which bifurcate supercritically are stable; subcritical solutions are unstable.
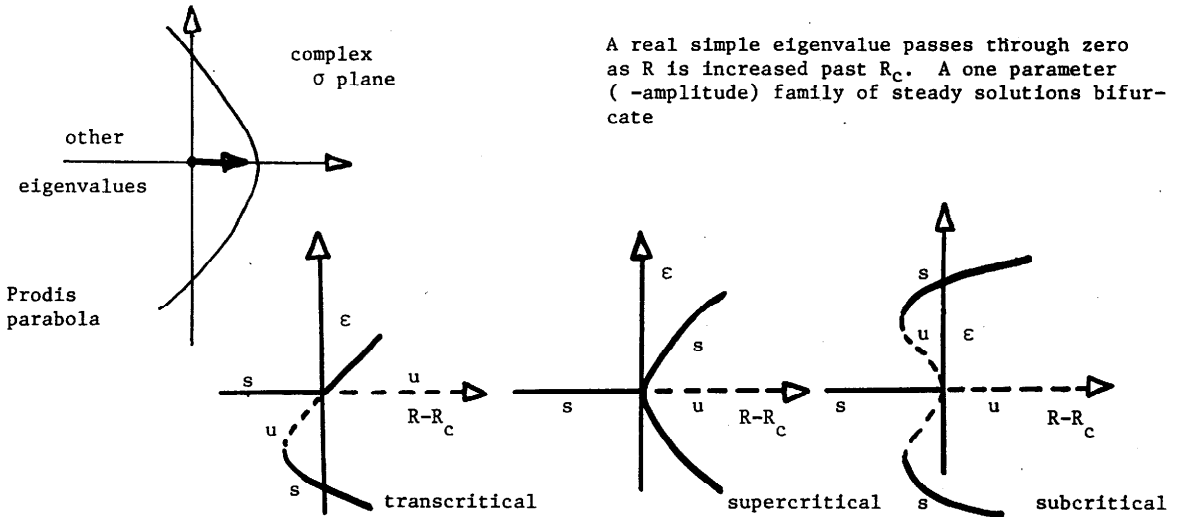
(II) A complex pair crosses. The quadratic projection vanishes automatically and we never get the transcritical case

$$R(\varepsilon) = R(-\varepsilon) \ ,$$

$$\omega(\varepsilon) = \omega(-\varepsilon) \ ,$$

$$\frac{d\underline{u}}{dt} = F(\underline{u},R), \text{ study the stability of } \underline{u} = 0 \ ,$$

$$\frac{d\underline{v}}{dt} = F_u(R|\underline{v}), \ \underline{v} = e^{\sigma t}\underline{\zeta} \ , \ \sigma\underline{\zeta} = F_u(R|\underline{\zeta}) \ .$$

A real simple eigenvalue passes through zero as R is increased past R_c. A one parameter (-amplitude) family of steady solutions bifurcate

Branching results are restricted. For the complete story we need global results. For example even in one dimension we can get isolated solutions like $F_2 = 0$ and $F_3 = D$ in Figure 1.
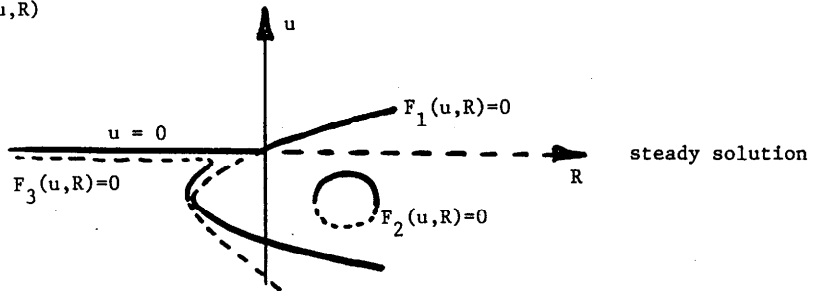
$$\frac{du}{dt} = uF_1(u,R)F_2(u,R)F_3(u,R)$$

Figure 1. Bifurcation of steady solutions. Dotted lines are unstable

As R is increased, new steady solutions, with different patterns of symmetry may bifurcate. After some number of these steady bifurcations a periodic solution will typically bifurcate.

IV. FLOQUIT THEORY AND THE STABILITY OF PERIODIC SOLUTIONS

Now we ask what happens when a periodic solution bifurcates. There are, in general, two possibilities: another periodic solution with a longer period may bifurcate, or a doubly periodic solution with two frequencies may bifurcate. For simplicity, suppose that we have a periodic solution with velocity $\underline{u}(t,R) = \underline{u}(t+T,R)$, periodic with fixed period T. Typically such solutions arise from forced T-periodic data. In bifurcation problems the period $T = T(R)$ changes with R. We suppose T is independent of R at the expense of some fine points, but the qualitative results are nearly the same. A small disturbance $\underline{v}$ of $\underline{V}$ satisfies the linearized equation

$$\frac{d\underline{v}}{dt} = F_u(R,\underline{u}(t,R)|\underline{v})$$

which can be studied by the method of Floquet. We may represent $\underline{v}(t)$ solving the linearized equations as

$$\underline{v} = e^{\sigma(R)t}\underline{\zeta}(R,t), \quad \underline{\zeta}(R,t) = \underline{\zeta}(R,t+T)$$

where $\sigma(R) = \xi(R) + i\Omega(R)$, the Floquet exponent, and $\underline{\zeta}(R,t)$ are eigenvalues and eigen-

vector of the operator

$$\mathbb{J} = -\frac{d}{dt} + F_u(R,\underline{u}(t,R)\mid\ )$$

whose domain is of T-periodic functions and $\mathbb{J}\underline{\zeta} = \sigma\underline{\zeta}$. For each and every σ in the spectrum of $\mathbb{J}$ there is a Floquet multiplier

$$\lambda(R,T) = e^{\sigma(R)T}$$

If $R < R_0$, where R_0 is the critical R for which $\xi(R_0) = 0$ for some exponent σ, then, $\xi(R) < 0$ for σ and

$$\lambda(R,nT) = e^{\sigma(R)nT} < 1$$

and, in fact tends to zero as $n \to \infty$. This is the stable case. At criticality $R = R_0$ and we assume that $\xi'(R_0) > 0$ for the eigenvalue for which $\zeta(R_0) = 0$. This gives rise to the situation exhibited in Fig. 2 in which a pair of multipliers

$$\lambda_0 = e^{\pm\ i\Omega_0 T}$$

escapes from the unit disk

λ plane

$\lambda_0 = e^{i\Omega_0 T}$

$\xi_0 = 0$

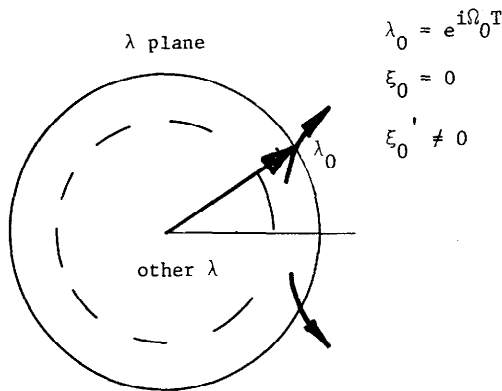$\xi_0' \neq 0$

λ_0

other λ

Figure 2: A conjugate pair of multipliers escapes from the Floquet circle at criticality [INSTABILITY]

V.   BIFURCATION OF PERIODIC SOLUTIONS

    Now we see what bifurcates; that is, we classify all the possibilities for bifurcation in the simple eigenvalue situation depicted in Figure 2. The classification is parameterized

by the points on the Floquet circle at which the conjugate multipliers escape. All points on circle

$$\lambda_0 = e^{i\Omega_0 T}$$

are given in terms of

$$r = \frac{\Omega_0}{(2\pi/T)}\ ,\quad 0 \le r < 1\ ,$$

the frequency ratio at criticality. At criticality solutions of the bifurcation problem are in the form

$$\hat{\underline{v}}_0(t_1,t_2) = \hat{\underline{v}}_0(\Omega_0 t,\frac{2\pi}{T}\ t) = e^{i\Omega_0 t}\ \underline{\zeta}_0(t)$$

where $\underline{\zeta}_0(t) = \underline{\zeta}_0(t+T)$ and $\Omega_0 = 2\pi r/T$. $\hat{\underline{v}}_0(t_1,t_2)$ is doubly periodic, periodic with period 2π, jointly in t_1 and t_2.

    (I) $\hat{\underline{v}}_0(t_1,t_2)$ is quasi-periodic if r, $0 < r < 1$, is irrational.

    (II) $\hat{\underline{v}}_0(t_1,t_2)$ is nT-periodic if $r = m/n < 1$, $0 \le m/n < 1$, is rational.

    These properties (I) and (II) <u>are not preserved</u> when the linear problem is perturbed with the nonlinear terms. We construct an asymptotic (at least) approximation

$$\hat{\underline{v}}(\Omega(\varepsilon)t,\frac{2\pi}{T}\ t),$$

doubly periodic with period 2π in each place, with a smooth $\Omega(\varepsilon)$ which bifurcates for each fixed r, $0 \le r < 1$, even for $r = m/n$, provided only that $n \neq 1, 2, 3, 4$. At these special rational values we get subharmonic, nT periodic solutions with $\Omega(\varepsilon) = 2\pi mt/nT$, independent of ε. We call the points $n = 1, 2, 3, 4$ where subharmonic solutions bifurcate, points of strong resonance. The T-periodic bifurcating solution (n = 1) is transcritical; it bifurcates on both sides of criticality. The 2T-periodic solution (n = 2) bifurcates either entirely to one side or to the other. And in both cases supercritical bifurcating solutions (R > R_0) are stable and subcritical solutions (R < R_0) are unstable. The 3T-periodic solution is transcritical and it is unstable on <u>both</u> sides of criticality. Two 4T-periodic bifurcate (if a certain inequality holds) and if the two bifurcate supercritically, one of the two is unstable.

$$r = \frac{\Omega_0}{(2\pi/T)}\ ,\quad 0 \le r < 1$$
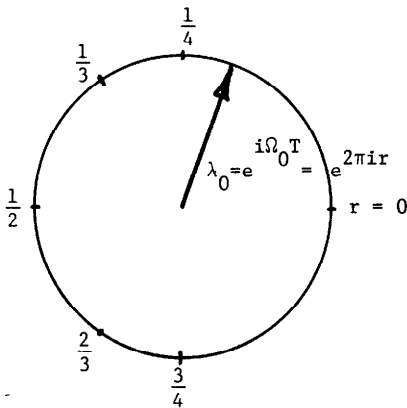
Frequency ratio at criticality

Figure 3: Classification of points on the Floquet circle. The Floquet circle at criticality is given by $\lambda_0 = e^{2\pi i r}$. (I) r is irrational, (II) $r = m/n < 1$ is rational $\lambda_0^n = e^{2\pi i m} = 1$.

Some elementary methods of analysis of bifurcation of periodic solutions are given in [13]. To get such results one may use an extension of the "method of averaging" to reduce the T-periodic problem to an autonomous one. First we decompose the bifurcating solution into a part spanning the null space of $\mathbf{J} - \sigma$ and the other part $\underline{w}(R,t)$:

$$\underline{u}(R,t) = z(R,t)\underline{\zeta}(R,t) + \bar{z}(R,t)\underline{\bar{\zeta}}(R,t) + \underline{w}(R,t)$$

where $\underline{\zeta}(R,t) = \underline{\zeta}(R,t+T)$ is the eigenfunction of $\mathbf{J}$ with eigenvalue

$$\sigma = i\Omega_0 + \mu\hat{\sigma}(\mu), \quad \mu = R - R_0 .$$

Then we do may changes of variables which shove $\underline{w}$ into higher order terms. The main action is in the projection described by the equations governing $z(R,t)$, rather $y(R,t)$ where y is introduced by a change of variables in the form

$$z = y + \sum_{p+q \geq 2}^{N} z^p \bar{z}^q \gamma_{pq}(t,\mu) .$$

The truncation number N is completely arbitrary, it may be as large as one likes. It is necessary, however, to retain higher order terms because we haven't proved convergence and don't yet know if and when these expressions converge. In the expressions below we suppress the higher order terms of $O(|y|^{N+1})$, but they are understood. If the frequency ratio r at criticality is irrational we get an autonomous equation

$$\dot{y} = \mu\hat{\sigma}(\mu)y + \sum_{q \geq 1} y|y|^{2q} a_q .$$

If the frequency ratio $r = m/n = \Omega_0/(2\pi/T)$ is rational, and $n \geq 5$ we put

$$y = xe^{i\Omega_0 t}$$

and find another autonomous equation

$$\dot{x} = \mu\hat{\sigma}(\mu) + \sum_{q \geq 1} x|x|^{2q} a_q +$$

$$\sum_{k>0} \sum_{q \geq 0} |x|^{2q}(x^{1+kn}a_{qk} + \bar{x}^{kn-1}a_{q,-k})$$

$$= \mu\hat{\sigma}(\mu)x + x|x|^2 a_1 + x|x|^4 a_2$$

$$+ \bar{x}^{n-1}a_{o,-1} + \ldots$$

VI. THE BIFURCATING TORUS AND QUASIPERIODIC SOLUTIONS

We may deduce the following conclusions from these autonomous equations.

(1) The cross-section of the two dimensional torus is a closed curve $\rho(\theta,\varepsilon)$ in the (ρ,θ) plane of mean radius

$$\varepsilon = \frac{1}{2\pi} \int_0^{2\pi} \rho(\theta,\varepsilon)d\theta .$$

(2) If r is irrational, then

$$\rho(\theta,\varepsilon) = \varepsilon \quad \text{is a circle}$$

and

$$\mu(\varepsilon) = \mu_2\varepsilon^2 + \mu_4\varepsilon^4 + \ldots$$

(3) If $r = m/n$, then $\mu(\varepsilon)$ is as above and

$$\rho = \varepsilon + \varepsilon^{n-3}\rho_{n-3}(\theta) + \varepsilon^{n-2}\rho_{n-2}(\theta) + \ldots$$

where

$$\rho_\ell(\theta) = \rho_\ell(\theta + \frac{2\pi}{n}) , \quad \int_0^{2\pi} \rho_\ell(\theta)d\theta = 0 .$$

In this case the torus is an n lobed figure of mean radius ε (Fig. 4).

(4) Steady solutions x correspond to nT-periodic $\underline{u}(t,R)$. If $n \geq 5$, exceptional conditions are needed to realize steady solutions. If the exceptional conditions are realized, there are two steady solutions on the torus, one of which is unstable (Fig. 4).

(5) Periodic solutions $x(s) = x(s + 2\pi)$, $s = \varepsilon^2 \hat{\Omega}(\varepsilon^2)t$ correspond to doubly periodic solutions $\underline{u}(t,R)$ with a smooth (in ε) frequency ratio $(\Omega_0 + \varepsilon^2 \hat{\Omega}(\varepsilon^2))/(2\pi/T)$.
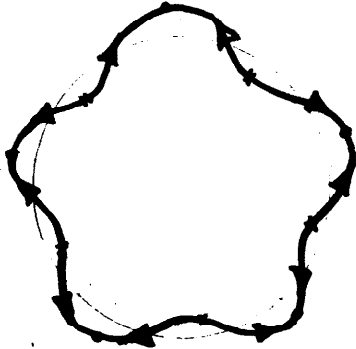

Figure 4: The cross-section of the torus in the case $n = 5$. If the ratio $a_1/\hat{\sigma}_0$ of complex numbers is real (an exceptional condition), the "trajectories" on the torus are two sets of five fixed points, one of which is unstable.

VII. THE CONJECTURES OF LANDAU AND HOPF

After having established conditions under which we get bifurcation into an invariant two dimensional torus with two frequencies, it is natural to inquire about the conditions under which solutions with three frequencies bifurcate from those with two. Indeed Landau [18] and Hopf [12] conjectured that turbulence arises through such a sequence of bifurcations with new frequencies introduced at each point of bifurcation as the Reynolds number is increased. In their view turbulence is multi-periodic flow with a finite number of discrete frequencies varying continuously with amplitude. In the end, as R tends to infinity, the number of frequencies could tend to infinity leading then to "almost periodic" turbulence. Exact mathematical conditions which suffice to guarantee the bifurcation from an n-dimensional to an n+1 dimensional torus have given by Chenciner and Iooss [2], Haken [10] and Sell [27]. These conditions are not inevitably realized in fluid motions, though three frequencies have been observed in convection [8] and four frequencies with some noise in the flow between rotating spheres [31].

VIII. BIFURCATION INTO NONPERIODIC (STRANGE) ATTRACTIONS

The point against turbulence as almost periodic is that unlike true turbulence almost periodic "turbulence" is not phase mixing. That if $\underline{u}(t)$ is a fluctuation with mean value zero

and is almost periodic, then

$$\underline{u}(t) \sim \sum_{-\infty}^{\infty} \underline{u}_n e^{-i\lambda_n t}, \quad \lambda_n \neq 0 .$$

The autocorrelation for this is

$$g(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_0^T \underline{u}(t + \tau)\underline{u}(t)dt$$

$$= \sum_{-\infty}^{\infty} |\underline{u}_n|^2 e^{-i\lambda_n \tau}$$

and $g(\tau)$ does not vanish for solutions of the Landau-Hopf type, as it must for true turbulence. In true turbulence events at distant times are presumably uncorrelated. In some experiments [20] a noisy part of spectrum coexists with a peaked part. In these cases the autocorrelation function will decay as the noisy part of power spectrum grows larger, but it will not decay to zero.

Lorenz [21] and Ruelle-Takens [25] suggested that turbulence could occur after a finite number of bifurcations. This picture is in much better agreement with experiments.

IX. POWER SPECTRA AND TRANSITION TO TURBULENCE

To detect the dynamical events leading to turbulence experimenters use power spectra. These are obtained from Fourier transforms of measured data. Usually the data is the time sequence of some component of the velocity. The use of power spectra in experiments on turbulence has a long history. But power spectra were not used to detect bifurcation events leading to turbulence until the important work of Swinney and Gollub [5]. In the short time since the appearance of their paper, the method of power spectra analysis of bifurcation has become standard. The way which power spectra can be used to reveal the qualitative structure of observed flows can be partly understood by studying Fig. 5.

By turbulent flow I mean a flow (in the ordinary sense of fluid mechanics and also in the more general sense of trajectories for an arbitrary dynamical system) which has the following properties.

(1) It is sensitive to initial conditions. Two turbulent flows with nearly equal initial conditions will undergo different time evolutions. No two realizations of turbulence are exactly the same, even if they have some common average value. This sensitivity to initial conditions is in sharp contrast to laminar flow at low Reynolds numbers where all initial conditions are attracted by one flow.

(2) It has a continuous spectrum. We sometimes say such a spectrum is "noisy" and

Small box of fluid   Flow between rotating cylinders and spheres
heated below
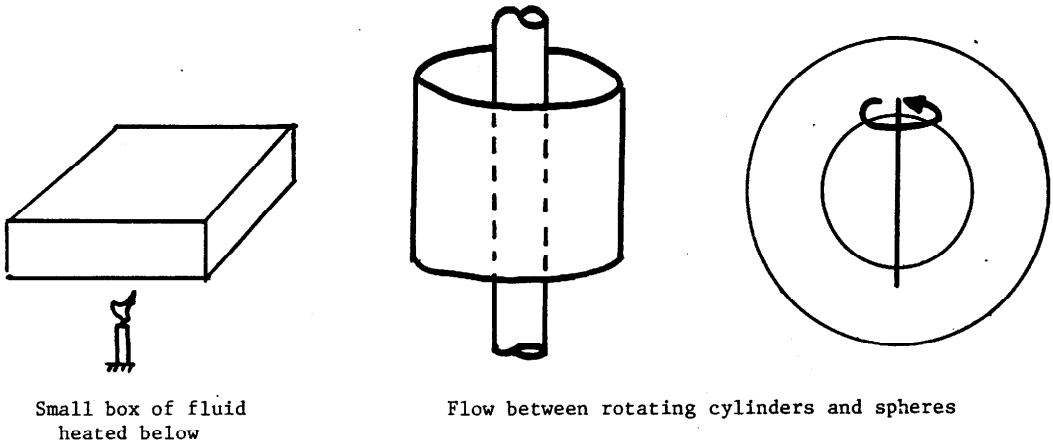
In some experiments [1], [5], [7], [8], [9], [22], [29], [31] a component of the velocity at a point
is monitored. Then power spectra is obtained from Fourier analysis of the time series.

Power

(a) Periodic solution with one sharp frequency plus
     harmonics

Frequency

ω 2ω 3ω 4ω

(b) Doubly periodic solution in which all the sharp
     spectral lines are of the form $m\omega + n\Omega$, where m
     and n are positive integers

$m\omega + n\Omega$

Ω ω

(c) Continuous spectra. In such flows the autocorrelation
     decays to zero

(d) Continuous spectra with sharp spectral lines. The
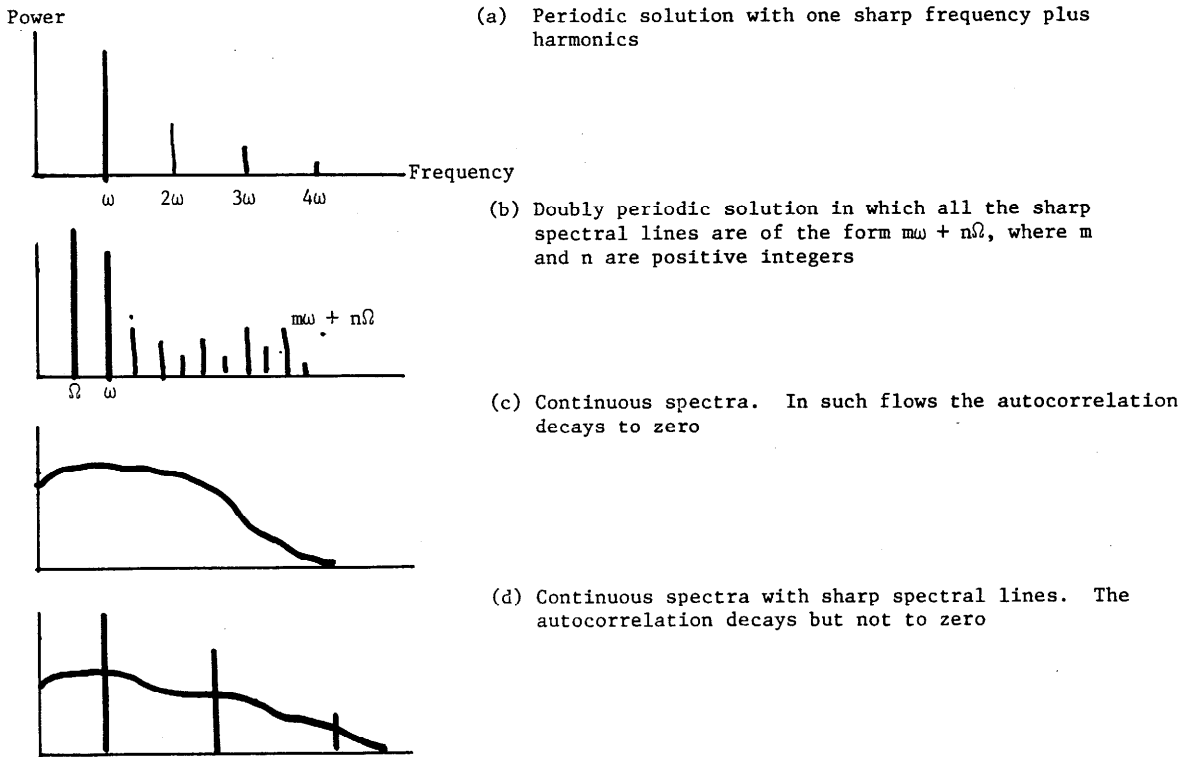   autocorrelation decays but not to zero

Figure 5: Power spectra associated with different kinds of flow. Flows with continuous spectra, even
in the presence of sharp peaks are turbulent. The four types of spectra exhibited can be found in
experiments on convection and Couette flow between cylinders and spheres and in other experiments.

mean dynamical rather than accidental or background noise.

(3) It is "mixing"; that is, it has a decaying autocorrelation $g(\tau)$. In the absence of sharp spectral features $g(\tau) \to 0$, and two velocity fluctuations which occur a long time from one another are uncorrelated.

X. MANY ROUTES TO TURBULENCE

There are bifurcations which frequently occur before the transition to turbulence. Bifurcation into periodic solutions followed by bifurcation of doubly periodic solutions with two continuously varying frequencies is often observed in experiments. Bifurcation of T periodic solutions into NT periodic ones with N = 2 and 4 is sometimes observed and bifurcation into a three frequency solution has been observed in one experiment [9]. All persons in fluid mechanics have some awareness of the sudden direct transition to turbulence from steady shear flow in pipes, boundary layers and the Couette flow problem when the outer cylinder rotates. Heikes and Busse [11] have given a theory and reported an experiment on rotating layers of fluids undergoing convection which give yet another type of chaotic behavior called weak turbulence. A few details of some typical experiments exhibiting different sequences of bifurcations into turbulence are given below.

Yavorskaya, Beleyaev, Monakov and Scherbakov [20] have carried out bifurcation experiments for the problem of flow between rotating spheres when the inner sphere rotates and the gap is wide. In Figure 6 I have sketched the frequency versus Reynolds number graph given as Figure 1 of the paper. They get their results by monitoring the fluctuating velocity at a point and they also measure the autocorrelation function.
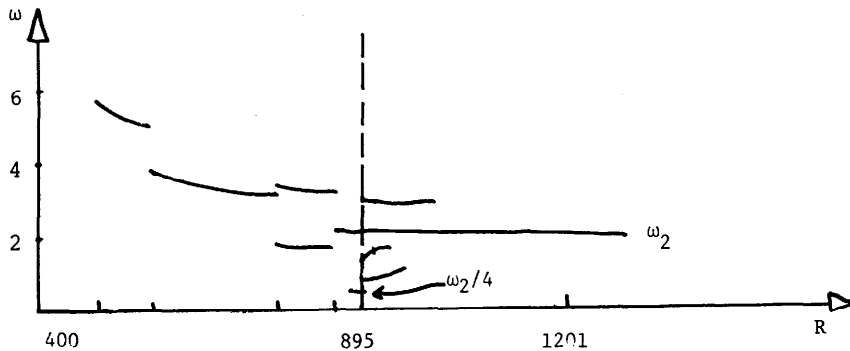

Figure 6: The flow between spheres is periodic when there is one frequency at a given R. The solution is doubly periodic when there are two frequencies present. Just before R = 895 where the autocorrelation function starts to decay there is $4(2\pi/\omega_2)$ subharmonic solution. The first decay of the autocorrelation never does decay fully because the sharp spectral component coexists with dynamic noise for the range of R considered.

Gollub and Benson [9] and Maurer and Libchaber [22] have done many experiments on bifurcation of convection in box of fluid heated from below. In the French experiments with liquid helium a first frequency ω_1 associated with oscillating rolls appears for a Rayleigh number around 2×10^4, then at about 2.7×10^4 a second frequency ω_0, much smaller is observed, two frequency locking regimes are observed, with hysteresis, for frequency ratios $\omega_1/\omega_2 = 6.5$ and $\omega_1/\omega_2 = 7$. The transition to turbulence in the experiments of Libchaber and Maurer [19] is triggered by the generation of frequencies $\omega_2/2$, $\omega_2/4$, $\omega_2/8$, $\omega_2/16 \to$ turbulence. A mathematical model for repeated 2T-periodic bifurcation into turbulence has been discussed by Tomita and Kai [28] and Ito [14]. A general theory of repeated 2T-periodic bifurcations of one-dimensional maps has recently been given by M. Feigenbaum [4]. His results seem to apply to mathematical models involving differential equations like those of Tomita and Kai [28] and to the equations of Lorenz [20]. Some features of Feigenbaum's predictions have been observed in the experiment of Gollub, Benson and Steinman [7].

The number of possible routes to turbulence is large and the possible routes cannot now be classified. The sequences of dynamic bifurcations which lead to turbulence are different when the spatial organization of flow is different. Since nonunique solutions are typical in hydrodynamics the number of possible sequences may be at least as great as the number of spatially distinct motions at a given Reynolds number.

People in fluid mechanics believe that flows are always turbulent when R is large enough. This type of property is not true for simple dynamical systems like those governed by

the equations of Lorenz. In these equations a turbulent attractor can degenerate into a limit cycle as R is increased. More typically one observes a noisy spectrum with sharp peaks which can be matched to one, two or three frequencies [31], [29], [20].

XI. FINITE DIMENSIONAL (GALERKIN) APPROXIMATION TO THE EQUATIONS WHICH GOVERN TURBULENCE

In our book [13], Iooss and I studied the problems which could be projected into $\mathbb{R}^2$. That is, we confined our attention to cases in which at most conjugate eigenvalues pass through criticality. Even for this case there are many possible types of bifurcation ranging from steady symmetry bifurcation to two dimensional bifurcating tori containing asymptotically quasiperiodic solutions and frequency locked solutions. In three dimensions, even apparently benign systems of three nonlinear ordinary differential equations, like the Lorenz equations [21], give rise to very complicated dynamics with turbulent-like attracting sets which defy description in simple terms. It is fairly obvious that the uniqueness of solutions, which prevents transverse intersections of trajectories in $\mathbb{R}^2$, has much less force in $\mathbb{R}^3$ where attracting sets of fractal dimension smaller than $\mathbb{R}^3$ can be quite complicated.

The problems which are encountered in dimensions higher than two, when we have turbulent attracting sets, are evidently not so easily treated by analytical methods which work well in problems which can be projected into two dimensions. For these problems various methods of topological dynamics are of theoretical interest. From a practical point of view, many problems of interest can be studied by Galerkin methods. The method is to expand the spatial structure in a complete set of functions with time dependent coefficients $a_n(t)$, truncate the set at a finite number N, and solve the initial value problem for the coefficients $a_n(t)$ in $\mathbb{R}^N$. This method sometimes leads to very good results with even astonishing agreements with observed bifurcations. Such agreements may be found in the comparison of the 14 node truncation of Curry [3] with the observations of convection by Gollub and Benson [8] or the comparison of truncated models for Taylor flow [30] with experiments by Swinney, et al. [5].

It is necessary to caution the reader that agreement between a finite dimensional approximation in $\mathbb{R}^N$ with experiments does not imply that the approximations are valid. In the case of the higher dimensional Lorenz equations the good agreement between experiments and the N = 14 mode model of [3] disappears as N is increased. Approximations should not change by much when N is increased past a certain value. This type of consideration suggests two questions: (1) Can finite dimensional approximations model infinite dimensional problems? (2) If it is possible to have finite dimen-

sional approximations, what is the dependence of the truncation number N(R) on R?

In an important paper on issues raised in the last paragraph, O. A. Ladyzhenskaya [17] has established that for the two dimensional Navier Stokes equations, with prescribed forces and zero data on the boundary of a bounded domain there is a compact set in an appropriate space on which the Navier Stokes equations form a dynamical system. The given set includes all limiting ($t \to \infty$) requires. Moreover, there is a certain sense in which this compact set is finite dimensional. For example, Ladyzhenskaya proves that if the solution of the finite dimensional is almost periodic then so is the solution of the full problem. In general, the dimension N of the finite dimensional subspace on which the problem may be faithfully projected is an increasing function of Reynolds number. Results of this same general nature, going further in some directions and less far in others have been given by Ruelle [26] using results of Mallet-Paret [23] and Foias-Temam [6].

Even if it is possible to state with certainty that the hydrodynamic problem can be modeled by ODE's in $\mathbb{R}^N$ there is no reason to expect that a small N will suffice in all problems. As an example, we can follow Orzag and Kell [29] and consider flow induced by a pressure drop ΔP down a plane channel. When ΔP is small the flow is laminar, and the velocity is unidirectional and varies across the cross-section like a parabola. At larger pressure drops there are alternate patches of laminar and turbulent flow, at still larger ΔP the flow is turbulent throughout. The bifurcation diagram of Figure 7 is for an idealized two-dimensional problem in which disturbances are assumed to be spatially periodic. The bifurcation is subcritical. On general theoretical grounds we expect the bifurcation diagram to recover stability when the amplitude is large (see "Factorization theorems" in [16]). In fact Orszag and Kell [24] calculate a curve like shown in Figure 7. They integrate the initial-value problem in $\mathbb{R}^N$ by brute force, using interesting Galerkin methods. They show that the large amplitude branch, which is stable for two dimensional disturbances, is unstable to three dimensional disturbances. Their numerical results are in agreement with experimental observations but they require N = 33792.

This stable branch is
unstable to three dimensional
disturbance

Mass flux
minus mass
flux in
laminar
flow

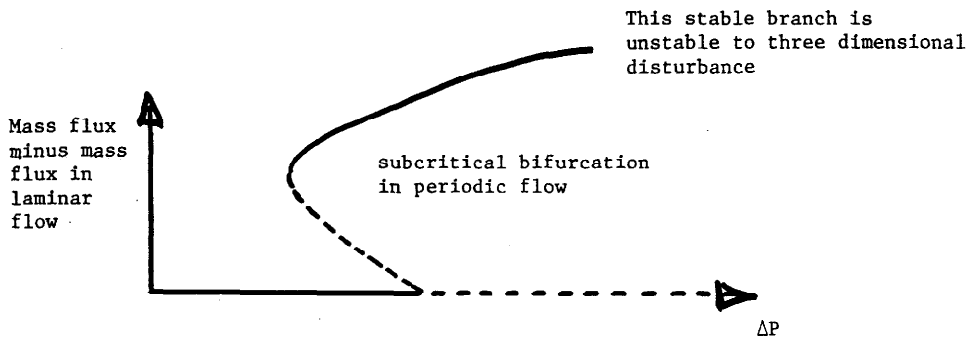subcritical bifurcation
in periodic flow

ΔP

Figure 7:   Direct bifurcation into turbulence in Poiseuille flow. The diagram is for
            the two dimensional problem.

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