An Integral Invariant for Jets of Liquid into Air

DANIEL D. JOSEPH

Dedicated to Clifford Ambrose Truesdell on his sixtieth birthday

A liquid is forced to move from left to right (x increasing) down a round pipe of length L by high pressure imposed at the entrance $x=-L$ of the pipe. The flow is assumed to be axisymmetric but the pressure and velocity which is prescribed at $x=-L$ is otherwise arbitrary. At $x=0$ the liquid is extruded into a zero gravity field, and an axially-symmetric capillary jet of radius $h(x)$ is formed. It is assumed that the wind shear is negligible, and that far downstream ($x\to\infty$) the flow must become rectilinear, with uniform velocity U_f and final jet diameter h_f, thus

(1) $$U_f=\lim_{x\to\infty} \boldsymbol{e}_x\cdot \boldsymbol{U}(x,r), \qquad h_f=\lim_{x\to\infty} h(x).$$

The mass flux is constant and invariant for the whole flow ($-L\leqq x<\infty$).

Making no assumption about the constitutive equation of the liquid other than incompressibility, I am going to prove the existence of an integral invariant of the axial momentum of the jet. This invariant is a locally conserved quantity which can be manipulated to produce the equation governing the global conservation of the axial momentum of the jet (see Part III, JOSEPH, 1974). The invariant should be useful in the analysis of die swell.

The motion of the liquid is governed by the following system of equations,

$$\boldsymbol{U}=\boldsymbol{e}_r w+\boldsymbol{e}_x u \qquad \text{(velocity)},$$

$$\boldsymbol{T}=-p\boldsymbol{l}+\boldsymbol{S} \qquad \text{(stress)},$$

(2) $$\frac{1}{r}\,\partial_r(rw)+\partial_x u=0$$

(3) $$\rho(w\,\partial_r u+u\,\partial_x u)=\partial_x T_{xx}+\frac{1}{r}\,\partial_r(r\,T_{rx})$$

and

(4) $$\rho(w\,\partial_r w+u\,\partial_x w)=\partial_r T_{rr}+\partial_x T_{xr}+\frac{1}{r}\,(T_{rr}-T_{\theta\theta}).$$

The velocity of the fluid in the pipe is bounded on the centerline and vanishes at the pipe wall

$$u(x, h_0)=w(x, h_0)=0. \tag{5}$$

The free boundary conditions, listed below, state, respectively, that the normal component of the velocity, the shear stress, the difference between the jump in the normal stress and the surface tension force (σ = coefficient of surface tension), the difference between the jet radius at $x=0$ and the pipe radius, and the slip of the free surface as $x \to \infty$ all vanish: thus

$$\begin{aligned} w-h'u=h'(T_{rr}-h'T_{xr})+(T_{xr}-h'T_{xx})&=0\\ T_{rr}-h'T_{xr}-\sigma\{h''-(1+h'^2)/h\}/(1+h'2)^{3/2}&=0\\ h(0)-h_0=h'(\infty)&=0 \end{aligned} \tag{6}$$

where $h' \equiv dh/dx$. It is convenient to write the surface tension equation $(6)_3$ as

$$T_{rr}-h'T_{xr}=-\frac{\sigma}{hh'}\frac{d}{dx}\left(\frac{h}{(1+h'^2)^{1/2}}\right). \tag{7}$$

The stress and velocity are continuous throughout the flow; in particular, they are continuous across the exit plane $x=0$. I do not exclude the possibility that the stress at the exit lip $x=0$, $r=h$ has a singularity. At the entrance $x=-L$ of the pipe there holds

$$(u, w, T_{xx})=(\hat{u}(r), \hat{w}(r), \hat{T}_{xx}(r)). \tag{8}$$

As $x \to \infty$ *the flow is essentially uniform and rectilinear in a cylinder of constant radius* h_f, with an isotropic stress which balances the hoop stress induced by surface tension. Thus

$$\lim_{x\to\infty}(u, w, h, \boldsymbol{T})=(U_f, 0, h_f, -\sigma \boldsymbol{l}/h_f). \tag{9}$$

The integral invariant ((12) below) to which I have already alluded is defined in terms of the *axial thrust*

$$\mathcal{M}(x)=\int_0^{h(x)} r(\rho u^2-T_{xx})\,dr. \tag{10}$$

It is also of interest to introduce the *radial thrust*

$$\mathcal{R}(x)=\int_0^{h(x)} r(\rho u w-T_{xr})\,dr. \tag{11}$$

Theorem 1. *The axial thrust*

$$\mathcal{M}(x)-\frac{\sigma h(x)}{[1+h'^2(x)]^{1/2}}=\frac{1}{2}(\rho U_f^2 h_f^2-\sigma h_f) \tag{12}$$

is conserved in the jet; that is, (12) *holds for all* $x \geqq 0$. *And in the pipe itself,*

$-L \leqq x \leqq 0$, *the rate of change of* $\mathcal{M}(x)$ *is proportional to the wall shear stress*

(13) $$\frac{d\mathcal{M}(x)}{dx} = h_0 T_{rx}(x, h_0).$$

The radial thrust satisfies

(14) $$\frac{d\mathcal{R}(x)}{dx} + \frac{\sigma}{h'}\frac{d}{dx}\left(\frac{h}{(1+h'^2)^{1/2}}\right) = -\int_0^{h(x)} T_{\theta\theta}\,dr$$

in the jet. And in the pipe, the rate of change of $\mathcal{R}(x)$ *is governed by*

(15) $$\frac{d\mathcal{R}(x)}{dx} = h_0 T_{rr}(x, h_0) - \int_0^{h_0} T_{\theta\theta}\,dr.$$

The proof of this theorem follows from an easy calculation: we first introduce the notation $[A] = A(x, h(x))$ for any function $A(x, r)$. We then form the area average of (3) and (4), using (2), and find that

$$r w\,\partial_r u + r u\,\partial_x u = \partial_r(r w u) + r\,\partial_x u^2,$$

$$\int_0^{h(x)} r(w\,\partial_r u + \tfrac{1}{2}\partial_x u^2)\,dr = \frac{d}{dx}\int_0^{h(x)} r u^2\,dr + [h u(w - h' u)],$$

$$\int_0^{h(x)} \{\partial_r(r T_{rx}) + \partial_x(r T_{xx})\}\,dr = \frac{d}{dx}\int_0^{h(x)} r T_{xx}\,dr + [h(T_{rx} - h' T_{xx})],$$

$$r w\,\partial_r w + u r\,\partial_x w = \partial_r(r w^2) + \partial_x(r u w),$$

$$\int_0^{h(x)} (\partial_r(r w^2) + \partial_x(r u w))\,dr = \frac{d}{dx}\int_0^{h(x)} r w u\,dx + [h w(w - h' u)],$$

and

$$\int_0^{h(x)} \{\partial_r(r T_{rr}) + \partial_x(r T_{xr})\}\,dr = \frac{d}{dx}\int_0^{h(x)} r T_{xr}\,dr + [h(T_{rr} - h' T_{xr})].$$

Inserting these expressions in the averaged equations of motion gives the differential equations satisfied by $\mathcal{M}(x)$ and $\mathcal{R}(x)$. In the region $x \geqq 0$, (6) and (7) hold, whence

(16) $$\lim_{x\to\infty}\int_0^{h(x)} T_{xx}(x, r)\,r\,dr = \int_0^{h_f} -\frac{\sigma}{h_f}\,r\,dr = -\frac{\sigma h_f}{2}$$

and the averaged equations of motion reduce to (12) and (13).

The relations given by Theorem 1 are of interest in the study of die swell. For example, it is frequently assumed in approximate studies of die swell that the main mechanism for the die swell is due to elastic recovery. I think that this assumption means that when the swell is large, say $h_f > 2h_0$, the average axial stress at the exit is a tension; that is,

(17) $$\int_0^{h_0} T_{xx}(0, r)\,r\,dr > 0.$$

I believe in (17) because of the inequality (18), given in

Theorem 2.

$$\frac{\rho}{2} h_f^2 U_f^2 \left(\frac{h_f^2}{h_0^2}-1\right)+\sigma\left(\frac{h_f}{2}-\frac{h_0}{(1+h'(0)^2)^{1/2}}\right) \leqq \int_0^{h0} T_{xx}(0,r)\,r\,dr. \tag{18}$$

To prove (18), we first define the average velocity

$$\bar{u}=\frac{2}{h_0^2}\int_0^{h_0} u(0,r)\,r\,dr.$$

Conservation of mass requires that

$$\int_0^{h(x)} u(x,r)\,r\,dr=Q$$

be independent of x for $-L \leqq x<\infty$. Hence,

$$\bar{u}=\frac{h_f^2}{h_0^2} U_f=\frac{2}{h_0^2} Q$$

and

$$0 \leqq \int_0^{h_0} r\left(u(0,r)-\bar{u}\right)^2 dr=\int_0^{h_0} u^2(0,r)\,r\,dr-\frac{h_f^4}{2h_0^2} U_f^2. \tag{19}$$

Evaluation of (12) on $x=0$ gives

$$\rho \int_0^{h_0} u^2(0,r)\,r\,dr=\int_0^{h_0} T_{xx}(0,r)\,r\,dr+\tfrac{1}{2}\rho U_f^2 h_f^2+\sigma\left(\frac{h_0}{(1+h'(0)^2)^{1/2}}-\frac{h_f}{2}\right). \tag{20}$$

Taken together, (19) and (20) imply (18).

The inequality (18) shows that the assumptions

$$\sigma=0=\int_0^{h_0} T_{xx}(0,r)\,r\,dr, \tag{21}$$

which are sometimes invoked to explain swelling of viscoelastic jets (see JOSEPH, 1974, p. 126), cannot lead to swelling since (12) and (18) imply that $h_f \leqq h_0$.

Equality holds in (18) when $Q=0$, for if $Q=0$ then by the uniqueness theorem on p. 123 of JOSEPH (1974) we have $h_0=h_f$, $h'(0)=0$ and $T_{xx}=-p=-\sigma/h_j$.

In Newtonian fluids the average axial stress may be expressed in terms of the reaction pressure,

$$\int_0^{h_0} T_{xx}(0,r)\,r\,dr=-\int_0^{h_0} p(0,r)\,r\,dr. \tag{22}$$

To prove (22) we note that $T_{xx}=-p+2\mu\,\partial_x u$, $u(0,h_0)=0$ and

$$\int_0^{h_0} [\partial_x u(x,r)]_{x=0}\,dr=\left[\frac{d}{dx}\int_0^{h(x)} u(x,r)\,r\,dr\right]_{x=0}-h'(0)\,u(0,h_0)=\frac{dQ}{dx}=0.$$

To obtain the equation expressing the global conservation of momentum of the fluid in the jet and pipe (cf. § 12, JOSEPH, 1974), we integrate (13) to obtain

$$\mathcal{M}(0)-\mathcal{M}(-L)=h_0 \int_{-L}^{0} T_{rx}(x, h_0)\,dx, \tag{23}$$

where

$$\mathcal{M}(-L)=\int_0^{h_0} r(\rho \hat{u}^2-\hat{T}_{xx})\,dr$$

and $\mathcal{M}(0)$ is given by (12). Equation (23) connects the flow at the entrance to the pipe at $x=-L$ with the asymptotic upstream flow.

The analysis given in this paper applies equally, with only slight and obvious changes, to the problem of jets extruded form coaxial cylinders. Theorem 1 also holds when the pressure p is replaced by $p+\psi$, *where* ψ *is a force potential. A different form for the "Bernoulli" constant on the right side of* (12) *may be required when* $\psi \neq 0$.

This work was supported by the U.S. Army Research Office and under NSF grant 19047.

Reference

JOSEPH, D. D., Slow motion and viscometric motion. Arch. Rational Mech. Anal. **56**, 99–157 (1974).

Dept. of Aerospace Engineering
University of Minnesota
Minneapolis, MN 55455

(Received January 15, 1978)