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Short Communication

A NORMAL STRESS AMPLIFIER FOR THE SECOND NORMAL STRESS DIFFERENCE

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1. Introduction

When a viscoelastic fluid flows down a tilted trough, the free surface bulges upward in the middle. The amount of bulge is proportional to the second normal stress difference of the fluid. Wineman and Pipkin [1] were the first to suggest that the deformation of this free surface could be used to obtain information about the normal stress difference. Since then several authors [2—6] have tried to develop the theory into a useful rheometric device. The last of these used domain perturbations and carried out the calculations through order four in the tilt angle, β, of the trough. The actual amount of bulging predicted was very small (only about 0.013 mm for STP in a 2 cm wide trough at a slope of 30 deg.) and it would be difficult to get accurate values of the second normal stress difference from such an apparatus.

The magnitude of the height rise and the usefullness of the device can be enhanced by using two different fluids in the trough. As in the rod climbing problem [7], the magnitude of the height rise is roughly proportional to the reciprocal of the density difference at the interface and floating one fluid on top of another of nearly the same density can greatly increase the bulging of the interface.

2. Description of the problem

The configuration is as sketched in Fig. 1. There is a layer of a viscoelastic fluid on top of an infinitely deep layer of a second fluid which may be either

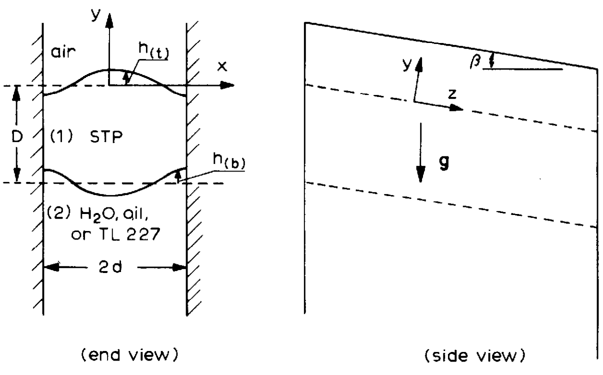

Fig. 1. A layer of STP flows down a trough on top of a second layer of fluid. The second fluid may be either Newtonian (e.g., water or oil) or non-Newtonian (e.g., TL227).

Newtonian or non-Newtonian. The top surface of the viscoelastic layer is open to the atmosphere and will be assumed to be a normal, stress-free, free surface. As the fluids flow down the channel, the variation in shear rate across the channel gives rise to a normal stress in the non-Newtonian fluids. The normal stress is, in general, not continuous across the interface, and the interface moves up or down to equilibrate the normal stresses across the interface.

The problem will be solved using a domain perturbation as in [6]. The notation used and the mathematical details are nearly the same as in that paper and the interested reader is referred to it for details of the derivation. Only the important equations and new results will be recorded here.

We first expand the variables in a series of powers of the tilt angle, β:

$$w = w^{[1]}\beta + w^{[3]}\beta^3/3! + \ldots, \tag{1}$$

$$\Phi = \Phi^{[2]}\beta^2/2! + \Phi^{[4]}\beta^4/4! + \ldots, \tag{2}$$

$$h = h^{[2]}\beta^2/2! + h^{[4]}\beta^4/4! + \ldots, \tag{3}$$

where w is the down-channel velocity, $\Phi = p + \rho g y \cos\beta$ is the pressure head and h is the height of the interface above its mean value. A secondary flow (x and y components of the velocity) does not develop until higher orders and will be neglected here. Symmetry of the problem with respect to changes of the sign of β have been incorporated into (1)—(3). Subscripts will be added when necessary to indicate whether the velocity and pressure are for fluids (1) or (2) and whether the height is that of the interface at the top (t) or bottom (b) of the viscoelastic layer.

The equation of motion at first order reduces to

$$\nabla^2 w_{(i)}^{[1]} + \rho_{(i)} g/\mu_{(i)} = 0, \qquad i = 1, 2 \tag{4}$$

and the shapes of the interfaces are determined from:

$$\sigma_{(t)} h_{(t),xx}^{[2]} - (\Delta\rho)_{(t)} g h_{(t)}^{[2]} = 2\tilde{N}_{2(t)} w_{(1),y}^{[1]\,2} - \Phi_{(1)}^{[2]} \tag{5}$$

and

$$\sigma_{(b)} h^{[2]}_{(b),xx} - (\Delta\rho)_{(b)} g h^{[2]}_{(b)} = (\Delta(2\tilde{N}_2 w^{[1]2}_{,y} - \Phi^{[2]}))_{(b)}, \tag{6}$$

where σ is the surface tension coefficient, $\Delta(\cdot)$ is the difference in $(\cdot)$ for the fluids below and above the interface, and $\tilde{N}_2 = 2\alpha_1 + \alpha_2$.

3. Solution for stress free interfaces

The solutions of (4), (5) and (6) are particularly simple when the interfaces are considered to be free of shear stresses. Then

$$w^{[1]}_{(i)} = V_{(i)} \tfrac{1}{2}(1 - \bar{x}^2), \tag{7}$$

where $V = \rho g d^2/\mu$ and $\bar{x} = x/d$. The zero-shear condition arises naturally at the STP—air or STP—water interface since the viscosity of the STP is so much greater than that of the air or water. However, eqn. (7) shows that the zero-shear condition can also be satisfied if the kinematic viscosities $\nu = \mu/\rho$ are equal since then the velocity distribution will be the same on both sides of the interface.

The first term on the right of (5) is zero because of the zero-shear-stress condition, and if it is assumed that the interface meets the wall perpendicularly, then

$$h^{[2]}_{(t)} = \frac{(\tilde{N}_2 V^2)_{(1)}}{(\sigma\zeta^2)_{(t)}} f(\bar{x}; \zeta_{(t)}) \tag{8}$$

and

$$h^{[2]}_{(b)} = \frac{(\tilde{N}_2 V^2)_{(2)} - (\tilde{N}_2 V^2)_{(1)}}{(\sigma\zeta^2)_{(b)}} f(\bar{x}; \zeta_{(b)}), \tag{9}$$

where

$$f(\bar{x}; \zeta) = 2\bar{x}^2 + \frac{4}{\zeta^2} - \frac{2}{3} - \frac{4\cosh \zeta\bar{x}}{\zeta \sinh \zeta}, \tag{10}$$

and

$$\zeta^2 = \Delta\rho g d^2/\sigma. \tag{11}$$

The behavior of $f(\bar{x}; \zeta)$ is sketched in Fig. 2 for various ζ. It can be seen that the shape of the interfaces are nearly identical — only the scales are different. Although the figure suggests that the maximum bulge occurs at the largest value of ζ, $f(\bar{x}; \zeta)$ must be divided by $\sigma\zeta^2$ to get the height. Thus, the height actually increases as ζ decreases. The relationship between the maximum bulge as given by $f(0; \zeta) - f(\pm 1; \zeta)$ and $(f(0; \zeta) - f(\pm 1; \zeta))/\zeta^2$ is shown in Table 1 for various ζ.

As an example of the magnitudes of bulges to be expected, the maximum

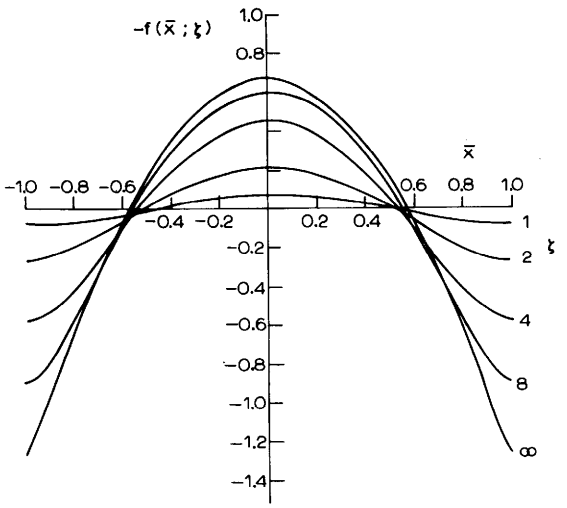

Fig. 2. Interface shapes for various dimensionless capillary radius ζ. Since $\tilde{N}_2$ is negative, the top surface will bulge upward as drawn. The bottom surface may bulge up or down as whether $(\tilde{N}_2V^2)_{(2)}$ is greater or less than $(\tilde{N}_2V^2)_{(1)}$.

bulge in a trough 2 cm wide and at an angle of $\pi/6$ radian is:

$$h(0)-h(1)=(h^{[2]}(0;\zeta)-h^{[2]}(\pm 1;\zeta))\beta^2/2!+O(\beta^4)$$

$$=\frac{\Delta(\tilde{N}_2V^2)}{\sigma\zeta^2}(f(0;\zeta)-f(\pm 1;\zeta))\beta^2/2!+O(\beta^4)$$

$$\doteq\begin{bmatrix} 0.0013\text{ cm} & \text{STP—air} \\ 0.0049\text{ cm} & \text{STP—water} \\ 0.0067\text{ cm} & \text{STP—oil} \\ 0.2421\text{ cm} & \text{STP—TL 227}\end{bmatrix}. \tag{12}$$

The maximum velocity in the STP at $\beta=\pi/6$ radians is

$$w_{\max}\doteq w^{[1]}_{\max}\beta=\tfrac{1}{2}V\beta=1.522\text{ cm/s}, \tag{13}$$

and the maximum shear rate is

$$w_{,x\max}\doteq V\beta=3.045\text{ s}^{-1}\ . \tag{14}$$

(Fluid properties used in these calculations are listed in Table 2.)

The above calculations have all used a surface tension coefficient of near 30 dynes/cm. Decreasing the surface tension has a marked effect on the bulging when the density difference is very small. As σ goes to zero, ζ goes to infinity and, from Table 1, $f(0;\zeta)-f(\pm 1;\zeta)$ goes to a maximum value of 2. The maximum bulge is then

$$h^{[2]}(0)-h^{[2]}(\pm 1)=\frac{2\Delta(\tilde{N}_2V^2)}{\sigma\zeta^2}=\frac{2\Delta(\tilde{N}_2V^2)}{\Delta\rho g d^2}$$

TABLE 1

Maximum bulge in a channel as a function of ζ

ζ	$f(\pm 1;\zeta) - f(0;\zeta)$	$\dfrac{f(\pm 1;\zeta) - f(0;\zeta)}{\zeta^2}$
0.00	0.000000	0.166667
0.10	0.001665	0.166500
0.20	0.006640	0.166003
0.30	0.014866	0.165180
0.40	0.026247	0.164042
0.42 *	0.028665	0.163801
0.50	0.040651	0.162603
0.59 **	0.056360	0.161033
0.80	0.100255	0.156649
1.00	0.151531	0.151531
2.00	0.476812	0.119203
2.05 ***	0.493655	0.117537
3.00	0.793136	0.088126
4.00	1.035972	0.064748
5.00	1.210709	0.048428
5.27 ****	1.249349	0.044909
6.00	1.336630	0.037129
8.00	1.500335	0.023443
10.00	1.600036	0.016000
15.00	1.733333	0.007704
20.00	1.800000	0.004500
25.00	1.840000	0.002944
30.00	1.866667	0.002074
40.00	1.900000	0.001187
60.00	1.933333	0.000537
80.00	1.950000	0.000305
∞	2.000000	0.000000

* STP—air interface.
** STP—water interface.
*** STP—oil interface.
**** STP—TL227 interface.

TABLE 2

Fluid properties used in the calculations

STP	$\alpha_1 =$	−1.475 g/cm
	$\alpha_2 =$	2.70 g/cm
	$\mu =$	150 g/cm s
	$\rho =$	0.896 g/cm^3
TL227	$\alpha_1 =$	−50 g/cm
	$\alpha_2 =$	83.9 g/cm
	$\mu =$	200 g/cm s
	$\rho =$	0.901 g/cm^3
$\sigma_{STP\text{—}air}$	=	31 dynes/cm
$\sigma_{STP\text{—}water}$	=	28 dynes/cm
$\sigma_{STP\text{—}oil}$	=	28 dynes/cm
$\sigma_{STP\text{—}227}$	=	28 dynes/cm
$\Delta\rho_{STP\text{—}oil}$	=	0.01 g/cm^3

which gets infinitely large as the density difference goes to zero. Therefore, if a surfactant were added to the interface to reduce the surface-tension coefficient, even greater bulges could be achieved.

4. Solution for continuous shear stresses at the interface

When both fluids are very viscous but their kinematic viscosities are different, the discontinuity of shear stresses should not be ignored. The solution to (4) is then

$$w^{[1]}_{(2)} = V_{(1)}\frac{1}{2}(1-\bar{x}^2) + \sum_{n=1}^{\infty} A_n \cos b_n\bar{x} \cosh b_n\bar{y}, \tag{15}$$

$$w^{[1]}_{(2)} = V_{(2)}\frac{1}{2}(1-\bar{x}^2) - \sum_{n=1}^{\infty} A_n \sinh b_n\bar{D} \cos b_n\bar{x} \exp[b_n(\bar{y}+\bar{D})], \tag{16}$$

where

$$A_n = -\frac{2(V_{(2)}-V_{(1)})(-1)^n}{b_n^3(\sinh b_n\bar{D} + \cosh b_n\bar{D})}, \tag{17}$$

$$b_n = (n-\tfrac{1}{2})\pi, \tag{18}$$

$\bar{D} = D/d$, and $\bar{y} = y/d$ is measured from the mean STP—air interface and is positive upwards. Note that the only difference between (15) and the trough of finite depth in [6] is the definition of the coefficient A_n. Accordingly, the equation for the shape of the STP—air interface is as given for the trough of finite depth in that paper.

The shape of the lower interface is considerably more complicated and is given by:

$$h^{[2]}_{(b)} = \frac{(\tilde{N}_2V^2)_{(2)} - (\tilde{N}_2V^2)_{(1)}}{(\sigma\zeta^2)_{(b)}} f(\bar{x}\,;\zeta_{(b)})$$
$$- \sum_{n=1}^{\infty} \frac{(\tilde{N}_2V)_{(1)}\cosh b_n\bar{D} + (\tilde{N}_2V)_{(2)}\sinh b_n\bar{D}}{\sigma_{(b)}\,g_n} 2A_n\left\{b_n\bar{x}\sin b_n\bar{x}\right.$$
$$\left. + \frac{1}{g_n}\left[(b_n^2-\zeta^2)\cos b_n\bar{x} + 2\zeta b_n(-1)^n\frac{\cosh\zeta\bar{x}}{\sinh\zeta}\right]\right\}$$
$$- \frac{\tilde{N}_{2(2)} - \tilde{N}_{2(1)}}{2\sigma_{(b)}} \sum_{n=1}^{\infty}\sum_{l=1}^{\infty} A_nA_lb_nb_l \sinh b_n\bar{D}\sinh b_l\bar{D}\left[\frac{\cos(b_n-b_l)\bar{x}}{r_{nl}}\right.$$
$$\left. + \frac{\cos(b_n+b_l)\bar{x}}{s_{nl}} - \frac{\delta_{nl}}{\zeta^2}\right]$$
$$- \sum_{n=1}^{\infty}\sum_{l=1}^{\infty} \frac{\tilde{N}_{2(1)}\cosh b_n\bar{D}\cosh b_l\bar{D} - \tilde{N}_{2(2)}\sinh b_n\bar{D}\sinh b_l\bar{D}}{2\sigma_{(b)}} A_nA_lb_nb_l$$
$$\left[\frac{\cos(b_n-b_l)\bar{x}}{r_{nl}} - \frac{\cos(b_n+b_l)\bar{x}}{s_{nl}} - \frac{\delta_{nl}}{\zeta^2}\right] \tag{19}$$

where

$$g_n = b_n^2 + \zeta^2,$$

$$r_{nl} = (b_n - b_l)^2 + \zeta^2,$$

$$s_{nl} = (b_n + b_l)^2 + \zeta^2,$$

$$\delta_{nl} = \begin{matrix} 1 & n = l \\ 0 & n \neq l. \end{matrix}$$

The first term is just the shape when shear stresses are not matched (eqn. (9)). The effect of the additional terms is to reduce the height rise by about 20% for D greater than about 2.

Acknowledgements

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