

6 The behaviour of solutions lying on an invariant 2-torus arising from the bifurcation of a periodic solution

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Dedicated to the memory of R. Bowen

6.1 Introduction

We are going to consider the problem of bifurcation of a periodic solution into an invariant two-dimensional torus, for the following autonomous differential equation in \mathbb{R}^k :

$$\frac{dV}{dt} = F(\mu, V), \quad (6.1)$$

where F is as smooth as we wish and μ is a real parameter. We assume that

$$V(t) = \hat{U}(\hat{\omega}(\mu)t, \mu) \quad (6.2)$$

is a periodic solution of (6.1) where the frequency $\hat{\omega}(\mu)$ is smooth in μ and $\hat{U}(\cdot, \mu)$ is 2π -periodic. The solution (6.2) exists for μ in a certain interval of $\mu_0 = 0$ if it exists at $\mu_0 = 0$ and the Floquet multiplier equal to 1 is simple. If the periodic solution (6.2) loses its stability as μ is increased past zero then, in general, an invariant two-dimensional torus will bifurcate, either to the supercritical ($\mu > 0$) or subcritical ($\mu < 0$) side. The supercritical torus attracts nearby trajectories and the subcritical torus repels them.

The results roughly summarized in the last paragraph were derived independently by Neimark [15], Sacker [17] and Ruelle and Takens [16]. A precise statement of these results can be formulated as follows. We introduce the linear operator

$$J(\mu) = -\hat{\omega}(\mu) \frac{d}{ds} + D_\nu F[\mu, \hat{U}(s, \mu)] \quad (6.3)$$

acting in the space $C^0(T^1; \mathbb{C}^k)$ of continuous 2π -periodic functions taking values in \mathbb{C}^k . The domain of $J(\mu)$ is $C^1(T^1; \mathbb{C}^k)$ and $J(\mu)$ is a closed Fredholm operator of index zero. (This follows directly from Floquet theory.) The spectrum of $J(\mu)$ consists of eigenvalues of finite multiplicities. The eigenvalues $\sigma(\mu)$ of $J(\mu)$ are Floquet exponents and $\zeta(\cdot, \mu)$ is an eigenvector belonging to σ which satisfies

$$\hat{\omega} \frac{d\zeta}{ds} + \sigma\zeta = D_\nu F[\mu, \hat{U}(s, \mu)]\zeta, \quad \zeta \in C^1(T^1; \mathbb{C}^k).$$

The number $\lambda(\mu) = e^{\sigma(\mu)2\pi/\hat{\omega}(\mu)}$ is called a *Floquet multiplier*. Equation (6.1) is invariant to translations of the origin of t . So if $\hat{U}(s, \mu)$ is a solution, so is $\hat{U}(s + \delta, \mu)$ and differentiating (6.1) with respect to δ , at $\delta = 0$, using $V(t) = \hat{U}(s + \delta, \mu)$, $s = \hat{\omega}(\mu)t$, we obtain

$$\left. \frac{d\hat{U}}{d\delta}(s + \delta, \mu) \right|_{\delta=0} = \frac{d\hat{U}}{ds}(s, \mu)$$

and

$$J(\mu) \frac{d\hat{U}}{ds}(\cdot, \mu) = 0. \quad (6.4)$$

Zero is always an eigenvalue with eigenfunction $d\hat{U}/ds$ of $J(\mu)$.

If the Floquet exponents $p i \hat{\omega}$, $p \in \mathbb{Z}$ (corresponding each and every one to the unique multiplier 1) are simple eigenvalues of J , and if all the remaining part of the spectrum of J is on the left side of the complex plane, then the periodic orbit \hat{U} is stable. If some Floquet exponent has a positive real part, the periodic orbit \hat{U} is unstable.

Having laid down requisite preliminaries, we now give a precise statement of the theorem about bifurcation of the invariant two-dimensional torus.

Theorem 6.1 Assume that zero and $i\eta_0$ are simple eigenvalues of $J(0)$ such that

$$0 < \eta_0/\hat{\omega}_0 < 1; \quad \eta_0/\hat{\omega}_0 \neq \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}; \quad \hat{\omega}_0 \stackrel{\text{defn}}{=} \hat{\omega}(0) \quad (6.5)$$

and that the transversality condition

$$\text{Re}(d\sigma(\mu)/d\mu)_{\mu=0} > 0$$

holds at criticality for the eigenvalue $\sigma(\mu)$ of $J(\mu)$ where $\sigma(0) = i\eta_0$. Then, in general, a two-dimensional torus which is invariant under (6.1) bifurcates either for $\mu > 0$ or for $\mu < 0$ and is such that the torus and the closed orbit \hat{U} coincide at $\mu = 0$. If the torus exists when $\mu > 0$ (supercritical) it is stable and attracts nearby trajectories. If the torus exists when $\mu < 0$ (subcritical) it is unstable and repels nearby trajectories.

Remark 1 'In general' means that $\text{Re } a_1(0) \neq 0$, where $a_1(0)$ is defined by (6.50).

Remark 2 If σ is a Floquet exponent $\sigma + pi\hat{\omega}$ is also a Floquet exponent for any $p \in \mathbb{Z}$. Without losing generality we may restrict our consideration to $0 < \eta_0 < \hat{\omega}_0$, η_0 is not a multiple of $\hat{\omega}_0$, and exclude the other cases of strong resonance mentioned under (6.5). In these cases the critical Floquet multiplier is a root of unity of order ≤ 4 and subharmonic solutions may bifurcate. These subharmonic solutions have a period close to a multiple of $2\pi/\hat{\omega}(\mu)$ and in general they do not lie on an invariant 2-torus. (It is well to note here that subharmonic solutions at points of weak resonance of order ≥ 5 do lie on an invariant torus. Moreover, in some special circumstances there can be a torus at order 4.)

Remark 3 Further details about the problem of this paper and related mathematical results can be found in our book on elementary bifurcation and stability theory [12].

Invariant 2-tori are surely common in nature, but only recently have they become an object of special attention in science and mathematics. A bifurcated 2-torus was observed and its principal part was calculated in a study [4] of a problem in electricity. Hydrodynamical systems also have such dynamics, but the study of the dynamics is still mostly experimental.

In the problem of incompressible flow between two concentric cylinders (the Taylor problem), with the inner one in steady rotation with angular speed Ω and the outer one fixed, a sequence of bifurcation events leading to a 2-torus have been observed first by Swinney and Gollub and then, with more careful and detailed measurements by Fenstermacher, Swinney and Gollub (see [5] and [6]) and Cognet (see [20]). Analogous observations of sequences of bifurcations leading to a 2-torus in small boxes of fluid heated from below (the Bénard problem) have been reported by many authors [1, 5, 7, 8, 14]. In these experiments the 2-torus is identified in the following way: the power spectrum of some fluctuating quantity, say a velocity component, is obtained. When the solution is strictly periodic the power spectrum exhibits peaks which can be identified with harmonics of the fundamental frequency ω_1 . After bifurcation of the periodic solution it is no longer possible to identify the peaks of the power spectrum with harmonics of ω_1 . Two frequencies ω_1, ω_2 are required for this and each frequency, as well as the ratio ω_1/ω_2 of frequencies, are continuous functions of μ . In some experiments [14, 8] there are intervals of μ , not very close to the bifurcation point, on which ω_1/ω_2 is constant and rational. The solutions corresponding to these constant values are periodic and subharmonic and are said to exhibit 'frequency locking'. These 'lock-ins' seem to play an important role in the sequence of events leading to turbulence.

In fact, if it be granted that the flow takes place on an invariant 2-torus, in a neighbourhood of the periodic orbit \hat{U} , then a theorem of Kolmogorov (cited in [18]) completed by the regular conjugation of M.

Herman (see [11]) asserts that doubly periodic flow with incommensurate periods occurs only for special values of μ , even when μ is small.

So, the problem is clearly to analyse the nature of the limit flow, lying on the invariant 2-torus. We will show that, in any event, such limit flows may be approximated near criticality by doubly periodic ones; that is, of the form $V(\omega_1 t, \omega_2 t)$, where V is 2π -periodic in both arguments and ω_1 and ω_2 depend on μ .

For a more precise analysis we introduce the Poincaré map Φ_μ on a hyperplane transverse to the trajectory $\hat{U}(\cdot, \mu)$ at the point $\hat{U}(0, \mu)$. Φ_μ is the map of first return of a trajectory starting from such a hyperplane. An invariant 2-torus for (6.1) corresponds to an invariant circle under Φ_μ . This invariant circle is a closed curve diffeomorphic to the circle T^1 and the restriction f_μ of the map Φ_μ to this invariant circle is a diffeomorphism of T^1 on which we may define the rotation number $\rho(\mu)$. This number represents the asymptotic rotation equivalent to f_μ when it is iterated infinitely many times (see (6.78)). The theorem of Kolmogorov (completed by the regular conjugation of M. Herman) says that the flow on the invariant torus is quasiperiodic for values of μ such that $\rho(\mu)$ is an irrational number 'badly approximated' by rationals. For values of μ such that $\rho(\mu)$ is rational, there are periodic orbits on the torus and one of them is attractive. So the limit flow is periodic. The rotation number $\rho(\mu)$ is a continuous function of μ but it is differentiable only at certain special points.

In this chapter we are going to obtain asymptotic representation of the torus and the trajectories on it which are doubly periodic or, in exceptional circumstances, to be specified precisely, we obtain subharmonic solutions, which correspond to closed orbits on the torus.

We wish to thank A. Chenciner for many useful discussions.

6.2 Normal forms

To simplify the computation we change variables. The bifurcated torus, the trajectories on the torus, and the law of movement on these trajectories are more easily expressed in the new variables.

6.2.1 Reduction to a non-autonomous problem in \mathbb{R}^{k-1}

We first introduce new coordinates (τ, u) defined by

$$V = \hat{U}(\tau, \mu) + u, \quad (6.6)$$

where

$$(u, X^*(\tau)) = 0, \quad (6.7)$$

$X^*(\tau)$ being any 2π -periodic smooth vector function satisfying

$$\left(\frac{d\hat{U}}{d\tau}(\tau, \mu), X^*(\tau)\right) \neq 0. \quad (6.8)$$

With a suitable choice of coordinates in the hyperplane defined by (6.7), we have u in \mathbb{R}^{k-1} .

It is not difficult to derive the equations governed by τ and u ; we obtain

$$\frac{d\tau}{dt} = f(\mu, \tau, u) \quad \text{in } \mathbb{R}, \quad (6.9a)$$

$$\frac{du}{dt} = \tilde{F}(\mu, \tau, u) \quad \text{in } \mathbb{R}^{k-1}, \quad (6.9b)$$

where

$$f(\mu, \tau, u) = (F[\mu, \hat{U}(\tau, \mu) + u], X^*(\tau)) \times \left[\left(\frac{d\hat{U}}{dt}(\tau, \mu), X^*(\tau) \right) - \left(u, \frac{dX^*}{d\tau}(\tau) \right) \right]^{-1} \quad (6.10)$$

$$\tilde{F}(\mu, \tau, u) = F[\mu, \hat{U}(\tau, \mu) + u] - f(\mu, \tau, u) \frac{d\hat{U}}{d\tau}(\tau, \mu). \quad (6.11)$$

We shall agree that τ in (6.9) is defined in the torus $T^1 = \mathbb{R}/2\pi\mathbb{Z}$ even though in (6.9) we have τ in \mathbb{R} . This means for instance that f and \tilde{F} are 2π -periodic in τ . Moreover, by construction we have

$$\begin{aligned} f(\mu, \tau, 0) &= \hat{\omega}(\mu), \\ \tilde{F}(\mu, \tau, 0) &= 0. \end{aligned} \quad (6.12)$$

Changing variables from t to τ in (6.9b) we find that

$$\frac{du}{d\tau} = G(\mu, \tau, u) = \frac{\tilde{F}(\mu, \tau, u)}{f(\mu, \tau, u)}, \quad (6.13)$$

where G is 2π -periodic in τ and such that

$$G(\mu, \tau, 0) = 0, \quad (6.14a)$$

$$D_u G(\mu, \tau, 0) = \frac{1}{\hat{\omega}(\mu)} \{ D_v F[\mu, \hat{U}(\tau, \mu)] - \frac{d\hat{U}}{d\tau}(\tau, \mu) \cdot D_u f(\mu, \tau, 0) \}. \quad (6.14b)$$

Remark Equation (6.13) lies in \mathbb{R}^{k-1} . We pay for this reduction in dimensionality by the 2π -periodicity of G . From (6.14b) we see that $D_u G(\mu, \tau, 0)$ is a linear operator in \mathbb{R}^k . However, since the operator \tilde{J}_μ defined by (6.15) leaves invariant the hyperplane defined by (6.7), we may consider it in \mathbb{R}^{k-1} .

The idea now is to work with solutions of equation (6.13), to determine the bifurcated invariant torus and the trajectories on it. To obtain the law

of movement however we need to solve (6.9a) with u as a known function of τ and μ .

We begin the analysis of (6.13) by defining a linear operator

$$\tilde{J}(\mu) = -\frac{d}{d\tau} + D_u G(\mu, \tau, 0) \quad (6.15)$$

acting in $C^0(T^1; \mathbb{C}^{k-1})$ and with domain $C^1(T^1, \mathbb{C}^{k-1})$. To use the information about the spectrum of $J(\mu)$ specified under Theorem 6.1, we write a lemma comparing the spectra of $J(\mu)$ and $\tilde{J}(\mu)$

Lemma 6.1 *If 0 is a simple eigenvalue of $J(\mu)$ (it is always an eigenvalue) then the spectrum of $\tilde{J}(\mu)$ is identical with $1/\hat{\omega}(\mu)$ times the spectrum of $J(\mu)$ after deleting the eigenvalues $\sigma = 0 + pi\hat{\omega}$, $p \in \mathbb{Z}$ from the spectrum of $J(\mu)$. Moreover, the multiplicities of the eigenvalues $\sigma(\mu) \neq 0 + pi\hat{\omega}$ of $J(\mu)$ are the same as those of $\sigma(\mu)/\hat{\omega}(\mu)$ of $\tilde{J}(\mu)$.*

Proof Let us consider an eigenvector Γ of $J(\mu)$ belonging to the eigenvalue $\sigma(\mu)$. It is convenient to decompose

$$\Gamma(\tau) = \alpha(\tau) \frac{d\hat{U}}{d\tau}(\tau, \mu) + \tilde{\Gamma}(\tau), \quad (6.16)$$

where $(\tilde{\Gamma}(\tau), X^*(\tau)) = 0$. We have

$$J(\mu)\Gamma = \sigma(\mu)\Gamma, \quad (6.17)$$

which leads to

$$J(\mu)\tilde{\Gamma} + \alpha J(\mu) \frac{d\hat{U}}{d\tau} - \hat{\omega}(\mu) \frac{d\alpha}{d\tau} \frac{d\hat{U}}{d\tau} = \sigma(\mu)\tilde{\Gamma} + \sigma(\mu)\alpha \frac{d\hat{U}}{d\tau}.$$

Now, thanks to (6.4), and using the definition (6.15), we find

$$\hat{\omega}(\mu)\tilde{J}(\mu)\tilde{\Gamma} = \sigma(\mu)\tilde{\Gamma}, \quad (6.18a)$$

$$\hat{\omega}(\mu) \frac{d\alpha}{d\tau} + \sigma(\mu)\alpha(\tau) = D_u f(\mu, \tau, 0)\tilde{\Gamma}(\tau). \quad (6.18b)$$

Equation (6.18a) shows that if $\tilde{\Gamma} \neq 0$ then $\sigma(\mu)/\hat{\omega}(\mu)$ is an eigenvalue of $\tilde{J}(\mu)$. If $\tilde{\Gamma} = 0$, which means that

$$\Gamma(\tau) = \alpha(\tau) \frac{d\hat{U}}{d\tau}(\tau, \mu),$$

we obtain

$$\hat{\omega}(\mu) \frac{d\alpha}{d\tau} + \sigma(\mu)\alpha(\tau) = 0.$$

This has 2π -periodic solutions only if $\sigma(\mu) = pi\hat{\omega}(\mu)$ for an integer p . Conversely, if $\tilde{\Gamma}$ is an eigenvector of $\tilde{J}(\mu)$ belonging to the eigenvalue $\sigma(\mu)/\hat{\omega}(\mu)$ we may compute a 2π -periodic solution $\alpha(\tau)$ of (6.18b)

whenever $i\sigma/\hat{\omega}$ is not an integer. But (6.18) is equivalent to (6.16), (6.17), so this means that $\sigma(\mu)$ is an eigenvalue of $J(\mu)$. It is also easy to show that if $\sigma/\hat{\omega} = pi$ is an eigenvalue of \tilde{J} for an integer p , then $pi\hat{\omega}$ is a multiple eigenvalue of J , which is excluded by the assumptions. We may show in an analogous way that the multiplicities of $\sigma(\mu)$ for $J(\mu)$ and $\sigma(\mu)/\hat{\omega}(\mu)$ for $\tilde{J}(\mu)$ are the same. The relation (6.16) defines the correspondence between the eigenvectors of $J(\mu)$ and those of $\tilde{J}(\mu)$. ■

6.2.2 Decomposition of equation (6.13)

Thanks to the assumptions of Theorem 6.1, we know that $\pm i\eta_0/\hat{\omega}_0$ are simple eigenvalues of $\tilde{J}(0)$. Moreover, we assume a strict crossing condition for the simple eigenvalues $\sigma(\mu)/\hat{\omega}(\mu)$ and $\bar{\sigma}(\mu)/\hat{\omega}(\mu)$ which reduce to $\pm i\eta_0/\hat{\omega}_0$ when $\mu = 0$. We then define eigenvectors $\zeta(\mu, \cdot)$ and $\bar{\zeta}(\mu, \cdot) \in C^1(T^1; \mathbb{C}^{k-1})$ belonging to these two eigenvalues of $\tilde{J}(\mu)$ and we introduce the adjoint operator with the same domain, in the same space:

$$\tilde{J}^*(\mu) = \frac{d}{d\tau} + [D_u G(\mu, \tau, 0)]^* \quad (6.19)$$

This adjoint operator is also a Fredholm operator and has properties analogous to those of $\tilde{J}(\mu)$. We may define the eigenvectors $\zeta^*(\mu, \cdot)$ and $\bar{\zeta}^*(\mu, \cdot)$ belonging to eigenvalues $\bar{\sigma}(\mu)/\hat{\omega}(\mu)$ and $\sigma(\mu)/\hat{\omega}(\mu)$ respectively, which are such that

$$\begin{aligned} (\zeta(\mu, \tau), \zeta^*(\mu, \tau)) &= 1 \quad (\text{independent of } \tau), \\ (\bar{\zeta}(\mu, \tau), \zeta^*(\mu, \tau)) &= 0. \end{aligned} \quad (6.20)$$

We may decompose u as follows:

$$u = z\zeta(\mu, \tau) + \bar{z}\bar{\zeta}(\mu, \tau) + w, \quad (6.21)$$

where

$$\begin{aligned} (w, \zeta^*(\mu, \tau)) &= (w, \bar{\zeta}^*(\mu, \tau)) = 0, \\ z &= (u, \zeta^*(\mu, \tau)). \end{aligned}$$

It is then easy to verify that equation (6.13) may be written as

$$\frac{dz}{d\tau} = \frac{\sigma(\mu)z}{\hat{\omega}(\mu)} + b(\tau, \mu, z, \bar{z}, w), \quad (6.22a)$$

$$\frac{dw}{d\tau} = D_u G(\mu, \tau, 0)w + B(\tau, \mu, z, \bar{z}, w), \quad (6.22b)$$

with the estimate

$$|b(\tau, \mu, z, \bar{z}, w)| + \|B(\tau, \mu, z, \bar{z}, w)\| = O(|z| + \|w\|)^2. \quad (6.23)$$

We remark that $w(\tau)$ lies in a (μ, τ) -dependent subspace of \mathbb{R}^{k-1} , of dimension $k-3$. The decomposition used here is an application of Floquet theory which may be extended to the infinite-dimensional case.

Moreover, in the subspace of w 's the linearization of (6.22b),

$$\frac{d\bar{w}}{d\tau} = D_u G(\mu, \tau, 0)\bar{w},$$

leads to an exponential decay of the norm of $\bar{w}(\tau)$, when $\tau \rightarrow \infty$. This decay is caused by the elimination of the part of the spectrum of $\tilde{J}(\mu)$ which crosses the imaginary axis.

For the next computations we write

$$\sigma(\mu)/\hat{\omega}(\mu) = \bar{\sigma}(\mu) = i\eta_0/\hat{\omega}(0) + \mu(\bar{\xi}(\mu) + i\hat{\omega}(\mu)). \quad (6.24)$$

The transversality condition under (6.5) leads to

$$\bar{\xi}_1 = \bar{\xi}(0) > 0. \quad (6.25)$$

6.2.3 Normal form of the system (6.22)

The idea is now to change variables to simplify the system of equations (6.22). We first need to write the nonlinear terms in (6.22) in a particular way:

$$\begin{aligned} b(\tau, \mu, z, \bar{z}, w) &= b(\tau, \mu, z, \bar{z}, 0) + b_1(\tau, \mu, z, \bar{z}, w), \\ B(\tau, \mu, z, \bar{z}, w) &= B(\tau, \mu, z, \bar{z}, 0) + B_1(\tau, \mu, z, \bar{z}, w), \end{aligned}$$

with

$$|b_1(\tau, \mu, z, \bar{z}, w)| + \|B_1(\tau, \mu, z, \bar{z}, w)\| = O(|z| \|w\| + \|w\|^2) \quad (6.26)$$

and

$$\begin{aligned} b(\tau, \mu, z, \bar{z}, 0) &= \sum_{p+q \geq 2}^N z^p \bar{z}^q \hat{b}_{pq}(\tau, \mu) + O(|z|^{N+1}), \\ B(\tau, \mu, z, \bar{z}, 0) &= \sum_{p+q \geq 2}^N z^p \bar{z}^q \hat{B}_{pq}(\tau, \mu) + O(|z|^{N+1}). \end{aligned} \quad (6.27)$$

Our change of variables will be such that the terms of order less than $|z|^N$ in $B(\tau, \mu, z, \bar{z}, 0)$ and in $b(\tau, \mu, z, \bar{z}, 0)$ will either be suppressed or simplified.

The change of variables is defined by

$$y = z + \gamma(\tau, \mu, z, \bar{z}), \quad (6.28a)$$

$$Y = w + \Gamma(\tau, \mu, z, \bar{z}), \quad (6.28b)$$

where

$$(Y, \zeta^*(\tau, \mu)) = (\Gamma, \zeta^*(\tau, \mu)) = 0,$$

and where γ and Γ are to be determined smooth functions, 2π -periodic in τ , at least quadratic in (z, \bar{z}) . We shall construct γ and Γ as polynomials

of degree N in z, \bar{z} with 2π -periodic coefficients:

$$\gamma(\tau, \mu, z, \bar{z}) = \sum_{p+q \geq 2}^N z^p \bar{z}^q \gamma_{pq}(\tau, \mu), \tag{6.29}$$

$$\Gamma(\tau, \mu, z, \bar{z}) = \sum_{p+q \geq 2}^N z^p \bar{z}^q \Gamma_{pq}(\tau, \mu).$$

We may express z as a function of y, \bar{y} by inverting (6.28a) in a neighbourhood of zero. After eliminating z, \bar{z} and w in (6.22) with y, \bar{y} and Y we get

$$\frac{dy}{d\tau} = \bar{\sigma}y + \sum_{p+q \geq 2}^N y^p \bar{y}^q \left\{ \frac{d\gamma_{pq}}{d\tau} + [(p-1)\bar{\sigma} + q\bar{\sigma}] \gamma_{pq} + b_{pq} \right\} + \bar{b}_1(\tau, \mu, y, \bar{y}, Y), \tag{6.30}$$

$$\begin{aligned} \frac{dY}{d\tau} &= D_u G(\mu, \tau, 0)Y + \sum_{p+q \geq 2}^N y^p \bar{y}^q \\ &\times \left\{ \frac{d}{d\tau} \Gamma_{pq} - D_u G(\mu, \tau, 0)\Gamma_{pq} + (p\bar{\sigma} + q\bar{\sigma})\Gamma_{pq} + B_{pq} \right\} \\ &+ \bar{B}_1(\tau, \mu, y, \bar{y}, Y), \end{aligned} \tag{6.31}$$

where

$$|\bar{b}_1(\tau, \mu, y, \bar{y}, Y)| + \|\bar{B}_1(\tau, \mu, y, \bar{y}, Y)\| = O(|y|^{N+1} + |y| \|Y\| + \|Y\|^2),$$

and where b_{pq} and B_{pq} are functions of γ_{ln} and Γ_{ln} with $l+n \leq p+q-1$, and 2π -periodic in τ .

We want to choose γ_{pq} and Γ_{pq} to simplify (6.30) and (6.31). For this simplification we need to prove two more lemmas.

Lemma 6.2 *The problem is to find $\Gamma \in C^{l+1}(T^1; \mathbb{C}^{k-1})$ satisfying*

$$\frac{d\Gamma}{d\tau} - D_u G(\mu, \tau, 0)\Gamma + (p\bar{\sigma} + q\bar{\sigma})\Gamma + B = 0, \tag{6.32}$$

where B is given in $C^l(T^1; \mathbb{C}^{k-1})$ and satisfies

$$(B(\tau), \zeta^*(\mu, \tau)) = (B(\tau), \bar{\zeta}^*(\mu, \tau)) = 0. \tag{6.33}$$

This problem admits a unique solution such that

$$(\Gamma(\tau), \zeta^*(\mu, \tau)) = (\Gamma(\tau), \bar{\zeta}^*(\mu, \tau)) = 0. \tag{6.34}$$

Lemma 6.3 *The problem is to find $\gamma \in C^{l+1}(T^1; \mathbb{C})$ satisfying*

$$\frac{d\gamma}{d\tau} + (\bar{\sigma}(p-1) + \bar{\sigma}q)\gamma + b = 0, \tag{6.35}$$

where b is given in $C^l(T^1; \mathbb{C})$. In the case when $(p-q-1)\eta_0/\hat{\omega}(0) \notin \mathbb{Z}$,

there is a unique solution γ smooth in μ close to 0. In the case when $(p-q-1)\eta_0/\hat{\omega}(0) = -l_0 \in \mathbb{Z}$, there is a solution γ smooth in μ only if the Fourier coefficient

$$b_{l_0} = \frac{1}{2\pi} \int_0^{2\pi} b(\tau) e^{-il_0\tau} d\tau = 0. \tag{6.36}$$

Moreover, if we require $\gamma_{l_0} = 0$, the solution is unique.

Proof of Lemma 6.2 This lemma rests on the fact that $p\bar{\sigma} + q\bar{\sigma}$ is not a Floquet exponent of the operator $\bar{J}(\mu)$ in the invariant subspace of functions orthogonal to $\zeta^*(\mu, \tau)$ and $\bar{\zeta}^*(\mu, \tau)$. Let us denote the fundamental solution matrix by $S_\mu(\tau, s)$, where

$$S_\mu(\tau, \tau) = \mathbb{I} \quad (\text{identity}),$$

$$\frac{\partial}{\partial \tau} S_\mu(\tau, s) = D_u G(\mu, \tau, 0) S_\mu(\tau, s). \tag{6.37}$$

Equation (6.32) is equivalent to

$$\begin{aligned} \Gamma(\tau) &= e^{-(p\bar{\sigma} + q\bar{\sigma})\tau} S_\mu(\tau, 0) \Gamma(0) \\ &\quad - \int_0^\tau e^{-(p\bar{\sigma} + q\bar{\sigma})(\tau-s)} S_\mu(\tau, s) B(s) ds. \end{aligned} \tag{6.38}$$

Requiring now that $\Gamma(\tau) = \Gamma(\tau + 2\pi)$, we get

$$\begin{aligned} [1 - e^{-(p\bar{\sigma} + q\bar{\sigma})2\pi} S_\mu(2\pi, 0)] \Gamma(0) \\ = - \int_0^{2\pi} e^{-(p\bar{\sigma} + q\bar{\sigma})(2\pi-s)} S_\mu(2\pi, s) B(s) ds. \end{aligned} \tag{6.39}$$

For $\mu = 0$, $e^{(p\bar{\sigma} + q\bar{\sigma})2\pi} = e^{2i\pi\eta_0(p-q)/\hat{\omega}(0)}$ belongs to the unit circle. The spectrum of $S_\mu(2\pi, 0)$ is formed by the two simple eigenvalues $\lambda(\mu) = \exp[2\pi\sigma(\mu)/\hat{\omega}(\mu)]$ and $\bar{\lambda}(\mu)$ and other values located inside a disc of radius less than one independent of μ when μ is small. It follows that the subspace orthogonal to $\zeta^*(\mu, 0), \bar{\zeta}^*(\mu, 0)$ (eigenvectors of $S_\mu^*(2\pi, 0)$ belonging to $\bar{\lambda}(\mu)$ and $\lambda(\mu)$) is invariant under $S_\mu(2\pi, 0)$. In this subspace the spectrum of $S_\mu(2\pi, 0) \exp[-(p\bar{\sigma} + q\bar{\sigma})2\pi]$ is entirely inside a disc of radius < 1 independent of μ . The condition (6.39) leads to a second member of (6.39) in this subspace (see Chapter V.4 in [11]). So (6.39) is solvable with respect to $\Gamma(0)$ in this subspace. Hence (6.38) gives a smooth $\Gamma(\tau)$ which automatically satisfies (6.34). ■

Proof of Lemma 6.3 A Fourier decomposition of (6.35) leads to

$$[il + (p-1)\bar{\sigma} + q\bar{\sigma}]\gamma_l + b_l = 0. \tag{6.40}$$

For $\mu = 0$, $(p-1)\bar{\sigma} + q\bar{\sigma} = i\eta_0/\hat{\omega}(0)[p-q-1]$, hence (6.36) is a necessary condition for solvability and γ , given by its Fourier series, is unique if $\gamma_{l_0} = 0$. Moreover, the differential equation (6.35) may be solved explicitly and the solution $\gamma(\tau, \mu)$ is smooth in μ provided that (6.36) is satisfied.

Lemmas 6.2 and 6.3 are now used to choose γ_{pq} and Γ_{pq} to simplify (6.30) and (6.31). We find that

$$\frac{dY}{d\tau} = D_u G(\mu, \tau, 0)Y + \tilde{B}_1(\tau, \mu, y, \bar{y}, Y) \tag{6.41}$$

and

$$\frac{dy}{d\tau} = \tilde{\sigma}y + \sum_{q \geq 1}^{2q+1 \leq N} y^{q+1} \bar{y}^q a_q(\mu) + \tilde{b}_1(\tau, \mu, y, \bar{y}, Y) \tag{6.42}$$

if $\eta_0/\hat{\omega}(0) \notin \mathbb{Q}$, or

$$\begin{aligned} \frac{dy}{d\tau} = & \tilde{\sigma}y + \sum_{q \geq 1}^{2q+1 \leq N} y^{q+1} \bar{y}^q a_q(\mu) \\ & + \sum_{k > 0} \sum_{q \geq 0}^{2q-1+kn \leq N} [y^{q+1+kn} \bar{y}^q a_{q,k}(\mu) e^{-ikm\tau} \\ & + y^q \bar{y}^{q-1+kn} a_{q,-k}(\mu) e^{ikm\tau}] + \tilde{b}_1(\tau, \mu, y, \bar{y}, Y) \end{aligned} \tag{6.43}$$

if $\eta_0/\hat{\omega}(0) = m/n \in \mathbb{Q}$, $n \geq 5$ (because of the assumption of Theorem 6.1). The coefficients $a_q, a_{q,k}$ and $a_{q,-k}$ may be easily computed from the original coefficients $\hat{b}_{pq}, \hat{B}_{pq}$. We cannot suppress all coefficients of $y^p \bar{y}^q$ in (6.42), (6.43) (see Lemma 6.3). We choose γ_{pq} such that $(\gamma_{pq})_0 = 0$ if $(p-q-1)\eta_0/\hat{\omega}(0) = -l_0$, which is realized for $p=q+1$ and also if $\eta_0/\hat{\omega}(0) = m/n$ for $p-q-1 = kn$, $k \in \mathbb{Z}$ (hence $l_0 = -km$).

The form (6.41), (6.42) or (6.43) of the differential equation, together with the estimate below (6.31), is called the *normal form* of the system.

In fact a more suitable form can be framed in polar coordinates. In the rational case,

$$y = e^{in_0\tau/\hat{\omega}_0} \rho e^{i\theta} \tag{6.44}$$

Hence (6.43) becomes

$$\begin{aligned} \frac{d\rho}{d\tau} = & \rho \left\{ \mu \tilde{\xi}(\mu) + \sum_{q \geq 1}^{2q+1 \leq N} \text{Re } a_q(\mu) \rho^{2q} \right. \\ & + \sum_{k > 0} \sum_{q \geq 0}^{2q+1+kn \leq N} \rho^{2q+kn} \text{Re}(a_{q,k}(\mu) e^{ikn\theta}) \\ & \left. + \sum_{k > 0} \sum_{q \geq 0}^{2q-1+kn \leq N} \rho^{2q-2+kn} \text{Re}(a_{q,-k}(\mu) e^{-ikn\theta}) \right\} \\ & + R_1(\tau, \mu, \rho, \theta, Y), \end{aligned} \tag{6.45}$$

$$\begin{aligned} \rho \frac{d\theta}{d\tau} = & \rho \left\{ \mu \tilde{\omega}(\mu) + \sum_{q \geq 1}^{2q+1 \leq N} \text{Im } a_q(\mu) \rho^{2q} \right. \\ & + \sum_{k > 0} \sum_{q \geq 0}^{2q+1+kn \leq N} \rho^{2q+kn} \text{Im}(a_{q,k}(\mu) e^{ikn\theta}) \\ & \left. + \sum_{k > 0} \sum_{q \geq 0}^{2q-1+kn \leq N} \rho^{2q-2+kn} \text{Im}(a_{q,-k}(\mu) e^{-ikn\theta}) \right\} \\ & + R_2(\tau, \mu, \rho, \theta, Y), \end{aligned} \tag{6.46}$$

where

$$|R_j(\tau, \mu, \rho, \theta, Y)| = O(|\rho|^{N+1} + |\rho| \|Y\| + \|Y\|^2), \tag{6.47}$$

$j = 1, 2.$

When $\eta_0/\hat{\omega}(0)$ is irrational we put

$$y = \rho e^{i\theta} \tag{6.48}$$

and instead of (6.45), (6.46) we have

$$\frac{d\rho}{d\tau} = \rho \left\{ \mu \tilde{\xi}(\mu) + \sum_{q \geq 1}^{2q+1 \leq N} \text{Re } a_q(\mu) \rho^{2q} \right\} + R_1(\tau, \mu, \rho, \theta, Y), \tag{6.45'}$$

$$\rho \frac{d\theta}{d\tau} = \rho \left\{ \eta_0/\hat{\omega}(0) + \mu \tilde{\omega}(\mu) + \sum_{q \geq 1}^{2q+1 \leq N} \text{Im } a_q(\mu) \rho^{2q} \right\} + R_2(\tau, \mu, \rho, \theta, Y). \tag{6.46'}$$

It follows from (6.45), (6.46) or (6.45'), (6.46') that if we truncate by suppressing R_1 and R_2 , we shall obtain an autonomous system without Y . (By 'autonomous' we mean 'independent of τ ' where τ is not the time!) The idea is to compute the trajectories of the truncated equations, and then to give an estimate relating the real trajectories on the invariant torus to the trajectories of the truncated equation.

Remark (6.41), (6.45), (6.46) contain the equations which were derived by the method of averaging by Chow and Mallet-Paret [4] as a special case in a low order truncation. Our method can be regarded as an extension and improvement of the method of averaging used in [4].

6.3 The bifurcated torus

We first consider the irrational case, using equations (6.41), (6.45'), (6.46'). If we suppress R_1 and R_2 in (6.45'), (6.46'), we obtain an autonomous system. In fact, for this truncated system we obtain an invariant circle

$$\begin{aligned} \rho_0 &= \epsilon, \\ \mu &= \mu(\epsilon) = \mu(-\epsilon) \end{aligned}$$

solving

$$\mu \tilde{\xi}(\mu) + \sum_{q \geq 1}^{2q+1 \leq N} \text{Re } a_q(\mu) \epsilon^{2q} = 0. \tag{6.49}$$

And, if we assume

$$\text{Re } a_1(0) \neq 0, \tag{6.50}$$

then, recalling that $\tilde{\xi}_1 > 0$ (the transversality condition (6.25)), we have

$$\mu(\epsilon) = -\frac{\text{Re } a_1(0)}{\tilde{\xi}_1} \epsilon^2 + O(\epsilon^4) = \sum_{p=1}^{2p \leq N} \mu_{2p} \epsilon^{2p} + O(\epsilon^{N+1}). \tag{6.51}$$

Changing variables,

$$\rho = \varepsilon(1 + \varepsilon^{N-3}\tilde{\rho}), \quad Y = \varepsilon^N \tilde{Y}, \quad (6.52)$$

we obtain the new system

$$\frac{d\tilde{\rho}}{d\tau} = -2\tilde{\xi}_1 \tilde{\rho} \mu_2 \varepsilon^2 + \varepsilon^3 \tilde{R}_1(\tau, \varepsilon, \tilde{\rho}, \theta, \tilde{Y}), \quad (6.53a)$$

$$\frac{d\tilde{Y}}{d\tau} = D_u G(\mu(\varepsilon), \tau, 0) \tilde{Y} + \varepsilon \tilde{B}_2(\tau, \varepsilon, \tilde{\rho}, \theta, \tilde{Y}), \quad (6.53b)$$

$$\frac{d\theta}{d\tau} = \eta_0 \hat{\omega}(0) + \varepsilon^2 \Omega(\varepsilon^2) + \varepsilon^N \tilde{R}_2(\tau, \varepsilon, \tilde{\rho}, \theta, \tilde{Y}), \quad (6.53c)$$

where \tilde{R}_1 , \tilde{B}_2 , \tilde{R}_2 are bounded when $\varepsilon \rightarrow 0$.

The system (6.53) gives rise to an invariant torus, due to the strength of the contraction for the linear terms in (6.53a) and (6.53b) with respect to the norm of nonlinear terms. On the torus, $\|\tilde{Y}\| = O(\varepsilon)$, $|\tilde{\rho}| = O(\varepsilon^3)$ (see [13] or [11, Chapter III]).

Hence, in the irrational case ($\eta_0/\hat{\omega}(0)$ irrational), we have the invariant bifurcated torus of (6.1) in the form

$$\begin{aligned} V &= \hat{U}[\tau, \mu(\varepsilon)] + u, \\ u &= z\zeta(\mu(\varepsilon), \tau) + \bar{z}\bar{\zeta}(\mu(\varepsilon), \tau) + w, \\ z &= \varepsilon e^{i\theta} + \sum_{p+q \geq 2}^N \varepsilon^{p+q} e^{i(p-q)\theta} \gamma'_{pq}[\tau, \mu(\varepsilon)] + O(\varepsilon^{N+1}), \\ w &= \sum_{p+q \geq 2}^N \varepsilon^{p+q} e^{i(p-q)\theta} \Gamma'_{pq}[\tau, \mu(\varepsilon)] + O(\varepsilon^{N+1}), \\ \mu &= \sum_{p \geq 1}^{2p \leq N} \mu_{2p} \varepsilon^{2p} + O(\varepsilon^{N+1}), \end{aligned} \quad (6.54)$$

where γ'_{pq} , Γ'_{pq} are like γ_{pq} , Γ_{pq} and are defined by the inversion of formulae (6.28), (6.29). The torus is here parametrized by τ and θ and it reduces to the periodic solution $\hat{U}(\cdot, 0)$ when $\varepsilon = 0$. It bifurcates supercritically ($\mu > 0$) if $\mu_2 > 0$ and subcritically ($\mu < 0$) if $\mu_2 < 0$.

We next consider the rational case ($\eta_0/\hat{\omega}(0) = m/n$) and change variables to derive a new system like (6.53). It is more difficult because of the extra terms in (6.45) and (6.46) which depend on θ . We may follow Iooss [11, Chapter III] and use successive changes of variables to derive the system which is equivalent to (6.53). A more economical procedure is to neglect R_j , $j = 1, 2$, from the outset and to look for $\rho_0(\theta, \varepsilon)$, $\mu(\varepsilon)$ solving

$$\rho_0 f(\mu, \rho_0, \theta) = \frac{d\rho_0}{d\theta} g(\mu, \rho_0, \theta), \quad (6.55)$$

where f and g are the functions defined by equations (6.45), (6.46)

written on the form

$$\begin{aligned} \frac{d\rho}{d\tau} &= \rho f(\mu, \rho, \theta) + R_1(\tau, \mu, \rho, \theta, Y), \\ \rho \frac{d\theta}{d\tau} &= \rho g(\mu, \rho, \theta) + R_2(\theta, \mu, \rho, \theta, Y). \end{aligned} \quad (6.45')$$

The parameter ε is here defined by

$$\varepsilon = \frac{1}{2\pi} \int_0^{2\pi} \rho_0(\theta, \varepsilon) d\theta \quad (6.56)$$

and for the computation we expand all coefficients in power series in μ , up to order N , and put

$$\begin{aligned} \rho_0(\varepsilon, \theta) &= \varepsilon \rho_1(\theta) + \varepsilon^2 \rho_2(\theta) + \dots, \\ \mu(\varepsilon) &= \varepsilon \mu_1 + \varepsilon^2 \mu_2 + \dots, \end{aligned} \quad (6.57)$$

where the mean value of $\rho_l(\theta)$ is 1 if $l = 1$, or 0 otherwise.

The assumption (6.50) and the transversality condition (6.25) allow us to identify powers of ε in (6.55). We obtain

$$\rho_1(\theta) \equiv 1, \quad \mu_1 = 0, \quad \tilde{\xi}_1 \mu_2 + \text{Re } a_1(0) = 0, \quad (6.58)$$

and so on. (Details of this computation may be found in our book on elementary bifurcation theory; see Chapter X of [12].) So the side of the bifurcation is determined as in the irrational case, by the sign of $\text{Re } a_1(0)$. The results are as follows:

(i) if $\eta_0/\hat{\omega}(0) = m/n$ with $n = 2\nu + 3$, $\nu \geq 1$,

$$\begin{aligned} \rho_0(\theta, \varepsilon) &= \varepsilon + \sum_{k \geq 1} \varepsilon^{2\nu+k-1} \sum_{q=0}^{k-2q>0} [g_{kq} e^{n(k-2q)i\theta} \\ &\quad + \bar{g}_{kq} e^{-n(k-2q)i\theta}] + O(\varepsilon^{N+1}); \end{aligned} \quad (6.59)$$

(ii) if $\eta_0/\hat{\omega}(0) = m/n$ with $n = 2\nu + 4$, $\nu \geq 1$,

$$\begin{aligned} \rho_0(\theta, \varepsilon) &= \varepsilon + \sum_{k \geq 0} \varepsilon^{2\nu+1+2k} \sum_{q=0}^k [g_{kq} e^{n(k+1-q)i\theta} \\ &\quad + \bar{g}_{kq} e^{-n(k+1-q)i\theta}] + O(\varepsilon^{N+1}); \end{aligned} \quad (6.60)$$

and in all cases

$$\begin{aligned} \mu(\varepsilon) &= \sum_{p \geq 1} \mu_{2p} \varepsilon^{2p} + O(\varepsilon^{N+1}), \quad \mu(\varepsilon) = \mu(-\varepsilon), \\ \mu_2 &= -\text{Re } a_1(0)/\tilde{\xi}_1. \end{aligned} \quad (6.61)$$

Now, a change of variables defined by

$$\begin{aligned} \rho &= \rho_0(\varepsilon, \theta)[1 + \varepsilon^{N-3}\tilde{\rho}], \\ Y &= \varepsilon^N \tilde{Y}, \end{aligned} \quad (6.62)$$

leads to a system of the form (6.53), except for the last equation which is

$$\frac{d\theta}{d\tau} = \sum_{q=0}^{2q+2 \leq N-1} \Omega_{2q} \varepsilon^{2q+2} + \varepsilon^{2\nu+1} \psi_{2\nu+1}^*(\theta) + \varepsilon^{2\nu+2} \psi_{2\nu+2}^*(\theta) + \dots + \varepsilon^N \tilde{R}_2(\tau, \varepsilon, \tilde{\rho}, \theta, Y) \quad (6.63)$$

if $\eta_0/\hat{\omega}(0) = m/n$ with $n = 2\nu + 3$, $\nu \geq 1$, or

$$\frac{d\theta}{d\tau} = \sum_{q=0}^{2q+2 \leq N-1} \Omega_{2q} \varepsilon^{2q+2} + \varepsilon^{2\nu+2} \psi_{2\nu+2}^*(\theta) + \dots + \varepsilon^N \tilde{R}_2(\tau, \varepsilon, \tilde{\rho}, \theta, \tilde{Y}) \quad (6.64)$$

if $\eta_0/\hat{\omega}(0) = m/n$ with $n = 2\nu + 4$, $\nu \geq 1$, and where the functions ψ_i^* are polynomials in $e^{\pm ni\theta}$ and satisfy

$$\frac{1}{\pi} \int_0^{2\pi} \psi_i^*(\theta) d\theta = 0, \quad \text{for all } l.$$

It is important to notice that if $n = 2\nu + 3$

$$\psi_{2\nu+1}^*(\theta) = \sum_{q=0}^{l-2q>0} (A_{lq} e^{n(l-2q)i\theta} + \bar{A}_{lq} e^{-n(l-2q)i\theta}), \quad l \geq 1, \quad (6.65)$$

and if $n = 2\nu + 4$

$$\psi_{2\nu+2+2l}^*(\theta) = \sum_{q=0}^l (A_{lq} e^{n(l+1-q)i\theta} + \bar{A}_{lq} e^{-n(l+1-q)i\theta}). \quad (6.66)$$

So, by the same argument used in the irrational case, we obtain a bifurcated torus, invariant under (6.1), in the form

$$\begin{aligned} V &= \hat{U}[\tau, \mu(\varepsilon)] + u, \\ u &= z\zeta(\mu(\varepsilon), \tau) + \bar{z}\bar{\zeta}(\mu(\varepsilon), \tau) + w, \\ z &= \rho_0(\theta, \varepsilon) e^{i(\eta_0\tau/\hat{\omega}_0 + \theta)} + \sum_{p+q \geq 2}^N [\rho_0(\theta, \varepsilon)]^{p+q} \gamma'_{pq}[\tau, \mu(\varepsilon)] \\ &\quad \times e^{i(p-q)(\eta_0\tau/\hat{\omega}_0 + \theta)} + O(\varepsilon^{N+1}), \\ w &= \sum_{p+q \geq 2}^N [\rho_0(\theta, \varepsilon)]^{p+q} \Gamma'_{pq}[\tau, \mu(\varepsilon)] e^{i(p-q)(\eta_0\tau/\hat{\omega}_0 + \theta)} \\ &\quad + O(\varepsilon^{N+1}), \\ \mu &= \sum_{p \geq 1} \mu_{2p} \varepsilon^{2p} + O(\varepsilon^{N+1}), \end{aligned} \quad (6.67)$$

where γ'_{pq} and Γ'_{pq} are related to γ_{pq} and Γ_{pq} by the inversion of formulae (6.28), (6.29). The torus is parametrized by τ and $\theta' = \eta_0\tau/\hat{\omega}_0 + \theta$. We observe that $\rho_0(\theta' - \eta_0\tau/\hat{\omega}_0, \varepsilon)$ is 2π -periodic in τ and in θ' because of (6.59), (6.60).

6.4 Trajectories on the torus

Trajectories on the torus are determined by the equation for $d\theta/d\tau$, (6.53c) or (6.63) or (6.64), where $\tilde{\rho}$ and \tilde{Y} in \tilde{R}_2 are evaluated on the torus (as functions of $\tau, \theta, \varepsilon$).

In the irrational case, it follows from (6.53c) that

$$\theta(\tau) = \eta_0\tau/\hat{\omega}_0 + \varepsilon^2 \Omega(\varepsilon^2)\tau + \chi(\tau, \varepsilon), \quad (6.68)$$

where

$$\frac{\partial \chi}{\partial \tau}(\tau, \varepsilon) = O(\varepsilon^N).$$

We cannot bound χ uniformly in τ because of possible secular terms.

The solution of (6.63) or (6.64) in the rational case is more complicated than (6.68). We first introduce a change of variable defining

$$\tilde{\theta} = \theta + \varepsilon^{n-4} h_{n-4}(\theta) + \varepsilon^{n-3} h_{n-3}(\theta) + \dots + \varepsilon^{N-1} h_{N-1}(\theta), \quad (6.69)$$

where $h_i(\theta)$ are to-be-determined functions satisfying

$$h_i(\theta) = h_i(\theta + 2\pi/n), \quad \frac{1}{2\pi} \int_0^{2\pi} h_i(\theta) d\theta = 0. \quad (6.70)$$

We determine the functions $h_i(\theta)$ so that to within terms of order N , $d\tilde{\theta}/d\tau$ is constant.

After an easy computation we find that if $n = 2\nu + 3$, $\nu \geq 1$, then

$$\begin{aligned} h_l &= 0 \quad \text{for } l \leq 2\nu - 2, \\ h_{2\nu-2+p}(\theta) &= \sum_{l=0}^{p \geq 2l} (b_{pl} e^{in\theta(p-2l)} + \bar{b}_{pl} e^{-in\theta(p-2l)}), \quad p \geq 1, \end{aligned} \quad (6.71)$$

and if $n = 2\nu + 4$, $\nu \geq 1$, then

$$\left. \begin{aligned} h_l &= 0 \quad \text{for } l \leq 2\nu - 2, \\ h_{2l+1} &= 0, \\ h_{2\nu+2p}(\theta) &= \sum_{l=0}^p (b_{pl} e^{in\theta(p+1-l)} + \bar{b}_{pl} e^{-in\theta(p+1-l)}), \quad p \geq 0. \end{aligned} \right\} \quad (6.72)$$

It is possible to compute the functions h_i only if certain coefficients are not zero. When $n = 2\nu + 3$, we must assume that the polynomial occurring in (6.63),

$$\sum_{q=0}^{q \leq \nu-1} \Omega_{2q} \varepsilon^{2q+2}, \quad (6.73)$$

is not identically zero. (We had to assume $\Omega_0 \neq 0$ to obtain formulae (6.71).)

When $n = 2\nu + 4$, we must assume either that the polynomial (6.73) occurring in (6.64) is not identically zero or, if it is identically zero, that an additional inequality holds for the coefficients of (6.66):

$$|\Omega_{2\nu}| > 2|A_{00}|. \quad (6.74)$$

Otherwise, we enter into the cases of *weak resonance*.

Assuming now that coefficients (6.73) do not vanish we obtain an equation for $\tilde{\theta}$ of the form

$$\frac{d\tilde{\theta}}{d\tau} = \varepsilon^2 \Theta(\varepsilon^2) + \varepsilon^N R_3(\tau, \varepsilon, \tilde{\theta}). \quad (6.75)$$

So, the trajectories on the torus satisfy

$$\theta + \varepsilon^{n-4} h_{n-4}(\theta) + \dots + \varepsilon^{N-1} h_{N-1}(\theta) = \varepsilon^2 \Theta(\varepsilon^2) \tau + \chi(\tau, \varepsilon), \quad (6.76)$$

where

$$\frac{\partial \chi}{\partial \tau}(\tau, \varepsilon) = O(\varepsilon^N)$$

and $\Theta(\varepsilon^2)$ is a polynomial in ε^2 .

To interpret this asymptotic result we consider the Poincaré map restricted to the invariant torus. (This map is defined in section 6.1.) We look at a point of the circle in the cross-section of the torus corresponding to initial data at $\tau = 0$. At $\tau = 2\pi$ the trajectory on the torus hits the circle in the cross-section once again defining a map

$$\phi \mapsto f_\varepsilon(\phi), \quad \phi \in T^1.$$

It is not hard to see that there is a change of variables which leads to a parametrization of the circle $\tau = 0$, such that the map $f_\varepsilon(\phi)$ satisfies

$$f_\varepsilon(\phi) = \phi + [\eta_0/\hat{\omega}(0) + \varepsilon^2 \Theta(\varepsilon^2)] 2\pi + g_\varepsilon(\phi), \quad (6.77)$$

where $g_\varepsilon(\phi) = O(\varepsilon^N)$ is 2π -periodic in ϕ , and where $\Theta(\varepsilon^2) \equiv \Omega(\varepsilon^2)$ in the irrational case. Recalling that the *rotation number* of such a diffeomorphism of \mathbb{R} (projected on T^1) is defined by the uniform limit

$$\rho(\varepsilon) = \frac{1}{2\pi} \lim_{k \rightarrow \infty} \frac{f_\varepsilon^k(\phi) - \phi}{k} \quad (6.78)$$

and is independent of ϕ , we now have

$$\rho(\varepsilon) = \eta_0/\hat{\omega}(0) + \varepsilon^2 \Theta(\varepsilon^2) + O(\varepsilon^N). \quad (6.79)$$

Equation (6.79) can be regarded as an estimate of the rotation number of the Poincaré map (see Fig. 6.1). For a fixed ε , the rotation number $\rho(\varepsilon)$ lies in a region of breadth $O(\varepsilon^N)$.

We know from general results that $\rho(\varepsilon)$ is a continuous function and that it is differentiable at special points where $\rho(\varepsilon)$ is irrational and badly approximated by rationals in a certain sense (Brunovsky cited in [10]). It is not in general differentiable at points where $\rho(\varepsilon)$ becomes rational, but it stays constant on a closed interval at these rational values.

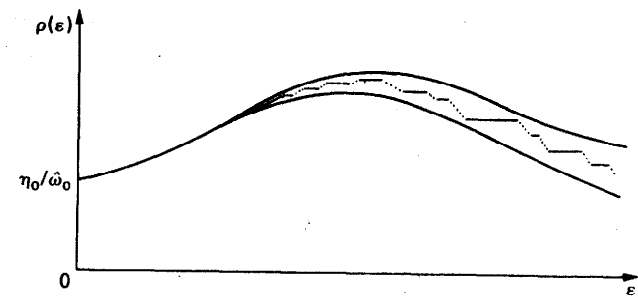


Fig. 6.1

A moment's consideration of the form of the trajectories on the invariant torus will persuade the reader that if we suppress the terms $\chi(\tau, \varepsilon)$ and $O(\varepsilon^{N+1})$ we obtain a family of solutions of the form

$$V \sim \mathcal{U}(\tau, \eta_0 \tau / \hat{\omega}_0 + \varepsilon^2 \Theta(\varepsilon^2) \tau + \theta_0), \quad \tau = \tau(t), \quad (6.80)$$

where \mathcal{U} is doubly 2π -periodic. In fact this exact doubly periodic function cannot generally represent the solutions on the torus for (6.1) because this implies that the rotation number is very smooth in ε (in fact, a polynomial), which is not true in general. It is, however, possible to find a change of variables leading to (6.80) when the rotation number is *irrational* (a theorem of Bohl cited in Hale [9]).

We turn next to the cases of weak resonance. In the case $n = 2\nu + 3$, if $\Omega_{2q} = 0$ for $q = 0, \dots, \nu - 1$ (ν conditions), we cannot compute the change of variables $\theta \mapsto \tilde{\theta}$. This is a consequence of the fact that $d\theta/d\tau$ may vanish for a certain θ_0 in (6.63). When the conditions for weak resonance are satisfied (6.63) reduces to

$$\frac{d\theta}{d\tau} = \varepsilon^{2\nu+1} \psi_{2\nu+1}^*(\theta) + O(\varepsilon^{2\nu+2}). \quad (6.63')$$

Using (6.65), we find that if $\theta = \theta_0$ satisfies

$$\psi_{2\nu+1}^*(\theta) = 0, \quad (6.81)$$

then all solutions of (6.81) are of the form

$$\theta = \pm \theta_0 + \frac{2k\pi}{n}, \quad k = 0, 1, \dots, n-1. \quad (6.82)$$

Now we use the implicit function theorem to determine fixed points of (6.63') which are close to θ_0 when $|\varepsilon|$ is small. It is not hard to find that there is $\theta_0(\varepsilon) = \theta_0 + O(\varepsilon)$ such that $d\theta/d\tau = 0$ provided only that $A_{10} \neq 0$. A fixed point of this type, with the deduced family of fixed points $\theta_0 + 2k\pi/n$, $k = 1, \dots, n-1$, corresponds to a *closed orbit on the torus*, hence to a periodic (subharmonic) solution. So we have proved that there

are, in this case, two periodic solutions on the invariant torus. One of the periodic solutions is attracting and the other one is repelling [12, Chapter X; 11, Chapter III].

A trajectory on the torus will therefore be attracted by the attracting periodic solution and will spiral in towards it. In this subharmonic case the rotation number $\rho(\epsilon)$ is constant and equal to $\rho(0) = \eta_0/\hat{\omega}_0$.

In the even case when $n = 2\nu + 4$, and $\Omega_{2q} = 0$ for $q = 0, \dots, \nu - 1$, we have (6.64) in the form

$$\frac{d\theta}{d\tau} = [\Omega_{2\nu} + \psi_{2\nu+2}^*(\theta)]e^{2\nu+2} + O(\epsilon^{2\nu+4}). \tag{6.64'}$$

If $\Omega_{2\nu} + \psi_{2\nu+2}^*(\theta) = 0$ has a solution θ_0 , then we cannot change variables to transform (6.64') into (6.75), otherwise we can (see [11, Chapter III]). The condition (6.74) is enough to allow the good change of variables. So if

$$|\Omega_{2\nu}| < 2|A_{00}|, \tag{6.83}$$

then there are solutions θ_0 of

$$\Omega_{2\nu} + \psi_{2\nu+2}^*(\theta_0) = 0 \tag{6.84}$$

which are of the form $\theta_0^{(j)} + 2k\pi/n$, $k = 0, 1, \dots, n-1$, $j = 1, 2$. Again there are two periodic solutions on the torus, one of which is unstable. So the main qualitative results hold in both the even and the odd cases.

Our study of weak resonance casts some light on the work of Arnold [2, 2']. Arnold (implicitly) considers the Floquet multiplier $\lambda(\mu) = e^{2\pi\sigma(\mu)/\hat{\omega}(\mu)}$ as a parameter, the other coefficients being fixed (nothing essential is modified if they depend on λ ; that is, on two real parameters). In our frame, Arnold's problem corresponds to fixing all of the parameters except $\lambda(\mu)$ at their $\mu = 0$ values. Since $\lambda(\mu)$ is complex-valued, Arnold's analysis corresponds to two-parameter systems of equations. Our weakly resonant subharmonic solutions correspond to the 'tongues' shown in Fig. 5 of Arnold's paper (our Fig. 6.2). The exceptional conditions required for subharmonic solutions at points of weak resonance determine the direction and thickness of the tongues. For example,

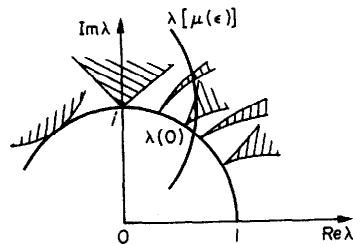


Fig. 6.2

the first condition, $\Omega_0 = \mu_2\tilde{\omega}_1 + \text{Im } a_1(0) = 0$, together with $\mu_2\tilde{\xi}_1 + \text{Re } a_1(0) = 0$ implies that

$$\frac{\tilde{\omega}_1}{\tilde{\xi}_1} = \frac{\text{Im } a_1(0)}{\text{Re } a_1(0)},$$

which determines the direction of the tongue. In general, there are ν conditions, or ν conditions plus an inequality where ν and n (of $\lambda^n(0) = 1$) are related by $n = 2\nu + 3$ or $n = 2\nu + 4$. We can draw a curve 'of general orientation' $\lambda = \lambda(\mu(\epsilon))$ in the complex λ -plane. Arnold asserts that a curve of general orientation crosses an infinite number of tongues, the rotation number being constant and rational in each tongue. There is no proof, at present, of this crossing, though such crossing would give a good explanation of phase locking and the steps in curve $\rho(\epsilon)$ of Fig. 6.1.

6.5 Law of the movement on the trajectories

It is of interest to study the motion along trajectories. We already know what happens in the case of weak resonance, except that the value of the period of the solution is as yet unknown. In fact (6.9a) and (6.12) lead to

$$\frac{d\tau}{dt} = \hat{\omega}(\mu(\epsilon)) + O(\epsilon).$$

Hence the period

$$T(\epsilon) = \frac{2\pi n}{\hat{\omega}(0)} + O(\epsilon)$$

is not, in general, exactly n times the period $T(\mu) = 2\pi/\hat{\omega}(\mu)$ of the basic periodic solution. We can get a better asymptotic representation of the period by integration of (6.9a) using the asymptotic representation of $u(\tau, \epsilon)$. Thus

$$T(\epsilon) = \int_0^{2n\pi} \frac{d\tau}{f(\mu, \tau, u(\tau, \epsilon))} \tag{6.85}$$

with $\mu(\epsilon)$, $u(\tau, \epsilon)$ defined by the computations in section 6.4.

Suppose that at a resonant point with $n \geq 5$ the conditions for weak resonance are not satisfied. Then we may replace $u(\tau, \epsilon)$ in (6.9a) and obtain an equation of the form

$$\begin{aligned} \frac{d\tau}{dt} = & \hat{\omega}[\mu(\epsilon)] + \sum_{k \geq 1}^N \epsilon^k T_k[\tau, \tilde{\Omega}(\epsilon^2)\tau + \theta_0] \\ & + \chi_2(\tau, \epsilon) + O(\epsilon^{N+1}), \end{aligned} \tag{6.86}$$

where the T_k 's are doubly 2π -periodic and

$$\tilde{\Omega}(\epsilon^2) = \eta_0/\hat{\omega}(0) + \epsilon^2\Theta(\epsilon^2), \quad \frac{\partial \chi_2}{\partial \tau} = O(\epsilon^{N+1}).$$

There are secular terms in general in χ_2 , but the T_k 's have finite Fourier decompositions in the second argument, all higher harmonics going into terms $O(\varepsilon^{N+1})$. In the irrational case as well as in the rational cases, the finiteness of the Fourier decomposition of doubly periodic functions T_k allows us to change variables $\tau \mapsto \tilde{\tau} = \tau + \varepsilon h(\varepsilon, \tau, \tilde{\Omega}\tau)$ using the same form as in section 6.4 for the change of variables $\theta \mapsto \tilde{\theta}$. The equation governing $\tilde{\tau}$ is

$$\frac{d\tilde{\tau}}{dt} = \omega(\varepsilon) + \chi_3(\tilde{\tau}, \varepsilon), \quad (6.87)$$

where $\omega(\varepsilon) = \hat{\omega}(0) + O(\varepsilon)$ is determined after the change of variables and is $\neq \hat{\omega}(\mu)$ in general.

When the conditions for subharmonic bifurcation at points of weak resonance are not satisfied we may rewrite (6.80) in terms of real time:

$$V(t) \sim \mathcal{U}(\omega(\varepsilon)t, \omega(\varepsilon)\tilde{\Omega}(\varepsilon^2)t), \quad (6.88)$$

which is valid asymptotically for times $O(1)$, and where \mathcal{U} is doubly 2π -periodic. Of course, the asymptotic forms need not be valid for all time. The form (6.88) is stronger than the form (6.80). In fact, a theorem of Kolmogorov¹ (see [18]) says that if the rotation number $\rho(\varepsilon)$ is irrational and badly approximated by rationals in a certain sense, then the flow is quasiperiodic on the torus. In fact, a recent result of Herman [10] guarantees that if $\rho(\varepsilon)$ is not identically constant then the set of ε 's (or μ 's) for which the flow is quasiperiodic has a nonzero Lebesgue measure.

So theory tells us that as ε is varied we will observe (i) true quasiperiodic motions with a Fourier spectrum composed of two fundamental peaks (two frequencies) and their harmonics and (ii) true subharmonic motions. The question is whether or not there are other flows which are neither quasiperiodic nor subharmonic. In experiments the flows appear to be doubly periodic even at rational values of $\rho(\varepsilon)$ when ε is near zero. The ratio of frequencies observed in the experiments is our $\rho(\varepsilon)$ and our study shows that, in any case, steps at rational values of $\rho(\varepsilon)$ have to be smaller than any power of ε when ε goes to zero. Steps (frequency locking) in $\rho(\varepsilon)$ at rational values p/q , with small p and q , when ε is not close to zero, have been observed in experiments reported in [6, 14, 8]. It may be hard to observe frequency locking with larger p and q when ε is small. The intervals of ε over which $\rho(\varepsilon)$ can be constant must tend to zero with ε . (We draw the reader's attention to the very interesting conjectures of Bowen [19] about transition to turbulence.)

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¹ See p. 94.

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