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MOTIONS PERTURBING STATES OF REST OF VISCOELASTIC SOLIDS

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1. Summary

Our goal is to derive the canonical forms of the stress and equations of motion governing the motions which perturb the rest state (of elastostatic deformation) and the natural (unstressed and undeformed) state of viscoelastic solids. In this theory nonlinear elasticity appears as a special case of nonlinear viscoelasticity which arises when the prescribed data is steady. The domain of deformations on which the constitutive equation for viscoelastic solids reduces to the constitutive equation for elasticity is the set of all kinematically admissible states of rest. We find the forms of the stress and discuss some properties of the equations of motion which perturb states of rest (elastostatics). There are too many unknown functions in the theory of perturbations of states of rest of viscoelastic solids to make the theory attractive to material scientists interested in rheometrical measurements. For this purpose, the theory of perturbations of the natural state of viscoelastic solids is more attractive. We develop a detailed theory of perturbations of the natural state and derive equations governing perturbations of zero displacements which may be solved sequentially. At each stage there are three equations for each component of displacement when the material is compressible and, when incompressible, there is an additional unknown, the reaction pressure, and an additional differential equation expressing the incompressibility. We identify the material parameters which must be measured to distinguish one solid from another. In the second order theory for compressible materials there are six elastic constants and twelve viscoelastic material functions. The number of constants is reduced to two and the number of material functions to three

in the incompressible case. For incompressible solids the leading operator which must be inverted at each stage in the perturbation is characterized by one constant and one function. If the constant and function satisfy some mild and physically natural conditions the solutions of the perturbation equations will be stable and unique. We show how to use the perturbation equations for material studies by deriving several problems of possible application in the design of rheometers for viscoelastic solids in motions which perturb the natural state.

In the last part of the paper we derive the linearized equations governing viscoelastic perturbations of elastostatic solutions with an eye to potential applications in the dynamic theory of stability and bifurcation. We give a heuristic and completely physical argument that solids undergoing static deformations cannot bifurcate into time-periodic motions. We argue that critical eigenvalues for the stability of elastostatic solutions cannot be complex-valued and suggest this principle: any choice of the material parameters leading to complex-valued critical values is the wrong choice. Finally, as an example of the nature of the application of the linearized theory, we derive the spectral problem governing the stability of Rivlin's (1949) universal solution of the torsion of an elastic cylinder.

We have used the same symbol for functions and their values. Sometimes for brevity we have suppressed some or all of the arguments of some of the functions. But they are easily understood from the context.

Relation to previous work

Our work falls in the framework of what elasticians call "small on large". This means small unknown deformations are superposed on large known ones. The unknown deformations are generally treated as elastic but in at least two important works, Coleman and Noll (1961) and Pipkin (1964), the unknown deformations are presumed to be viscoelastic. Some of our best results do not fit in framework of "small on large" or even of "small on small" but, instead, fall into the frame of "small on zero". In the theory of solids, treated here, the "small" is arbitrary and viscoelastic and the "zero" corresponds to the natural state of the body. Many of the results of Green & Rivlin (1957), Coleman and Noll (1961), Pipkin and Rivlin (1961) and Pipkin (1964) can also be interpreted as falling under the theory of "small on zero".

We can also argue that the results given by the authors mentioned in the last paragraph are contributions to the asymptotic theory of the stress in simple materials. In such studies the norm of some measure of deformations is presumed small and one seeks canonical forms of the stress which perturb the zero norm. Naturally, these canonical forms are ordered in powers of the relevant measure of deformation expressed in terms of multilinear functionals simplified to the degree required by considerations of material symmetry.

The purpose of the asymptotic theory is to specify relatively simple forms of the stress when the motion is one which is small in the appropriate norm. The asymptotic theory is approximate, since the exact conditions allowed by nature, under which the appropriate norm is small, is left unspecified.

In contrast, we have tried to establish the consequences of the fact that the small norm must somehow arise from small prescribed data, like small external forcing, filtered through the equations of motion. So we come up with an ordered asymptotic sequence of boundary value problems in which redundant terms are purged from the canonical forms of the stress, and we derive algorithms in which the ordered computation of motion and deformation is reduced to a recipe. So our theory is a formally exact asymptotic theory in which the prescribed data rather than the unknown motion is presumed small.

The heart of our theory is the perturbation equations of motion. So we are obliged to consider the usual interesting questions of existence, stability and uniqueness. Naturally, these questions cannot come up in studies which are abandoned at the point where the stress is arranged into a suitably invariant series of powers of the deformation.

In solids it is possible to choose several different measures of deformation to express the stress. These different choices lead to different but possibly equivalent expressions for the stress. We have thought the choice of measure of deformation to be important and we followed Coleman and Noll (1961) because in their formulation the constitutive equation for all static deformations of viscoelastic materials obviously and easily reduces to elasticity. So there is no difference between nonlinear elastostatics and nonlinear viscoelastostatics. There is no doubt that the theory of Green & Rivlin (1957), Pipkin and Rivlin (1961) and Pipkin (1964) also reduces to elastostatics for static deformations but the reduction is less obvious. Suprisingly, this obvious connection between elasticity and viscoelasticity, which is certainly well-known to people working in mechanics, seems not to have been stressed (or even mentioned) by any of the authors cited above.

The parts of the theory of Coleman & Noll (1961) which we use are summarized in the introduction. In this formulation the stress depends on the left Cauchy-Green tensor and the history of the right relative Cauchy-Green tensor. Green and Rivlin (1957), Pipkin and Rivlin (1961) and Pipkin (1964) express the stress in terms of the history of the right Cauchy - Green tensor (not the relative tensor) which is defined is

$$\underline{\underline{C}}(\underline{X},\tau) = \underline{\underline{F}}(\tau)^T F(\tau)$$

where $\underline{\underline{F}}$ is defined in the first equation of the introduction. For some reason that we don't understand Pipkin works with the time derivative $\dot{\underline{\underline{C}}}$. The time derivative of kinematic tensors is good for fluids but not for solids (Joseph & Beavers, 1977).

2. Introduction

The viscoelastic solids which we study are simple isotropic materials of the type discussed in §33 of the treatise by Truesdell and Noll (1965). The material properties of such solids are independent of time and the state of stress is determined by the first spatial gradient of the deformation

$$\underline{\underline{F}} = \nabla \underline{x}, \quad F_{ij} = \partial x_i / \partial X_j$$

where $\underline{X}_i$ are the cartesian coordinates of a particle $\underline{X}$ of the body in the undeformed isotropic state and $x_i(\underline{X},\tau)$ are the cartesian coordinates of the position $\underline{x}$ of the same particle at the time τ. The Cauchy stress at a particle is given by an expression of the form

$$(2.1) \qquad \underline{\underline{T}} = \underset{s=0}{\overset{\infty}{\mathfrak{F}}} [\underline{\underline{B}};\ \underline{\underline{G}}(s)]$$

where

$$\underline{\underline{B}} = \underline{\underline{F}}\ \underline{\underline{F}}^T$$

is the left Cauchy-Green strain tensor at the present time t,

$$\underline{\underline{G}}(s) = \underline{\underline{C}}_t(\tau) - \underline{\underline{1}}, \quad s = t - \tau, \quad -\infty < \tau \leq t.$$

$\underline{\underline{C}}_t(\tau)$ is the right, relative Cauchy-Green strain tensor;

$$\underline{\underline{C}}_t(\tau) = \underline{\underline{F}}^T{}_t(\tau)\ \underline{\underline{F}}_t(\tau), \quad \underline{\underline{C}}_t(t) = \underline{\underline{1}}$$

where

$$\underline{\underline{F}}_t(\tau) = \nabla_x\, \chi_t(\underline{x},\tau), \quad \underline{\underline{F}}_t(t) = \underline{\underline{1}}$$

is the relative deformation gradient tensor defined in terms of the relative position vector $\chi_t(\underline{x},\tau)$ of a particle which at the time $\tau = t$ is at place $\underline{x} = \chi_t(\underline{x},t)$. $\underline{\underline{\mathfrak{F}}}$ is a functional of the history $\underline{\underline{G}}(s)$ of a particle depending on the tensor parameter $\underline{\underline{B}}$ and is such that

(2.2) $\underline{\underline{T}} = \underline{\underline{\mathfrak{F}}}[\underline{\underline{1}}, \underline{\underline{0}}] = \underline{\underline{0}}.$

Eq. (2.2) says that there is no stress in the undeformed state of the body. The stress-free, undeformed state of the body is called the natural state of the body.

Nonlinear elasticity arises from (2.1) when $\underline{\underline{G}}(s)$ is put to zero:

(2.3) $\underline{\underline{T}}(t) = \underline{\underline{\mathfrak{F}}}[\underline{\underline{B}}(t), \underline{\underline{0}}]$.

It is possible and, in some asymptotic limits, it is useful to regard (2.3) as defining the dynamical response of a nonlinearly elastic body. But we also note that an elastostatics

(2.4) $\underline{\underline{T}} = \underline{\underline{\mathfrak{F}}}[\underline{\underline{B}}, \underline{\underline{0}}]$

of viscoelastic solids arises automatically from (2.1) for every deformation such that $\underline{x} = \underline{x}(\underline{X}, t)$ is independent of t. (If $\underline{x}$ is independent of t, then $\underline{\underline{F}}_t(\tau) = \underline{\underline{1}}$ and $\underline{\underline{G}}(s) = \underline{\underline{0}}$.) We think of the class of t-independent deformations defining the rest state of a viscoelastic solid as coinciding with elasticity; that is, all solids are at least viscoelastic but the constitutive equation for viscoelastic solids reduces to elasticity when the deformations are restricted to t-independent ones. So from the point of view of material science we do not think it useful to admit dynamic elasticity as a viable subject. After a time vibrating solid bodies always come to rest, unless forced, and when they are in motion these bodies satisfy a constitutive equation which is at least as complicated as (2.1). So we think that nonlinear dynamic viscoelasticity and nonlinear elastostatics are not different subjects but just different realizations of the same governing equations corresponding to unsteady or steady solutions which arise in response to the given data: the initial and boundary conditions and the prescribed forcing. In any case, that is the nature of the theory which we shall now develop.

3. Frechét expansion of the stress

In the state of rest (elastic deformation) $\underline{\underline{G}}(s) = \underline{\underline{0}}$. We assume that the stress perturbing states of rest is expressible as a Frechét expansion of $\underline{\underline{\mathfrak{F}}}$ in powers of $\underline{\underline{G}}(s)$. Thus,

$$(3.1)\qquad \underline{\underline{\mathfrak{F}}}[\underline{\underline{B}}, \underline{\underline{G}}(s)] = \underline{\underline{\mathfrak{F}}}[\underline{\underline{B}},\underline{\underline{0}}] + \underline{\underline{\mathfrak{F}}}_1[\underline{\underline{B}},\underline{\underline{0}}|\underline{\underline{G}}(s)] + \frac{1}{2}\underline{\underline{\mathfrak{F}}}_2[\underline{\underline{B}},\underline{\underline{0}}|\underline{\underline{G}}(s_1)|\underline{\underline{G}}(s_2)] + o(\|\underline{\underline{G}}(s)\|^3)$$

where $\underline{\underline{\mathfrak{F}}}_1[\underline{\underline{B}},\underline{\underline{0}}|\cdot]$ is a linear operator and $\underline{\underline{\mathfrak{F}}}_2[\underline{\underline{B}},\underline{\underline{0}}|\cdot|\cdot]$ a bilinear operator evaluated on the zero history. Green and Rivlin (1957) assumed an expansion in the form (3.1) with the Frechét derivatives expressed as multiple integrals. They appealed to the Stone-Weirestrass theorem for functionals for mathematical justification. Coleman and Noll (1961) also arrived at an integral expansion. They introduced a Hibert space of histories endowed with a weighted scalar product (fading memory) and appealed to the Riesz representation theorem to justify an integral representation of the first term.

We follow the authors just named and assume that the terms in (3.1) can be expressed as integrals.

$$(3.2)\qquad \underline{\underline{T}} = \underline{\underline{f}}(\underline{\underline{B}}) + \int_0^\infty \underline{\underline{K}}(s, \underline{\underline{B}}(t))\underline{\underline{G}}(s)ds + \int_0^\infty\int_0^\infty \underline{\underline{\Gamma}}(s_1,s_2,\underline{\underline{B}}(t))\underline{\underline{G}}(s_1)\underline{\underline{G}}(s_2)ds_1ds_2 + o(\|\underline{\underline{G}}\|^3)$$

where $\underline{\underline{K}}(s,\underline{\underline{B}})$ is an isotropic tensor function of $\underline{\underline{B}}$ of order four whose components $K_{ijk\ell}$ are symmetric in successive pairs of indices, $\underline{\underline{\Gamma}}(s_1,s_2,\underline{\underline{B}}) = \underline{\underline{\Gamma}}(s_2,s_1,\underline{\underline{B}})$ is an isotropic tensor function of $\underline{\underline{B}}$ of order six whose components $\Gamma_{ijk\ell mn}$ are symmetric in successive pairs of indices.

If the integrals in (3.2) are set to zero we are left with

$\underline{\underline{T}} = \underline{\underline{f}}(\underline{\underline{B}})$ which is supposed to be the response of bodies which are said to be purely elastic. In mathematical studies various conditions are proposed about $\underline{\underline{f}}(\underline{\underline{B}})$ to insure appropriate properties of existence and uniqueness of the solutions of the equations governing the dynamic response of nonlinearly elastic bodies. In our study of nonlinear viscoelasticity we are also required to introduce small nonlinear effects of $\underline{\underline{f}}(\underline{\underline{B}})$. But in our study special assumptions are not required. Instead, the determination of some nonlinear properties of $\underline{\underline{f}}(\underline{\underline{B}})$ is left as an open question for rheometrical measurements and experiments.

The methods for finding the most general form of isotropic tensor-valued functions of many tensors have been given by Wineman and Pipkin (1964), based on earlier work of Rivlin, Smith and Spencer (see R.S. Rivlin, 1969; Truesdell & Noll, 1965, §13). Dixit (1979) applied these methods to (3.2). The reduction of (3.2) to isotropic form is like the Hamilton-Cayley reduction of a tensor polynomial of degree m > 2 to m = 2. Suppose we have a tensor* $\underline{\underline{A}}$ which is a function of tensors $\underline{\underline{B}}$, $\underline{\underline{C}}$, ...

$$\underline{\underline{A}} = \underline{\underline{g}}(\underline{\underline{B}}, \underline{\underline{C}}, \ldots)$$

The dependence is such that $\underline{\underline{g}}$ satisfies

$$\underline{\underline{Q}}\,\underline{\underline{g}}(\underline{\underline{B}},\underline{\underline{C}},\ldots)\underline{\underline{Q}}^T = \underline{\underline{g}}(\underline{\underline{Q}}\,\underline{\underline{B}}\,\underline{\underline{Q}}^T, \underline{\underline{Q}}\,\underline{\underline{C}}\,\underline{\underline{Q}}^T,\ldots), \quad \underline{\underline{Q}} \in \mathcal{O} .$$

$\mathcal{O}$ is the set of all orthogonal tensors.†

The method of finding the most general form of $\underline{\underline{g}}$ is as follows: First introduce an auxiliary second-order tensor $\underline{\underline{\phi}}$. Let $\alpha = \underline{\underline{\phi}} \cdot \underline{\underline{g}}(\underline{\underline{B}},\underline{\underline{C}},\ldots)$. Now α is a scalar invariant of

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*Here we consider only the second-order tensors. But the method is applicable even for tensors of other orders.

†More generally, $\mathcal{O}$ is the symmetry-group of the material.

tensors $\underline{\underline{\phi}}$, $\underline{\underline{B}}$, $\underline{\underline{C}}$... . A set of scalar invariants $H_j(\underline{\underline{\phi}}, \underline{\underline{B}}, \underline{\underline{C}},\ldots)$, $j = 1, \ldots k$ is called <u>a functional basis</u> if every scalar invariant of $\underline{\underline{\phi}}$, $\underline{\underline{B}}$, $\underline{\underline{C}}$.........can be expressed as a function of H_j, $j = 1, \ldots k$.

If $\underline{\underline{g}}$ is a polynomial in $\underline{\underline{B}}$, $\underline{\underline{C}}$, ..., then α is a polynomial scalar invariant of $\underline{\underline{\phi}}$, $\underline{\underline{B}}$, $\underline{\underline{C}}$, ... A set of polynomial scalar invariants $I_j(\underline{\underline{\phi}}, \underline{\underline{B}}, \underline{\underline{C}}, \ldots)$, $j = 1 \ldots, n$ is called <u>an integrity basis</u> if every polynomial scalar invariant of $\underline{\underline{\phi}}$, $\underline{\underline{B}}$, $\underline{\underline{C}}$...can be expressed as a polynomial in I_j, $j = 1,\ldots,n$.

Wineman and Pipkin (1964) have shown that an integrity basis is also a functional basis.

An integrity basis for an arbitrary number of symmetric second-order isotropic tensors was given by Spencer, Smith & Rivlin (see Rivlin, 1969). Integrity bases for an arbitrary number of tensors and vectors and for the case in which the symmetry-group is not the group of all orthogonal tensors was given by Spencer, Smith, Rivlin, Adkins and Weyl (see Wineman and Pipkin, 1964).

Once an integrity basis for the tensors $\underline{\underline{\phi}}$, $\underline{\underline{B}}$, $\underline{\underline{C}}$,... has been found, the elements which are functions of $\underline{\underline{B}}$, $\underline{\underline{C}}$, ... alone are singled out. Call these I_γ, $\gamma = 1,\ldots m$. (These form an integrity basis for the tensors $\underline{\underline{B}}$, $\underline{\underline{C}}$,... .) Then the elements which are linear in $\underline{\underline{\phi}}$ are selected. Each such invariant is of the form $\phi_{ij} f_{ij}^{(\beta)}(\underline{\underline{B}}, \underline{\underline{C}},\ldots)$, $\beta = 1,\ldots \ell$. Then

$$\underline{\underline{g}} = \sum_{\beta=1}^{\ell} F_\beta(I_1, I_2,\ldots I_m)\underline{\underline{f}}^{(\beta)}(\underline{\underline{B}}, \underline{\underline{C}},\ldots)\,.$$

Applying this method to (3.2) we find that

(3.3) $$\underline{\underline{f}}(\underline{\underline{B}}) = f_0\underline{\underline{1}} + f_1\underline{\underline{B}} + f_2\underline{\underline{B}}^2 ,$$

and

(3.4) $$\underline{\underline{K}}(s,\underline{\underline{B}})\underline{\underline{G}}(s) = \text{tr}[(\phi_{00}\underline{\underline{1}} + \phi_{01}\underline{\underline{B}} + \phi_{02}\underline{\underline{B}}^2)\underline{\underline{G}}(s)]\underline{\underline{1}}$$
$$+ \text{tr}[(\phi_{10}\underline{\underline{1}} + \phi_{11}\underline{\underline{B}} + \phi_{12}\underline{\underline{B}}^2)\underline{\underline{G}}(s)]\underline{\underline{B}}$$
$$+ \text{tr}[(\phi_{20}\underline{\underline{1}} + \phi_{21}\underline{\underline{B}} + \phi_{22}\underline{\underline{B}}^2)\underline{\underline{G}}(s)]\underline{\underline{B}}^2$$
$$+ (\phi_{30}\underline{\underline{1}} + \phi_{31}\underline{\underline{B}} + \phi_{32}\underline{\underline{B}}^2)\underline{\underline{G}}(s)$$
$$+ \underline{\underline{G}}(s)(\phi_{30}\underline{\underline{1}} + \phi_{31}\underline{\underline{B}} + \phi_{32}\underline{\underline{B}}^2)$$

where the f_i are functions of the three principal invariants of $\underline{\underline{B}}$, $I_B = \text{tr}\underline{\underline{B}}$, $II_B = \frac{1}{2}[(\text{tr}\underline{\underline{B}})^2 - \text{tr}\underline{\underline{B}}^2]$, $III_B = \det \underline{\underline{B}}$, and the ϕ_{ij} are functions of the same three invariants and the time lag $s = t - \tau$. The isotropic form of $\underline{\underline{\Gamma}}(s_1,s_2;\underline{\underline{B}})\underline{\underline{G}}(s_1)\underline{\underline{G}}(s_2)$ is lengthy and will not be given here (see Dixit, 1979).

The forms of the stress which perturb the natural state have a simpler structure than the forms (3.3,4) which perturb states of rest (elastic deformation). The natural state is a state of rest in which the body is undeformed and unstressed so that $\underline{\underline{G}}(s) = \underline{\underline{0}}$, $\underline{\underline{B}}(t) = \underline{\underline{1}}$ and $\mathfrak{F}[\underline{\underline{1}},\underline{\underline{0}}] = \underline{\underline{0}}$. To compute stresses relative to the natural state it is convenient to expand the tensor functions of $\underline{\underline{B}}(t)$ in (3.2) into a series of powers of the perturbation tensor

(3.5) $$\underline{\underline{b}}(t) = \underline{\underline{B}}(t) - \underline{\underline{1}} .$$

This procedure reduces the problem of finding the most general isotropic forms of (3.2) to a problem of finding isotropic tensor coefficients for multilinear forms. At the end of the analysis one finds that

$$\underline{\underline{f}} = \beta\underline{\underline{b}} + \beta^{[1]}\underline{\underline{1}}\ \mathrm{tr}\underline{\underline{b}} + \beta^{[2]}\ \underline{\underline{bb}} + \beta^{[3]}\underline{\underline{1}}\ (\mathrm{tr}\underline{\underline{b}})^2 + \beta^{[4]}\underline{\underline{1}}\ \mathrm{tr}(\underline{\underline{bb}}) + \beta^{[5]}\ (\mathrm{tr}\underline{\underline{b}})\underline{\underline{b}} + O(|\underline{\underline{b}}|^3). \tag{3.6}$$

$$\begin{aligned}\underline{\underline{K}}\big(s,\underline{\underline{B}}(t)\big)\underline{\underline{G}}(s) = {} & \zeta(s)\underline{\underline{G}}(s) + \zeta^{[1]}\underline{\underline{1}}\ \mathrm{tr}\underline{\underline{G}}(s) \\ & + \zeta^{[2]}(s)\ \{\underline{\underline{b}}(t)\underline{\underline{G}}(s) + \underline{\underline{G}}(s)\underline{\underline{b}}(t)\} \\ & + \zeta^{[3]}(s)\underline{\underline{G}}(s)\ \mathrm{tr}\underline{\underline{b}}(t) \\ & + \zeta^{[4]}(s)\underline{\underline{b}}(t)\ \mathrm{tr}\underline{\underline{G}}(s) \\ & + \zeta^{[5]}(s)\underline{\underline{1}}[\mathrm{tr}\underline{\underline{b}}(t)][\mathrm{tr}\underline{\underline{G}}(s)] \\ & + \zeta^{[6]}(s)\underline{\underline{1}}\ \mathrm{tr}[\underline{\underline{b}}(t)\underline{\underline{G}}(s)] \\ & + O(|\underline{\underline{b}}|^2|\underline{\underline{G}}|),\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}\underline{\underline{\Gamma}}\big(s_1,s_2,\underline{\underline{B}}(t)\big)\underline{\underline{G}}(s_1)\underline{\underline{G}}(s_2) = {} & \alpha(s_1,s_2)\underline{\underline{G}}(s_1)\underline{\underline{G}}(s_2) \\ & + \alpha^{[1]}(s_1,s_2)\underline{\underline{1}}\ [\mathrm{tr}\underline{\underline{G}}(s_1)][\mathrm{tr}\underline{\underline{G}}(s_2)] \\ & + \alpha^{[2]}(s_1,s_2)\underline{\underline{1}}\ \mathrm{tr}[\underline{\underline{G}}(s_1)\underline{\underline{G}}(s_2)] \\ & + \alpha^{[3]}(s_1,s_2)\underline{\underline{G}}(s_1)\mathrm{tr}\underline{\underline{G}}(s_2) \\ & + \alpha^{[4]}(s_1,s_2)\underline{\underline{G}}(s_2)\mathrm{tr}\underline{\underline{G}}(s_1) \\ & + O(|\underline{\underline{b}}||\underline{\underline{G}}|^2).\end{aligned} \tag{3.8}$$

When the solid is incompressible the density is a constant and

(3.9) $\det \underline{\underline{F}} = 1$.

In this case the stress is constitutively determined only up to a scalar field p

(3.10) $\underline{\underline{T}} = - p\, \underline{\underline{1}} + \underset{s=0}{\overset{\infty}{\underline{\underline{\mathfrak{F}}}}}[\underline{\underline{B}}(t), \underline{\underline{G}}(s)]$.

The scalar field p is an additional unknown and (3.9) is the additional equation necessary to determine this field.

The forms of $\underline{\underline{\mathfrak{F}}}$ perturbing the rest state and the natural state are the ones already derived for the compressible case with two differences. The first difference is that all the terms in the expansions (3.3) through (3.8) which are proportional to $\underline{\underline{1}}$ may be grouped with p. We may regard the new coefficient of $\underline{\underline{1}}$ in $\underline{\underline{T}}$ as a new "pressure", say π, which is constitutively indeterminate and is to be determined ultimately from the solutions of the equations of motion. So in the incompressible case we take the forms of $\underline{\underline{\mathfrak{F}}}$ given by (3.3) through (3.8) modulo terms proportional to $\underline{\underline{1}}$. A second difference between the stress in the compressible and incompressible case arises as a consequence of (3.9). This second difference will be discussed in §7.

4. Equations of motion for the perturbations of the natural state

In solid bodies the natural state is important because the elastic stresses are measured relative to the undeformed, unstressed state of the body. So if $\underline{t}_n$ is the traction vector on the boundary $\partial\mathcal{V}$ of the region of space occupied by the deformed body, then

(4.1) $$\int_{\partial\mathcal{V}} \underline{t}_n \, da = \int_{\partial\mathcal{V}_0} \underline{\underline{S}}^T \cdot \underline{N} \, dA$$

where $\mathcal{V}_0$ is the region occupied by the undeformed body and $\underline{N}$ is the outward normal on $\partial\mathcal{V}_0$; $\underline{n}$ is the outward normal on $\partial\mathcal{V}$ and

(4.2) $$\underline{n} \, da = \det \underline{\underline{F}} \, (\underline{\underline{F}}^T)^{-1} \cdot \underline{N} \, dA \; .$$

The Piola-Kirchoff stress $\underline{\underline{S}}^T$ is given in terms of the Cauchy stress by

(4.3) $$\underline{\underline{S}}^T = \underline{\underline{T}}^T \, (\underline{\underline{F}}^T)^{-1} \det \underline{\underline{F}} = \underline{\underline{T}}(\underline{\underline{F}}^T)^{-1} \det \underline{\underline{F}} \; .$$

The balance of momentum in any small part of $\mathcal{V}$ (also called $\mathcal{V}$) may be written as

(4.4) $$\int_{\mathcal{V}} \rho \, \ddot{\underline{u}} \; d\mathcal{V} = \int_{\mathcal{V}} \underline{b} \, d\mathcal{V} + \int_{\partial\mathcal{V}} \underline{t}_n \, da$$

where $\underline{b}$ is the body force per unit mass,

(4.5) $$\underline{u} = \underline{x} - \underline{X}$$

is the displacement vector of the partical $\underline{X}$,

$$\ddot{\underline{u}} = \partial^2 \underline{u}(\underline{X},t)/\partial t^2 ,$$

the acceteration, is a derivative following the particle (at fixed $\underline{X}$). In a loose notation, we use the symbol $\underline{u}(\underline{x},t)$ and $\underline{u}(\underline{X},t)$ for different functions whose values $\underline{u}$ are identical when $\underline{X}$ is the particle presently in the place $\underline{x}$. The density $\rho(\underline{x},t)$ in (4.4) is related to the density ρ_o of the same particle in the natural state by

$$\rho_o(\underline{X}) = \rho(\underline{x},t) \det \underline{\underline{F}}(t) . \tag{4.6}$$

Eq. (4.6) implies

$$\rho \, d\mathcal{V} = \rho_o d\mathcal{V}_0 . \tag{4.7}$$

Inserting (4.1) and (4.7) into (4.4) we find the Piola-Kirchoff equations of motion

$$\rho_o(\underline{X})\ddot{\underline{u}}(\underline{X},t) = \rho_o \, \underline{b}(\underline{X},t) + \operatorname{div} \underline{\underline{S}}^T(\underline{X},t) . \tag{4.8}$$

Solutions of (4.8) are driven by the prescribed data: the force field $\underline{b}$, the boundary conditions and the initial history. Our purpose is to develop an algorithm to compute solutions of (4.8) which perturb the zero data giving rise to the natural state. And in the usual way we serve our purpose by requiring that the prescribed data be proportional to a small parameter ϵ so that solutions of (4.8) with $\epsilon \neq 0$ reduce to the natural state in which $\underline{u}$ and $\underline{\underline{S}}$ both vanish when $\epsilon = 0$. For example, we may say that the deformations are driven by $\underline{b}(\epsilon, \underline{X}, t) = \epsilon \underline{\mathfrak{b}}(\underline{X}, t)$. Naturally the computation of fields at $\epsilon = 0$ means that the perturbation problems are all posed on the domain $\mathcal{V}_0$ of the natural state. It is perhaps of interest to remark that our method of solution introduces the natural state automatically through the data and there is no particular advantage gained by starting with the Piola-Kirchoff equations of motion. We arrive at exactly the same equations of motion if we start with Cauchy's equations. In fact, it is more natural to prescribe conditions on $\partial\mathcal{V}$, the boundary of the deformed body, than on $\partial\mathcal{V}_0$, the boundary of the body in the natural state.

Turning now to the aforementioned boundary conditions we declare our interest in boundary value problems of the mixed type. In specifiying "mixed type" boundary conditions we decompose the boundary of the deformed body into two parts.

(4.9) $$\partial\mathcal{V}(t,\epsilon) = \partial\mathcal{V}_1(t,\epsilon) \cup \partial\mathcal{V}_2(t,\epsilon)$$

where the deformation is prescribed on $\partial\mathcal{V}_1(t,\epsilon)$,

(4.10) $$\underline{x} \in \partial\mathcal{V}_1(t,\epsilon) \text{ is prescribed } ;$$

and the traction vector is prescribed on $\partial\mathcal{V}_2(t,\epsilon)$,

(4.11) $$\underline{\underline{T}} \cdot \underline{n}(\underline{x},t,\epsilon) = \underline{t}_n(\underline{x},t,\epsilon) \text{ is prescribed for}$$
$$\underline{x} \in \partial\mathcal{V}_2(t,\epsilon) \text{ where } \underline{t}_n(\underline{x},t,0) = \underline{0} \, .$$

The attentive reader will notice that the prescription of the traction vector in (4.11) is given in terms of the Cauchy stress rather than the Piola-Kirchoff stress. We have already noted that in our local theory the distinction between the Cauchy and Piola-Kirchoff stress is downgraded because both forms lead to exactly the same perturbation equations.

In the same spirit it is convenient to prescribe displacements of the boundary $\partial\mathcal{V}_1(t,\varepsilon)$ of the deformed body where for simplicity we require that

(4.12) $$\underline{x} - \underline{X} = \varepsilon\underline{U}(\underline{X}, t, \varepsilon), \qquad \underline{X} \in \partial\mathcal{V}_{10}, \ \underline{x} \in \partial\mathcal{V}_1(t,\varepsilon)$$

where

$$\partial\mathcal{V}_{10} = \partial\mathcal{V}_1(t,0)$$

is a portion of the boundary

$$\partial\mathcal{V}_0 = \partial\mathcal{V}_{10} \cup \partial\mathcal{V}_{20}$$

of the body in the natural state. Of course $\partial\mathcal{V}_{10}$ and $\partial\mathcal{V}_{20}$ are independent of time. Equation (4.12) says that the set of boundary points for which displacements are prescribed is a material set and no new material points, points on $\partial\mathcal{V}_{20}$, can enter this set as ε is varied.

To complete the prescription of the data for the initial-history problem we prescribe the initial history :

$$\underline{u}_0(\underline{X},t) \text{ is prescribed for } \underline{X} \in \mathcal{V}_0, \ t \leq 0$$

and

(4.13) $$\underline{u}(\underline{X},t,\varepsilon) = \varepsilon\underline{u}_0(\underline{X},t) \text{ for } \underline{X} \in \mathcal{V}_0, \ t \leq 0 .$$

5. Kinematics for perturbations of the natural state

In our perturbation we develop a sequence of equations which may be systematically associated with a perturbation of data giving rise to the natural state. The data is all important and when we perturb it we induce a perturbation of the kinematics as well as of the constitutive equation. The perturbation formulas for the kinematic variables are easy to derive. Only the results are listed below.

(5.1) $$\underline{u}(\underline{X},\tau,\varepsilon) = \varepsilon\underline{u}^{<1>}(\underline{X},\tau) + \varepsilon^2\,\underline{u}^{<2>}(\underline{X},\tau) + 0(\varepsilon^3)\ .$$

(5.2) $$\underline{\underline{F}}(\underline{X},\tau,\varepsilon) = \underline{\underline{1}} + \nabla\underline{u}(\underline{X},\tau) = \underline{\underline{1}} + \varepsilon\underline{\underline{F}}^{<1>}(\underline{X},\tau) + \varepsilon^2\underline{\underline{F}}^{<2>}(X,\tau) + 0(\varepsilon^3)$$

where

$$\underline{\underline{F}}^{<n>}(\underline{X},\tau,\varepsilon) = \nabla\underline{u}^{<n>}(\underline{X},\tau),(F_{ij}^{<n>} = \partial u_i^{<n>}/\partial X_j)\ .$$

(5.3) $$\underline{\underline{F}}^{-1} = \underline{\underline{1}} - \underline{\underline{F}}^{<1>}\varepsilon + (-\underline{\underline{F}}^{<2>} + \underline{\underline{F}}^{<1>}\ \underline{\underline{F}}^{<1>})\varepsilon^2 + 0(\varepsilon^3)\ .$$

(5.4) $$\underline{\underline{G}}(s,\varepsilon) = \underline{\underline{F}}_t^T(\tau,\varepsilon)\ \underline{\underline{F}}_t(\tau,\varepsilon) - \underline{\underline{1}} = \varepsilon\underline{\underline{G}}^{<1>}(s) + \varepsilon^2\underline{\underline{G}}^{<2>}(s) + 0(\varepsilon^3)$$

where

$$\underline{\underline{G}}^{<1>}(s) = 2\{\underline{\underline{E}}^{<1>}(t-s) - \underline{\underline{E}}^{<1>}(t)\}\ ,$$

$$\underline{\underline{G}}^{<2>}(s) = 2\{\underline{\underline{E}}^{<2>}(t-s) - \underline{\underline{E}}^{<2>}(t)\} + \underline{\underline{\xi}}^{<2>}(t,s)\ ,$$

$$\underline{\underline{E}}^{<n>} = \frac{1}{2}(\underline{\underline{F}}^{<n>} + \underline{\underline{F}}^{T<n>})\ ,$$

and

$$\boldsymbol{\mathcal{E}}^{<2>}(t,s) = \underline{\underline{F}}^{T<1>}(t-s)\underline{\underline{F}}^{<1>}(t-s) + \underline{\underline{F}}^{T<1>}(t)\underline{\underline{F}}^{<1>}(t) - 2\underline{\underline{F}}^{T<1>}(t)\underline{\underline{E}}^{<1>}(t-s)$$

$$+ \underline{\underline{F}}^{<1>}(t)\underline{\underline{F}}^{<1>}(t) + \underline{\underline{F}}^{T<1>}(t)\underline{\underline{F}}^{T<1>}(t) - 2\underline{\underline{E}}^{<1>}(t-s)\underline{\underline{F}}^{<1>}(t).$$

(5.5) $$\underline{\underline{B}}(t,\varepsilon) = \underline{\underline{F}}(t,\varepsilon)\underline{\underline{F}}^{T}(t,\varepsilon) = \underline{\underline{1}} + 2\varepsilon\underline{\underline{E}}^{<1>}(t) + \varepsilon^{2}\{2\underline{\underline{E}}^{<2>}(t) + \underline{\underline{F}}^{<1>}(t)\underline{\underline{F}}^{T<1>}(t)\} + 0(\varepsilon^{3}$$

(5.6) $$\det \underline{\underline{F}} = 1 + \varepsilon \operatorname{tr} \underline{\underline{F}}^{<1>} + \varepsilon^{2}\{\operatorname{tr} \underline{\underline{F}}^{<2>} + \tfrac{1}{2}[\operatorname{tr} \underline{\underline{F}}^{<1>}]^{2} - \tfrac{1}{2} \operatorname{tr}[\underline{\underline{F}}^{<1>}\underline{\underline{F}}^{<1>}]\} + 0(\varepsilon^{3}) .$$

$$\rho(\underline{x},t,\varepsilon) = \rho_{o}(\underline{X}) + \varepsilon\rho^{<1>}(\underline{X},t) + \varepsilon^{2}\rho^{<2>}(\underline{X},t) + 0(\varepsilon^{3}) .$$

Since ρ_o is independent of ε we may expand $\rho \det \underline{\underline{F}} = \rho_o$ in powers of ε. Identifying independent powers of ε we find that

(5.7) $$\rho^{<1>} + \rho_{o} \operatorname{tr} \underline{\underline{F}}^{<1>} = 0$$

and

(5.8) $$\rho^{<2>} + \rho^{<1>} \operatorname{tr} \underline{\underline{F}}^{<1>} + \rho_{o} \operatorname{tr} \underline{\underline{F}}^{<2>} + \frac{\rho_{o}}{2}[\operatorname{tr} \underline{\underline{F}}^{<1>}]^{2} - \frac{\rho_{o}}{2} \operatorname{tr} [\underline{\underline{F}}^{<1>}\underline{\underline{F}}^{<1>}] = 0 .$$

We shall also need a formula for the pertrubation of the normal

(5.9) $$\underline{n} = \underline{N} + \varepsilon\underline{n}^{<1>} + 0(\varepsilon^{2}) .$$

To find $\underline{n}^{<1>}$ we write $\tilde{J} = da/dA$ and

$$\underline{n}\tilde{J} = \det \underline{\underline{F}}(\underline{\underline{F}}^{T})^{-1}\cdot\underline{N} .$$

Combining (5.3), (5.6), (5.9) and $\tilde{J} = 1 + \varepsilon \tilde{J}^{<1>} + 0(\varepsilon^2)$ we get

$$\underline{n}^{<1>} + \underline{N}\,\tilde{J}^{<1>} = (\det \underline{\underline{F}}^{<1>})\underline{N} - \underline{\underline{F}}^{T\,<1>}\cdot\underline{N}\ .$$

Since $\underline{n}$ is a unit vector, $\underline{N}\cdot\underline{n}^{<1>} = 0$ and

$$\tilde{J}^{<1>} = \det \underline{\underline{F}}^{<1>} - \underline{N}\cdot\underline{\underline{F}}^{T\,<1>}\cdot\underline{N} \quad .$$

Hence

(5.10) $$\underline{n}^{<1>} = (\underline{N}\cdot\underline{\underline{F}}^{T\,<1>}\cdot\underline{N})\underline{N} - \underline{\underline{F}}^{T<1>}\cdot\underline{N} \quad .$$

Using (5.10), we may write prescribed conditions for the traction vector $\underline{t}_n(\underline{x},t,\varepsilon)$ for $\underline{x} \in \partial\mathcal{V}_2(t,\varepsilon)$ in terms of perturbed Cauchy stresses $\underline{\underline{T}}^{<n>}(\underline{X},t)$ for $X \in \partial\mathcal{V}_{20}$. Thus

(5.11) $$\begin{aligned} \underline{t}_n = \underline{\underline{T}}\cdot\underline{n} &= (\varepsilon\underline{\underline{T}}^{<1>} + \varepsilon^2\underline{\underline{T}}^{<2>} + \ldots)\cdot(\underline{N} + \varepsilon\underline{n}^{<1>} + \ldots) \\ &= \varepsilon\ \underline{\underline{T}}^{<1>}\cdot\underline{N} + \varepsilon^2(\underline{\underline{T}}^{<2>}\cdot\underline{N} + \underline{\underline{T}}^{<1>}\cdot\underline{n}^{<1>}) + 0(\varepsilon^3) \\ &= \varepsilon\ \underline{\underline{T}}^{<1>}\cdot\underline{N} + \varepsilon^2\{(\underline{\underline{T}}^{<2>} - \underline{\underline{T}}^{<1>}\ \underline{\underline{F}}^{T<1>})\cdot\underline{N} \\ &\quad + (\underline{N}\cdot\underline{\underline{F}}^{T\,<1>}\cdot\underline{N})\underline{\underline{T}}^{<1>}\cdot\underline{N}\} + 0(\varepsilon^3) \end{aligned}$$

gives the series expansion of $\underline{t}_n$ on $\partial\mathcal{V}_2$ in terms of $\underline{\underline{T}}^{<n>}$ and geometric quantities defined on $\partial\mathcal{V}_{20}$.

6. Canonical forms for the perturbation stresses and equations of motion for compressible solids

The canonical forms of the Cauchy stress for perturbations of the natural state

$$(6.1) \qquad \underline{\underline{T}} = \varepsilon \underline{\underline{T}}^{<1>} + \varepsilon^2 \underline{\underline{T}}^{<2>} + 0(\varepsilon^3)$$

can be obtained by identification by combining (3.2, 6, 7, 8) with (5.4) and (5.5). We find that

$$(6.2) \qquad \underline{\underline{T}}^{<1>}(t) = \underline{\underline{\tau}}[\underline{u}^{<1>}(t)] \overset{\text{def}}{=} 2\beta \underline{\underline{E}}^{<1>}(t) + 2\beta^{[1]} \operatorname{div} \underline{u}^{<1>}(t)\underline{\underline{1}}$$

$$+ \int_0^\infty \{\zeta(s) 2[\underline{\underline{E}}^{<1>}(t-s) - \underline{\underline{E}}^{<1>}(t)]$$

$$+ 2\zeta^{[1]}(s) \operatorname{div} [\underline{u}^{<1>}(t-s) - \underline{u}^{<1>}(t)]\underline{\underline{1}}\} ds ,$$

and

$$(6.3) \qquad \underline{\underline{T}}^{<2>} = \underline{\underline{\tau}}[\underline{u}^{<2>}] + \underline{\underline{h}}[\underline{u}^{<1>}]$$

where

$$\underline{\underline{h}}[\underline{u}^{<1>}] \overset{\text{def}}{=} \beta\, \underline{\underline{F}}^{<1>} \underline{\underline{F}}^{T<1>} + \beta^{[1]}\, \underline{\underline{1}} \operatorname{tr}\left[\underline{\underline{F}}^{<1>}\, \underline{\underline{F}}^{T<1>}\right]$$

$$+ \int_0^\infty \{\zeta(s) \underline{\underline{\mathcal{E}}}^{<2>}(t,s) + \underline{\underline{1}}\zeta^{[1]}(s) \operatorname{tr} \underline{\underline{\mathcal{E}}}^{<2>}(t,s)\} ds$$

$$+ \beta^{[2]}\, \underline{\underline{B}}^{<1>}\, \underline{\underline{B}}^{<1>} + \underline{\underline{1}}\, \beta^{[3]} (\operatorname{tr} \underline{\underline{B}}^{<1>})^2$$

$$+ \beta^{[4]}\, \underline{\underline{1}} \operatorname{tr} [\underline{\underline{B}}^{<1>} \underline{\underline{B}}^{<1>}] + \beta^{[5]} \underline{\underline{B}}^{<1>} \operatorname{tr} \underline{\underline{B}}^{<1>}$$

$$+ \int_0^\infty \{\zeta^{[2]}(s)\,[\underline{\underline{B}}^{<1>}(t)\underline{\underline{G}}^{<1>}(s) + \underline{\underline{G}}^{<1>}(s)\underline{\underline{B}}^{<1>}(t)]$$

$$+ \zeta^{[3]}(s)\ \underline{\underline{G}}^{<1>}(s)\ \mathrm{tr}\ \underline{\underline{B}}^{<1>}(t) + \zeta^{[4]}(s)\ \underline{\underline{B}}^{<1>}(t)\mathrm{tr}\ \underline{\underline{G}}^{<1>}(s)$$

$$+ \zeta^{[5]}(s)\ \underline{\underline{1}}\ [\mathrm{tr}\ \underline{\underline{B}}^{<1>}(t)][\mathrm{tr}\ \underline{\underline{G}}^{<1>}(s)]$$

$$+ \zeta^{[6]}(s)\underline{\underline{1}}\ \mathrm{tr}\ [\underline{\underline{B}}^{<1>}(t)\underline{\underline{G}}^{<1>}(s)]\}ds$$

$$+ \int_0^\infty\int_0^\infty \{\alpha(s_1,s_2)\ \underline{\underline{G}}^{<1>}(s_1)\underline{\underline{G}}^{<1>}(s_2)$$

$$+ \alpha^{[1]}(s_1,s_2)\ \underline{\underline{1}}\ [\mathrm{tr}\ \underline{\underline{G}}^{<1>}(s_1)][\mathrm{tr}\ \underline{\underline{G}}^{<1>}(s_2)]$$

$$+ \alpha^{[2]}(s_1,s_2)\underline{\underline{1}}\ \mathrm{tr}\ [\underline{\underline{G}}^{<1>}(s_1)\ \underline{\underline{G}}^{<1>}(s_2)]$$

$$+ \alpha^{[3]}(s_1,s_2)\ \underline{\underline{G}}^{<1>}(s_1)\ \mathrm{tr}\ \underline{\underline{G}}^{<1>}(s_2)$$

$$+ \alpha^{[4]}(s_1,s_2)\ \underline{\underline{G}}^{<1>}(s_2)\ \mathrm{tr}\ \underline{\underline{G}}^{<1>}(s_1)\}ds_1\ ds_2 \quad .$$

The Piola-Kirchoff stress tensor is now given by (4.3), (5.3), (5.6) and (6.1,2,3) as

$$\text{(6.4)} \qquad \underline{\underline{S}}^T = \varepsilon\underline{\underline{T}}^{<1>} + \varepsilon^2\{\underline{\underline{T}}^{<2>} - \underline{\underline{T}}^{<1>}\ \underline{\underline{F}}^{T<1>} + \underline{\underline{T}}^{<1>}\ \mathrm{tr}\ \underline{\underline{F}}^{<1>}\} + 0(\varepsilon^3) \ .$$

To characterize the motion of a particular compressible viscoelastic solid at first order we need values for

2 elastic constants β and $\beta^{[1]}$

and

2 material functions $\zeta(s)$ and $\zeta^{[1]}(s)$.

To characterize the motion of a particular compressible viscoelastic solid at second order we need values for

6 elastic constants β; $\beta^{[n]}$, $n = 1,2,3,4,5$

and

12 material functions $\zeta(s)$; $\zeta^{[n]}(s)$, $n = 1,2,3,4,5,6$; $\alpha(s_1,s_2)$ and $\alpha^{[\ell]}(s_1,s_2)$, $\ell = 1,2,3,4$. Obviously, the rheometrical problem of material characterization in the second order theory of motions of viscoelastic solids perturbing the natural state is very hard because there are so many material functions and constants.

We have seen that the expansion of $\underline{u}(\underline{X},\tau,\varepsilon)$ in powers of ε ultimately induces an expansion of the stresses $\underline{\underline{T}}$ and $\underline{\underline{S}}^T$ in powers of ε. The expansion of $\underline{u}(\underline{X},\tau,\varepsilon)$ for $\underline{X} \in \mathcal{V}_0$ was presumed given, but it is not given; it must be determined from solutions of the equations which perturb the natural state. The perturbation of the data driving the motion is given and it induces the expansion of all the other interlocked quantities.

The equations which perturb the natural state arise from (4.8), (4.11), (4.12), (4.13) with $\underline{b} = \varepsilon \underline{\mathfrak{b}}$ by identification when all of the variables are expanded in powers of ε. The expansion of the stress is given by (6.1-3) and the expansion of the kinematic variables by (5.1-11). At first order

$$\rho_o \ddot{\underline{u}}^{<1>} = \rho_o \underline{\mathfrak{b}} \qquad + \operatorname{div} \underline{\underline{\tau}}[\underline{u}^{<1>}] \quad \text{for} \quad \underline{X} \in \mathcal{V}_0,\ t > 0 \text{;}$$

$$\underline{u}^{<1>}(\underline{X},t) \quad \text{is prescribed for } \underline{X} \in \partial\mathcal{V}_{10},\ t > 0\ ;$$

(6.5)
$$\underline{\underline{T}}^{<1>}(\underline{X},t)\cdot\underline{N} = \underline{t}_n^{<1>}(\underline{X},t) \text{ is prescribed for } \underline{X} \in \partial\mathcal{V}_{20},\ t > 0\ ;$$

$$\underline{u}^{<1>}(\underline{X},t) \text{ is prescribed for } \underline{X} \in \mathcal{V}_0,\ t \le 0\ .$$

For orders $n > 1$ we find that in $\mathcal{V}_0$ and for $t > 0$

$$\rho_o\, \ddot{\underline{u}}^{<n>} = \operatorname{div} \underline{\underline{\tau}}[\underline{u}^{<n>}] + \text{terms of lower order} \tag{6.6}$$

where

$$\underline{u}^{<n>}(\underline{X},t) \text{ is prescribed in terms of lower order for}$$

$$X \in \partial\mathcal{V}_{10},\quad t > 0\ ;$$

$$\underline{\underline{T}}^{<n>}(\underline{X},t)\cdot\underline{N} \text{ is prescribed in terms of lower order}$$

$$\text{for } X \in \partial\mathcal{V}_{20},\ t > 0\ ;$$

and

$$\underline{u}^{<n>}(\underline{X},t) = \underline{0} \quad \text{for } X \in \mathcal{V}_0,\quad t \le 0\ .$$

For example when $n = 2$, the terms of lower order in $(6.6)_1$ are

$$\operatorname{div}\{\underline{\underline{h}}[\underline{u}^{<1>}] - \underline{\underline{T}}^{<1>}\,\underline{\underline{F}}^{T<1>} + \underline{\underline{T}}^{<1>}\operatorname{tr}\underline{\underline{F}}^{<1>}\}\ . \tag{6.7}$$

The perturbation equations can be solved sequentially and at each step of the sequence there are three equations for the three unknown components of $\underline{u}^{<n>}$.

Changes in density due to deformation may be expressed by (5.7) and (5.8). Similar formulas hold at higher orders.

The practical utility of a theory which requires knowing the value of 6 elastic constants and 12 material functions is debatable. Pipkin (1964) noticed that there is a big reduction in the number of unknown material parameters when the material is incompressible.

7. Canonical forms for the perturbation stresses and equations of motion for incompressible solids

Incompressible solids have been discussed in §3. We have already explained that in the incompressible case we may group all terms of

$$\underline{\underline{T}} = -p\underline{\underline{1}} + \underline{\underline{\mathfrak{F}}}\,[\underline{\underline{B}}(t), \underset{s=0}{\overset{\infty}{\underline{\underline{G}}(s)}}]$$

which are proportional to $\underline{\underline{1}}$ with $-p$. The $\underline{\underline{\mathfrak{F}}}$ part of $\underline{\underline{T}}$ is constitutively determined while the spherically symmetric $p\underline{\underline{1}}$ part of $\underline{\underline{T}}$ is to be determined from the equations of motions. So one simplification in the form of $\underline{\underline{\mathfrak{F}}}$ comes from dumping terms of (3.6,7,8) proportional to $\underline{\underline{1}}$ into $-p$. A second simplification comes from setting det $\underline{\underline{F}} = 1$ in (5.6). Then

(7.1) $$\text{tr}\,\underline{\underline{F}}^{<1>} = \text{div}\,\underline{u}^{<1>} = 0 ,$$

and

(7.2) $$\text{tr}\,\underline{\underline{F}}^{<2>} = \text{div}\,\underline{u}^{<2>} = \frac{1}{2}\,\text{tr}[\underline{\underline{F}}^{<1>}\,\underline{\underline{F}}^{<1>}] .$$

It follows from (5.4), (5.5) and (7.1) that

(7.3) $$\text{tr}\,\underline{\underline{G}}^{<1>}(s) = \text{tr}\,\underline{\underline{B}}^{<1>} = 0 ,$$

and from (5.5) and (7.2) that

(7.4) $$\text{tr}\,\underline{\underline{B}}^{<2>} = 2\,\text{div}\,\underline{u}^{<2>} + \text{tr}[\underline{\underline{F}}^{<1>}\underline{\underline{F}}^{T<1>}]$$

$$= \text{tr}[\underline{\underline{F}}^{<1>}\underline{\underline{F}}^{<1>} + \underline{\underline{F}}^{<1>}\underline{\underline{F}}^{T<1>}] ,$$

and from (5.4) and (7.2) that

$$(7.5) \qquad \operatorname{tr} \underline{\underline{G}}^{<2>}(s) = 2 \operatorname{div} [\![\underline{u}^{<2>}]\!] + \operatorname{tr} \underline{\underline{\mathcal{E}}}^{<2>}(t,s)$$

$$= \operatorname{tr}[\![\underline{\underline{F}}^{<1>}\underline{\underline{F}}^{<1>}]\!] + \operatorname{tr} \underline{\underline{\mathcal{E}}}^{<2>}(t,s)$$

where $[\![\cdot]\!]$ is a jump operator on whose domain are functions a of t.

$$[\![a]\!] \overset{\text{def}}{=} a(t-s) - a(t) \; .$$

The perturbed stresses are given by

$$(7.6) \qquad \underline{\underline{T}}^{<1>} = -p^{<1>}\underline{\underline{1}} + 2\beta\underline{\underline{E}}^{<1>}(t) + 2\int_0^\infty \zeta(s)\, [\![\underline{\underline{E}}^{<1>}]\!] ds$$

$$= -p^{<1>}\underline{\underline{1}} + 2\gamma\underline{\underline{E}}^{<1>}(t) + 2\int_0^\infty \zeta(s)\underline{\underline{E}}^{<1>}(t-s) ds$$

where

$$(7.7) \qquad \gamma = \beta - \int_0^\infty \zeta(s) ds \quad ,$$

and

$$(7.8) \qquad \underline{\underline{T}}^{<2>} = -\pi^{<2>}\underline{\underline{1}} + 2\gamma\underline{\underline{E}}^{<2>}(t) + 2\int_0^\infty \zeta(s)\underline{\underline{E}}^{<2>}(t-s) ds$$

$$+ \beta\underline{\underline{F}}^{<1>}\underline{\underline{F}}^{T<1>} + \beta^{[2]}\underline{\underline{B}}^{<1>}\underline{\underline{B}}^{<1>} + \int_0^\infty \zeta(s)\underline{\underline{\mathcal{E}}}^{<2>}(t,s) ds$$

$$+ \int_0^\infty \zeta^{[2]}(s)\,[\underline{\underline{B}}^{<1>}(t)\underline{\underline{G}}^{<1>}(s) + \underline{\underline{G}}^{<1>}(s)\underline{\underline{B}}^{<1>}(t)] ds$$

$$(7.8) \qquad + \int_0 \int_0 \alpha(s_1,s_2)\,\mathbf{G}^{<1>}(s_1)\;\mathbf{G}^{<1>}(s_2)\,ds_1\,ds_2$$

where

$$(7.9) \qquad \pi^{<2>} = p^{<2>} - \beta^{[1]}\,\mathrm{tr}(\mathbf{F}^{<1>}\mathbf{F}^{<1>} + \mathbf{F}^{<1>}\mathbf{F}^{T<1>}) - \beta^{[4]}\mathrm{tr}[\mathbf{B}^{<1>}\mathbf{B}^{<1>}]$$

$$- \int_0^\infty \zeta^{[1]}(s)\{\mathrm{tr}\,[\![\mathbf{F}^{<1>}\mathbf{F}^{<1>}]\!] + \mathrm{tr}\;\mathfrak{E}^{<2>}(t,s)\}ds$$

$$- \int_0^\infty \zeta^{[6]}(s)\;\mathrm{tr}[\mathbf{B}^{<1>}(t)\mathbf{G}^{<1>}(s)]ds$$

$$- \int_0^\infty\int_0^\infty \alpha^{[2]}(s_1,s_2)\;\mathrm{tr}\;[\mathbf{G}^{<1>}(s_1)\mathbf{G}^{<1>}(s_2)]\;ds_1\,ds_2 \quad .$$

To characterize the motion of a particular incompressible viscoelastic solid at first order we need values for

1 elastic constant β

and

1 material function $\zeta(s)$.

To characterize the motion of an incompressible solid at second order we need

2 elastic constants β and $\beta^{[2]}$

and

3 material functions $\zeta(s)$, $\zeta^{[2]}(s)$ and $\alpha(s_1,s_2)$.

The material constants appearing in $\pi^{<2>}$ are constitutively undetermined since $\pi^{<n>}$ is to be determined as one of the four unknown fields in the canonical problems governing the perturbation displacements.

Turning next to these canonical problems we find that

$$(7.10) \qquad \mathbb{J}\underline{u}^{<1>} + \nabla p^{<1>*} = \underline{0}\ , \quad \text{div}\ \underline{u}^{<1>} = 0 \quad \text{in} \quad \mathcal{V}_0\ , \ t > 0\ ;$$

$$\underline{u}^{<1>}(\underline{X},t) \text{ is prescribed for } \underline{X} \in \partial\mathcal{V}_{10}\ , \ t > 0\ ;$$

$$\underline{t}_n^{<1>} = \underline{\underline{T}}^{<1>}\cdot\underline{N} \text{ is prescribed for } \underline{X} \in \partial\mathcal{V}_{20}\ , \ t > 0\ ;$$

$$\underline{u}^{<1>}(\underline{X},t) = \underline{u}_0(\underline{X},t) \quad \text{for} \quad \underline{X} \in \mathcal{V}_0\ , \quad t \leq 0\ .$$

In (7.10) and elsewhere

$$(7.11) \qquad \mathbb{J}(\cdot) \overset{\text{def}}{=} \rho_0\ (\ddot{\cdot}) - \gamma\nabla^2(\cdot) - \nabla^2 \int_0^\infty \zeta(s)(\cdot)(t-s)ds\ .$$

At second order

$$(7.12) \qquad \mathbb{J}\underline{u}^{<2>} + \nabla\pi^{<2>} = \underline{M}_2, \quad \text{div}\ \underline{u}^{<2>} = \theta_2 \quad \text{in} \quad \mathcal{V}_0\ , \ t > 0\ ;$$

$$\underline{u}^{<2>}(\underline{X},t) \text{ is prescribed for } \underline{X} \in \partial\mathcal{V}_{10},\ t > 0\ ;$$

$$\underline{t}_n^{<2>} = \underline{\underline{T}}^{<2>}\cdot\underline{N} + \underline{\underline{T}}^{<1>}\cdot\underline{n}^{<1>} = (\underline{\underline{T}}^{<2>} - \underline{\underline{T}}^{<1>}\ \underline{\underline{F}}^{T<1>})\cdot\underline{N}$$

$$+ (\underline{N}\cdot\underline{\underline{F}}^{T<1>}\cdot\underline{N})\underline{\underline{T}}^{<1>}\cdot\underline{N} \quad \text{is prescribed for} \quad \underline{X} \in \partial\mathcal{V}_{20}, t>0\ ;$$

and

$$\underline{u}^{<2>}(\underline{X},t) = \underline{0} \quad \text{for} \quad X \in \mathcal{V}_0\ , \quad t \leq 0\ .$$

In (7.12)

*Henceforth, we neglect the body force.

$$\theta_2 = \frac{1}{2}\,\mathrm{tr}[\underline{\underline{F}}^{<1>}\underline{\underline{F}}^{<1>}] ,$$

and

(7.13) $$\underline{M}_2 = \underline{\underline{F}}^{T<1>}\cdot\nabla p^{<1>} + \mathrm{div}\,\{\beta^{[2]}\,\underline{\underline{B}}^{<1>}\underline{\underline{B}}^{<1>}$$

$$+ \int_0^\infty \zeta(s)\,[-\underline{\underline{G}}^{<1>}(s)\;\underline{\underline{B}}^{<1>}(t)+2[\![\underline{\underline{F}}^{T<1>}]\!]\;\underline{\underline{E}}^{<1>}(t-s)]ds$$

$$+ \int_0^\infty \zeta^{[2]}(s)\,[\underline{\underline{B}}^{<1>}(t)\underline{\underline{G}}^{<1>}(s)+\underline{\underline{G}}^{<1>}(s)\;\underline{\underline{B}}^{<1>}(t)]ds$$

$$+ \int_0^\infty\int_0^\infty \alpha(s_1,s_2)\underline{\underline{G}}^{<1>}(s_1)\;\underline{\underline{G}}^{<1>}(s_2)ds_1\,ds_2\} .$$

In deriving $\underline{M}_2$ we made use of the identity

$$\frac{1}{2}\nabla\mathrm{tr}\;\underline{\underline{F}}^{<1>}\underline{\underline{F}}^{<1>} = \mathrm{div}\,\{\underline{\underline{F}}^{T<1>}\underline{\underline{F}}^{T<1>}\}$$

which holds whenever $\mathrm{div}\;\underline{u}^{<1>} = 0$. Many terms in the expression $\mathrm{div}\,\{\underline{\underline{T}}^{<2>} - \underline{\underline{T}}^{<1>}\;\underline{\underline{F}}^{T<1>}\}$ vanish.

At every order $n \geq 1$ we find that $\underline{u}^{<n>} = \underline{v}$ and $\pi^{<n>} = \pi$ satisfy

(7.14) $$\left.\begin{aligned}\mathbb{J}\underline{v} + \nabla\pi &= \underline{f}_1(\underline{X},t)\\ \mathrm{div}\;\underline{v} &= f_2(\underline{X},t)\end{aligned}\right\}\quad \text{for } \underline{X}\in\mathcal{V}_0 ,\quad t > 0 ;$$

$$\underline{v}(\underline{X},t) = \underline{f}_3(\underline{X},t) \quad \text{for} \quad \underline{X}\in\partial\mathcal{V}_{10} ,\quad t>0 ;$$

$$(-\pi\underline{\underline{1}} + 2\gamma\;\underline{\underline{E}}[\underline{v}(t)] + 2\int_0^\infty \zeta(s)\underline{\underline{E}}[\underline{V}(t-s)]ds)\cdot\underline{N}$$

$$= \underline{f}_4(\underline{X},t) \quad \text{for } \underline{X}\in\partial\mathcal{V}_{20} ,\; t > 0 ;$$

$$\underline{v}(\underline{X},t) = \underline{f}_5(\underline{X},t) \text{ for } \underline{X} \in \mathcal{V}_0, \; t \leq 0 \quad .$$

where $\underline{\underline{E}}[\underline{v}(t)] = \frac{1}{2}(\nabla\underline{v}(\underline{X},t) + \text{transpose})$ and the right hand sides of (7.14) are known from the prescribed data and lower order solutions. At each order we must solve four equations for the fields $\underline{v}(X,t)$ and $\pi(\underline{X},t)$ in $\mathcal{V}_0$.

8. Stability and uniqueness of solutions of the canonical equations perturbing the natural state

The main aim of this section is to show unique solvability for the sequence of perturbation problems (7.14) for incompressible, viscoelastic solids. We do not consider the problem of existence, however, and restrict ourselves to a discussion of uniqueness and the related problem of stability. It is probable that an existence and uniqueness theory of the type recently given by Slemrod (1977) for Joseph's (1976) theory of motions which perturb the rest state of simple fluids can be adapted to the present problem. But here we follow a different path.

To motivate the analysis we remind the reader that the theory of slow flow of Navier-Stokes fluids is a relatively uncomplicated subject because when the flow is slow (or the Reynolds number is small) there is just one solution in the long run and it is uniquely determined by the boundary conditions and body forces, independent of initial conditions. The unique solution is globally stable in the sense that all disturbances, small or large, of this solution ultimately decay. So we expect to observe in nature what we calculate from the equations when the flow is slow. And we do. But this simplicity is lost when the flow is not slow because there are many solutions for prescribed boundary conditions and body forces and many of these are unstable.

The situation of viscoelastic fluids and solids is not so different. In any event, when the forcing data is small, the rest state or natural state is stable in the linearized approximation provided only that material parameters and their derivatives have the expected sign . For larger forcing data the problem of stability is probably at least as complicated as in the Navier-Stokes theory. In fact, viscoelastic materials can exhibit shock-ups and loss of existence of smooth solutions without parallel in the Navier-Stokes theory.

Of course, stability can never be asserted on the basis of linearized equations alone because linearized equations do not govern the evolution of large disturbances. So our statements about stability are at best conditional, subject to the restriction that disturbances are sufficiently small. In fact, since conditional stability theorems are not known for viscoelastic materials, it has to be assumed that the analysis of the linearized equations applies to the nonlinear equations when the nonlinear part is small.

The linearized stability problem for the stability of the natural state may be obtained from the linearization of the initial history problem (4.8), (4.11), (4.12), (4.13) and (3.9) for an infinitesimal disturbance $\underline{v}$ of $\underline{u} \equiv \underline{0}$:

$$
\begin{aligned}
&\left.\begin{aligned} &\mathbb{J}\underline{v} + \nabla\pi = \underline{0} \\ &\text{div } \underline{v} = 0 \end{aligned}\right\} \quad \text{for } \underline{X} \in \mathcal{V}_0, t > 0; \\
&\underline{v}(\underline{X},t) = \underline{0} \text{ for } \underline{X} \in \mathcal{V}_{10},\ t > 0; \\
&\{-\pi\underline{\underline{1}} + 2\gamma\underline{\underline{E}}[\underline{v}(t)] + 2\int_0^\infty \zeta(s)\ \underline{\underline{E}}[\underline{v}(t-s)]ds\} \cdot \underline{N} \\
&\qquad = \underline{0} \text{ for } \underline{X} \in \mathcal{V}_{20},\ t > 0; \\
&\underline{v}(\underline{X},t) = \underline{v}_0(\underline{X},t) \text{ is prescribed for } \underline{X} \in \mathcal{V}_0 \text{ and } t \le 0.
\end{aligned} \tag{8.1}
$$

We are interested in finding the conditions under which $\underline{v} \to 0$ as $t \to \infty$.

The problem (8.1) is identical to the problem which governs the uniqueness of solutions of the initial-history problem (7.14) if $\underline{v}_0(\underline{X},t)$ is set to zero. To study uniqueness we consider two solutions of (7.14) with same prescribed data and initial histories. The difference between these two solutions satisfies (8.1) with $\underline{v}_0(\underline{X},t) = \underline{0}$.

Uniqueness theorems for linearized initial-history problems for viscoelastic solids under slightly different conditions and constitutive assumptions have been given by Edelstein and Gurtin (1964), Odeh and Tadjbakhsh(1965), Gurtin and Sternberg (1962), Breuer and Onat (1962) and Onat and Breuer (1963). If $\underline{v} = \underline{0}$ is asymptotically stable then the solution $\underline{v} = \underline{0}$ corresponding to a zero initial history is automatically unique. On the other hand uniqueness for the initial history problem does not imply stability since new solutions with $\underline{v} \neq \underline{0}$ may arise from the loss of stability of $\underline{v} = \underline{0}$. Loss of stability is associated with the evolution of disturbances, new initial conditions, which are the inevitable results of fluctuations in the prescribed data.

Unique solvability is intimately connected with stability and has almost no relation to the problem of uniqueness of the initial-history problem. In the present circumstances unique solvability comes down to a verification that $\mathfrak{J}$ is uniquely invertible, and new solutions with $\underline{v} \neq \underline{0}$ cannot bifurcate.

Existence, uniqueness and asymptotic stability of generalized solutions of equations like ours have been given by Dafermos (1970) for problems in which displacements are prescribed on the entire boundary $\partial\mathcal{V}_0$ of $\mathcal{V}_0$. In the problem treated by Dafermos the vector field $\underline{v}$ is not necessarily solenoidal and $\pi = 0$. Our problem can be reduced to the one considered by Dafermos by projection techniques used in mathematical studies of the Navier-

Stokes equations (Fujita and Kato, 1964 ; Ladyzhenskaya, 1963). In this method one introduces a Hilbert space H of vectors with scalar product

$$<\underline{a},\underline{b}> = \int_{\mathcal{V}_0} \underline{a}(\underline{X}) \cdot \underline{b}(\underline{X}) d\mathcal{V}_0 ,$$

and norm $||a|| = <\underline{a},\underline{a}>^{1/2}$ by completing the $C^\infty(\mathcal{V}_0)$ vectors with compact support. The compact support is natural for problems like the one which governs $\underline{v}$ when $\partial\mathcal{V}_{10} = \partial\mathcal{V}_0$ so that $\underline{v} = \underline{0}$ for $\underline{X} \in \partial\mathcal{V}_0$, in which the function is prescribed to be zero on the boundary $\partial\mathcal{V}_0$ of $\mathcal{V}_0$. For such problem it is possible to decompose H into orthogonal subspaces of solenoidal vectors(H_1) and gradients(H_2), $H = H_1 \oplus H_2$. There is then the orthogonal projection $\mathbb{P}$ which commutes with $\mathbb{J}$ and annihilates gradients. So we get

(8.2)

$$\mathbb{J}\mathbb{P}\underline{v} = \underline{0} \text{ in } \mathcal{V}_0, \; t > 0;$$

$$\mathbb{P}\underline{v} = \underline{0} \text{ for } \underline{X} \in \partial\mathcal{V}_0, \; t > 0;$$

$$\mathbb{P}\underline{v} = \mathbb{P}\underline{v}_0 \text{ is prescribed in } \mathcal{V}_0, \; t \leq 0.$$

This problem, (8.2), falls in the frame of the study of Dafermos (1970) who shows that $<\dot{\underline{v}},\dot{\underline{v}}>(t) \to 0$ and $<\nabla\underline{v},\nabla\underline{v}>(t) \to 0$ provided that

(8.3)

1. $\beta > 0$,

2a. $\zeta, \dot{\zeta} \in C^0[0,\infty) \cap L^1[0,\infty)$,

2b. $\zeta(s) \leq 0$ for $s \in [0,\infty)$,

2c. $\dot{\zeta}(s) \geq 0$ for $s \in [0,\infty)$,

2d. ζ does not vanish identically .

The conditions required by Dafermos for asymptotic stability do not disagree with conditions which rheologists would require on physical grounds using experience and intuition.

The conditions on β and $\zeta(s)$ derived by Dafermos are the desired conditions which guarantee unique solvability of (7.14) when displacements are prescribed over all of $\partial\mathcal{V}_0$ ($\partial\mathcal{V}_{10}=\partial\mathcal{V}_0$, $\partial\mathcal{V}_{20} = 0$). We therefore turn next to the mixed problem using elementary methods in which the source of restrictions on β and $\zeta(s)$ will be easy to interpret.

We start by deriving a spectral problem for solutions of (8.1) using the method of the exponential time factor:

$$\underline{v}(\underline{X},t) = e^{\sigma t}\,\underline{v}(\underline{X}) \;, \tag{8.4}$$

$$\pi(\underline{X},t) = e^{\sigma t}\pi(\underline{X})$$

where

$$\sigma = \xi + i\omega \;,$$

and σ, $v_i(\underline{X})$ and $\pi(\underline{X})$ satisfy equations obtained from (8.1) and (8.4).

$$\rho_o\sigma^2 v_i(\underline{X}) = \kappa(\sigma)\nabla^2 v_i(\underline{X}) - \frac{\partial\pi}{\partial X_i}(\underline{X}) \;,$$

and

$$\partial v_i/\partial X_i = 0$$

in $\mathcal{V}_0$ and

$$v_i(\underline{X}) = 0 \text{ for } \underline{X}\in\partial\mathcal{V}_{10},$$

and

$$\{-\pi\delta_{ij} + \kappa(\sigma)(\frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i})\} \quad N_j = 0 \text{ for } \underline{X}\in\partial\mathcal{V}_{20}$$

where

$$\kappa(\sigma) = \beta + \int_0^\infty \zeta(s)\{e^{-\sigma s}-1\}ds = \gamma + \int_0^\infty \zeta(s)e^{-\sigma s}ds.$$

After introducing new variables

$$\lambda = -\kappa(\sigma)/\rho_o\sigma^2; \quad \tilde{\pi}(\underline{X}) = \pi(\underline{X})/\rho_o\sigma^2$$

we may rewrite the spectral problem as

$$\underline{v} + \lambda\nabla^2\underline{v} + \nabla\tilde{\pi} = \underline{0}, \text{ div } \underline{v} = 0 \text{ in } \mathcal{V}_0 ,$$

(8.5) $$v_i = 0 \qquad \text{for } \underline{X} \in \partial\mathcal{V}_{10} ,$$

$$\{\tilde{\pi}\delta_{ij} + \lambda(\frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i})\} \quad N_j = 0 \text{ for } \underline{X} \in \partial\mathcal{V}_{20}.$$

Eqs. (8.5) define a spectral problem. We seek the values of λ for which (8.5) has solutions $(\underline{v},\tilde{\pi}) \neq (\underline{0},0)$. The following properties characterize the spectrum of (8.5):

(1) The spectrum $\Sigma(\lambda)$ of (8.5) is a pure point spectrum, that is, λ are eigenvalues of (8.5) .

(2) The eigenvalues λ of (8.5) are real-valued .

(3) The number of eigenvalues is countably infinite. They are of finite multiplicity, all semi-simple and may be arranged as a decreasing sequence clustering at zero:

$$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \ldots \geq 0.$$

If the measure of $\partial\mathcal{V}_{10}$ is greater than zero then $\lambda_1 < \infty$.

Proof: To prove the asserted properties we show that the λ_i are the critical points of the functional

(8.6) $$\lambda[\underline{v}] \overset{def}{=} \frac{\langle|\underline{v}|^2\rangle}{\langle|\underline{\underline{A}}(\underline{v})|^2\rangle}$$

where $\langle\cdot\rangle = \int_{\mathcal{V}_0} (\cdot)\, d\mathcal{V}_0$,

$$|\underline{v}|^2 = v_i v_i, |\underline{\underline{A}}(\underline{v})|^2 = A_{ij}(\underline{v}) A_{ij}(\underline{v}) ,$$

$$A_{ij}(\underline{v}) = 2E_{ij}(\underline{v}) = \frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i} ,$$

among functions

$$\underline{v} \in H = [\underline{v}: \text{div } \underline{v} = 0, \underline{v}\Big|_{\partial\mathcal{V}_{10}} = \underline{0}, \langle|\underline{\underline{A}}(\underline{v})|^2\rangle < \infty] .$$

Every solution of (8.5) satisfies

$$\langle\underline{v}, \underline{v} + \lambda \text{div } \underline{\underline{A}}(\underline{v}) + \nabla\tilde{\pi}\rangle = \langle|\underline{v}|^2\rangle + \lambda\langle v_i \frac{\partial A_{ij}(\underline{v})}{\partial X_j}\rangle$$

$$+ \langle v_i \frac{\partial \tilde{\pi}}{\partial X_i}\rangle = \langle|\underline{v}|^2\rangle - \lambda\langle A_{ij}(\underline{v}) A_{ij}(\underline{v})\rangle$$

$$+ \int_{\partial\mathcal{V}_{20}} v_i N_j [\tilde{\pi}\delta_{ij} + \lambda A_{ij}(\underline{v})] d\Sigma$$

$$= \langle|\underline{v}|^2\rangle - \lambda\langle|\underline{\underline{A}}(\underline{v})|^2\rangle = 0$$

where the scalar product $\langle,\rangle$ is as defined before. The eigenvalues $\lambda_n \neq \lambda$ are critical points of the Rayleigh quotient

(8.7) $$\lambda_n = \max_{H_n} \lambda[\underline{v}]$$

where H_n is the subspace of the space H which is orthogonal to eigensubspaces of the first n-1 eigenvectors. The asserted properties are all known consequences of the variational characterization of eigenvalues (see, for example, Reisz-Nagy, 1955, page 232). The condition measure $\partial\mathcal{V}_{10} > 0$ rules out the solution $\underline{v} = \text{const} \neq \underline{0}$ and guarantees the existence of a finite least upper bound for $\lambda[\underline{v}]$, $\underline{v} \in H$.

Next, we show that solutions of (8.7) are also the eigenvalues of Eqs. (8.5). We reformulate (8.7) as

(8.8) $\lambda_1 = \max_{\underline{v} \in \mathcal{H}} \hat{\lambda}[\underline{v}]$ where

(8.9) $\mathcal{H} = [\underline{v}: \underline{v}|_{\partial \mathcal{V}_{10}} = \underline{0}, <|\underline{\underline{A}}(\underline{v})|^2> < \infty]$,

(8.10) $\hat{\lambda}[\underline{v}] = \{<|\underline{v}|^2> - 2<q \text{ div } \underline{v}>\}/<|\underline{\underline{A}}(\underline{v})|^2>$,

and

$q \in C^1(\mathcal{V}_0)$ is a Lagrange multiplier.

Let $\underline{v}_1$ be the maximizing function. It satisfies the constraint div $\underline{v}_1 = 0$. For any $\underline{\phi} \in \mathcal{H}$ and any real ε

(8.11) $$\frac{d}{d\varepsilon} \hat{\lambda}[\underline{v}_1 + \varepsilon\underline{\phi}]\Big|_{\varepsilon=0} = \frac{2}{<|\underline{\underline{A}}(\underline{v}_1)|^2>} \{<\underline{v}_1, \underline{\phi}> - <q \text{ div } \underline{\phi}> - \lambda_1 <\underline{\underline{A}}(\underline{v}_1) \cdot \underline{\underline{A}}(\underline{\phi})>\} = 0 \quad .$$

Hence

(8.12) $$<\underline{v}_1, \underline{\phi}> - <q \text{ div } \underline{\phi}> - \lambda_1 <\underline{\underline{A}}(\underline{v}_1) \cdot \underline{\underline{A}}(\underline{\phi})> = 0 \ .$$

After some integration by parts, we get

(8.13) $$<(\underline{v}_1 + \lambda_1 \nabla^2 \underline{v}_1 + \nabla q), \underline{\phi}> - \int_{\partial \mathcal{V}_{20}} N_j \phi_i [q\delta_{ij} + \lambda_1 A_{ij}(\underline{v}_1)] = 0 \ .$$

Eq. (8.13) vanishes for all $\underline{\phi} \in \mathcal{H}$; in particular for $\underline{\phi}$ such that $\underline{\phi} = 0$ on $\partial \mathcal{V}_{20}$. The fundamental lemma of the calculus of variation implies that

$$\underline{v}_1 + \lambda_1 \nabla^2 \underline{v}_1 + \nabla q = \underline{0} \text{ in } \mathcal{V}_0 \quad .$$

Now $\int_{\partial \mathcal{V}_{20}} N_j \phi_i [q\delta_{ij} + \lambda_1 A_{ij}(\underline{v}_1)] = 0$ for arbitrary ϕ.
Hence the boundary condition $(8.5)_3$

$$[q \, \underline{\underline{1}} + \lambda_1 \underline{\underline{A}}(\underline{v}_1)] \cdot \underline{N} = \underline{0}$$

arises as a natural boundary condition for the variational problem.

It follows now that (8.5) are Euler equations for $\hat{\lambda}[\underline{v}]$, $\underline{v} \in \mathcal{H}$ subject to the constraint div $\underline{v} = 0$. In a similar fashion it can be shown that solutions of Eq. (8.7) are also the eigenvalues of Eqs. (8.5). Or equivalently all the eigenvalues of Eqs. (8.5) may be characterized variationally by Eq. (8.7).

Having determined the eigenvalues $\lambda_n > 0$ of (8.5) we return to problem of stability of $\underline{u} = \underline{0}$. In the context of the spectral problem $\underline{u} = \underline{0}$ is stable if Re $\sigma = \xi < 0$, neutrally stable if $\xi = 0$ and unstable if $\xi > 0$. The determination of the sign of ξ may be made by analysis of the functional equation

$$- \frac{\rho_o \sigma^2}{\kappa(\sigma)} = 1/\lambda > 0, \; \sigma = \xi + i\omega,$$

that is,

$$(8.19) \qquad m\{\beta + \int_0^\infty \zeta(s)[e^{-\sigma s} - 1]ds\} = - \sigma^2$$

where $m = 1/(\rho_o \lambda) > 0$.

We need to determine the conditions on β and $\zeta(s)$ which will guarantee $\xi < 0$ for all $m > 0$. If there is only elasticity, and no viscoelasticity, ($\zeta(s) = 0$), then (8.19) reduces to $m\beta = - \sigma^2$. If $\beta < 0$ then $\sigma = \xi = \pm \sqrt{m|\beta|}$ so that $\underline{u} = 0$ is unstable if $\beta < 0$. Hence, for stability $\beta > 0$. This is condition (1) of Dafermos.

The other conditions of Dafermos also follow from the analysis of (8.19). When $\omega = 0$, Eq. (8.19) reduces to

$$(8.20) \qquad m\{\beta + \int_0^\infty \zeta(s)[e^{-\xi s} - 1]ds\} = - \xi^2 \quad .$$

If $\zeta(s) \leq 0$ for $s \in [0,\infty)$ then $\xi \geq 0$ cannot be a solution of (8.20). $\xi \geq 0$ makes the left hand side of (8.20) positive but the right hand side of (8.20) is always nonpositive.

When $\omega \neq 0$, we decompose Eq. (8.19) into real and imaginary parts:

$$m\{\beta + \int_0^\infty \zeta(s)[e^{-\xi s}\cos\omega s - 1]ds\} = -(\xi^2 - \omega^2) ,$$

(8.21)

$$m\{\zeta(s)e^{-\xi s}\sin\omega s\} = 2\xi\omega .$$

If $\zeta(s) \leq 0$ for $s \in [0,\infty)$ and $|\zeta(s)| \to 0$ as $s \to \infty$ monotonically then $m \int_0^\infty \zeta(s)e^{-\xi s}\frac{\sin\omega s}{\omega} ds$ is negative for $\xi \geq 0$. This is so because $\zeta(s)e^{-\xi s}\frac{\sin\omega s}{\omega}$ is negative when s is small, it changes sign at each zero of $\sin\omega s$ and the contribution to the value of the integral on each interval is of decreasing magnitude. The negative contributions are therefore larger than the positive ones. This shows that Eq. $(8.21)_2$ is not satisfied for $\xi \geq 0$.

It is better for understanding to proceed less generally and to determine the sign of ξ for a relaxation modulus

(8.22) $\quad \zeta(s) = -\mu e^{-\nu s}, \ \mu > 0, \ \nu > 0$

of the Maxwell type. We find that

(i) finite solution of (8.19) and (8.22) have

(8.23) $\quad \nu + \xi > 0$,

(ii) finite solutions of (8.19) and (8.22) have at least one and at most three real values of $\sigma = \sigma_n$ for $m = 1/(\rho_o\lambda_n)$ and

(8.24) $\quad \xi_n < 0.$

We may restate these results as follows: Given $\beta > 0$, $\zeta(s) < 0(\mu > 0)$, $\dot{\zeta}(s) > 0(\nu > 0)$ there are a countably infinite number of finite solutions $\sigma_n = \xi_n + i\omega_n$ of (8.19), and $\xi_n < 0$ for all such solutions.

Proof: Substitution of (8.22) into (8.19) leads to

$$(8.25) \qquad m\{\beta - \mu\int_0^\infty [e^{-(\nu+\sigma)s} - e^{-\nu s}]ds\} = -\sigma^2 .$$

If $\nu + \xi \leq 0$ the integral diverges. To prevent this we admit as solutions only those σ for which (8.23) holds. Assuming now that $\nu + \xi > 0$ we evaluate (8.25) as

$$(8.26) \qquad -\sigma^2 = m\{\beta + \mu\sigma/\nu(\nu + \sigma)\}.$$

Eq. (8.26) is a cubic in σ which has to be solved subject to the constraint (8.23). When $\mu = 0$, $\sigma = \pm i\sqrt{m\beta}$. When μ is small, these two roots split into a conjugate pair with $\omega \neq 0$. The real and imaginary parts of (8.26) are

$$(8.27) \qquad \xi^2 - \omega^2 = -m\{\beta + \frac{\mu}{\nu}\frac{\xi^2+\xi\nu+\omega^2}{[(\nu+\xi)^2+\omega^2]}\} ,$$

and

$$(8.28) \qquad 2\xi\omega = -\mu m\omega/[(\nu+\xi)^2 + \omega^2] .$$

When $\omega \neq 0$, (8.28) shows that $\xi < 0$. When $\omega = 0$, (8.27) reduces to

$$(8.29) \qquad \xi^2 + m\beta = -m\mu\xi/\nu(\nu+\xi) .$$

Since $\nu + \xi > 0$, (8.29) shows that $\xi < 0$. When μ is small, there is only one real root of (8.29).

Now we will give a formal argument, based on Laplace Transforms, to show that the criteria $\xi < 0$ for all eigenvalues σ in the spectrum of (8.5) implies that $\underline{u} = 0$ is asympotically stable. We first rewrite the problem (8.1) as

$$\rho_0 \ddot{\underline{v}} - \gamma \nabla^2 \underline{v} - \nabla^2 \int_0^t \zeta(s) \underline{v}(\underline{X}, t-s) ds - \nabla^2 \int_{-\infty}^0 \zeta(t-\tau) \underline{v}(\underline{X}, \tau) d\tau$$

$$+ \nabla \pi = \underline{0} \quad ,$$

(8.30) $$\operatorname{div} \underline{v} = 0 \quad ,$$

$$\underline{v}(\underline{X}, t) \Big|_{\partial \mathcal{V}_{10}} = \underline{0} \quad ,$$

$$\{-\pi \underline{\underline{1}} + 2\gamma \underline{\underline{E}}[\underline{v}(t)] + 2\int_0^t \zeta(s) \, \underline{\underline{E}}[\underline{v}(t-s)] \, ds$$

$$+ 2 \int_{-\infty}^0 \zeta(t-\tau) \underline{\underline{E}}[\underline{v}(\tau)] d\tau\} \cdot \underline{N} \Big|_{\partial \mathcal{V}_{20}} = \underline{0} \quad .$$

Since $\underline{v}(\underline{X}, \tau) = \underline{v}_0(\underline{X}, \tau)$ for $\underline{X} \in \mathcal{V}_0, \tau \leq 0$ is known $\nabla^2 \int_{-\infty}^0 \zeta(t-\tau) \underline{v}(\underline{X}, \tau) d\tau$ and $2 \int_{-\infty}^0 \zeta(t-\tau) \underline{\underline{E}}[\underline{v}(\tau)] d\tau$ are known.

Let

(8.31) $$\nabla^2 \int_{-\infty}^0 \zeta(t-\tau) \underline{v}(\underline{X}, \tau) d\tau = \underline{f}_1(\underline{X}, t) \quad ,$$

$$\{2 \int_{-\infty}^0 \zeta(t-\tau) \underline{\underline{E}}[\underline{v}(\tau)] d\tau\} \cdot \underline{N} \Big|_{\partial \mathcal{V}_{20}} = - \underline{f}_2(\underline{X}, t) \quad .$$

Now we have

$$\rho_0 \ddot{\underline{v}} - \gamma \nabla^2 \underline{v} - \nabla^2 \int_0^t \zeta(s) \underline{v}(\underline{X}, t-s) ds + \nabla \pi = \underline{f}_1(\underline{X}, t) \quad ,$$

(8.32) $$\operatorname{div} \underline{v} = 0 \quad ,$$

$$\underline{v}(\underline{X}, t) \Big|_{\partial \mathcal{V}_{10}} = \underline{0} \quad ,$$

$$\{-\pi \underline{\underline{1}} + 2\gamma \underline{\underline{E}}[\underline{v}(t)] + 2 \int_0^t \zeta(s) \underline{\underline{E}}[\underline{v}(t-s)] ds\} \cdot \underline{N} \Big|_{\partial \mathcal{V}_{20}}$$

$$= \underline{f}_2(\underline{X}, t) \quad .$$

Now we assume that $\underline{v}$, $\ddot{\underline{v}}$, div $\underline{v}$, $\underline{\underline{E}}[\underline{v}]$, $\nabla^2 \underline{v}$, π, ζ, $\underline{f}_1$ and $\underline{f}_2$ possess Laplace transforms.

Let

$$(8.33)\qquad \begin{bmatrix} \underline{V}(\underline{X},\hat{\sigma}) \\ \hat{\pi}(\underline{X},\hat{\sigma}) \\ \hat{\zeta}(\hat{\sigma}) \\ \underline{F}_1(\underline{X},\hat{\sigma}) \\ \underline{F}_2(\underline{X},\hat{\sigma}) \end{bmatrix} = \int_0^\infty e^{-\hat{\sigma}t} \begin{bmatrix} \underline{v}(\underline{X},t) \\ \pi(\underline{X},t) \\ \zeta(t) \\ \underline{f}_1(X,t) \\ \underline{f}_2(\underline{X},t) \end{bmatrix}.$$

Then we have

$$\rho_o\ \hat{\sigma}^2\underline{V} - \kappa(\hat{\sigma})\nabla^2\underline{V} + \nabla\hat{\pi} = \underline{F}_1(\underline{X},\hat{\sigma}) + [\hat{\sigma}\underline{v}(\underline{X},0) + \underline{\dot{v}}(\underline{X},0)]\rho_o\ ,$$

$$\text{div}\ \underline{V} = 0\ ,$$

$$(8.34)\qquad \underline{V}(\underline{X},\hat{\sigma})\Big|_{\partial\mathcal{V}_{10}} = 0\ ,$$

$$\{-\hat{\pi}\ \underline{\underline{1}} + 2\kappa(\hat{\sigma})\underline{\underline{E}}[\underline{V}]\}\cdot\underline{N}\Big|_{\partial\mathcal{V}_{20}} = \underline{F}_2(\underline{X},\hat{\sigma})\ .$$

In deriving (8.34), we have used the convolution property

$$\int_0^\infty [\int_0^t \zeta(s)\underline{v}(\underline{X},t-s)ds]e^{-\hat{\sigma}t}\,dt = \hat{\zeta}(\hat{\sigma})\underline{V}(\underline{X},\hat{\sigma})\ ,$$

and

$$\int_0^\infty \underline{\ddot{v}}\ e^{-\hat{\sigma}t}\,dt = \hat{\sigma}^2 V(\underline{X},\hat{\sigma}) - \hat{\sigma}\ \underline{v}(\underline{X},0) - \underline{\dot{v}}(\underline{X},0)\ .$$

Equations (8.34) can be rewritten as

$$\underline{V} + \lambda\nabla^2\underline{V} + \nabla p = F_3(\underline{X},\hat{\sigma})\ ,$$

$$(8.35)\qquad \text{div}\ \underline{V} = 0\ ,$$

$$\underline{V}(\underline{X},\hat{\sigma})\Big|_{\partial\mathcal{V}_{10}} = 0\ ,$$

$$\{p\underline{\underline{1}} + 2\lambda\underline{\underline{E}}[\underline{V}]\}\cdot\underline{N}\Big|_{\partial\mathcal{V}_{20}} = \underline{F}_4(\underline{X},\hat{\sigma})$$

where

$$\lambda = \kappa(\hat{\sigma})/(-\rho_o\hat{\sigma}^2) ,$$

$$p = \hat{\pi}/\rho_o \hat{\sigma}^2 ,$$

$$\underline{F}_3(\underline{X},\hat{\sigma}) = [\underline{F}_1(\underline{X},\hat{\sigma})/\rho_o + \hat{\sigma}\underline{v}(\underline{X},0) + \dot{\underline{v}}(\underline{X},0)]/\hat{\sigma}^2 ,$$

$$\underline{F}_4(\underline{X},\hat{\sigma}) = \underline{F}_2(\underline{X},\hat{\sigma})/(-\rho_\sigma \hat{\sigma}^2) .$$

The spectrum of the linear operator defined by (8.35) is the collection of complex values $\hat{\sigma} = \sigma_n$ for which (8.35) is not uniquely invertible with inverse depending continuously on $\underline{v}_0(\underline{X},\tau)$, $\tau \leq 0$ through $\underline{F}_3(\underline{X},\hat{\sigma})$ and $\underline{F}_4(\underline{X},\hat{\sigma})$. These are the eigenvalues σ_n which we have already characterized variationally through the functional equation (8.19). We learned that $Re\sigma_n = \xi_n < 0$ for all σ_n when $\beta > 0$, $\zeta(s) \leq 0$ and $|\zeta(s)| \to 0$ as $s \to \infty$ monotonically. It follows that Eqs. (8.33) hold for all $\hat{\xi}$ such that $\hat{\xi} > \xi_1$.

For the other values of $\hat{\sigma}$, not in the spectrum of (8.35), (8.35) is uniquely invertible and

(8.36) $$\underline{V}(\underline{X},\hat{\sigma}) = \underline{\underline{R}}_{\hat{\sigma}}\, \underline{F}_3(\underline{X},\hat{\sigma}) + \underline{\underline{S}}_{\hat{\sigma}}\, \underline{F}_4(\underline{X},\hat{\sigma})$$

depends continuously on $\underline{v}_0(\underline{X},\tau)$, $\tau \leq 0$ through, $\underline{F}_3(\underline{X},\hat{\sigma})$, $\underline{F}_4(\underline{X},\hat{\sigma})$ and matrix-valued resolvent operators $\underline{\underline{R}}_{\hat{\sigma}}$ and $\underline{\underline{S}}_{\hat{\sigma}}$. The values $\hat{\sigma}$ not in the spectrum are said to be in the resolvent set.

Given $\underline{V}(\underline{X},\hat{\sigma})$, we may use Laplace inversion integral to compute

(8.37) $$\underline{v}(\underline{X},t) = \frac{1}{2\pi i} \int_{\hat{\xi}-i\infty}^{\hat{\xi}+i\infty} e^{\hat{\sigma}t}\, \underline{V}(\underline{X},\hat{\sigma})d\hat{\sigma}$$

$$= \frac{1}{2\pi i} \int_{\hat{\xi}-i\infty}^{\hat{\xi}+i\infty} e^{\hat{\sigma}t} [\underline{\underline{R}}_{\hat{\sigma}}\underline{F}_3(\underline{X},\hat{\sigma}) + \underline{\underline{S}}_{\hat{\sigma}}\underline{F}_4(\underline{X},\hat{\sigma})]d\hat{\sigma} .$$

Eq. (8.37) holds for any value of $\hat{\sigma} = \hat{\xi} + i\hat{\omega}$ for which Eqs. (8.33) hold; that is for $\hat{\xi} > \xi_1$. Since $\xi_1 < 0$, we may choose $\hat{\xi} < 0$. Then (8.37) shows that $\underline{v}(\underline{X},t)$ is asymptotically stable with exponential decay. Eigenvalues associated with (8.35) appear as singularities of the resolvent operators $\underline{\underline{R}}_{\hat{\sigma}}$ and $\underline{\underline{S}}_{\hat{\sigma}}$. If all the eigenvalues are simple then (8.37) may be evaluated by residues.

$$(8.38) \qquad \underline{v}(\underline{X},t) = \sum_n e^{\hat{\sigma}_n t} a_n[\underline{v}_0(\underline{X},\tau)\Big|_{\tau=-\infty}^{0}, \hat{\sigma}_n]$$

where the coefficients a_n are functionals of the initial history $\underline{v}_0(\underline{X},\tau)$, $X \in \mathcal{V}_0$, $\tau \leq 0$.

9. Free surface problems perturbing the natural state

Many problems in elastostatics and viscoelastic dynamics can be solved using the equations derived in §7. From these, we have selected two problems in which the second order theory is required for the computation of the change in the shape of a stress-free surface due to nonlinear effects of inertia and stress. Free surface problems are of interest to material scientists because the distortion of the free surface due to deformation can be a sensitive mirror into the state of stress and the measurement of the distorted shape may provide a rheometrical device for measuring material constants and material functions. This hope we have for solids is a fact in fluids (see Joseph and Beavers, 1977).

The problems to be derived here involve distortion due to deformation in viscoelastic solids which are right circular cylinders in the natural state. The effects we compute may be regarded as analogous to Weissenberg effects in fluids and as in the fluids problem the Weissenberg effects appear first at second order. The second order problems require that we solve certain fourth order linear partial differential equations which in the simplest cases reduce to the biharmonic ones. These problems are probably best suited to analysis in biorthogonal series. Solving the problems is a major job which requires considerable analysis unrelated to the physical problem being studied here. So we defer the computation of solutions to a later work and concentrate on deriving the boundary-value problems which need solving under general circumstances.

The problems we treat are axisymmetric. Cylindrical coordinates are natural to such problems and it is necessary to compute the components of displacements

$$(9.1) \qquad \underline{x} = \underline{X} + \underline{u}(\underline{X},t,\varepsilon) = r\underline{e}_r(\theta) + z\underline{e}_z,$$

where $\underline{e}_r(\theta)$ and $\underline{e}_z$ are a cylindrical basis in the coordinates (r,θ,z) of the distorted configuration, relative to a cylindrical basis $(\underline{e}_R, \underline{e}_\Theta, \underline{e}_Z)$, in the natural state

$$(9.2) \qquad \underline{X} = \underline{e}_R(\Theta)R + \underline{e}_Z\, Z.$$

The components of $\underline{u}(\underline{X},t,\varepsilon)$ are independent of Θ:

$$(9.3) \qquad \begin{aligned} r &= R + r^{<1>}(R,Z,t)\varepsilon + r^{<2>}(R,Z,t)\varepsilon^2 + O(\varepsilon^3), \\ \theta &= \Theta + \theta^{<1>}(R,Z,t)\varepsilon + \theta^{<2>}(R,Z,t)\varepsilon^2 + O(\varepsilon^3), \\ z &= Z + z^{<1>}(R,Z,t)\varepsilon + z^{<2>}(R,Z,t)\varepsilon^2 + O(\varepsilon^3). \end{aligned}$$

To expand $\underline{u}(\underline{X},t,\varepsilon)$ it is necessary to compute the expansion of

$$(9.4) \qquad \underline{e}_r(\theta) = \underline{e}_R + \varepsilon\, \underline{e}_\Theta \theta^{<1>} + \varepsilon^2\{\underline{e}_\Theta \theta^{<2>} - \underline{e}_R \theta^{<1>^2}/2\} + O(\varepsilon^3)$$

induced by $(9.3)_2$. The term $-\varepsilon^2 e_R \theta^{<1>^2}/2$ can be regarded as an effect of "inertia". Combining (9.1,2,3,4) we find that

$$\underline{u} = \underline{u}^{<1>} + \varepsilon^2 \underline{u}^{<2>} + O(\varepsilon^3)$$

where

$$(9.5) \qquad \underline{u}^{<1>} = r^{<1>}\underline{e}_R + R\theta^{<1>}\underline{e}_\Theta + z^{<1>}\underline{e}_Z \quad ,$$

and

$$(9.6) \qquad \underline{u}^{<2>} = (r^{<2>} - R\theta^{<1>^2}/2)\underline{e}_R + (R\theta^{<2>} + r^{<1>}\theta^{<1>})\underline{e}_\Theta + z^{<2>}\underline{e}_Z \, .$$

The components of $\underline{\underline{F}}^{<1>}$ in the basis $(\underline{e}_R, \underline{e}_\Theta, \underline{e}_Z)$ are

$$(9.7) \qquad [\underline{\underline{F}}^{<1>}] = \begin{pmatrix} r^{<1>}_R & -\theta^{<1>} & r^{<1>}_Z \\ (R\theta^{<1>})_R & r^{<1>}/R & R\theta^{<1>}_Z \\ z^{<1>}_R & 0 & z^{<1>}_Z \end{pmatrix} .$$

Here and elsewhere $(\cdot)_R$ and $(\cdot)_Z$ denote partial derivatives of $(\cdot)$ with respect to R and Z respectively.

The computation of components in a cartesian basis is slightly less involved because "inertia" terms are absent.

To proceed further it is necessary to be more specific about the two problems under consideration.

(I) <u>Distortion of the cylindrical free surface on a viscoelastic right circular cylinder induced by torsional oscillation of rigidly bonded end plates</u>

A right circular cylinder of viscoelastic solid material of radius a is bonded to rigid parallel plates separated always by distance 2. The plates may rotate around the axis of the cylinder in a more or less arbitrary manner. The natural state is sketched in Fig. 9.1.a and the distorted shape is shown in Fig. 9.1.b.

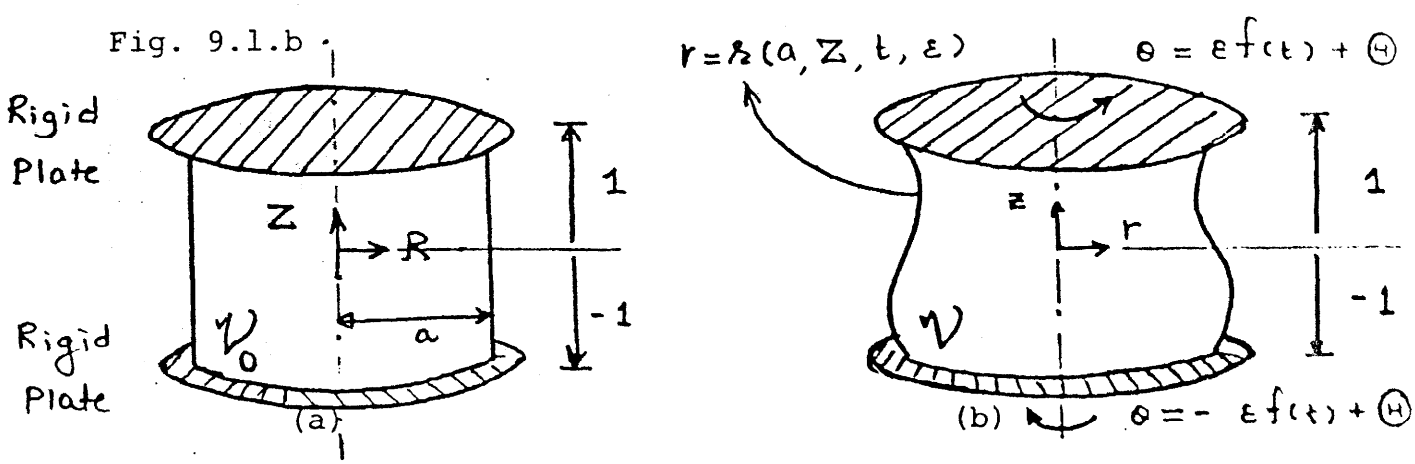


Fig. 9.1 A viscoelastic cylinder is sheared by the rotation of two end plates: (a) is the natural state

$$\mathcal{V}_0 = \{R, Z: \ 0 \le R < a, \ -1 < Z < 1\},$$

and (b) is the deformed state

$$\mathcal{V} = \{r, z: \ 0 \le r < \mathcal{R}(a, Z, t, \varepsilon), \ -1 < z < 1\}$$

where $\mathcal{R}(a, Z, t, \varepsilon) \overset{\text{def}}{=} r(R{=}a, Z, t, \varepsilon)$ is the radial displacement function given by $(9.3)_1$.

Conservation of volume in the deformed incompressible viscoelastic solid may be expressed through the requirement that

$$(9.8) \qquad \frac{1}{2} \int_{-1}^{1} \mathscr{r}^2(a,Z,t,\varepsilon)\,dZ = a \quad .$$

The displacement boundary condition are as follows:

$$(9.9) \qquad \theta(R, \pm 1, t, \varepsilon) = \pm\varepsilon f(t) + \Theta \ ,$$

$$r(R, \pm 1, t, \varepsilon) = R \ .$$

It follows then that

$$(9.10) \qquad r^{<n>}(R, \pm 1, t) = 0, \quad n = 1,2,3...,$$

$$\theta^{<1>}(R, \pm 1, t) = \pm f(t) \ ,$$

and

$$\theta^{<n>}(R, \pm 1, t) = 0, \quad n \geq 2.$$

The stress on $r = \mathscr{r}(a,Z,t,\varepsilon)$ must vanish.

If $f(t) = 1$, the cylinder undergoes a steady displacement and the problem falls in the class of universal deformations in nonlinear elastostatics found by Rivlin (1949). These deformations are independent of the constitutive equation provided that the material is undergoing elastic deformation; then, globally, without perturbations we get

$$(9.11) \qquad r = R, \ z = Z, \ \theta = \Theta + \varepsilon Z.$$

So the free surface will not change shape under a static twist. The distortion shown in 9.1.b is entirely dynamic in this (but not other) problems.

Now we shall solve the equations (7.10) governing the first order perturbation

$$p^{<1>} = r^{<1>} = z^{<1>} = 0,$$

(9.12) $$\theta^{<1>}(R,Z,t) = \phi(Z,t) \text{ is independent of } R.$$

Then $(7.10)_1$ reduces to

$$\rho_o R\ddot{\phi} = \hat{L}\{\nabla^2(R\phi) - \phi/R\} = R\hat{L}(\partial^2\phi/\partial Z^2)$$

where

(9.13) $$\hat{L}(\cdot) \overset{def}{=} \gamma(\cdot)(t) + \int_0^\infty \zeta(s)(\cdot)(t-s)ds ,$$

so that

(9.14)
$$\rho_o\ddot{\phi} = \partial^2\hat{L}(\phi)/\partial Z^2 ,$$
$$\phi(\pm 1,t) = \pm f(t) .$$

All other conditions on the first order perturbation are satisfied identically when (9.12) and (9.14) hold; in particular $\underline{\underline{T}}^{<1>} \cdot \underline{N} = \underline{\underline{T}}^{<1>} \cdot \underline{e}_R = \underline{0}$ on $R = a$ where

(9.15) $$\underline{\underline{T}}^{<1>} = 2\hat{L}(\underline{\underline{E}}^{<1>}) = (\underline{e}_\Theta\underline{e}_Z + \underline{e}_Z\underline{e}_\Theta)\ R\hat{L}(\partial\phi/\partial Z).$$

To solve the second order perturbation problem we must compute the inhomogeneous terms in (7.12). All of these terms may be computed from

(9.16) $$[\underline{\underline{F}}^{<1>}] = \begin{bmatrix} 0 & -\phi & 0 \\ \phi & 0 & R\partial\phi/\partial Z \\ 0 & 0 & 0 \end{bmatrix} ,$$

and

(9.17) $$\underline{u}^{<2>} = (r^{<2>} - R\phi^2/2)\underline{e}_R + R\theta^{<2>}\underline{e}_\Theta + z^{<2>}\underline{e}_Z$$

using the expressions given in §5.

The equation $(7.12)_2$ expressing the conservation of volume simplifies to

(9.18) $$\frac{1}{R}\frac{\partial(Rr^{<2>})}{\partial R} + \frac{\partial z^{<2>}}{\partial Z} = 0 ,$$

because the term proportional to ϕ^2 in (9.17) cancels Θ_2. To compute $\underline{M}_2$ in (7.13) we need only to note that $p^{<1>} = 0$ and the rest follows from simple operations with (9.16). We find that the components (M_{2R}, $M_{2\Theta}$ and M_{2Z}) of $\underline{M}_2$ are given by

$$(9.19)\quad M_{2R} = - \beta^{[2]} R\phi'^2(t) + R\int_0^\infty \zeta(s)\,\{[\phi'(t-s)[\![\phi]\!]]' + \phi'(t)[\![\phi']\!]\}ds - 2R\int_0^\infty \zeta(s)^{[2]}\phi'(t)[\![\phi']\!]ds - R\int_0^\infty\int_0^\infty \alpha(s_1,s_2)[\phi'(t-s_1) - \phi'(t)][\phi'(t-s_2) - \phi'(t)]ds_1 ds_2 \;,$$

$$M_{2\Theta} = 0,$$

$$M_{2Z} = \beta^2 R^2[\phi'^2(t)]' + R^2\int_0^\infty \zeta(s)\,([\![\phi']\!]^2)'ds + 2R^2\int_0^\infty \zeta^{[2]}(s)\,[\phi'(t)[\![\phi']\!]]'ds + R^2\int_0^\infty\int_0^\infty \alpha(s_1,s_2)\{[\phi'(t-s_1) - \phi'(t)][\phi'(t-s_2) - \phi'(t)]\}'ds_1 ds_2$$

where prime denotes partial derivative w. r. t. Z. The R, Θ and Z components of $(7.12)_1$ may be written as

$$(9.20)\quad \rho_o\ddot{r}^{<2>} - \rho_o R\,\overline{\ddot{\phi}^2}/2 = -\partial\pi^{<2>}/\partial R + \hat{L}(\nabla^2 r^{<2>} - r^{<2>}/R^2) - \hat{L}(R[\phi\phi']') + M_{2R} \;,$$

$$(9.21)\quad \rho_o R\ddot{\theta}^{<2>} = \hat{L}[\nabla^2(R\theta^{<2>}) - \theta^{<2>}/R] \;,$$

and

$$(9.22)\quad \rho_o\ddot{z}^{<2>} = - \partial\pi^{<2>}/\partial Z + \hat{L}(\nabla^2 z^{<2>}) + M_{2Z}.$$

The displacement boundary conditions are

$$(9.23) \qquad r^{<2>}(R, \pm 1, t) = \theta^{<2>}(R, \pm 1, t) = z^{<2>}(R, \pm 1, t) = 0.$$

To form the stress boundary conditions $(7.12)_4$ we note that

$$(9.24) \qquad \underline{N} = \underline{e}_R, \quad \underline{\underline{T}}^{<1>} \cdot \underline{N} = \underline{0}, \quad \underline{\underline{T}}^{<1>} \underline{\underline{F}}^{T<1>} \cdot \underline{N} = - R\phi \hat{L}(\phi') \underline{e}_Z \quad .$$

$\underline{\underline{T}}^{<2>}$ is given by (7.8). To compute $\underline{\underline{T}}^{<2>} \cdot \underline{e}_R$ we note that $\underline{\underline{E}}^{<2>} = \frac{1}{2}[\nabla \underline{u}^{<2>} + (\nabla \underline{u}^{<2>})^T]$ and using (9.17) find

$$[2\underline{\underline{E}}^{<2>}] = \begin{bmatrix} 2r_R^{<2>} - \phi^2 & R\theta_R^{<2>} & r_Z^{<2>} + z_R^{<2>} - R\phi\phi' \\ R\theta_R^{<2>} & 2r^{<2>}/R - \phi^2 & R\theta_Z^{<2>} \\ r_Z^{<2>} + z_R^{<2>} - R\phi\phi' & R\theta_Z^{<2>} & 2z_Z^{<2>} \end{bmatrix},$$

so that

$$(9.25) \qquad 2\underline{\underline{E}}^{<2>} \cdot \underline{e}_R = 2r_R^{<2>} \underline{e}_R + R\theta_R^{<2>} \underline{e}_\Theta + (r_Z^{<2>} + z_R^{<2>}) \underline{e}_Z$$

$$- (\phi^2 \underline{e}_R + R\phi\phi' \underline{e}_Z) \quad .$$

The remaining terms of $\underline{\underline{T}}^{<2>} \cdot \underline{e}_R$ are formed from simple manipulations using (9.16) .

$$(9.26) \qquad \underline{\underline{T}}^{<2>} \cdot \underline{e}_R = \underline{e}_R \{-\pi^{<2>} + \beta\phi^2 + \int_0^\infty \zeta(s) [\![\phi^2]\!] ds\}$$

$$+ \underline{e}_Z \{R \int_0^\infty \zeta(s) \phi'(t-s) [\![\phi]\!] ds\} + \hat{L}(2\underline{\underline{E}}^{<2>} \cdot \underline{e}_R)$$

$$= \underline{e}_R [-\pi^{<2>} + \hat{L}(2r_R^{<2>})] + \underline{e}_\Theta R\hat{L}(\theta_R^{<2>})$$

$$+ \underline{e}_Z \{\hat{L}(r_Z^{<2>} + z_R^{<2>}) - R\phi\hat{L}(\phi')\} \quad .$$

Combining (9.24) and (9.26), we get

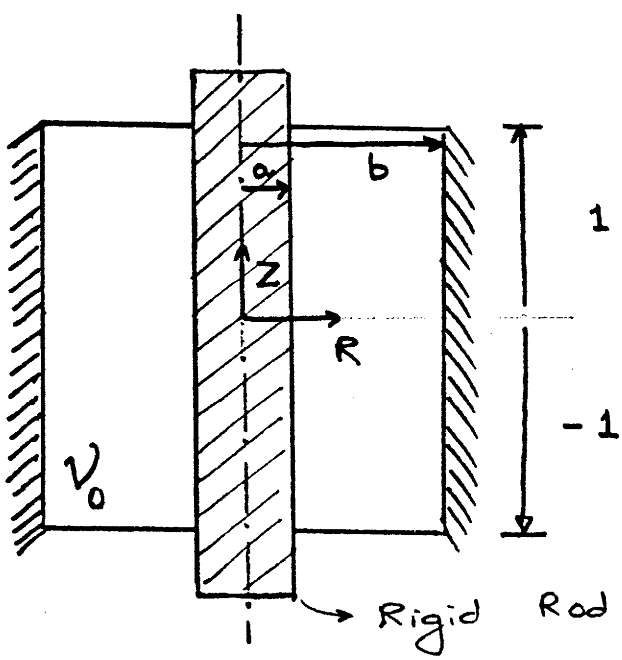
$$(9.27) \qquad \underline{t}_n^{<2>} = \underline{e}_R [-\pi^{<2>} + \hat{L}(2r_R^{<2>})] + \underline{e}_\Theta R\hat{L}(\theta_R^{<2>})$$

$$+ \underline{e}_Z \hat{L}(r_Z^{<2>} + z_R^{<2>}) = \underline{0} \quad \text{at } R = a.$$

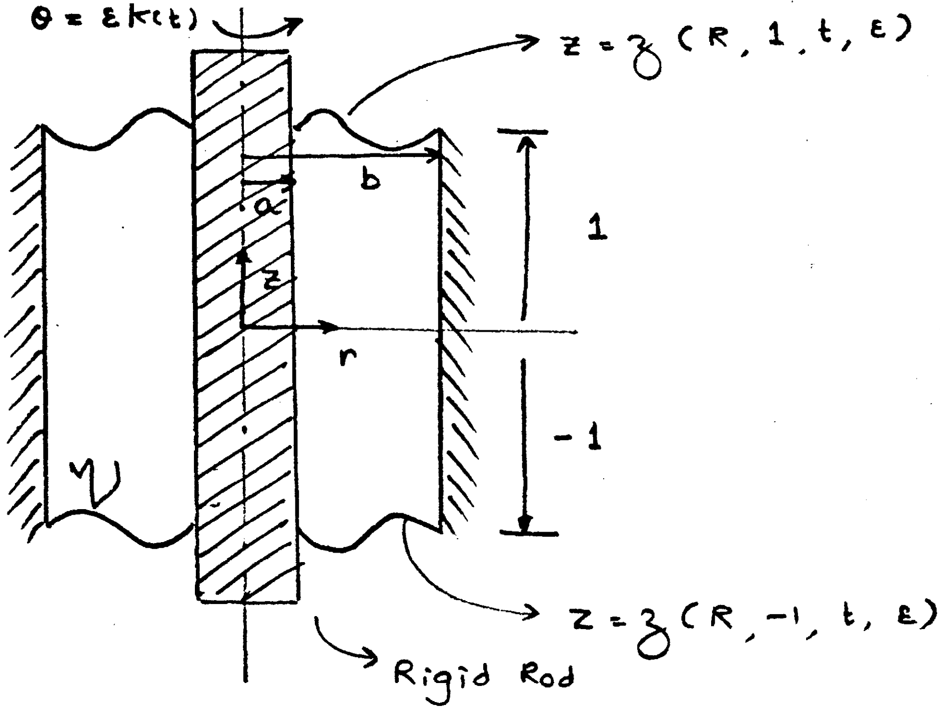
We note that the equation (9.21) governing $\theta^{<2>}$ is homogeneous ($M_{2\Theta} = 0$) with homogeneous boundary conditions [equations (9.23) and (9.27)]. Hence $\theta^{<2>} \equiv 0$. To find $r^{<2>}$ and $z^{<2>}$, we must solve (9.18), (9.20), (9.22) subject to the boundary conditions (9.23) and (9.27).

(II) Distortion of the plane surfaces perpendicular to the axis of viscoelastic cylindrical annulus rigidly bonded at the inner and outer radii and undergoing torsional oscillations at the inner radius

A right circular cylindrical annulus of viscoelastic material of inner radius a and outer radius b is rigidly bonded at both the radii. Its initial length is 2. Both the radii remain fixed during the motion. The rigid rod to which the annulus is bonded at the inner radius rotates around the axis of the annulus in a more or less arbitrary manner. The natural state is sketched in Fig. 9.2.a and the distorted shape is shown in Fig. 9.2.b.



(a)



(b)

Fig. 9.2 A viscoelastic cylindrical annulus is sheared by the rotation of the rigid rod, bonded to the annulus at the inner radius: (a) is the natural state

$$\mathcal{V}_0 = \{R,Z: \quad a < R < b, \ -1 < Z < 1\} \ ,$$

and (b) is the deformed state

$$\mathcal{V} = \{r,z: \quad a < r < b, \ \zeta(R,-1,t,\varepsilon) < z < \zeta(R,1,t,\varepsilon)\}$$

where $\zeta(R,\pm 1,t,\varepsilon) \overset{\text{def}}{=} z(R,Z = \pm 1, t,\varepsilon)$ is the axial displacement function given by $(9.3)_3$.

Conservation of volume in the deformed incompressible viscoelastic solid may be expressed through the requirement that

$$(9.28) \qquad \int_a^b [\zeta(R,1,t,\varepsilon) - \zeta(R,-1,t,\varepsilon)] \, RdR = (b^2-a^2) \ .$$

The displacement boundary conditions are as follows:

$$(9.29) \qquad \begin{aligned} \theta(a,Z,t,\varepsilon) &= \varepsilon k(t) + \Theta \ , \\ \theta(b,Z,t,\varepsilon) &= \Theta \ , \\ z(R,Z,t,\varepsilon) &= Z \quad \text{at } R = a \ \& \ b \ . \end{aligned}$$

It follows that

(9.30) $\theta^{<1>}(a,Z,t) = k(t)$,

$\theta^{<n>}(a,Z,t) = 0$ for $n \geq 2$,

$\theta^{<n>}(b,Z,t) = 0$ for $n = 1,2,3...$,

$z^{<n>}(R,Z,t) = 0$ at $R = a$ & b for $n = 1,2,3...$.

The stress on $z = (R, \pm 1,t,\varepsilon)$ must vanish.

Now we shall solve the equations (7.10) governing the first order perturbation .

(9.31) $p^{<1>} = r^{<1>} = z^{<1>} = 0$,

$\theta^{<1>}(R,Z,t) = \phi(R,t)$ is independent of Z.

Then (7.10) reduces to

(9.32) $$\rho_o R\ddot{\phi} = \hat{L}[\nabla^2(R\phi) - \phi/R)] = \hat{L}[\frac{\partial^2(R\phi)}{\partial R^2} + \frac{1}{R}\frac{\partial(R\phi)}{\partial R} - \frac{\phi}{R}] ,$$

$\phi(a,t) = k(t)$,

$\phi(b,t) = 0$

where $\hat{L}(\cdot)$ is as defined by (9.13). All other conditions on the first order perturbation are satisfied identically when (9.31) and (9.32) hold; in particular $\underline{\underline{T}}^{<1>} \cdot \underline{N} = \underline{\underline{T}}^{<1>} \cdot \underline{e}_Z = \underline{0}$ on $Z = +1$ (and $\underline{\underline{T}}^{<1>} \cdot \underline{N} = - \underline{\underline{T}}^{<1>} \cdot \underline{e}_Z = \underline{0}$ on $z = -1$) where

(9.33) $$\underline{\underline{T}}^{<1>} = 2\hat{L}(\underline{\underline{E}}^{<1>}) = (\underline{e}_R\underline{e}_\Theta + \underline{e}_\Theta\underline{e}_R)\ \hat{L}(R\partial\phi/\partial R) .$$

To solve the second order perturbation problem we must compute the nonhomogeneous terms in (7.12). All of these may be computed from

(9.34) $$[\underline{\underline{F}}^{<1>}] = \begin{bmatrix} 0 & -\phi & 0 \\ \phi+R\partial\phi/\partial R & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} ,$$

and

(9.35) $$\underline{u}^{<2>} = (r^{<2>} - R\phi^2/2)\underline{e}_R + R\theta^{<2>}\underline{e}_\Theta + z^{<2>}\underline{e}_Z$$

using the expressions given in §5.

The equation $(7.12)_2$ expressing the conservation of volume simplifies to

(9.36) $$\frac{1}{R}\frac{\partial(Rr^{<2>})}{\partial R} + \frac{\partial z^{<2>}}{\partial Z} = 0$$

because the term proportional to ϕ^2 in (9.35) cancels θ_2. To compute $\underline{M}_2$ in (7.13), we need only to note that $p^{<1>} = 0$ and the rest follows from simple operations with (9.34). We find that the components $(M_{2R}, M_{2\Theta}, M_{2Z})$ of $\underline{M}_2$ are given by

(9.38) $$M_{2R} = \frac{d}{dR}\{\beta^{[2]}c(t)^2 + \int_0^\infty \zeta(s)[c(t-s)[\![c+a]\!] - c(t)[\![c]\!]]ds$$

$$+ 2\int_0^\infty \zeta^{[2]}(s)[(c(t)[\![c]\!]]\,ds$$

$$+ \int_0^\infty \int_0^\infty \alpha(s_1,s_2)[c(t-s_1)-c(t)]$$

$$[c(t-s_2)-c(t)]ds_1 ds_2\}$$

$$+ \frac{1}{R}\{\int_0^\infty \zeta(s)c(t-s)([\![c+a]\!] + [\![a]\!])ds\},$$

$$M_{2\Theta} = M_{2Z} = 0$$

where

(9.39) $$c(t) = R\frac{\partial\phi(R,t)}{\partial R},$$

$$a(t) = \phi(R,t).$$

The R, Θ and Z components of $(7.12)_1$ may be written as

(9.40) $\rho_o \ddot{r}^{<2>} - \rho_o R \overline{\ddot{\phi}^2}/2 = - \partial\pi^{<2>}/\partial R + \hat{L}(\nabla^2 r^{<2>} - r^{<2>}/R^2)$

$$+ \hat{L}[R\phi\partial^2\phi/\partial R^2 + R(\partial\phi/\partial R)^2 + 3\phi\partial\phi/\partial R] + M_{2R} \quad ,$$

(9.41) $\rho_o R\ddot{\theta}^{<2>} = \hat{L}[\nabla^2(R\theta^{<2>}) - \theta^{<2>}/R]$,

and

(9.42) $\rho_o \ddot{z}^{<2>} = - \partial\pi^{<2>}/\partial Z + \hat{L}(\nabla^2 z^{<2>})$.

The displacement boundary conditions are

(9.43) $r^{<2>}(R,Z,t) = \theta^{<2>}(R,Z,t) = z^{<2>}(R,Z,t) = 0$ at $R = a$ & b.

To form the stress boundary conditions $(7.12)_4$, we note that $\underline{N} = \pm \underline{e}_Z$ (at $Z = \pm 1$), $\underline{\underline{T}}^{<1>} \cdot \underline{N} = \underline{0}$, $\underline{\underline{F}}^{<1>T} \cdot \underline{N} = \underline{0}$. Hence

(9.44) $\underline{\underline{T}}^{<1>} \cdot \underline{n}^{<1>} = \underline{0}$.

To compute $\underline{\underline{T}}^{<2>} \cdot \underline{e}_Z$, we note that $\underline{\underline{E}}^{<2>} = \frac{1}{2}[\nabla\underline{u}^{<2>} + (\nabla\underline{u}^{<2>})^T]$ and using (9.34) find

(9.45)

$$[2\underline{\underline{E}}^{<2>}] = \begin{pmatrix} 2r_R^{<2>} - \phi^2 - 2R\phi\phi_R & R\theta_R^{<2>} & r_Z^{<2>} + z_R^{<2>} \\ R\theta_R^{<2>} & 2r^{<2>}/R - \phi^2 & R\theta_Z^{<2>} \\ r_Z^{<2>} + z_R^{<2>} & R\theta_Z^{<2>} & 2z_Z^{<2>} \end{pmatrix} ,$$

so that

$$2\underline{\underline{E}}^{<2>} \cdot \underline{e}_Z = (r_Z^{<2>} + z_R^{<2>})\underline{e}_R + R\theta_Z^{<2>}\underline{e}_\Theta + 2z_Z^{<2>}\underline{e}_Z \quad .$$

The remaining terms of $\underline{\underline{T}}^{<2>} \cdot \underline{e}_Z$ are formed form simple manipulations using (9.16).

(9.46) $\underline{\underline{T}}^{<2>} \cdot \underline{e}_Z = - \pi^{<2>}\underline{e}_Z + \hat{L}(2\underline{\underline{E}}^{<2>} \cdot \underline{e}_Z)$

$$= \underline{e}_R \hat{L}(r_Z^{<2>} + z_R^{<2>}) + \underline{e}_\Theta R \hat{L}(\theta_Z^{<2>})$$

$$+ \underline{e}_Z [-\pi^{<2>} + \hat{L}(2z_Z^{<2>})] \quad .$$

combining (9.44) and (9.46), we get

$$(9.47) \qquad \underline{t}_n^{<2>} = \underline{e}_R \hat{L}(r_Z^{<2>} + z_R^{<2>}) + \underline{e}_\Theta R\hat{L}(\theta_Z^{<2>})$$

$$+ \underline{e}_Z [-\pi^{<2>} + \hat{L}(2z_Z^{<2>})] = \underline{0} \quad \text{at } Z = \pm 1.$$

We note that the equation (9.41) governing $\theta^{<2>}$ is homogeneous ($M_{2\Theta} = 0$) with homogeneous boundary conditions [equations (9.43) and (9.47)]. Hence $\theta^{<2>} \equiv 0$. To find $r^{<2>}$ and $z^{<2>}$, we must solve (9.36), (9.40), (9.42) subject to the boundary conditions (9.43) and (9.47).

10. Linearized theory of perturbation of the rest state

In this section we derive the first order equations of motion for incompressible solids for motions perturbing the rest state. Since the rest state contains all static deformations and, in fact, coincides with the set of all elastic deformations, the linearized equations derived here form the basis for the discussion of stability and bifurcation of elastostatic deformations of solids. The trouble we find when carrying out a correct analysis of the linearized theory is that so many (2) material constants and (7) functions are needed to characterize the material.

For incompressible materials det $\underline{\underline{F}} = 1$. In the rest state

(10.1) $$\det \underline{\underline{F}}^{<0>} = 1,$$

where

$$\underline{\underline{F}}(t) = \nabla\underline{x}(\underline{X},t), \quad \underline{\underline{F}}^{<0>} = \nabla\underline{u}^{\circ}(\underline{X}) + \underline{\underline{1}} \ ,$$

(10.2) $$\underline{x} = \underline{X} + \underline{u}^{\circ}(\underline{X}) + \varepsilon\underline{u}^{<1>}(\underline{X},t) + O(\varepsilon^2) .$$

In §5, we derived the perturbation formulas for the kinematic variables for perturbations of the natural state. But these are valid only when $\underline{\underline{F}}^{<0>} = \underline{\underline{1}}$. In the case of perturbations of the rest state, the derivation of the perturbation formulas for the kinematic variables is similar to the one in §5. Here we list only the results.

(10.3) $$\underline{\underline{F}}(\underline{X},\tau,\varepsilon) = \underline{\underline{F}}^{<0>}(\underline{X}) + \varepsilon\underline{\underline{F}}^{<1>}(\underline{X},t) + O(\varepsilon^2).$$

(10.4) $$\underline{\underline{F}}^{-1}(\tau,\varepsilon) = (\underline{\underline{F}}^{<0>})^{-1} - \varepsilon(\underline{\underline{F}}^{<0>})^{-1}(\underline{\underline{F}}^{<1>}(\tau))(\underline{\underline{F}}^{<0>})^{-1} + O(\varepsilon^2).$$

(10.5) $$\underline{\underline{G}}(s,\varepsilon) = \varepsilon\underline{\underline{G}}^{<1>}(s) + O(\varepsilon^3),$$

where

$$(10.6)\qquad \underline{\underline{G}}^{<1>}(s) = \underline{\underline{F}}^{<1>}(\tau)(\underline{\underline{F}}^{<0>})^{-1} + (\underline{\underline{F}}^{<0>^T})^{-1}\,\underline{\underline{F}}^{<1>^T}(\tau)$$

$$- \underline{\underline{F}}^{<1>}(t)(\underline{\underline{F}}^{<0>})^{-1} - (\underline{\underline{F}}^{<0>^T})^{-1}\,\underline{\underline{F}}^{<1>^T}(t).$$

$$(10.7)\qquad \underline{\underline{B}}(\tau,\varepsilon) = \underline{\underline{F}}^{<0>}\underline{\underline{F}}^{<0>^T} + \varepsilon(\underline{\underline{F}}^{<1>}\underline{\underline{F}}^{<0>^T} + \underline{\underline{F}}^{<0>}\underline{\underline{F}}^{<1>^T}) + O(\varepsilon^2).$$

$$(10.8)\qquad \det \underline{\underline{F}}(\tau,\varepsilon) = 1 + \varepsilon\{\mathrm{tr}[(\underline{\underline{F}}^{<0>})^{-1}\,\underline{\underline{F}}^{<1>}(\tau)]\} + O(\varepsilon^2)$$

where we have used the equation (10.1). We do not need the perturbation formula for ρ. To find the perturbation formula for $\underline{n}$, we use

$$(10.9)\qquad da^2 = (\det \underline{\underline{F}})^2(\underline{N}\cdot\underline{\underline{C}}^{-1}\cdot\underline{N})dA^2.$$

where $\underline{\underline{C}} = \underline{\underline{F}}^T\underline{\underline{F}}$ is the right Cauchy-Green strain tensor. This formula reduces to

$$(10.10)\qquad da^2 = (\underline{N}\cdot\underline{\underline{C}}^{-1}\cdot\underline{N})dA^2$$

because of (10.1). Expanding both sides of (10.10) into powers of ε and identifying independent powers of ε, we get

$$(10.11)\qquad da^{<0>} = [\underline{N}\cdot(\underline{\underline{C}}^{<0>})^{-1}\cdot\underline{N}]^{1/2}dA,$$

$$(10.12)\qquad da^{<1>} = -\underline{N}\cdot[(\underline{\underline{F}}^{<0>})^{-1}\,\underline{\underline{F}}^{<1>}(\underline{\underline{C}}^{<0>})^{-1} + (\underline{\underline{C}}^{<0>})^{-1}\,\underline{\underline{F}}^{<1>^T}(\underline{\underline{F}}^{<0>^T})^{-1}]$$

$$\cdot\,\underline{N}\,dA/2[\underline{N}\cdot(\underline{\underline{C}}^{<0>})^{-1}\cdot\underline{N}]^{1/2}.$$

Next we use

$$(10.13)\qquad \underline{n}\,da = (\det \underline{\underline{F}})(\underline{\underline{F}}^{-1})^T\cdot\underline{N}\,dA\ ,$$

and (10.1) to derive

$$(10.14)\qquad \underline{n}^{<0>} = (\underline{\underline{F}}^{<0>^T})^{-1}\cdot\underline{N}/[\underline{N}\cdot(\underline{\underline{C}}^{<0>})^{-1}\cdot\underline{N}]^{1/2}\ ,$$

(10.15) $$\underline{n}^{<1>} = \{\underline{N} \cdot [(\underline{\underline{F}}^{<0>})^{-1} \underline{\underline{F}}^{<1>}(\underline{\underline{C}}^{<0>})^{-1} + (\underline{\underline{C}}^{<0>})^{-1} \underline{\underline{F}}^{<1>^T} (\underline{\underline{F}}^{<0>^T})^{-1}] \cdot \underline{N}$$

$$(\underline{\underline{F}}^{<0>^T})^{-1} \cdot \underline{N}/2[\underline{N} \cdot (\underline{\underline{C}}^{<0>})^{-1} \cdot \underline{N}]$$

$$-(\underline{\underline{F}}^{<0>^T})^{-1} \underline{\underline{F}}^{<1>^T} (\underline{\underline{F}}^{<0>^T})^{-1} \cdot \underline{N}\}/[\underline{N} \cdot (\underline{\underline{C}}^{<0>})^{-1} \cdot \underline{N}]^{1/2}.$$

Finally

(10.16) $$\underline{t}_n = \underline{t}_n^{<0>} + \varepsilon \underline{t}_n^{<1>} + O(\varepsilon^2)$$

where

(10.17) $$\underline{t}_n^{<0>} = \underline{\underline{T}}^{<0>} \cdot \underline{n}^{<0>} ,$$

and

(10.18) $$\underline{t}_n^{<1>} = \underline{\underline{T}}^{<1>} \cdot \underline{n}^{<0>} + \underline{\underline{T}}^{<0>} \cdot \underline{n}^{<1>}.$$

We note that when $\underline{\underline{F}}^{<0>} = \underline{\underline{1}}$, all of these formulas reduce to those in §5.

To find the canonical forms for the perturbation stresses, we need to expand equations (3.3) and (3.4) into powers of ε. So we need perturbation formulas for f_i, $i = 0, 1, 2$ and ϕ_{ij}, $i = 0, 1, 2, 3$, $j = 0, 1, 2$.

(10.19) $$f_i = f_i^{<0>} + \varepsilon f_i^{<1>} + O(\varepsilon^2)$$

where

(10.20) $$f_i^{<0>} = f_i(\mathrm{tr}\, \underline{\underline{B}}^{<0>}, \mathrm{tr}\, \underline{\underline{B}}^{<0>^2})$$

and

(10.21) $$f_i^{<1>} = (\partial f_i/\partial I_B\big|_{\varepsilon=0})\,\mathrm{tr}\, \underline{\underline{B}}^{<1>} + 2(\partial f_i/\partial II_B\big|_{\varepsilon=0})\,\mathrm{tr}(\underline{\underline{B}}^{<0>}\underline{\underline{B}}^{<1>}).$$

(10.22) $$\phi_{ij} = \phi_{ij}^{<0>} + \varepsilon \phi_{ij}^{<1>} + O(\varepsilon^2)$$

where

$$(10.23)\qquad \phi_{ij}^{<0>} = \phi_{ij}(\mathrm{tr}\,\underline{\underline{B}}^{<0>},\ \mathrm{tr}\,\underline{\underline{B}}^{<0>^2},\ s)\ ,$$

and

$$(10.24)\qquad \phi_{ij}^{<1>} = (\partial\phi_{ij}/\partial I_B\Big|_{\varepsilon=0})\mathrm{tr}\underline{\underline{B}}^{<1>} + 2(\partial\phi_{ij}/\partial II_B\Big|_{\varepsilon=0})\ \mathrm{tr}(\underline{\underline{B}}^{<0>}\underline{\underline{B}}^{<1>})\,.$$

As stated earlier, we may group all terms of $\underline{\underline{T}}$ proportional to $\underline{\underline{1}}$ with $-p$. Another simplification comes from combining equations (10.1) and (10.8). Then

$$(10.25)\qquad \mathrm{tr}[(\underline{\underline{F}}^{<0>})^{-1}\ \underline{\underline{F}}^{<1>}\ (\tau)] = 0.$$

Hence

$$(10.26)\qquad \mathrm{tr}[\underline{\underline{G}}^{<1>}(s)] = 0.$$

But $\mathrm{tr}\ \underline{\underline{B}}^{<1>} \neq 0$.

The perturbed stresses are given by

$$(10.27)\qquad \underline{\underline{T}}^{<0>} = -p^{<0>}\ \underline{\underline{1}} + f_1^{<0>}\underline{\underline{B}}^{<0>} + f_2^{<0>}\underline{\underline{B}}^{<0>^2}\ ,$$

and

$$(10.28)\qquad \underline{\underline{T}}^{<1>} = -p^{<1>}\underline{\underline{1}} + (f_1^{<1>}\underline{\underline{B}}^{<0>} + f_1^{<0>}\underline{\underline{B}}^{<1>})$$
$$+ (f_2^{<1>}\underline{\underline{B}}^{<0>^2} + f_2^{<0>}\underline{\underline{B}}^{<1>}\underline{\underline{B}}^{<0>} + f_2^{<0>}\underline{\underline{B}}^{<0>}\underline{\underline{B}}^{<1>})$$
$$+ \int_0^\infty\{\mathrm{tr}[\phi_{11}^{<0>}\underline{\underline{B}}^{<0>} + \phi_{12}^{<0>}\underline{\underline{B}}^{<0>^2})\underline{\underline{G}}^{<1>}(s)]\underline{\underline{B}}^{<0>}$$
$$+ \mathrm{tr}[\phi_{21}^{<0>}\underline{\underline{B}}^{<0>} + \phi_{22}^{<0>}\underline{\underline{B}}^{<0>^2})\underline{\underline{G}}^{<1>}(s)]\underline{\underline{B}}^{<0>^2}$$
$$+ (\phi_{30}^{<0>}\underline{\underline{1}} + \phi_{31}^{<0>}\underline{\underline{B}}^{<0>} + \phi_{32}^{<0>}\underline{\underline{B}}^{<0>^2})\underline{\underline{G}}^{<1>}(s)$$
$$+ \underline{\underline{G}}^{<1>}(s)(\phi_{30}^{<0>}\underline{\underline{1}} + \phi_{31}^{<0>}\underline{\underline{B}}^{<0>} + \phi_{32}^{<0>}\underline{\underline{B}}^{<0>^2})\}ds^*\ .$$

*We note that when $\underline{\underline{F}}^{<0>} = \underline{\underline{B}}^{<0>} = \underline{\underline{1}}$, equation (10.25) reduces to (7.1). Hence $\mathrm{tr}\underline{\underline{B}}^{<1>} = 0$. It implies $f_i^{<1>} = 0$ for $i = 1,2$.

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Equation (10.28) shows that for the linearized theory of perturbation of the rest state of incompressible viscoelastic solids, we need 2 elastic constants: f_1 & f_2 and 7 viscoelastic material functions: $\phi_{ij}(s)$, $i = 1,2,3$, $j = 1,2$ and $\phi_{30}(s)$*.

To find the perturbed form of Piola-Kirchoff stress, we expand $\underline{\underline{S}}^T = \underline{\underline{T}}(\underline{\underline{F}}^T)^{-1}$ in powers of ε.

(10.29) $$\underline{\underline{S}}^T = \underline{\underline{S}}^{<0>^T} + \varepsilon \underline{\underline{S}}^{<1>^T} + O(\varepsilon^2)$$

where

(10.30) $$\underline{\underline{S}}^{<0>^T} = \underline{\underline{T}}^{<0>}(\underline{\underline{F}}^{<0>^T})^{-1} ,$$

and

(10.31) $$\underline{\underline{S}}^{<1>^T} = \underline{\underline{T}}^{<1>}(\underline{\underline{F}}^{<0>^T})^{-1} - \underline{\underline{T}}^{<0>}(\underline{\underline{F}}^{<0>^T})^{-1} \underline{\underline{F}}^{<1>}(\underline{\underline{F}}^{<0>^T})^{-1}.$$

Substitution of (10.27) and (10.28) into (10.31) yields:

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It reduces $\underline{\underline{T}}^{<1>}$ to

$$\underline{\underline{T}}^{<1>} = -p^{<1>}\underline{\underline{1}} + (f_1^{<0>} + 2f_2^{<0>})\underline{\underline{B}}^{<1>} + \int_0^\infty 2(\phi_{30}^{<0>} + \phi_{31}^{<0>} + \phi_{32}^{<0>})\underline{\underline{G}}^{<1>}(s)ds .$$

Here we have used equation (10.26). This form of $\underline{\underline{T}}^{<1>}$ is same as that given by equation (7.6) where

$$\beta = f_1^{<0>}(3,3) + 2f_2^{<0>}(3,3) ,$$

and

$$\zeta(s) = 2[\phi_{30}^{<0>}(3,3,s) + \phi_{31}^{<0>}(3,3,s) + \phi_{32}^{<0>}(3,3,s)].$$

Here we have used the fact that when $\underline{\underline{B}}^{<0>} = 1$, $tr\underline{\underline{B}}^{<0>} = tr\underline{\underline{B}}^{<0>^2} = 3$.

*Exactly the same number of material parameters arise in Pipkin's (1964) viscoelastic perturbation of elastostatic deformations. Elastic constants appear when the stress is evaluated on deformations which are independent of time.

(10.32) $$\underline{\underline{S}}^{<1>^T} = [p^{<0>}(\underline{\underline{F}}^{<0>^T})^{-1}\,\underline{\underline{F}}^{<1>^T}(\underline{\underline{F}}^{<0>^T})^{-1} - p^{<1>}(\underline{\underline{F}}^{<0>^T})^{-1}]$$

$$+ f_1^{<1>}\underline{\underline{F}}^{<0>} + f_1^{<0>}\underline{\underline{F}}^{<1>} + f_2^{<1>}\underline{\underline{B}}^{<0>}\underline{\underline{F}}^{<0>}$$

$$+ f_2^{<0>}(\underline{\underline{B}}^{<1>}\underline{\underline{F}}^{<0>} + \underline{\underline{B}}^{<0>}\underline{\underline{F}}^{<1>})$$

$$+ \int_0^\infty \{\mathrm{tr}[\phi_{11}^{<0>}\underline{\underline{B}}^{<0>} + \phi_{12}^{<0>}\underline{\underline{B}}^{<0>^2})\underline{\underline{G}}^{<1>}(s)\,]\underline{\underline{F}}^{<0>}$$

$$+ \mathrm{tr}[(\phi_{21}^{<0>}\underline{\underline{B}}^{<0>} + \phi_{22}^{<0>}\underline{\underline{B}}^{<0>^2})\underline{\underline{G}}^{<1>}(s)]\,\underline{\underline{B}}^{<0>}\underline{\underline{F}}^{<0>}$$

$$+ (\phi_{30}^{<0>}\underline{\underline{1}} + \phi_{31}^{<0>}\underline{\underline{B}}^{<0>} + \phi_{32}^{<0>}\underline{\underline{B}}^{<0>^2})\underline{\underline{G}}^{<1>}(s)(\underline{\underline{F}}^{<0>^T})^{-1}$$

$$+ \underline{\underline{G}}^{<1>}(s)(\phi_{30}^{<0>}(\underline{\underline{F}}^{<0>^T})^{-1} + \phi_{31}^{<0>}\underline{\underline{F}}^{<0>} + \phi_{32}^{<0>}\underline{\underline{B}}^{<0>}\underline{\underline{F}}^{<0>})\}ds$$

where we have used the identities

(10.33) $$\underline{\underline{B}}^{<0>}(\underline{\underline{F}}^{<0>^T})^{-1} = \underline{\underline{F}}^{<0>},$$

and

$$\underline{\underline{B}}^{<1>}(\underline{\underline{F}}^{<0>^T})^{-1} = \underline{\underline{F}}^{<1>} + \underline{\underline{F}}^{<0>}\underline{\underline{F}}^{<1>^T}(\underline{\underline{F}}^{<0>^T})^{-1}.$$

Now we can write down the linearized problem of perturbation of the rest state of incompressible viscoelastic solid:

(10.34) $$\left.\begin{aligned} \rho_o\,\ddot{\underline{u}}^{<1>} &= \mathrm{div}\,\underline{\underline{S}}^{<1>^T} \\ \mathrm{tr}[(\underline{\underline{F}}^{<0>})^{-1}\,\underline{\underline{F}}^{<1>}] &= 0 \end{aligned}\right\}\quad \text{in } \mathcal{V}_0 \text{ for } t > 0,$$

$\underline{u}^{<1>}$ is specified on $\partial\mathcal{V}_{10}$ for $t > 0$,

$\underline{t}_n^{<1>}$ is specified on $\partial\mathcal{V}_{20}$ for $t > 0$,

$\underline{u}^{<1>}$ is specified in $\mathcal{V}_0$ for $t \in (-\infty, 0]$,

where $\underline{\underline{S}}_1^{<1>^T}$ is given by (10.32), $\underline{\underline{F}}^{<1>} = \nabla\underline{u}^{<1>}(\underline{X},t)$, and $\underline{t}_n^{<1>}$ is given by (10.18).

11. The linearized theory and elastic stability

The linearized theory of perturbations of the rest state is a good place to start the study of stability and bifurcation of elastostatic solutions of viscoelastic problems. We have maintained that the use of elastic equations for unsteady motions of simple solids has no good justification and is probably unjustified, except as an approximation which is valid in certain asymptotic limits. It is often true that these asymptotic limits contain all the points at issue in certain studies. But as a matter of principle in the study of stability it is necessary at least to test the stability of a solution to small disturbances of arbitrary frequency. Such time dependent disturbances lead to the linearized equations derived in §10 and not to equations of "dynamic elasticity".

Some interesting points about the stability of elastic solutions of viscoelastic problems emerge from general considerations arising in the theory of stability and bifurcation. To develop these points it is necessary to assume that the stability criteria which are associated with the linearized equations are valid for small disturbances governed by the exact nonlinear equations. So if all solutions of the linearized equations are asymptotically stable then the rest state is stable to small disturbances (conditionally stable) but if one of these disturbances grows without bound the rest state is actually unstable. In the exact theory of stability one goes a step further. In this step, the linearized equations are replaced with spectral equations which arise formally from substituting solutions of the form

(11.1) $$\underline{u}^{<1>}(\underline{X},t) = e^{\sigma t}\underline{v}(\underline{X}),\ p^{<1>}(\underline{X},t) = e^{\sigma t}\ p^{<1>}(\underline{X})$$

into the linearized equations of motions. The values of $\sigma = \xi + i\omega$ for which the resulting problem has solutions are said to be in the spectrum of that problem. We say that the rest state is stable by criteria of the linearized theory if there are no values σ for which $\xi > 0$ and is unstable if there are some such values. In the exact theory one proves that stability and instability by spectral criteria imply actual stability and instability for the correct nonlinear problem when disturbances are small.

In the problems which come up in mechanics the spectral values $\sigma(k)$ depend on a parameter k. The value σ_1 with the largest real part is called the principal spectral value. The loss of stability is associated with a critical value $k = k_o$ at which $\xi_1(k) = \text{re}\ \sigma_1(k)$ passes through zero from negative to positive. In most of the problems studied in mechanics the spectrum which crosses over in this way is of eigenvalues.

In bifurcation theory we usually assume that the principal spectral value σ_1 is isolated and has only one eigenfunction $\underline{v}(\underline{X})$. In this case, if $d\xi_1(k_o)/dk \neq 0$, we get steady bifurcating solutions if $\sigma_1(k_o) = 0$ and time-periodic ones if $\sigma_1(k_o) = i\omega_o$.

Now we are going to assume all is good with the linearized theory of the stability of the rest state (elastostatics), and that the properties relating the spectral problem to true bifurcation are as in the general theory of bifurcation at a simple eigenvalue.

We may derive the spectral problem by substituting (11.1) into (10.34). It is easy to verify that

$$\underline{\underline{S}}^{<1>^T}(\underline{u}^{<1>}) = \underline{\underline{S}}^{<1>^T}(e^{\sigma t}\underline{v}) = e^{\sigma t}\underline{\underline{\mathcal{L}}}(\underline{v})$$

where $\underline{\underline{\mathcal{L}}}(\underline{v})$ is defined by (11.4), and

(11.2) $$\underline{t}_n^{<1>}(\underline{u}^{<1>}) = \underline{t}_n^{<1>}(e^{\sigma t}\underline{v}) = e^{\sigma t}\underline{\mathcal{B}}(\underline{v})$$

where $\underline{\mathcal{B}}(\underline{v})$ is defined by the operator which arises from (10.18) when $\underline{u}^{<1>} = e^{\sigma t}\underline{v}$. The spectral problem governing the stability of all elastostatic solutions is then

(11.3)
$$\left.\begin{aligned} \rho_o\sigma^2\underline{v} &= \operatorname{div}\underline{\underline{\mathcal{L}}}(\underline{v}) \\ \operatorname{tr}[(\underline{\underline{F}}^{<0>})^{-1}(\nabla\underline{v})] &= 0 \end{aligned}\right\} \text{ in } \mathcal{V}_0,$$

$$\underline{v} \text{ is specified on } \partial\mathcal{V}_{10},$$

$$\underline{\mathcal{B}}(\underline{v}) \text{ is specified on } \partial\mathcal{V}_{20},$$

where

(11.4)
$$\begin{aligned}
\underline{\underline{\mathcal{L}}}(\underline{v}) = {} & p^{<0>}(\underline{\underline{F}}^{<0>^T})^{-1}(\nabla\underline{v})^T(\underline{\underline{F}}^{<0>^T})^{-1} - p^{<1>}(\underline{\underline{F}}^{<0>^T})^{-1} \\
& + \hat{f}_1^{<1>}\underline{\underline{F}}^{<0>} + f_1^{<0>}\nabla\underline{v} + \hat{f}_2^{<1>}\underline{\underline{B}}^{<0>}\underline{\underline{F}}^{<0>} \\
& + f_2^{<0>}(\hat{\underline{\underline{B}}}^{<1>}\underline{\underline{F}}^{<0>} + \underline{\underline{B}}^{<0>}\nabla\underline{v}) \\
& + \int_0^\infty \{\operatorname{tr}[(\phi_{11}^{<0>}(s)\underline{\underline{B}}^{<0>} + \phi_{12}^{<0>}(s)\underline{\underline{B}}^{<0>^2})\hat{\underline{\underline{G}}}^{<1>}]\underline{\underline{F}}^{<0>} \\
& + \operatorname{tr}[\phi_{21}^{<0>}(s)\underline{\underline{B}}^{<0>} + \phi_{22}^{<0>}(s)\underline{\underline{B}}^{<0>^2})\hat{\underline{\underline{G}}}^{<1>}]\underline{\underline{B}}^{<0>}\underline{\underline{F}}^{<0>} \\
& + (\phi_{30}^{<0>}(s)\underline{\underline{1}} + \phi_{31}^{<0>}(s)\underline{\underline{B}}^{<0>} + \phi_{32}^{<0>}(s)\underline{\underline{B}}^{<0>^2}) \\
& \qquad \hat{\underline{\underline{G}}}^{<1>}(\underline{\underline{F}}^{<0>^T})^{-1} \\
& + \hat{\underline{\underline{G}}}^{<1>}(\phi_{30}^{<0>}(s)(\underline{\underline{F}}^{<0>^T})^{-1} + \phi_{31}^{<0>}(s)\underline{\underline{F}}^{<0>} \\
& \qquad + \phi_{32}^{<0>}(s)\underline{\underline{B}}^{<0>}\underline{\underline{F}}^{<0>})\}(e^{-\sigma s} - 1)ds,
\end{aligned}$$

$$\hat{f}_i^{<1>} = (\partial f_i/\partial I_B\Big|_{\varepsilon=0})\,\mathrm{tr}\,\hat{\underline{\underline{B}}}^{<1>} + 2(\partial f_i/\partial II_B\Big|_{\varepsilon=0})\,\mathrm{tr}(\underline{\underline{B}}^{<0>}\hat{\underline{\underline{B}}}^{<1>}),$$

$$\hat{\underline{\underline{B}}}^{<1>} = (\nabla\underline{v})\underline{\underline{F}}^{<0>^T} + \underline{\underline{F}}^{<0>}(\nabla\underline{v})^T,$$

and

$$\hat{\underline{\underline{G}}}^{<1>} = (\nabla\underline{v})(\underline{\underline{F}}^{<0>})^{-1} + (\underline{\underline{F}}^{<0>^T})^{-1}(\nabla\underline{v})^T.$$

All the quantities except the $\phi_{ij}^{<0>}(s)$ under the integral sign in (11.4) are independent of s; for example,

$$\int_0^\infty \mathrm{tr}[\phi_{11}^{<0>}(s)\ \underline{\underline{B}}^{<0>}\hat{\underline{\underline{G}}}^{<1>}]\underline{\underline{F}}^{<0>}ds$$

$$= \{\int_0^\infty \phi_{11}^{<0>}(s)ds\}\ \mathrm{tr}[\underline{\underline{B}}^{<0>}\hat{\underline{\underline{G}}}^{<1>}]\underline{\underline{F}}^{<0>}.$$

So the spectral problem contains the history in the convenient form of integrals over material functions, independent of the motion.

In the simplest of the rest states, the natural state studied in §8, it was possible to obtain explicit formulas, like (8.19) for σ and to use such results to infer properties of the material parameters. The problem (11.3) is much more difficult than the one in §8 because there are so many material parameters and because the equations (11.3) govern the stability of the whole class of elastostatic deformations of viscoelastic solids. Nonetheless, there may be a principle, equivalent to the requirement that the natural state of a solid should be stable against all disturbances, which can be used to characterize the material parameters appearing in the problem (11.3). Many states of elastostatic deformation of solids are unstable so that the simple criterion of stability has no force here.

The principle which we wish to consider is that instability of elastic deformations of viscoelastic solids cannot lead to bifurcation into self-sustained oscillations. It is difficult for us to imagine how a time-dependent motion of a viscoelastic solid could arise from a static deformation of that solid.

To illuminate some considerations behind our conjecture it is useful to compare the stability problems which arise in the stability of fluids with those which arise in the stability of solids. In the celebrated Taylor problem in hydrodynamics the flow between concentric cylinders is driven, say, by the steady rotation of the inner cylinder. At first there is a featureless flow (Couette flow) which is uniform like the data; at higher speeds Couette flow gives up its stability to a secondary flow which is arrayed in a set of Taylor vortices of approximately square cross-section. At still higher speeds these steady vortices bifurcate into a time-periodic motion in which a wave undulates around the vortices. The corresponding elastic solution is the torsional deformation of an incompressible elastic cylinder. The solution of this problem was found by Rivlin (1949). Green and Spencer (1959) have studied the problem using a static theory of elastic stability. Penn and Kearsley (1976) have demonstrated in experiments that Rivlin's solution is unstable when the deformation is sufficiently large. The symmetry-breaking bifurcation observed by Penn and Kearsley is in the form of spiral bands. We shall formulate the problem of stability of Rivlin's solution in the context of our viscoelastic theory in the Appendix.

The main point of comparison is that in the fluids problem the boundary data, though steady, does work and gives a continuous supply of energy which can be converted into permanent time-dependent motion. In the solids problem, the imposed steady twist does no work and does not supply energy which can be used to drive a motion.

If we now suppose that the spectral problem (11.3) has the same relevance to bifurcation as in the general theory of bifurcation, we may expect to find steady symmetry-breaking bifurcation, like the experiments, when the spectrum is of eigenvalues σ and the eigenvalue σ with the largest real part is real-valued at criticality. The other possibility is that $\sigma = i\omega$ is not zero when ξ is. Then $\pm i\omega$ are both eigenvalues at criticality and $e^{\pm i\omega t}$ is oscillatory. In the usual case this situation implies bifurcation into a time-periodic motion. If our intuition is correct we should not have time-periodic bifurcation in the elastic problem with steady forcing. So if our choice of material parameters and functions leads to complex-valued σ at criticality we have made a bad choice.

Appendix: The spectral problem for the stability of Rivlin's solution for torsional deformation of a viscoelastic cylinder

It is of interest to consider the stability theory discussed in §11 in the simplest possible nontrivial case. Perhaps this simplest nontrivial case is one of those (torsional deformation) found by Rivlin (1949).

Torsional deformation of a right circular cylinder (of radius a and height 2) of incompressible, initially isotropic, viscoelastic material is given by:

$$r^o = R,$$

$$\theta^o = \Theta + kZ,$$

$$z^o = Z,$$

where $X_I = \{R, \Theta, Z\}$ are coordinates of a material particle $\underline{X}$ in the natural state and $x_i^o = \{r^o, \theta^o, z^o\}$ are coordinates of $\underline{x}^o$, the position of $\underline{X}$ in the deformed state. It is necessary that some forces and torques be applied to the top and bottom surfaces to maintain this deformation and the constant height. The boundary conditions satisfied by this deformation are: $\theta = \Theta \pm k$ at $Z = \pm 1$ and the surface $r^o = a$ is stress-free.

To form the spectral problem (11.3) for the stability of Rivlin's solution we need to find the components of $\underline{\underline{F}}^{<0>}$, $(\underline{\underline{F}}^{<0>})^{-1}$, $\underline{\underline{B}}^{<0>}$ and $\underline{\underline{C}}^{<0>}$. $\underline{e}_I = \{\underline{e}_R, \underline{e}_\Theta, \underline{e}_Z\}$ is the orthogonal basis corresponding to the coordinate-system in the natural state. We introduce the orthonormal base vectors $\hat{\underline{e}}_I = \underline{e}_I/|\underline{e}_I|$. $|\underline{e}_I| = \{1, R, 1\}$. Similarly $\underline{e}_i = \{\underline{e}_{r^o}, \underline{e}_{\theta^o}, \underline{e}_{z^o}\}$ is the orthogonal

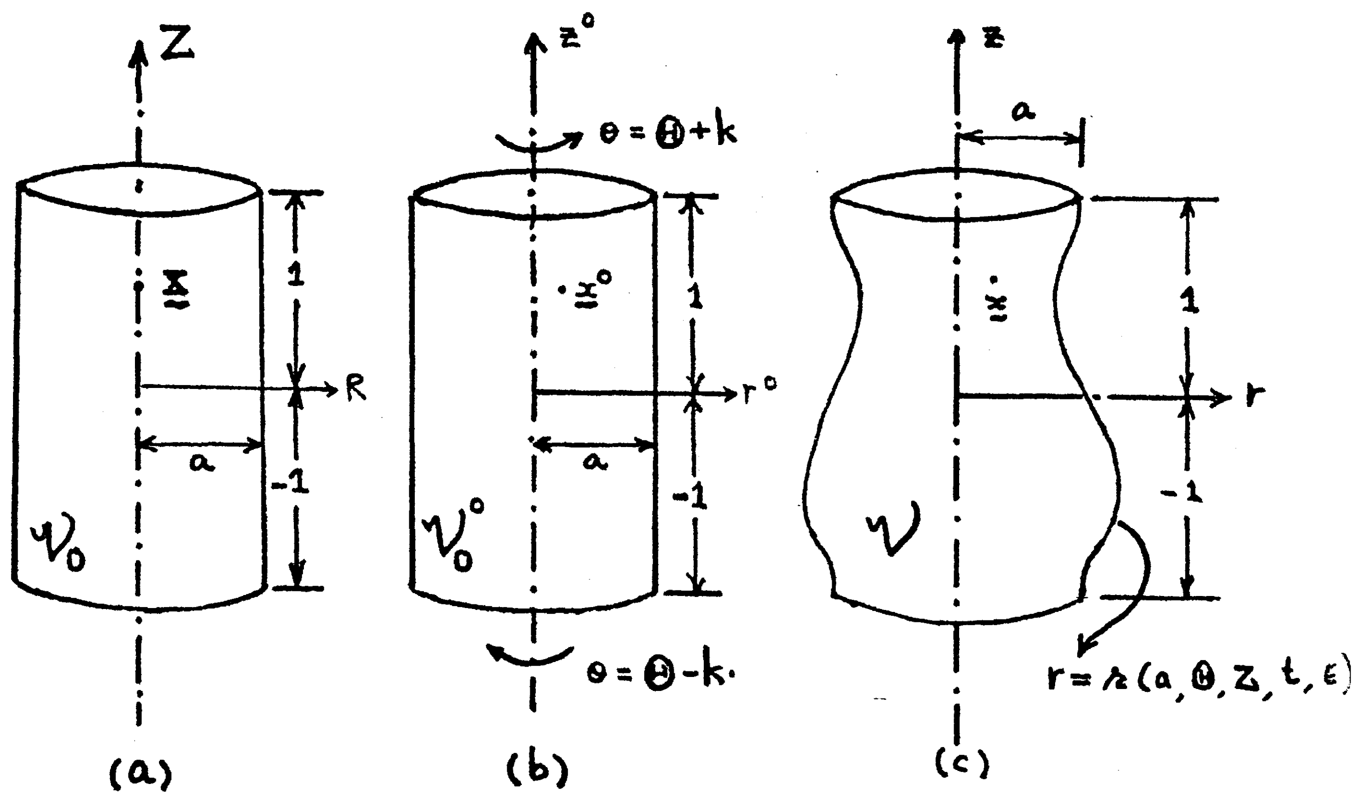

Fig. A.1 Perturbation of the rest (torsional) state of an incompressible, initially isotorpic, cylinder.

(a) The natural state

$$\mathcal{V}_0 = \{R,\Theta,Z:\ 0 \leq R < a,\ 0 \leq \Theta \leq 2\pi,\ -1 < Z < 1\}\quad ,$$

(b) The rest state

$$\mathcal{V}_0^o = \{r^o,\theta^o,z^o:\ 0 \leq r^o < a,\ 0 \leq \theta^o \leq 2\pi,\ -1<z^o<1\}\ ,$$

(c) The perturbed state

$$\mathcal{V} = \{r,\theta,z:\ 0 \leq r < \hbar(a,\Theta,Z,t,\varepsilon),\ 0 \leq \theta \leq 2\pi,\ -1<z<1\}$$

where $\hbar(a,\Theta,Z,t,\varepsilon)$ is the free surface.

basis in the rest state. The corresponding orthonormal basis is $\hat{\underline{e}}_i = \underline{e}_i/|\underline{e}_i|$ where $|\underline{e}_i| = \{1, r^o, 1\}$. Then $\underline{\underline{F}}^{<0>}$, $\underline{\underline{B}}^{<0>}$ and $\underline{\underline{C}}^{<0>}$ have the following representation:

$$\underline{\underline{F}}^{<0>} = F^{<0>}_{iJ}\, \hat{\underline{e}}_i \otimes \hat{\underline{e}}_J{}^*;\quad F^{<0>}_{iJ} = \frac{\partial x^o_i}{\partial X_J}\,\frac{|\underline{e}_i|}{|\underline{e}_J|} \text{ (no sum over i \& J);}$$

$$\underline{\underline{B}}^{<0>} = B^{<0>}_{ij}\, \hat{\underline{e}}_i \otimes \hat{\underline{e}}_j;\quad B^{<0>}_{ij} = F^{<0>}_{iK}\, F^{<0>}_{jK};$$

$$\underline{\underline{C}}^{<0>} = C^{<0>}_{IJ}\, \hat{\underline{e}}_I \otimes \hat{\underline{e}}_J;\quad C^{<0>}_{IJ} = F^{<0>}_{kI}\, F^{<0>}_{kJ}.$$

The matrices of components $F^{<0>}_{iJ}$, $B^{<0>}_{ij}$, $C^{<0>}_{IJ}$ are:

$$[F^{<0>}_{iJ}] = [\delta_{iJ}] + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & r^o k \\ 0 & 0 & 0 \end{pmatrix},$$

$$[B^{<0>}_{ij}] = [\delta_{ij}] + \begin{pmatrix} 0 & 0 & 0 \\ 0 & r^{o2} k^2 & r^o k \\ 0 & r^o k & 0 \end{pmatrix},$$

$$[C^{<0>}_{IJ}] = [\delta_{IJ}] + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & r^o k \\ 0 & r^o k & r^{o2} k^2 \end{pmatrix}.$$

The components of $(\underline{\underline{F}}^{<0>})^{-1}$ are easy to compute.

$$(\underline{\underline{F}}^{<0>})^{-1} = (F^{<0>})^{-1}_{Ij}\, \hat{\underline{e}}_I \otimes \hat{\underline{e}}_j$$

where the matrix $[(F^{<0>})^{-1}_{Ij}]$ is the inverse of the matrix $[F^{<0>}_{iJ}]$.

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*In this section, we use upper and lower case suffixes. This is to emphasize the fact that such components are either with respect to the basis in the natural state (upper case suffix) or with respect to the mixed basis (One upper and one lower case suffix). The usual summation convention applies to these suffices also unless the contrary is explicitely stated.

$$[(F^{<0>})^{-1}{}_{Ij}] = [\delta_{Ij}] - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & r^o k \\ 0 & 0 & 0 \end{pmatrix}.$$

From this we can compute $\underline{\underline{T}}^{<0>}$, $\underline{t}_n^{<0>}$ and $\underline{\underline{S}}^{<0>^T}$. $p^{<0>}$ is computed by solving the equation of motion div $\underline{\underline{S}}^{<0>^T} = \underline{0}$. It is easy to check that $\underline{n}^{<0>} = \underline{e}_{r^o} = \hat{\underline{e}}_1$.

Now we consider perturbation of the rest state given by:

$$r = r^o + \varepsilon r^{<1>} + O(\varepsilon^2) = R + \varepsilon r^{<1>} + O(\varepsilon^2),$$

$$\theta = \theta^o + \varepsilon\theta^{<1>} + O(\varepsilon^2) = \Theta + kZ + \varepsilon\theta^{<1>} + O(\varepsilon^2),$$

$$z = z^o + \varepsilon z^{<1>} + O(\varepsilon^2) = Z + \varepsilon z^{<1>} + O(\varepsilon^2) ,$$

where (r,θ, z) are coordinates of $\underline{x}$, the position of $\underline{X}$ in the perturbed state. We could treat (r,θ,z) as functions of either r^o,θ^o and z^o or R, Θ & Z. $\underline{x} = \underline{x}^o + \varepsilon\underline{u}^{<1>} + O(\varepsilon^2) = \underline{X} + \underline{u}^o + \varepsilon\underline{u}^{<1>} + O(\varepsilon^2)$. Then $\underline{u}^{<1>}$ has the representation:

$$\underline{u}^{<1>} = r^{<1>}\hat{\underline{e}}_1 + r^o\theta^{<1>}\hat{\underline{e}}_2 + z^{<1>}\hat{\underline{e}}_3 .$$

The components of $\partial\underline{u}^{<1>}/\partial\underline{x}^o$ in the basis $\hat{\underline{e}}_i$ are

$$[\partial\underline{u}^{<1>}/\partial\underline{x}^o] = \begin{pmatrix} \partial r^{<1>}/\partial r^o & \frac{1}{r^o}\,\partial r^{<1>}/\partial\theta^o - \theta^{<1>} & \partial r^{<1>}/\partial z^o \\ \partial(r^o\,\theta^{<1>})/\partial r^o & \partial\theta^{<1>}/\partial\theta^o + r^{<1>}/r^o & r^o\partial\theta^{<1>}/\partial z^o \\ \partial z^{<1>}/\partial r^o & \frac{1}{r^o}\,\partial z^{<1>}/\partial\theta^o & \partial z^{<1>}/\partial z^o \end{pmatrix}.$$

Now

$$\underline{\underline{F}}^{<1>} = \partial\underline{u}^{<1>}/\partial\underline{X} = (\partial\underline{u}^{<1>}/\partial\underline{x}^o)(\partial\underline{x}^o/\partial\underline{X}).$$

The representation of $\underline{\underline{F}}^{<1>}$ is:

$$\underline{\underline{F}}^{<1>} = F^{<1>}{}_{iJ}\, \hat{\underline{e}}_i \otimes \hat{\underline{e}}_J ; \quad F^{<1>}{}_{iJ} = \sum_{k=1}^{3} \left(\frac{\partial \underline{u}^{<1>}}{\partial \underline{x}^o}\right)_{ik} \frac{\partial x_k{}^o}{\partial X_J} \frac{|\underline{e}_k|}{|\underline{e}_J|} \quad \text{(no sum over J)}$$

$$= \left(\frac{\partial \underline{u}^{<1>}}{\partial \underline{x}^o}\right)_{ik} F^{<0>}{}_{kJ} \, .$$

The matrix of $F^{<1>}{}_{iJ}$ is given by :

(A.1)
$$[F^{<1>}{}_{iJ}] = \begin{bmatrix} \partial r^{<1>}/\partial r^o & (1/r^o)\partial r^{<1>}/\partial\theta^o - \overset{\cdot}{\theta}^{<1>} & \partial r^{<1>}/\partial z^o \\ \partial(r^o \overset{\cdot}{\theta}^{<1>})/\partial r^o & \partial\theta^{<1>}/\partial\theta^o + r^{<1>}/r^o & r^o \partial\overset{\cdot}{\theta}^{<1>}/\partial z^o \\ \partial z^{<1>}/\partial r^o & (1/r^o)\partial z^{<1>}/\partial\theta^o & \partial z^{<1>}/\partial z^o \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 & k(\partial r^{<1>}/\partial\theta^o - r^o\theta^{<1>}) \\ 0 & 0 & k(r^o\partial\theta^{<1>}/\partial\theta^o + r^{<1>}) \\ 0 & 0 & k(\partial z^{<1>}/\partial\theta^o) \end{bmatrix} .$$

Now we are in a position to find the components of $\underline{\underline{B}}^{<1>}$ and $\underline{\underline{G}}^{<1>}(s)$:

$$\underline{\underline{B}}^{<1>} = B^{<1>}{}_{ik}\, \hat{\underline{e}}_i \otimes \hat{\underline{e}}_k$$

where

(A.2)
$$[B^{<1>}{}_{ik}] = 2[E^{<1>}{}_{ik}] + r^o k \begin{bmatrix} 0 & F^{<1>}{}_{13} & 0 \\ F^{<1>}{}_{13} & 2F^{<1>}{}_{23} & F^{<1>}{}_{33} \\ 0 & F^{<1>}{}_{33} & 0 \end{bmatrix} ,$$

and

$$2E^{<1>}{}_{ik} = F^{<1>}{}_{iJ}\,\delta_{Jk} + F^{<1>}{}_{kJ}\,\delta_{Ji} ;$$

$$\underline{\underline{G}}^{<1>}(s) = G^{<1>}{}_{ik}(s)\, \hat{\underline{e}}_i \otimes \hat{\underline{e}}_k$$

where

$$[G^{<1>}{}_{ik}(s)] = [2\,[\![E^{<1>}{}_{ik}]\!]\,] - r^o k \begin{pmatrix} 0 & 0 & [\![F^{<1>}{}_{12}]\!] \\ 0 & 0 & [\![F^{<1>}{}_{22}]\!] \\ [\![F^{<1>}{}_{12}]\!] & [\![F^{<1>}{}_{22}]\!] & 2[\![F^{<1>}{}_{32}]\!] \end{pmatrix}$$

and

$$[\![a]\!] \equiv a(t-s) - a(t) \text{ for any scalar } a.$$

Now we can find components of $\underline{n}^{<1>}$, $\underline{\underline{T}}^{<1>}$, $\underline{t}_n{}^{<1>}$ and div $\underline{\underline{S}}^{<1>^T}$ in the basis of the rest state and components of

$$\underline{\underline{S}}^{<1>^T} = S^{<1>^T}{}_{iJ}\ \hat{\underline{e}}_i \otimes \hat{\underline{e}}_J$$

in the mixed basis.

Finally we note that $(\underline{\underline{F}}^{<0>})^{-1}\ \underline{\underline{F}}^{<1>}$ has the representation:

$$(\underline{\underline{F}}^{<0>})^{-1}\ \underline{\underline{F}}^{<1>} = (F^{<0>})^{-1}_{Ij}\ F^{<1>}{}_{jK}\ \hat{\underline{e}}_I \otimes \hat{\underline{e}}_K\ .$$

So

$$\mathrm{Tr}[(\underline{\underline{F}}^{<0>})^{-1}\ \underline{\underline{F}}^{<1>}] = (F^{<0>})^{-1}{}_{Ij}\ F^{<1>}{}_{jI}$$

$$= F^{<1>}{}_{11} + F^{<1>}{}_{22} + F^{<1>}{}_{33} - r^o k\ F^{<1>}{}_{32}.$$

Now to write the spectral problem (11.3) into the component form, we let

$$r^{<1>}(\underline{X},t) = e^{\sigma t}\ \hat{r}^{<1>}(\underline{X}),$$

$$\theta^{<1>}(\underline{X},t) = e^{\sigma t}\ \hat{\theta}^{<1>}(\underline{X}),$$

$$z^{<1>}(X,t) = e^{\sigma t}\ \hat{z}^{<1>}(\underline{X}).$$

Then

(A.3) $$\underline{v} = \hat{r}^{<1>}\ \hat{\underline{e}}_1 + r^o\ \hat{\theta}^{<1>}\ \hat{\underline{e}}_2 + z^{<1>}\ \hat{\underline{e}}_3\ .$$

$$\nabla\underline{v} = (\nabla\underline{v})_{iJ}\ \hat{\underline{e}}_i \otimes \hat{\underline{e}}_J\ ,$$

and

$$\hat{B}^{<1>} = \hat{B}^{<1>}_{ik}\, \hat{\underline{e}}_i \otimes \hat{\underline{e}}_k,$$

where the components $(\nabla\underline{v})_{iJ}$ and $\hat{B}^{<1>}_{ik}$ can be obtained from the equations (A.1) and (A.2) just by replacing $r^{<1>}$, $\theta^{<1>}$ and $z^{<1>}$ by $\hat{r}^{<1>}$, $\hat{\theta}^{<1>}$ and $\hat{z}^{<1>}$.

$$\hat{\underline{\underline{G}}}^{<1>} = \hat{G}^{<1>}_{ik}\, \hat{\underline{e}}_i \otimes \hat{\underline{e}}_k$$

where

$$[\hat{G}^{<1>}_{ik}] = [(\nabla\underline{v})_{iJ}\,\delta_{Jk} + (\nabla\underline{v})_{kJ}\,\delta_{Ji}] - r^o k \begin{bmatrix} 0 & 0 & (\nabla\underline{v})_{12} \\ 0 & 0 & (\nabla\underline{v})_{22} \\ (\nabla\underline{v})_{12} & (\nabla\underline{v})_{22} & 2(\nabla\underline{v})_{32} \end{bmatrix}.$$

Now we can find the components of $\text{div}\,\underline{\underline{\mathcal{L}}}(\underline{v})$ and $\underline{\mathcal{B}}(\underline{v})$ in the basis of the rest state. Finally

$$\text{Tr}[(\underline{\underline{F}}^{<0>})^{-1}(\nabla\underline{v})] = (\nabla\underline{v})_{11} + (\nabla\underline{v})_{22} + (\nabla\underline{v})_{33} - r^o k(\nabla\underline{v})_{32}.$$

Then the spectral problem (11.3) becomes:

$$\left.\begin{aligned} \rho_o\sigma^2\hat{r}^{<1>} &= [\text{div}\,\underline{\underline{\mathcal{L}}}(\underline{v})]_1 \\ \rho_o\sigma^2 r^o\hat{\theta}^{<1>} &= [\text{div}\,\underline{\underline{\mathcal{L}}}(\underline{v})]_2 \\ \rho_o\sigma^2\hat{z}^{<1>} &= [\text{div}\,\underline{\underline{\mathcal{L}}}(\underline{v})]_3 \\ (\nabla\underline{v})_{11} + (\nabla\underline{v})_{22} + (\nabla\underline{v})_{33} &= r^o k(\nabla\underline{v})_{32} \end{aligned}\right\} \text{ in } \mathcal{V}_0 ,$$

$$\hat{r}^{<1>} = \hat{\theta}^{<1>} = \hat{z}^{<1>} = 0 \qquad \text{on } Z=\pm 1 ,$$

$$\underline{\mathcal{B}}(\underline{v}) = \underline{0} \qquad \text{on } R = a ,$$

where

$$\underline{v} = \hat{r}^{<1>}\,\hat{\underline{e}}_1 + r^o\,\hat{\theta}^{<1>}\,\hat{\underline{e}}_2 + \hat{z}^{<1>}\,\hat{\underline{e}}_3$$

as given by (A.3).

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