Boundary conditions for thin lubrication layers

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In certain circumstances, the effects of a thin lubrication layer may be accommodated by a slip flow boundary condition with the gradient of the tangential component of the velocity at the wall proportional to the square of the tangential component there.

We consider the effects of a thin lubricating film between a flowing liquid and a solid surface. The analysis suggests that the effects of the thin lubrication layer may be accommodated by changing the boundary conditions on the tangential component $u(x,0)$ of velocity at the solid surface so that

$$\frac{\partial u(x,0)}{\partial y}=\frac{\mu_1 u^2(x,0)}{2\mu_2 Q} \quad \text{on } y=0\,, \tag{1}$$

where μ_1/μ_2 is the ratio of the viscosity of the lubricating fluid in the layer to the viscosity μ_2 of the bulk fluid above the layer, y is perpendicular to the boundary and Q is the constant volume flux of fluid in the thin layer (see Fig. 1). We present a boundary-layer theory for flows in the presence of such a thin film and solve one problem in the theory.

We are interested generally in the coupled motion of two fluids separated by an interface. In the general problem, we have the continuity of velocity across the interface and we require that the difference in stress across the interface be balanced by interfacial forces, like surface tension. But, here we do not aim at generality and instead seek the simplest theory we can get when the two fluids are incompressible and Newtonian and the fluid near the solid is confined to a very thin layer.

By a thin layer we shall understand a layer to which lubrication theory applies. The component $\hat{u}(x,y)$ of velocity parallel to the solid wall at $y=0$ satisfies

$$p'(x)=\frac{dp}{dx}(x)=\mu_1\frac{\partial^2\hat{u}(x,y)}{\partial y^2}\,, \tag{2}$$

where μ_1 is the viscosity of the fluid in the layer and the pressure gradient depends only on x. The velocity of the fluid above the layer is $u(x,y)=(u,v)$ and its viscosity is μ_2. The velocity is continuous across the interface at $y=h(x)$:

$$u[x,h(x)]=\hat{u}[x,h(x)]=V(x)\,. \tag{3}$$

The nature of the continuity of the stresses depends on the nature of the motion and the material. Here we neglect surface tension, for example, and assume that both fluids are Newtonian. The continuity of the shear stress then requires that

$$\mu_2\frac{\partial u}{\partial y}=\mu_1\frac{\partial \hat{u}}{\partial y} \quad \text{at } y=h(x)\,. \tag{4}$$

It follows that

$$\hat{u}(x,y)=\frac{p'(x)}{2\mu_1}[y^2-h(x)y]+\frac{V(x)y}{h(x)}\,, \tag{5}$$

where

$$\mu_2\frac{\partial u[x,h(x)]}{\partial y}=\frac{p'(x)}{2}h+\mu_1\frac{V(x)}{h(x)}\,. \tag{6}$$

The volume flux

$$Q_{\hat{u}}=\int_0^{h(x)}\hat{u}(y)\,dy=\frac{-p'(x)h^3(x)}{12\mu_1}+\frac{V(x)h(x)}{2} \tag{7}$$

is an unknown constant of the motion. Even if the left-hand side of (6) were known, it would not be possible to find three fields $V(x)$, $h(x)$, and $p(x)$ from the two Eqs. (6) and (7). We need to specify a condition for the jump in the normal component of stress.

In all that follows we assume that $p'(x)$ is continuous across the surface at $y=h(x)$. This means that we are assuming that the x derivatives $\mu\partial/\partial y(\partial v/\partial x)$ of the viscous part of the normal stress and the surface curvature are small relative to $p'(x)$.

It is central to the argument leading to (1) that $p'(x)$ is determined, as in boundary-layer theory, by dynamics which are independent or weakly dependent on the thickness $h(x)$ of the lubrication layer. In fact, the film thickness $h(x)$ of the lubrication layer is clearly related to the average thickness $\bar{h}=h(\bar{x})$ of the film over the length of interest and the average $\bar{h}$ is determined by the amount of fluid which is put into the film. For example, in flow over a greasy surface we may control the average depth of grease by controlling the amount of grease deposited on the surface. All this means that we can take $h(x)$ very small in (5), (6), and (7) without changing $p'(x)$ by much. So, for very thin films under the assumptions of the last two paragraphs we get

$$\hat{u}(x,y)=\frac{V(x)y}{h(x)}\,, \tag{8}$$

$$\mu_2\frac{\partial u(x,h)}{\partial y}=\mu_1\frac{V(x)}{h(x)}\,, \tag{9}$$

$$Q=\tfrac{1}{2}[V(x)h(x)]\,. \tag{10}$$

Eliminating $h(x)$ in (9) with (10), we find, using $V(x)$, $u(x,h)$, that

$$\mu_2\frac{\partial u(x,h)}{\partial y}=\frac{\mu_1}{2Q}u^2(x,h)\,, \tag{11}$$

and since $h(x)$ is nearly zero, we may put it to zero in (11), as in the boundary condition given by (1).

We note Eq. (11) is to be understood as a drag flow approximation, $u(x)/Q$ is positive and u has the same sign as $\partial u/\partial y$. It might be useful to write (1) as

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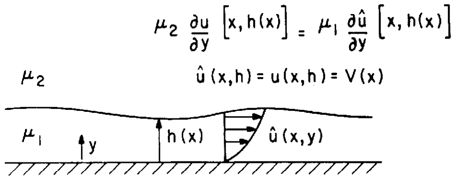

FIG. 1. Lubrication layer at $y=h(x)$. The velocity and shear stress are continuous across the layer.

$$\mu_2 \frac{\partial u(x,0)}{\partial y} = \frac{\mu_1}{2|Q|} u^2(x,0) \operatorname{sgn} \frac{\partial u}{\partial y}(x,0) .$$

The application of the drag flow approximation (8) to lubricating layers is not new and has been discussed for some interesting special problems, by Pearson.[1,2] The "slip flow" boundary condition (1) with gradient proportional to the square of velocity, and its application in theory of boundary layers seems not to have been noted before.

In boundary layers over lubricated walls the pressure gradient, the film thickness $h(x)/$ and the interfacial velocity $u[x, h(x)]$ depend on x. To study these problems we use Prandtl's equations and require that the velocity component $v[x, h(x)]$ normal to the interface of the film vanish and the tangential component of velocity $u[x, h(x)]$ and the shear stress are continuous across $y=h$. In the limit of small h, we have Prandtl's equations in the fluid

$$v(x,0)=0 , \tag{12}$$

and

$$\frac{\partial u}{\partial y} = \frac{\mu_1 u^2}{\mu_2 2Q} \quad \text{at} \quad y=0 . \tag{13}$$

Having solved these equations, given Q and p', we may determine $u(x,0)$ and, as an afterthought, compute $h(x) = 2Q/u(x,0)$.

It is possible to reconsider the traditional boundary layer problems in which the no-slip condition $u=0$ at $y=0$ is replaced with the lubricated wall condition (13). It is perhaps not so surprising that most of the similarity solutions of traditional theory do not carry over to the lubricated wall problem. In general, boundary layers at lubricated walls are nonsimilar. However, the similarity solution for flow into a converging nozzle does generalize to the lubricated wall case and we consider it next.

Our problem is framed in terms of the stream function

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}, \tag{14}$$

$$\frac{\partial \psi}{\partial y} \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi \partial^2 \psi}{\partial x \partial y^2} = U\frac{dU}{dx} + \mu_2 \frac{\partial^3 \psi}{\partial y^3} . \tag{15}$$

At the wall $y=0$, $\partial\psi/\partial x = 0$ and

$$\frac{\partial^2 \psi}{\partial y^2} = \frac{\mu_1}{2\mu_2 Q}\left(\frac{\partial \psi}{\partial y}\right)^2 , \tag{16}$$

and $\partial\psi/\partial y = U(x)$ at $y=\infty$. The potential flow

$$U = u_1/-x \quad u_1 > 0 \tag{17}$$

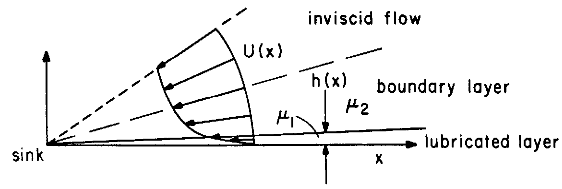

FIG. 2. Flow in a convergent channel with a lubricated wall.

represents a two-dimensional motion in a convergent channel with flat walls (sink flow, see Fig. 2). We next introduce the usual similarity variable

$$\eta = \frac{y}{x}\left(\frac{u_1}{\nu}\right)^{1/2} \quad \psi(x,y) = f(\eta)\,(\nu u_1)^{1/2} . \tag{18}$$

and from (14) we obtain

$$u = Uf', \quad v = -\left(uv_1\right)^{1/2} \eta f'/x , \tag{19}$$

where (15) reduces to

$$f''' - f'^2 + 1 = 0 \quad \text{for } \eta > 0 , \tag{20}$$

and (16) [or (13)] reduces to

$$f'' = \lambda f'^2 \quad \text{at } \eta = 0 , \tag{21}$$

where

$$\lambda = -\mu_1/2\mu_2 Q \geqslant 0 . \quad Q \leqslant 0 .$$

The condition $v=0$ at $y=0$ is satisfied automatically.

Upon multiplying (20) with f' and integrating we have

$$f''^2 - \tfrac{2}{3}(f'-1)^2(f'+2) = a , \tag{22}$$

where a is a constant of integration. Its value is zero, as $f'=1$ and $f''=0$ for $\eta \to \infty$. Now combine (21) with (22) evaluated at $\eta=0$ and find that

$$\lambda^2 f_0'^4 - \tfrac{2}{3}(f_0'-1)^2(f_0'+2) = 0 , \tag{23}$$

where $f_0' = f'(0) = u(x,y)/U(x)$. Recalling now that $f'(\eta) = u(x,y)/U(x)$ and that $f'(\eta)=1$ for $\eta \to \infty$, we restrict our consideration to positive $f' \leqslant 1$ and note that (23) gives

$$f_0' = 0 \quad \text{when } \lambda^2 \to \infty ,$$

$$f_0' = 1 \quad \text{when } \lambda^2 = 0 .$$

For each nonnegative λ^2, $0 \leqslant \lambda < \infty$ there is one and only one f_0', $0 \leqslant f_0' \leqslant 1$ satisfying (23).

The limiting case $\lambda^2 = \infty$ gives back the traditional sink flow. $\lambda^2 = \infty$ corresponds to a lubricating layer with an infinitely large viscosity, so the no-slip condition applies.

The limiting case

$$\lambda^2 = \frac{\mu_1^2}{4Q^2\mu_2^2} \to 0$$

corresponds to using a lot (large Q^2) of low viscosity (small μ_1/μ_2) lubricant in the layer. When $\lambda=0$, $f_0'=1$ and we find that

$$f'(\eta) = 1 \quad \text{for all } \eta .$$

That is, $u(x,y) = U(x)$, and the entire flow is a potential flow, all the way to the boundary.[3]

Given $f'_0 = u(x,0)/U(x)$, where $U(x) = -u_1/x$ we may compute the layer thickness as

$$h = \frac{2Q}{u(x,0)} = \frac{2|Q|}{u_1 f'_0} x .$$

Expressing $|Q|$ in terms of λ, we get

$$h = \frac{\mu_1}{\mu_2 u_1 f'_0 \lambda} x . \tag{24}$$

Returning now to (22) with $a=0$, we have

$$\frac{df'}{d\eta} = \left[\tfrac{2}{3}(f'-1)^2(f'+2)\right]^{1/2}$$

or

$$\eta = \int_{f'_0}^{f'} \frac{df'}{[\frac{2}{3}(f'-1)^2(f'+2)]^{1/2}} \tag{25}$$

in closed form, as

$$\eta = 2^{1/2} \tanh^{-1}\left(\frac{2+f'}{3}\right)^{1/2} - \tanh^{-1}\left(\frac{2+f'_0}{3}\right)^{1/2},$$

or, solving for $f' = u/U$,

$$f'(\eta) = 3\tanh^2\left[\frac{\eta}{2^{1/2}} + \tanh^{-1}\left(\frac{2+f'_0}{3}\right)^{1/2}\right] - 2 .$$

Winter et al.[3] have used wall lubrication in an experiment designed to achieve pure extensional motions of viscoelastic liquids. They wish to use the experiment to measure properties of the liquid in extensional motion. They apparently achieve a flow which is close to a potential stagnation flow (pure extension) by lubricating the walls of a channel whose boudaries form the streamlines of the potential flow. They make λ small by making $|Q|$ large and μ_1/μ_2 small. They have large Q and large $h(x)$, so "drag flow" may not apply. The shear stress at $y=h$ in the viscoelastic fluid is not given by $\mu_2 \partial u/\partial y(x,h)$, call it $\hat{\tau}$. Then, in the general case, we find that

$$\hat{\tau} = \frac{\mu_1}{2|Q|} u^2(x,0) \quad (\operatorname{sgn} \hat{\tau}) .$$

A result of Caswell[4] suggests that it may be possible to justify the expression $\hat{\tau} = \tau[\partial u(x,0)]/\partial y$, where $\tau(\kappa)$ is the shear viscosity function and κ is the shear.

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