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Stokes Flow in a Driven Sector by Two Different Methods

A biorthogonal series expansion and a numerical finite-difference approximation are applied to the problem of steady Stokes flow in a driven sector of 10° total angle, providing mutual support of the theoretical techniques. For this problem the method of biorthogonal series is faster, cheaper, and more accurate.

Introduction

In this paper we model the Stokes flow in a long driven sector, using finite differences and a biorthogonal series expansion to compare the results. The problem is chosen from a modified Couette flow including a sector cavity [1]. Our aim is to examine closely the results of the approximate finite difference solution and to advertise the biorthogonal series for solving biharmonic boundary-value problems in domains where separation of variables is possible (a very common problem in fluid mechanics and elasticity). The analytic method is elucidated in [2, 3]. New aspects concerning the computation are developed here.

Mathematical Formulation

The slow motion of a Newtonian liquid, neglecting gravity (Stokes flow) for two-dimensional flow is described by

$$\nabla^4\Psi = 0 \tag{1}$$

where Ψ is the stream function and ∇^2 is the Laplacian operator. Using polar coordinates (r, φ, z),

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}$$

and the velocity $\mathbf{v} = \operatorname{rot}(\Psi\mathbf{e}_z)$. In our model there shall be viscous nonslip at the solid walls $\varphi = \pm\beta$ and $r = r_0$. At the outer radius $r = r_1 = 1$ we prescribe the vorticity $\Omega = 1 - \sin^3(\pi\varphi/2\beta)$, where $\Omega = -\nabla^2\Psi$, and no flow through the surface shall be possible (Fig. 1). For our comparison we chose $r_0 = 0.05$ and $2\beta = 10°$.

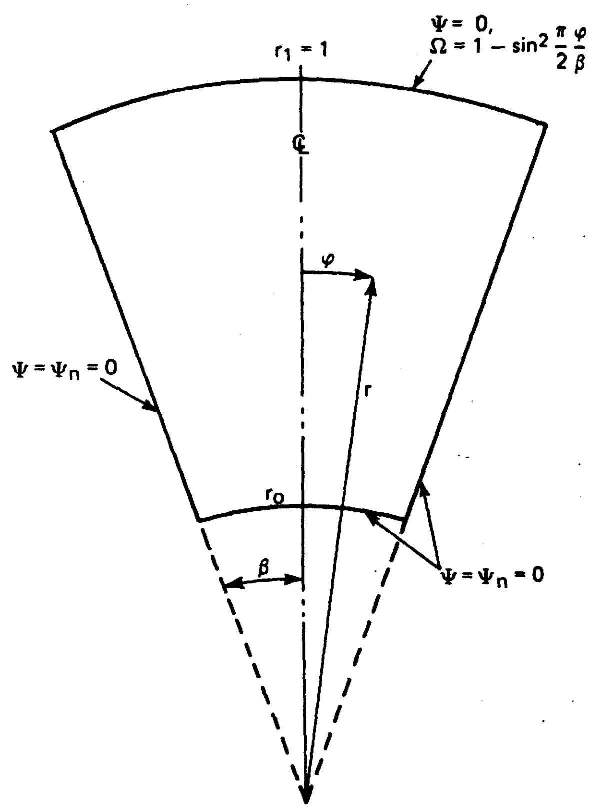

Fig. 1 The biharmonic sector problem

The Series Solution

The theory of biorthogonal series for biharmonic functions as described in [2, 3] allows us to write the solution of (1) in the form

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Table 1 The first five eigenvalues (note that $\lambda_{-n} = \bar{\lambda}_n$ where overbar denotes complex conjugate)

n	λ_n
1	25.14114414 + 12.86408537i
2	62.38088865 + 17.74998684i
3	98.82482881 + 20.31681729i
4	135.06392018 + 22.08005326i
5	171.21595479 + 23.42596613i

$$\Psi = \sum_{-\infty}^{\infty} \left(C_n r^{\lambda_n} + D_n r^{-\lambda_n+2} \right) \frac{\phi_1^{(n)}(\varphi)}{\lambda_n(\lambda_n - 2)} \tag{2}$$

where $\phi_1^{(n)}(\varphi) = \cos(\lambda_n - 2)\beta \cos \lambda_n \varphi - \cos \lambda_n \beta \cos(\lambda_n - 2)\varphi$, the λ_n are roots of $\sin[2\beta(\lambda_n - 1)] + (\lambda_n - 1)\sin 2\beta = 0$ (see Table 1) and $C_0 = D_0 = 0$. The boundary conditions at $\varphi = \pm\beta$ are already satisfied, so that the constants C_n and D_n will have to match the conditions at the inner and outer radius.

We introduce the biorthogonal sequence $\boldsymbol{\phi}^{(n)}, \boldsymbol{\psi}^{(n)}$, where

$$\boldsymbol{\phi}^{(n)} = \left(\phi_1^{(n)}, \phi_2^{(n)} \right) \text{ with } \phi_2^{(n)} \equiv \phi_1^{(n)''}(\varphi)/\lambda_n(\lambda_n - 2),$$

corresponding adjoint $\boldsymbol{\psi}^{(n)}$ with

$$\psi_1^{(n)} \equiv \frac{(\lambda_n - 2)}{\lambda_n} \cos(\lambda_n - 2)\beta \cos \lambda_n \varphi - \frac{\lambda_n}{\lambda_n - 2} \cdot \cos \lambda_n \beta \cos(\lambda_n - 2)\varphi; \; \psi_2^{(n)} \equiv \phi_1^{(n)}$$

such that

$$\langle \boldsymbol{\psi}^{(n)T} A \boldsymbol{\phi}^{(m)} \rangle \equiv \int_{-\beta}^{\beta} \boldsymbol{\psi}^{(n)T} A \boldsymbol{\phi}^{(m)} \, d\varphi = 0 \quad \text{for} \quad (\lambda_n - 1)^2 \neq (\lambda_m - 1)^2$$
$$= F_n \quad \text{for} \quad (\lambda_n - 1)^2 = (\lambda_m - 1)^2 \tag{3}$$

and the biorthogonality matrix

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}.$$

The "Fourier" coefficients C_n and D_n are determined by the biorthogonality condition

$$\left\langle \boldsymbol{\psi}^{(n)T} A \begin{pmatrix} \Psi_{rr} + \dfrac{1}{r}\Psi_r \\ \Psi_{\varphi\varphi} \end{pmatrix} \right\rangle \Bigg|_{r=r_1} = - \left\langle \boldsymbol{\psi}^{(n)T} A \begin{pmatrix} 1 - \sin^2\left(\dfrac{\pi\varphi}{2\beta}\right) \\ 0 \end{pmatrix} \right\rangle \tag{4a}$$

and

$$\left\langle \boldsymbol{\psi}^{(n)T} A \begin{pmatrix} \Psi_r \\ \Psi_{\varphi\varphi} \end{pmatrix} \right\rangle \Bigg|_{r=r_0} = 0. \tag{4b}$$

Further details of the theory can be found in [2]. We solve the linear system (4) by truncation, i.e., replace the "∞" sign in (2) by a finite number. At this point it is interesting to look at equations (4) in detail,

$$(C_n + D_n)F_n + \sum_{m=-\infty}^{\infty} \left(\frac{2}{\lambda_m - 2} C_m - \frac{2}{\lambda_m} D_m \right) \langle \phi_1^{(n)} \phi_1^{(m)} \rangle = - \left\langle \boldsymbol{\psi}^{(n)T} A \begin{pmatrix} 1 - \sin^2 \dfrac{\pi\varphi}{2\beta} \\ 0 \end{pmatrix} \right\rangle \tag{5a}$$

where

$$F_n = 4\left(\frac{\beta \cos^2 \lambda_n \beta}{\lambda_n} - \frac{\beta \cos^2(\lambda_n - 2)\beta}{(\lambda_n - 2)} - \frac{1}{\lambda_n(\lambda_n - 2)} \sin 2\beta \cos \lambda_n \beta \cos(\lambda_n - 2)\beta \right),$$

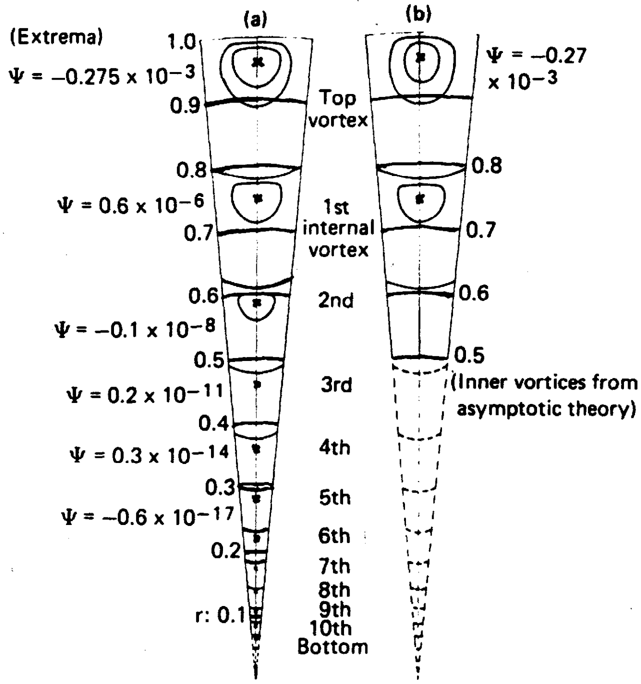

Fig. 2 Sector solutions (streamlines); extrema at vortex centers, X; (a) biorthogonal series solution; (b) approximate solution

Table 2 Coefficients in the biorthogonal series (scientific notation: the second number is the power of ten)

	Real	Imaginary
C_1 =	(0.208369 + 01,	−0.219573 + 00)
C_2 =	(−0.266598 + 00,	0.440204 + 00)
C_3 =	(−0.851842 − 01,	0.559270 − 01)
C_4 =	(−0.332293 − 01,	0.152098 − 01)
C_5 =	(−0.162217 − 01,	0.596175 − 02)
D_1 =	(−0.201026 − 62,	0.686494 − 62)
D_2 =	(0.760589 − 112,	0.229382 − 111)
D_3 =	(0.459605 − 159,	0.613469 − 159)
D_4 =	(−0.258214 − 206,	0.221233 − 206)
D_5 =	(0.697736 − 254,	0.252238 − 254)

and

$$\sum_{m=-\infty}^{\infty} r_0^{\lambda_m} C_m \left(2\langle \phi_1^{(n)} \phi_2^{(m)} \rangle + \frac{\langle \phi_1^{(n)} \phi_1^{(m)} \rangle}{r_0(\lambda_m - 2)} - \langle \psi_1^{(n)} \phi_2^{(m)} \rangle \right) + r_0^{-\lambda_m} D_m \left(2r_0^2 \langle \phi_1^{(n)} \phi_2^{(m)} \rangle - \frac{r_0}{\lambda_m} \langle \phi_1^{(n)} \phi_1^{(m)} \rangle - r_0^2 \langle \psi_1^{(m)} \phi_1^{(m)} \rangle \right) = 0. \tag{5b}$$

For the chosen β, the real parts of the eigenvalues λ_n are very large (Table 1), so that for $r_0 = 0.05$ the coefficients in (5b) suggest that the C_m are large compared with the D_m. Therefore (5a) or (4a) can be solved for the C_m, neglecting the D_m, which then can be easily found from (5b) or (4b) (or find $\tilde{D}_m = r_0^{-\lambda_m} D_m$). Thus the system (4) or (5) is split into two systems that can be solved consecutively. This reflects the fact that the boundary condition at r_0 does not have any significant influence on the flow, except very close to r_0 where the $D_n \cdot r^{-\lambda_n+2}$ term in (2) is dominating (even when the D_n are small). Note that the $D_n = 0$ for $r_0 = 0$.

Result. Sufficient accuracy of the truncated series can be obtained for five terms in the series. The coefficients C_n and D_n converge rapidly as n increases; see Table 2. The residual error in the boundary

Table 3 Boundary values at the inner and outer radius; a_1, b_1, c_1, and d_1 are the indicated values at the boundary; they are compared to: a_2 the prescribed vorticity at $r = 1$, $b_2 = c_2$ the zero stream function values and d_2 the zero gradient value at $r = 0.05$

φ	a_1 $\Omega(r = 1)$	a_2 $1 - \sin^2 \varphi\pi/10°$	b_1 $\Psi(r = 1)$	b_2 $\Psi(r = 1)$	c_1 $\Psi(r = 0.05)$	c_2 $\Psi(r = 0.05)$	d_1 $\partial\Psi/\partial r$ $(r = 0.05)$	d_2 $\partial\Psi/\partial r$ $(r = 0.05)$
0°	0.9990	1	$-0.659 \cdot 10^{-7}$	0	$0.141 \cdot 10^{-39}$	0	$0.231 \cdot 10^{-35}$	0
1°	0.9056	0.9045	$0.552 \cdot 10^{-7}$	0	$-0.149 \cdot 10^{-39}$	0	$-0.220 \cdot 10^{-35}$	0
2°	0.6530	0.6545	$0.204 \cdot 10^{-7}$	0	$0.156 \cdot 10^{-39}$	0	$0.182 \cdot 10^{-35}$	0
3°	0.3481	0.3455	$-0.452 \cdot 10^{-7}$	0	$-0.115 \cdot 10^{-39}$	0	$-0.103 \cdot 10^{-35}$	0
4°	0.0910	0.0955	$0.129 \cdot 10^{-6}$	0	$-0.337 \cdot 10^{-40}$	0	$-0.757 \cdot 10^{-36}$	0
5°	−0.0052	0	0	0	0	0	$0.407 \cdot 10^{-46}$	0

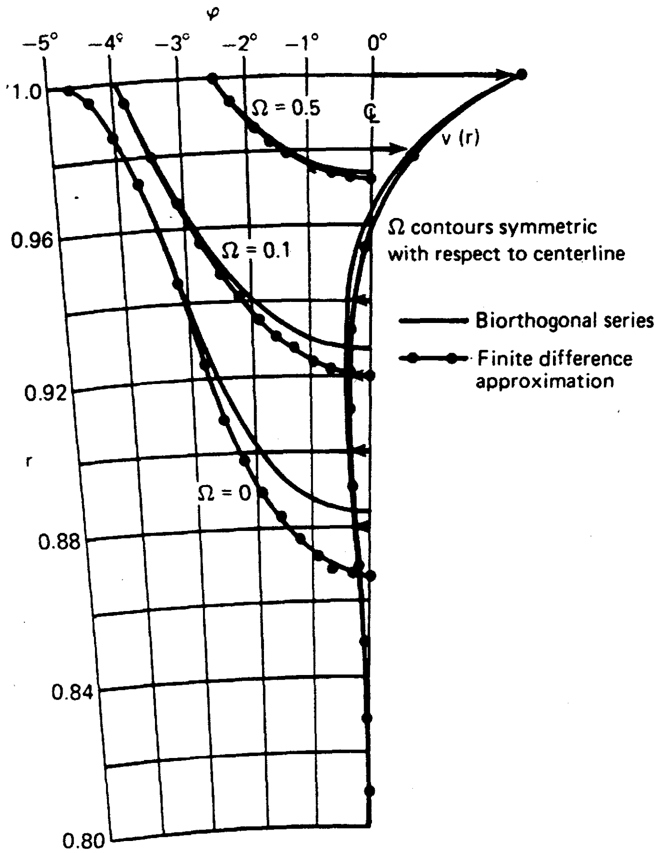

Fig. 3 Comparisons of the velocity and vorticity in the outer region; —— biorthogonal series; - - -0- - - approximate solution

conditions is insignificant; see Table 3. The Ψ boundary conditions are satisfied exactly on the sidewalls, $\beta = \pm 5°$. The stream function Ψ (Fig. 2(*a*)) and vorticity (Fig. 3) show details of the solution.

The Numerical Solution

Now the same problem is solved numerically using finite differences. A successive over-relaxation method is used, alternating between Ψ and Ω with a fixed relaxation factor for each as described in [4]. The relaxation factors were not optimized. In order to work in a rectangular plane a new radial coordinate $\eta = \ln r$ is introduced. Compromising between the desired accuracy and the cost of the computations, we use meshes of $h_\eta = 0.023404$ and $h_\varphi = 0.005454$.

Result. The stream function Ψ (Fig. 2(*b*)) and the vorticity Ω are calculated until their residual values are less than 10^{-9} and 10^{-6}, respectively. Asymptotic theory,[1] utilizing the first eigenvalue (after Moffatt [5] with Burggraf correction [6]), is used to fill in the inner part of the sector where Ω residuals exceed the functional values. Results are shown in Fig. 2(*b*) and Fig. 3.

Comparison

The profiles of the center-line velocity are compared in Fig. 3. The velocity at the center of the outside arc is 0.0201 for the approximate numerical solution and 0.0196 in the analytical result.

It is obvious that the result of the biorthogonal series solution is more accurate and because of the easy, straightforward computation its use should be preferred for similar problems. The computation of the numerical solution was carried out on an IBM 360/91 requiring about 66 sec of computing time compared to only fractions of a second for the series (on a Cyber 74). However, this test has shown that the numerical results may be good enough for many applications (within the two top vortices, where the liquid flows fastest, the streamline error lies within the mesh length) and the method can be applied to more general, nonseparable domains. The truncation error can be reduced by a finer mesh computation.

References

1 O'Brien, V., "Steady Circular Shear Flow Over a Sector Cavity," (submitted elsewhere).

2 Liu, C. H., and Joseph, D. D., "Stokes Flow in Wedge-Shaped Trenches," *Journal of Fluid Mechanics*, Vol. 80, Part 3, 1977, pp. 443–463.

3 Joseph, D. D., "A New Separation of Variables Theory for Problems of Stokes Flow and Elasticity," (Sept. 1977), to appear.

4 Erhlich, L. W., "Solving the Biharmonic Equation as Coupled Finite-Difference Equations," *SIAM Journal of Num. Anal.*, Vol. 8, 1971, pp. 278–287.

5 Moffatt, H. K., "Viscous and Resistive Eddies Near a Sharp Corner," *Journal of Fluid Mechanics*, Vol. 18, Part 1, 1964, pp. 1–18.

6 Burggraf, O. R., "Analytical and Numerical Studies of the Structure of Steady Separated Flows," *Journal of Fluid Mechanics*, Vol. 24, Part 1, 1966, pp. 113–151.

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[1] Neglecting the boundary condition at r_0 which cannot be satisfied.