# DIRECT AND REPEATED BIFURCATION INTO TURBULENCE

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This lecture is a review of the applications of the theory of bifurcation to the problem of transition to turbulence. Most of the material in this lecture can be found in detail in my recent review [11], in other reviews in the same volume and in the monographs [12]. We shall discuss some new results having to do with frequency-locked solutions and bifurcation into higher dimensional tori in the transition to turbulence which were not discussed in [11] and [12]. Some of these results are derived in the new book on bifurcation theory by Iooss and Joseph [9]. To keep the lecture and this written report of it discursive, I am not going to do much citing and attributing of old results; complete citations for the older work can be found in [11] and [12].

I will confine my remarks to a discussion of the bifurcation of solutions of the Navier-Stokes equations for an incompressible fluid when the velocity $\underline{V}_B(\underline{x},t)$ of the boundary B of the region $\mathcal{V}$ occupied by the fluid is prescribed together with field forces $\underline{G}(\underline{x},t)$:

$$\left.\begin{aligned}\frac{\partial \underline{V}}{\partial t} + \underline{V}\cdot\nabla\underline{V} &= -\nabla p + \frac{1}{R}\nabla^2\underline{V} + \underline{G}(\underline{x},t),\\ \operatorname{div}\underline{V} &= 0\ ,\end{aligned}\right\}\ \underline{x}\in\mathcal{V}$$

$$\underline{V} = \underline{V}_B(\underline{x},t),\quad \underline{x}\in B\ .$$

We call the prescribed values $\underline{V}_B(\underline{x},t)$ and $\underline{G}(\underline{x},t)$, the data. R is the Reynolds number, a dimensionless parameter composed of the product of a velocity times a length divided by the kinematic viscosity. We can think of it as a dimensionless speed.

The motion of the fluid must ultimately be determined by the data. When the Reynolds number is small the motion is uniquely determined by the data. The meaning of this is as follows: given an initial condition

(1.2) $$\underline{V}_0(\underline{x}) = \underline{V}(\underline{x},0),\ \underline{x}\in\mathcal{V}$$

we may suppose that the initial-boundary-value problem (1.1) and (1.2) have a unique solution. When R is small, each of these different solutions belonging to different $\underline{V}_0$, tend to a single one determined ultimately by $\underline{G}$ and $\underline{V}_B$ and not by $\underline{V}_0$. So when R small we ultimately get solutions which reproduce the symmetries of the data. Steady data gives rise to steady solutions, periodic data to periodic solutions.

When R is large solutions are not uniquely determined by the data. The relation between the data and solutions is subtle and elusive.

Let us consider what happens when the data is steady as we increase the Reynolds number. For technical reasons we suppose now and hereafter that $V$ is a bounded domain, or it can be made bounded by devices such as restricting solutions to spatially periodic ones which can be confined to a period cell. We suppose $U(R)$ is a steady solution which is the continuation of the unique steady solution which exists when $R$ is small. When the equation satisfied by this solution is subtracted from Navier-Stokes equation, we get equations for the disturbance $\underline{u}$ of $U(R)$ which has zero data and nonzero initial conditions

$$\frac{\partial \underline{u}}{\partial t} + \underline{U}\cdot\nabla\underline{u} + \underline{u}\cdot\nabla\underline{U} + \underline{u}\cdot\nabla\underline{u} = -\nabla p + \frac{1}{R}\nabla^2\underline{u} \quad \text{in } V$$

(1.3)
$$\operatorname{div} \underline{u} = 0$$
$$\underline{u} = 0, \ \underline{x} \in B$$

(1.4)
$$\underline{u}(\underline{x},0) \neq 0, \ \underline{x} \in V$$

If the null solution $\underline{u} = 0$ of (1.3) is stable, then $\underline{U}(R)$ is stable. It is stable when $R$ is small. We want to catalogue what can happen when $\underline{u} = 0$ loses stability as $R$ is increased.

For simplicity we first write (1.3) as an evolution equation in some space, say a Banach space

(1.5)
$$\frac{d\underline{u}}{dt} = \underline{F}(R,\underline{u}), \quad \underline{F}(R,0) = 0 .$$

It does no harm to think of (1.5) as a system of ordinary differential equations in $\mathbb{R}^n$.

To study the stability of $\underline{u} = 0$ we linearize (1.5) and introduce exponential solutions in order to derive the associated spectral problem:

$$\frac{d\underline{v}}{dt} = \underline{F}_u(R|\underline{v}) ,$$
$$\underline{v} = e^{\sigma t}\underline{\zeta} ,$$
$$\sigma = \xi(R) + i\eta(R) \in \textstyle\sum \underline{F}_u(R|\cdot) ,$$
$$\sigma\underline{\zeta} = \underline{F}_u(R|\underline{\zeta})$$

where $\sum \underline{F}_u$ means the spectrum of $\underline{F}_u$. If $\sigma$ is in the spectrum so is $\bar{\sigma}$. When $V$ is a bounded domain the spectrum of $\underline{F}_u$ is all of eigenvalues and when $R$ is small all of the eigenvalues are bounded by a parabola on the left hand side of the complex $\sigma$ plane. As $R$ is increased past its first critical value some eigenvalues cross into the right side of the complex $\sigma$ plane. In the usual case a single eigenvalue or a complex conjugate pair of eigenvalues cross over.

We state the foregoing conditions, which are sufficient for bifurcation, in a precise mathematical sense as follows. $R = R_c$ is the first critical value of $R$ such that $\xi(R) < 0$ for all eigenvalues belonging to $\underline{F}_u(R|\cdot)$ when $R < R_c$, $\xi(R_c) = 0$, $\sigma(R_c) = i\omega_0$ (where $\omega_0 = \eta(R_c)$) is an algebraically simple eigenvalue of $\underline{F}_u(R_c|\cdot)$ and the loss of stability of $\underline{u} = 0$ at $R_c$ is strict; that is,

$$\xi'(R_c) > 0 .$$

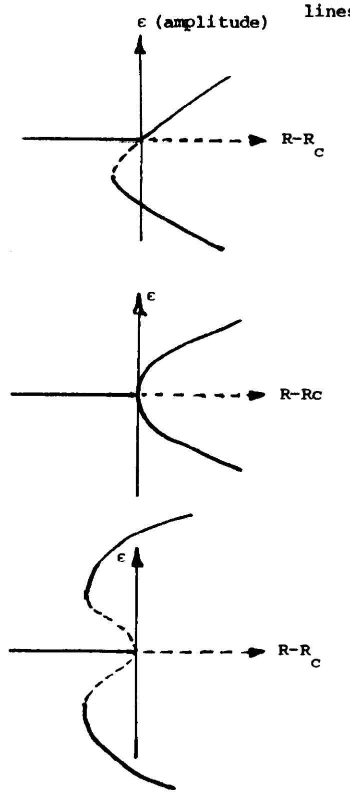


Given the assumptions made in the last paragraph there are two possibilities:

(I) $\omega_0 = 0$ and one real eigenvalue crosses at critically. A steady solution which breaks the spatial symmetry of the data, bifurcates.

It is usually enough to consider three possible types of bifurcation into steady solutions (see Figure 1). Transcritical bifurcation occurs when the projection of the quadratic part of the nonlinear terms into the null space of $F_u(R_c|\cdot)$ is nonvanishing. When this projection does vanish bifurcation is controlled by cubic terms. When these terms don't vanish there are two possibilities: bifurcation to the right (supercritical) and bifurcation to the left (subcritical). Solutions which bifurcate supercritically are stable; subcritical solutions are unstable

Figure 1: Bifurcation of Steady Solutions. Dotted lines indicate unstable solution.




(I) A real eigenvalue crosses at criticality and the projection of quadratic terms does not vanish. The bifurcation is transcritical.

The quadratic terms vanish, the cubic terms do not, and the bifurcation is supercritical.

Cubic terms don't vanish. Steady solutions bifurcate subcritically.

(II) A complex pair crosses. The quadratic projection vanishes automatically and we never get the transcritical case

$$R(\varepsilon) = R(-\varepsilon)$$
$$\omega(\varepsilon) = \omega(-\varepsilon)$$

As $R$ is increased, new steady solutions, with different patterns of symmetry, may bifurcate. After some number of these steady bifurcations a periodic solution will typically bifurcate.

Now we ask what happens when a periodic solution bifurcates? Suppose we have a stable periodic solution with velocity given by

$$\underline{V}(\omega(\varepsilon)(t+\delta_1),\varepsilon)$$

where $\varepsilon$ is the amplitude, $\omega(\varepsilon)=\omega(-\varepsilon)$ and $\delta_1$ is an arbitrary phase which may be set to zero by a suitable choice of the origin of time. A small disturbance $\underline{q}$ of $\underline{V}$ satisfies the linearized equation

$$\frac{d\underline{q}}{dt} = \underline{F}_u(R(\varepsilon),\underline{V}(\omega(\varepsilon)t,\varepsilon)|\underline{q}) .$$

We can derive a spectral problem for $\underline{F}_u(R(\varepsilon),\underline{V}(\omega(\varepsilon)t,\varepsilon)|\cdot)$ by the method of Floquet.

$$q = e^{\sigma t}\zeta(t),\ \underline{\zeta}\in \mathbb{P}_{\frac{2\pi}{\omega}}$$

$$\sigma\zeta = -\frac{d\zeta}{dt} + \underline{F}_u(R,\underline{V}|\underline{\zeta}) \stackrel{def}{=} \Pi\zeta$$

$$\text{dom } \Pi = \mathbb{P}_{\frac{2\pi}{\omega}}$$

where

$$\sigma = \xi(\varepsilon) + i\Omega(\varepsilon)\in \textstyle\sum\Pi$$

and

$$\lambda = e^{2\pi\sigma/\omega}$$

are the Floquet exponent and multiplier and $\mathbb{P}_{2\pi/\omega}$ is the space of $2\pi/\omega$ periodic functions.

We now suppose that the periodic solution $\underline{u}$ loses stability strictly when $\varepsilon=\varepsilon_1$, $R=R(\varepsilon_1)=R_1$; that is,

$$\xi(\varepsilon_1) = \xi_1 = 0,\ \xi'(\varepsilon_1) > 0$$

and further,

$$\sigma(\varepsilon_1) = \sigma_1 = i\Omega_1, \Omega_1 = \Omega(\varepsilon_1)$$

is a simple eigenvalue of the operator $\Pi$, where all the other eigenvalues of $\Pi$ have negative real parts $(\xi < 0)$. The critical multipliers

$$\lambda_1 = e^{2\pi i\Omega_1/\omega(\varepsilon_1)}$$

are on the unit disk and all the other multipliers are inside the unit disk (see Figure 2).

We can correlate the type of bifurcation with the properties of the multipliers which pass out of the unit disk at criticality. These properties are determined by the values of the frequency ratio $\Omega_1/\omega_1$. There are two possibilities: $\Omega_1/\omega_1$ is irrational or $\Omega_1/\omega_1$ is rational. We get all the rational points on the Floquet circle shown in Figure 3 if we take

$$\frac{\Omega_1}{\omega_1} = \frac{m}{n},\ 0 \le \frac{m}{n} < 1 .$$

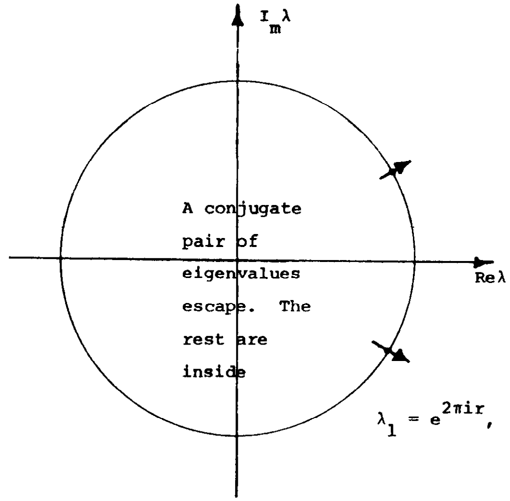


The rational points are called points of resonance and the irrational points are called quasiperiodic points. The resonant points are roots of unity, $\lambda_1^n = 1$. We further divide the resonant points into

(i) points of strong resonance: $n = 1,2,3,4$ and,

(ii) points of weak resonance: $n > 4$

Under the assumptions we have made we get bifurcation of periodic solutions. At points of strong resonance a subharmonic solution (a periodic solution) with a new period $2\pi/\Omega(\varepsilon)$. $(\Omega(\varepsilon_1) = \Omega_1)$ approximately $n(n = 1,2,3,4)$ times the old




$$\lambda_1 = e^{2\pi i r},\quad r = \Omega_1/\omega_1,\quad 0 \le r < 1$$

Figure 2: Floquet circle at criticality. $r$ is irrational at quasiperiodic points. The resonant points are fractions

$$r = \frac{m}{n} < 1,\ \lambda_1^n = 1$$

At points of strong resonance $(n = 1,2,3,4)$ we get subharmonic solutions with periods nearly $n$ times $2\pi/\omega$. In the other cases a torus of asymptotically doubly periodic flows bifurcate.

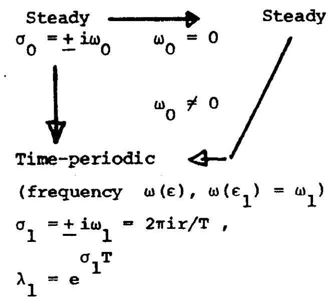
one, bifurcates when $\varepsilon$ is near to $\varepsilon_1$. If the period $2\pi/\omega$ of the old solution is independent of the amplitude as in the case of periodic forcing, the period of the new solution is also independent of the amplitude and the ratio $\Omega/\omega$ of frequencies is exactly $m/n$. For ease of description we will confine our remarks immediately below to the forced periodic case. The results for bifurcation of periodic solutions in autonomous problems are slightly different [9].

At quasiperiodic points we get bifurcation into asymptotically doubly-periodic solutions on a torus. The asymptotic expressions have two frequencies $\omega = 2\pi/T$ and $\Omega(\varepsilon)$ and $\Omega(\varepsilon)$ varies continuously, so that for almost all $\varepsilon$ in the range of these solutions the doubly periodic solution is quasiperiodic, with two frequencies, whilst for a dense set of rational points the two periods of the doubly periodic solution fit into a common period (see [9]).

Subharmonic solutions can bifurcate at points of weak resonance $(n \geq 5)$ only when exceptional conditions hold. In the usual case a torus of asymptotically quasi-periodic solutions bifurcates, even at points of weak resonance.

In Figures 3 and 4 we summarize the various possibilities for bifurcation which have been discussed so far.

Figure 3: Bifurcation of steady solutions into periodic ones and the bifurcation of periodic solutions




r is irrational. A torus $T^2$ bifurcates. Solutions on $T^2$ are asymptotically doubly-periodic with two frequencies.
r is rational, $r = m/n$, $\lambda_1^n = 1$. Case (i): $n = 1,2,3,4$, strong resonance. Subharmonic, nT-periodic solutions bifurcate. Case (ii): $n \geq 5$, weak resonance. A torus $T^2$ of asymptotically doubly periodic bifurcates. Case (iii): $n \geq 5$, weak resonance. Subharmonic solutions on $T^2$ will bifurcate if certain coefficients vanish.

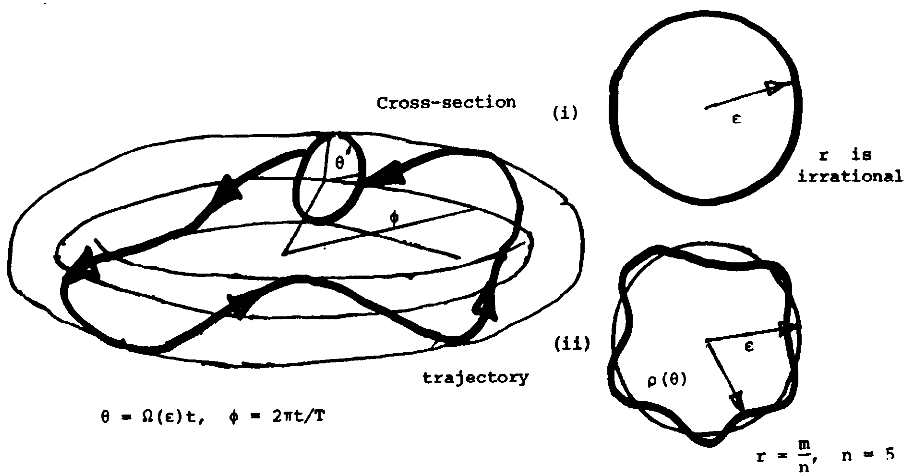
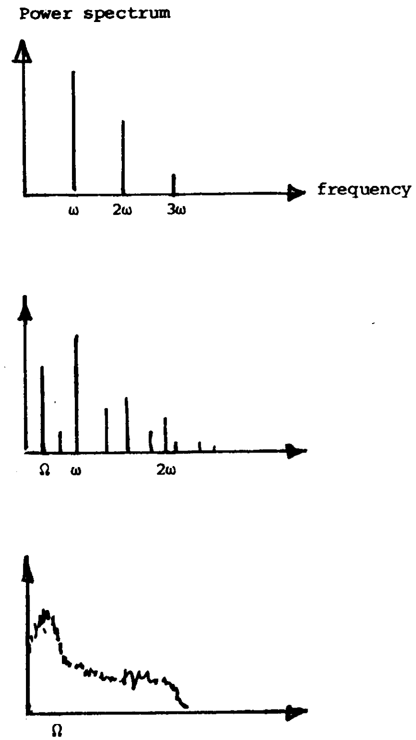





Figure 4: The bifurcating torus [9]. The amplitude $\varepsilon$ is the mean radius of the torus. The cross-section is circular when the Floquet multipliers escape at quasi-periodic points. At points of weak resonance $(n \geq 5)$ a torus, with n lobes bifurcates unless special conditions hold. Whenever there is a torus, the solutions on it are asymptotically doubly-periodic. When the special conditions hold for $n \geq 5$, subharmonic, nT-periodic solutions on the torus will bifurcate. When the Floquet multiplier crosses at a point of strong resonance, $n = 1,2,3,4$ an nT periodic solution, not on a torus, will bifurcate.

The circumstances under which the solutions on $T^2$ are exactly and not just asymptotically quasiperiodic are presently unknown. In our asymptotic result, and in the experiments, there are two frequencies $\omega(\varepsilon)$ and $\Omega(\varepsilon)$ which appear to vary smoothly with the amplitude $\varepsilon$, when $\varepsilon$ is not too large, and the asymptotic solutions are doubly-periodic, of the form $f(\omega t, \Omega t)$, where f is $2\pi$-periodic in both arguments. Higher dimensional tori $T^n$ may be associated with n frequencies in the same way.

A good way to determine the properties of solutions in experiments to get the Fourier transform of measured data, say the power spectrum of the fluctuating values of velocity at a point, see Figure 5. A periodic solution $T^1$ shows sharp peaks in the power spectrum and these peaks are harmonics of one frequency $\omega$. The power spectrum of a doubly periodic solution also has sharp peaks and all of them may be matched to linear combinations of two frequencies. And for multiperiodic solutions the same thing goes for n frequencies.




$T^1$: There is one frequency and harmonics.

$T^2$: All spectral lines are of the form $2\left(\frac{n}{\Omega}+\frac{m}{\omega}\right)$. If $\Omega/\omega$ is irrational the solution is quasiperiodic.

Nonperiodic (strange) attractor (dynamical noise) centered at $\Omega$.

Figure 5: Power spectrum for periodic, doubly periodic and nonperiodic solution.

As long as there is a periodic solution we can study its stability by Floquet theory. But the study of the stability and bifurcation of quasiperiodic solutions on a two-dimensional torus as R is increased is more complicated.

Three types of changes of $T^2$ are observed in experiments when the Reynolds number if increased.

(1) The solutions lock frequencies. A locked solution is a periodic solution on the torus in which the ratio of frequencies is rational. Locked in solutions are subharmonic; the time taken by N times one cycle is the same as M times the other cycle

$$\tau = \frac{2\pi N}{\omega} = \frac{2\pi M}{\omega}$$

The locked in solutions appear to be related to those which bifurcate at points of weak resonance. A locked in solution has the property that the ratio of frequencies on $T^2$ remains constant even when R varies.



(2) $T^2$ bifurcates directly into nonperiodic attractor. This is not well understood theoretically.

(3) $T^2$ bifurcates into $T^3$ (Gollub and Benson, unpublished). The bifurcation of $T^2$ into $T^3$ (and $T^n$ into $T^{n+1}$) was discussed by Landau and Hopf. Mathematical conditions which are sufficient to guarantee bifurcation of the torus $T^n$ into the torus $T^{n+1}$ have recently been given by Chenciner and Iooss [3], Haken [8] and Sell [18]. If the fluid systems satisfied the conditions set out by these authors we would get turbulence of the type proposed by Landau and Hopf.

According to Landau and Hopf we get turbulence by adding new frequencies through bifurcation as the Reynolds number is increased. With each frequency we have an associated arbitrary phase so the motion looks chaotic. In the triply periodic case we have a velocity at a point in the form

$$\underline{u}(t,\varepsilon) = \hat{\underline{u}}(\omega_1(\varepsilon)(t-\delta_1),\omega_2(\varepsilon)(t-\delta_2),\omega_3(\varepsilon)(t-\delta_3),\varepsilon)$$

where the $\omega_i$ are the frequencies and $\delta_i$ the phases. Turbulence then is always quasiperiodic with a finite number of discrete frequenceies.

Real turbulence is phase mixing, quasiperiodic turbulence is not phase mixing. If $\underline{u}(t)$ is a fluctuation with mean value zero and is almost periodic, then

$$\underline{u}(t) \sim \sum_{-\infty}^{\infty} \underline{u}_n e^{-i\lambda_n t}, \quad \lambda_n \neq 0 .$$

The autocorrelation for this is

$$g(\tau) = \lim_{T\to\infty} \frac{1}{T}\int_0^T \underline{u}(t+\tau)\underline{u}(t)\,dt = \sum_{-\infty}^{\infty} |\underline{u}_n|^2 e^{-i\lambda_n \tau}$$

and $g(\tau)$ does not vanish for solutions of the Landau-Hopf type, as it must for true turbulence. In true turbulence events at distant times are presumably uncorrelated. In some experiments [20] a noisy part of spectrum coexists with a peaked part. In these cases the autocorrelation function will decay as the noisy part of power spectrum grows larger, but it will not decay to zero.

Lorenz [14] and Ruelle-Takens [17] suggested that turbulence could occur after a finite number of bifurcations. Then there would be an attracting set of lower dimensionality in phase space in which solutions are:

(1) Sensitive to initial conditions. Two velocity fields which are initially close evolve into very different fields.

(2) Mixing, with a decaying autocorrelation function.

(3) Noisy with broad band components as well as sharp peaks in the spectrum.

Attracting sets of this type are sometimes called nonperiodic or strange.

Experiments favor Lorenz-Ruelle-Takens rather than Landau-Hopf. But it seems like there is no universal sequence of bifurcation into turbulence. We turn next to experiments.

In comparing bifurcation theory to experiments it is necessary to remember that bifurcation results are local and therefore do not cover all the possibilities in experiments. To make this point more strongly we consider the equilibrium solutions $F(R,\varepsilon) = 0$ of the evolution equation $\overset{\circ}{u} = F(R,u)$ in $\mathbb{R}$. We may imagine

$$F(R,u) = uF_1(R,u)F_2(R,u)\cdots F_n(R,u)$$

Each vanishing factor $F_\ell(R,\varepsilon) = 0$ gives a different solution as in Figure 6. Only the intersecting ones could in principle be studied by local bifurcation theory. Isolated solutions would escape analysis.

In experiments we frequently see the early sequence of bifurcating solutions predicted by bifurcation theory:

(1) Steady solutions bifurcate into time-periodic ones.

(2) Time-periodic ones bifurcate into subharmonic ones.

(3) Time-periodic ones bifurcate into doubly-periodic ones. But we also see the bifurcation of

(4) Steady solutions into nonperiodic ones as in the examples given below. This property holds for solutions of the Lorenz [14] equations as the Rayleigh number is increased.

(5) Doubly periodic ones $(T^2)$ into nonperiodic ones [1].

(6) Doubly periodic ones into triply periodic ones $(T^3)$ (Gollub and Benson, unpublished)

(7) Frequency locking followed by bifurcation into nonperiodic solutions [5], [6], [7], [13].

(8) Frequency locking followed by a cascade of repeated bifurcation of periodic doubling solution into turbulence, [13].

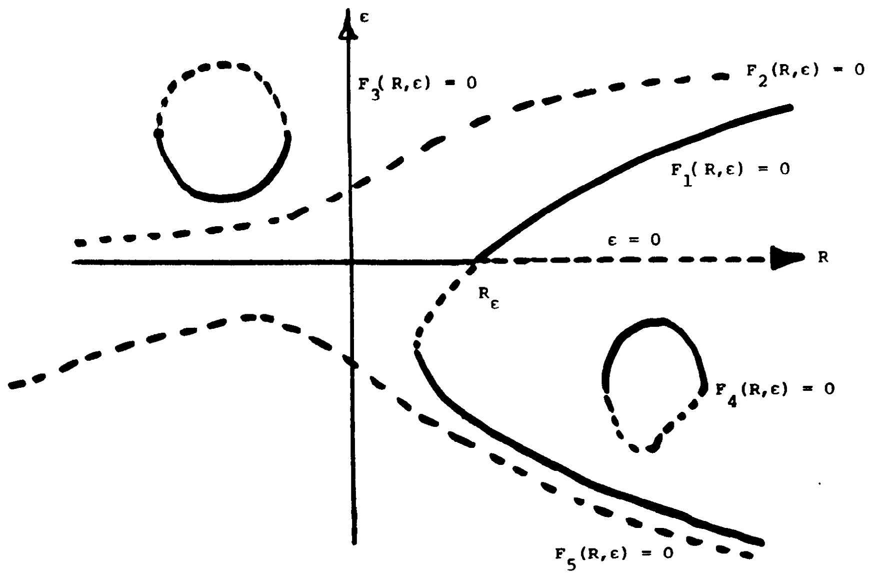



Figure 6: Bifurcation and stability of steady solutions $(R,\varepsilon)$ of $\overset{\circ}{u} = uF_1F_2F_3F_4F_5$



In convection experiments in boxes the sequence of observed bifurcating states is very dependent on the aspect ratio of the box and on the spatial form of the convection [15]. The observations are so varied and so recent that it is not possible or desirable to systematize them. Instead we can report on some interesting cases.

We start first by giving an account of some direct transitions to turbulence. Consider flow induced by a pressured drop $\Delta P$ down a plane channel. When $\Delta P$ is small the flow is laminar, and the velocity is unidirectional and varies across the cross-section like a parabola. At larger pressure drops there are alternate patches of laminar and turbulent flow, at still larger $\Delta P$ the flow is turbulent throughout. The bifurcation diagram of Figure 7 is for an idealized two-dimensional problem in which disturbances are assumed to be spatially periodic. The bifurcation is subcritical. On general theoretical grounds we expect the bifurcation diagram to recover stability when the amplitude is large (see "Factorization theorems" in [9]). In fact Orszag and Kelm [16] calculate a curve like shown in Figure 7. They integrate the initial-value problem by brute force, using interesting spectral methods. They show that the large amplitude branch which is stable for two dimensional disturbances is unstable to three dimensional disturbances. Their numerical results are in agreement with experimental observations.

Experiments in round pipes

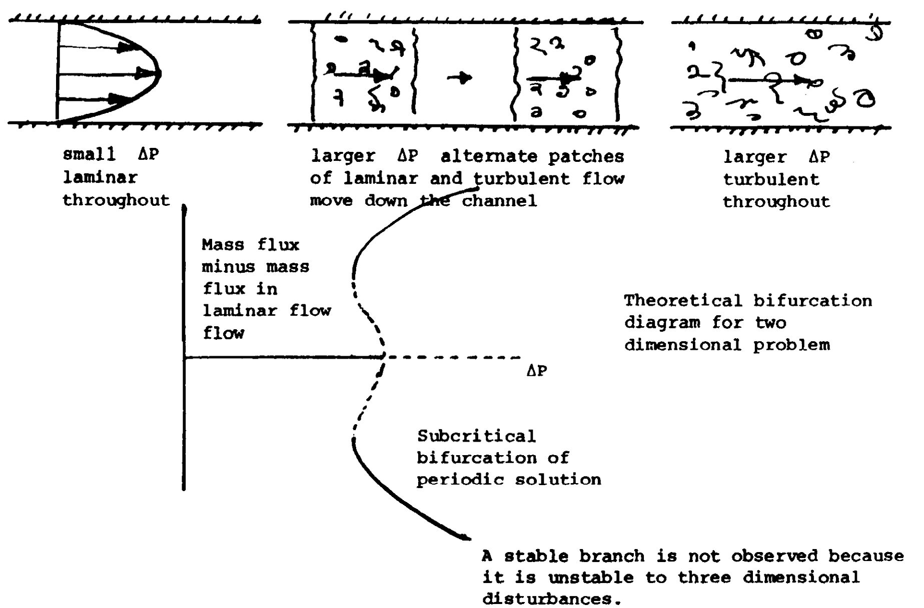
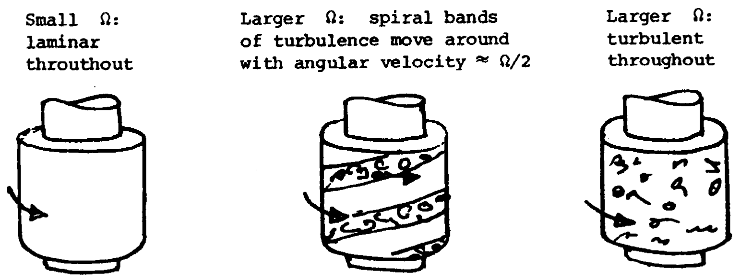



Figure 7: Direct bifurcation into turbulence in Poiseuille flow.

The same type of direct transition to turbulence occurs in Couette flow with the inner cylinder at rest and the outer in steady rotation with angular velocity $\Omega$

Figure 8: Spiral turbulence

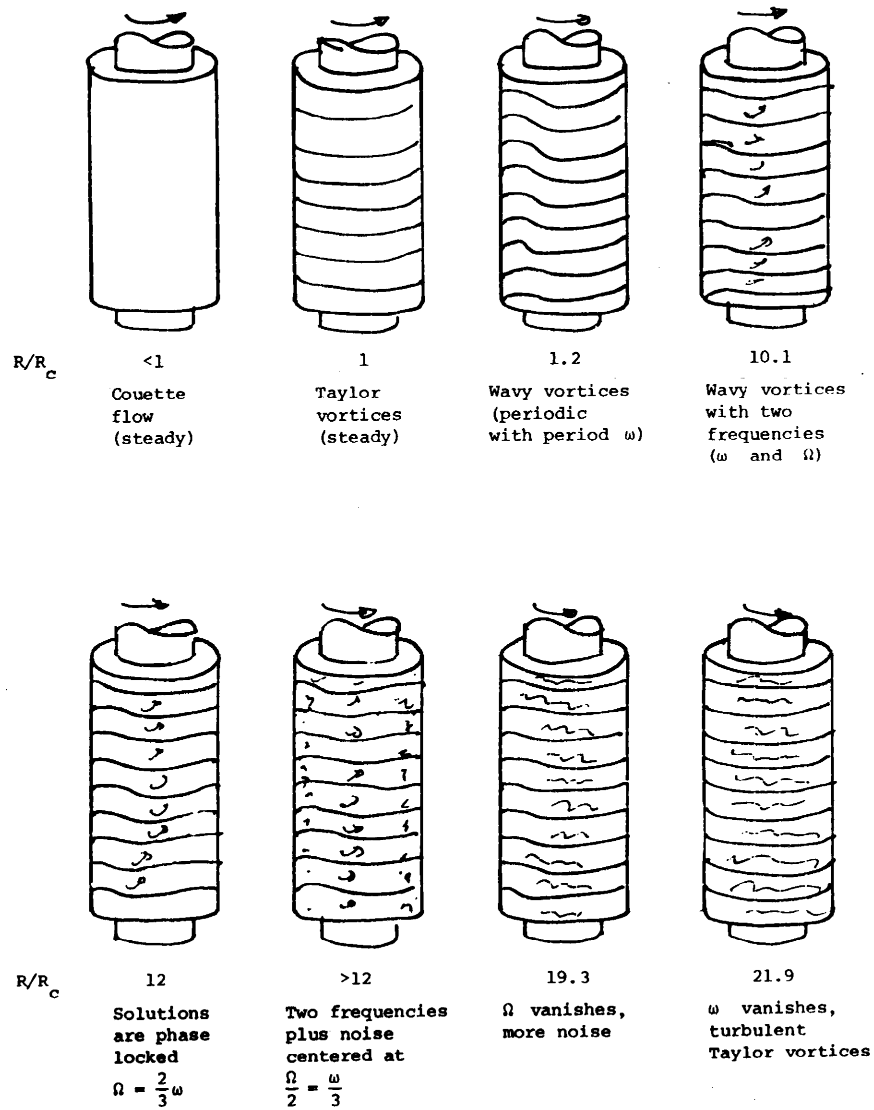



We turn next to experiments which exhibit repeated supercritical branching leading to turbulence after a finite number of bifurcations. The experiments which seem most interesting are in relatively small, enclosed volumes of fluid in which the eigenvalues in the spectrum of the linearized operator are widely separated. This separation seems to be associated with the fact that the dynamics of these fluid systems behave very nearly as if they were governed by ODE's in $\mathbb{R}^n$ with small values of n. This feature is of very great interest because it suggests that some features of turbulence are governed by a small number of ordinary differential equations. For example, the complicated sequence of bifurcations in a box of fluid heated from below [6] are well simulated in numerical solutions of 14 coupled ODE's [4] which arise by truncation into spatial Fourier modes with unknown time-dependent coefficients of solutions of the convection equations used by Lorenz [14].

The first experiments to report frequency data for different bifurcations leading to turbulence were done by Swinney-Gollub and the most recent and comprehensive report of developments coming from that work have been given by Fenstermacher, Swinney and Gollub. A summary of their observations are shown in Figure 9.



Figure 9: Couette flow, outer cylinder stationary, increasing $\Omega(R) \rightarrow$



Yavorskaya, Beleyaev, Monakov and Scherbakov [20] have carried out bifurcation experiments for the problem of flow between rotating spheres when the inner sphere rotates and the gap is wide. In Figure 10 I have sketched the frequency versus Reynolds number graph given as Figure 1 of their paper. They get their results by monitoring the fluctuating velocity at a point and they also measure the autocorrelation function.

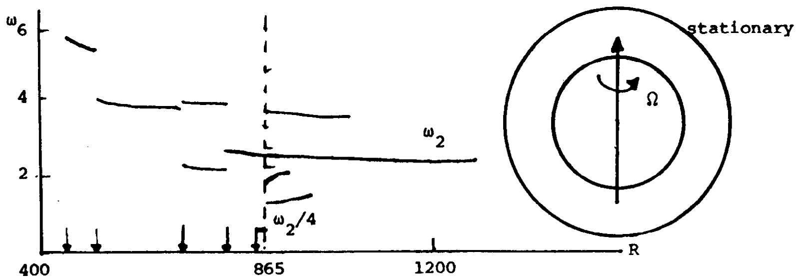



Figure 10: The flow between spheres is periodic when there is one frequency at a given R. The solution is doubly periodic when there are two frequencies present. Just before $R = 895$ where the autocorrelation function starts to decay there is $4(2\pi/\omega_2)$ subharmonic solution. The first decay of the autocorrelation at $R = 895$ is accompanied by the appearance of three new frequencies. The autocorrelation never does decay fully because the sharp spectral component coexists with dynamic noise for the range of R considered.

Gollub and Benson [7] and Maurer and Libchaber [15] have done many experiments on bifurcation of convection in box of fluid heated from below. In the French experiments with liquid helium a first frequency $\omega_1$ associated with oscillating rolls appears for a Rayleigh number around $2 \times 10^4$, then at about $2.7 \times 10^4$ a second frequency $\omega_0$, much smaller is observed, two frequency locking regimes are observed, with hysteresis, for frequency ratios $\omega_1/\omega_2 = 6.5$ and $\omega_1/\omega_2 = 7$. The transition to turbulence in the experiments of Libchaber and Maurer [13] is triggered by the generation of frequencies $\frac{\omega_2}{2}, \frac{\omega_2}{4}, \frac{\omega_2}{8}, \frac{\omega_2}{16} \rightarrow$ turbulence. A mathematical model for repeated 2T-periodic bifurcation into turbulence has been discussed by Tomita and Kai [19] and Ito [10].

The results reviewed in this lecture are astonishing in the sense that show that complicated hydrodynamical problems have dynamics which seem predictable from analysis of systems of nonlinear differential equations in $\mathbb{R}^n$, with small n. Concepts like frequency locking which have been well known to electrical engineers for many years, are now known to have an important connection to some types of turbulence. The idea of nonperiodic or strange attractors is a very major advance in the subject. On the other hand, experiments do not seem to suggest the transitor to turbulence can be characterized in any simple way.

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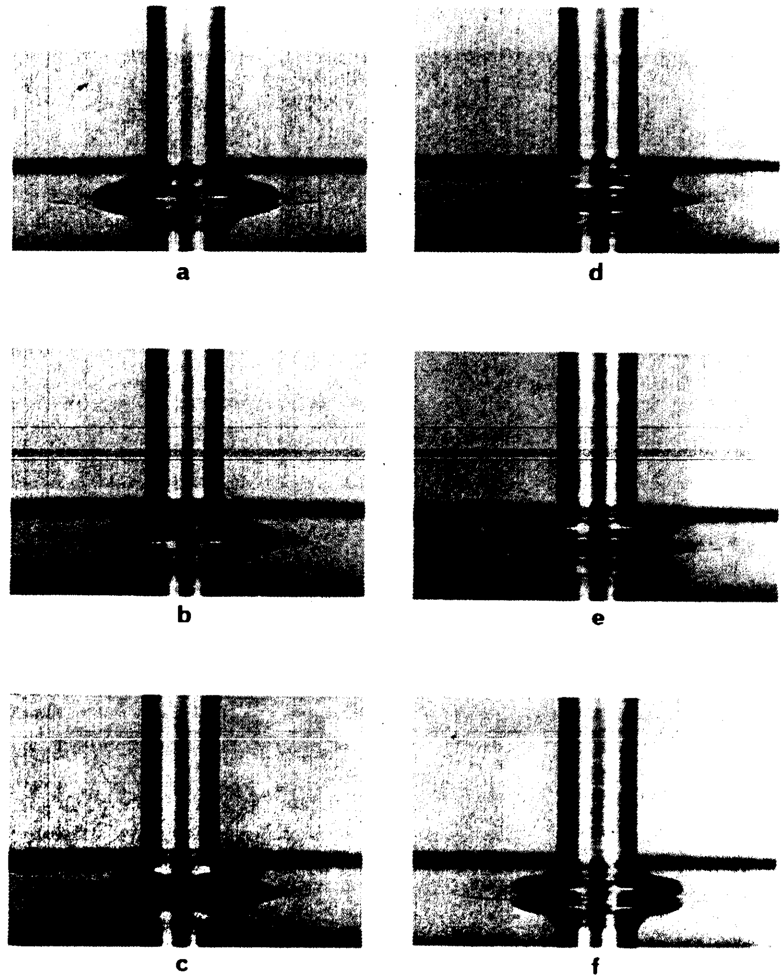
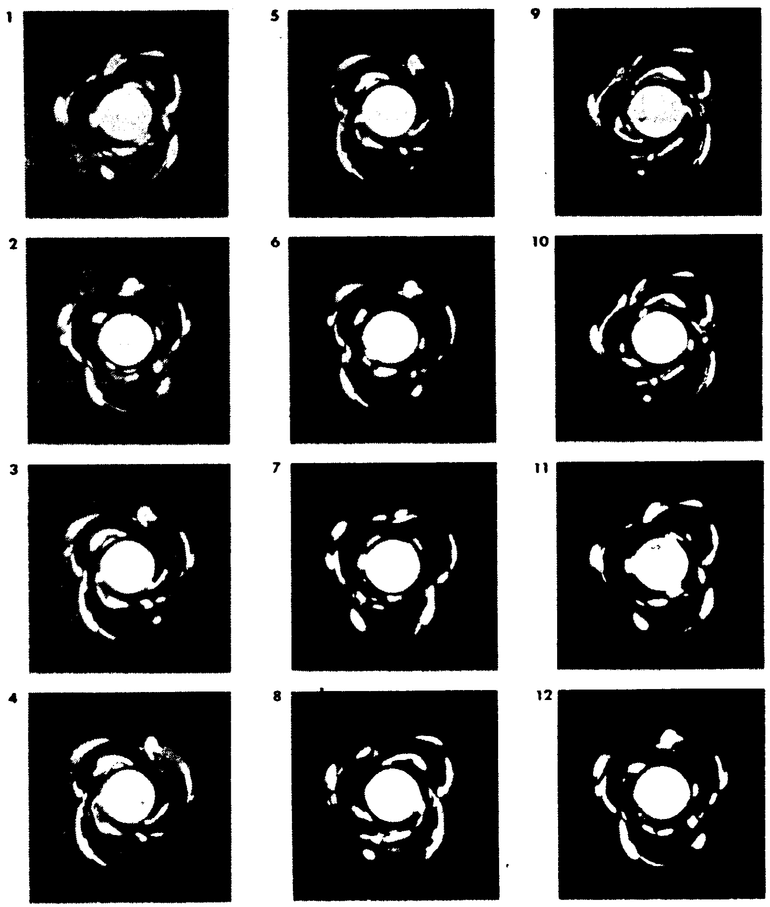



Time-periodic motion of the drop of climbing STP. Rod radius, 0.635; rotational speed, 13.3 rev $s^{-1}$; frequency of periodic motion, 0.4 cycles $s^{-1}$.

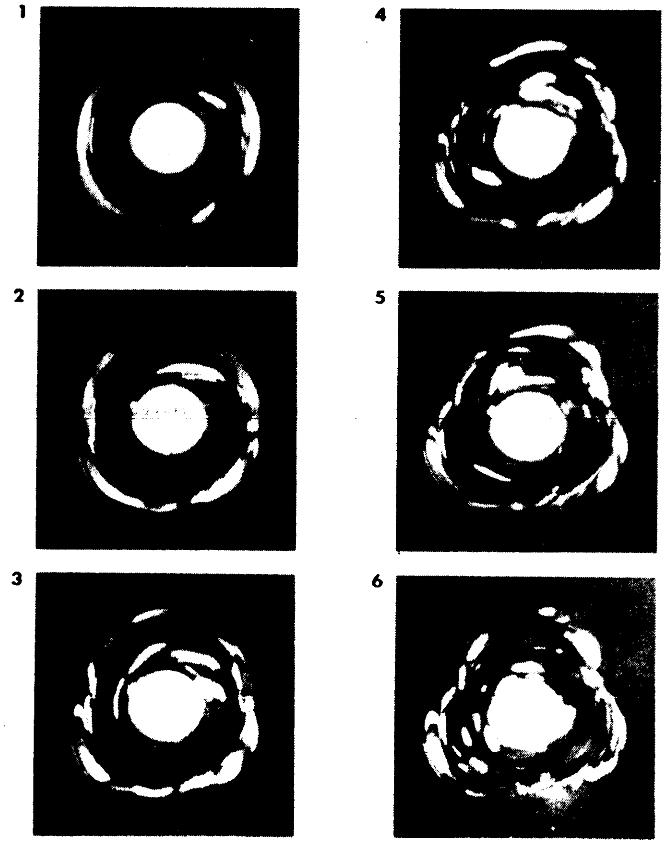
(From "The Rotating Rod Viscometer", by G. S. Beavers and D. D. Joseph, Journal of Fluid Mechanics, 69, 1975, pp. 475-511.)



The motion of the fully-developed three-lobe flower instability as the rod goes through one complete cycle. The time between frames is approximately 0.015 sec.

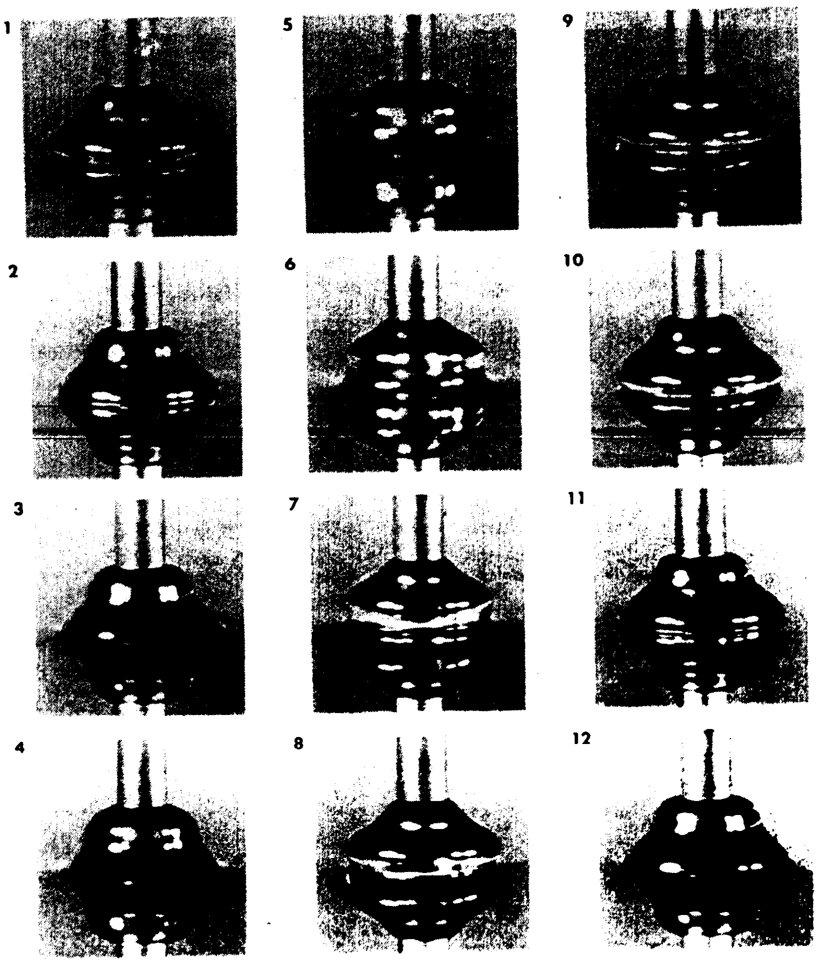
(From "Experiments on Free Surface Phenomena" by G. S. Beavers and D. D. Joseph, Journal of Non-Newtonian Fluid Mechanics, 5, 1979, pp. 323-352.)





The growth of the flower instability in TLA-227 as viewed from above. The angle of twist is 4 radians and the frequency of oscillation is 7 cycles per sec. The photographs show the rod at approximately the same position in its cycle. The time between photographs is approximately 1.1 sec.

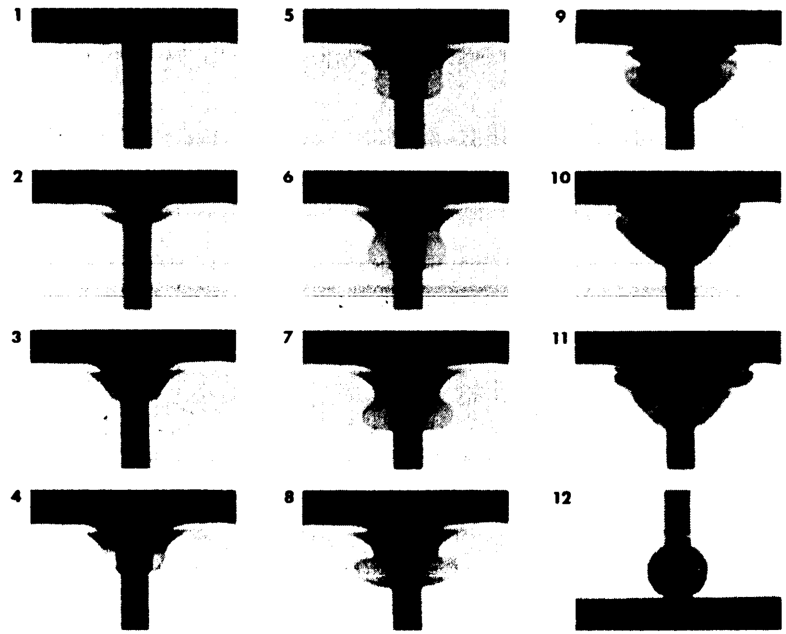
(From "Experiments on Free Surface Phenomena" by G. S. Beavers and D. D. Joseph, Journal of Non-Newtonian Fluid Mechanics, 5, 1979, pp. 323-352.)




The breathing instability of a bubble of STP on a rod of radius 0.476 cm rotating at 19 rev $s^{-1}$. Frames 1-10 represent one complete cycle. Time between frames = 0.15 sec.

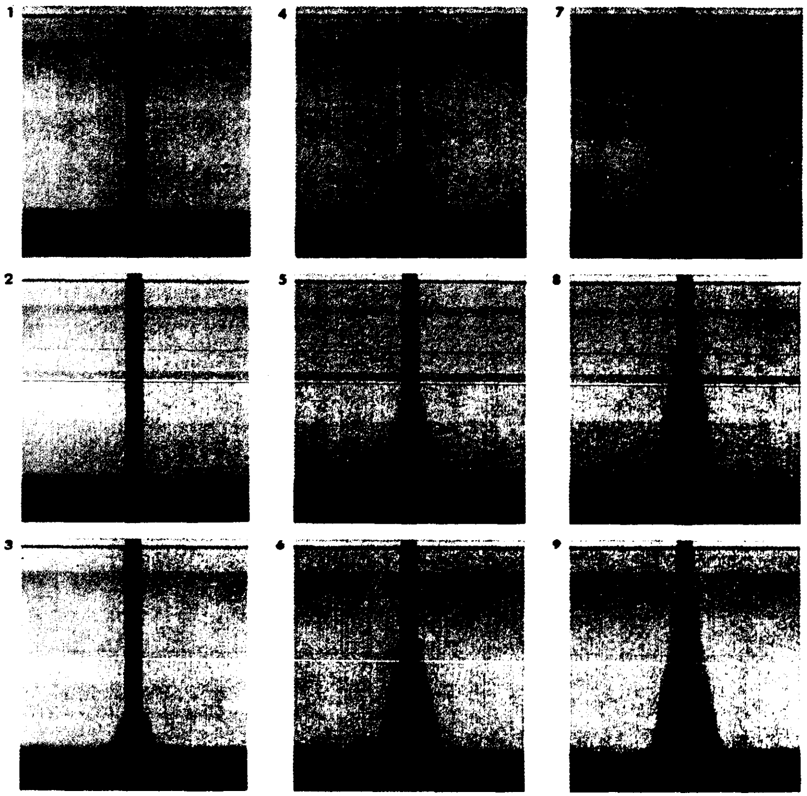
(From "Experiments on Free Surface Phenomena" by G. S. Beavers and D. D. Joseph, Journal of Non-Newtonian Fluid Mechanics, 5, 1979, pp. 323-352.)






The motion of the bubble of TLA-227 on a rod which rotates at 14.5 rev $s^{-1}$ in TLA-227 floating on water. The periodic motion of the bubble is controlled by the competing effects of normal stresses, inertia, surface tension and buoyancy forces. The bubble on the upper surface of the TLA-227 (frame 12) is steady and stable. The time between frames is 6 sec.

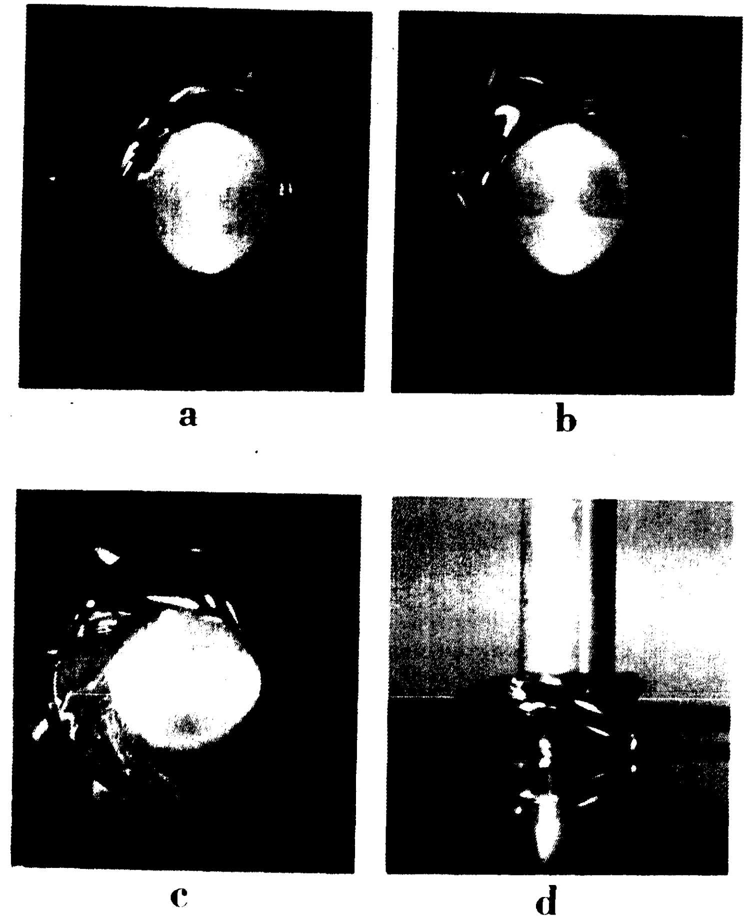
(From "Experiments on Free Surface Phenomena" by G. S. Beavers and D. D. Joseph, Journal of Non-Newtonian Fluid Mechanics, 5, 1979, pp. 323-352.



The climb of TLA-227 on a rod of radius 0.620 cm for a configuration in which STP floats on TLA-227. The steady rotational speed is 3.2 rev $s^{-1}$. The density difference at the STP/TLA-227 interface is 0.0005 g $cm^{-3}$ and the difference in climbing constants is approximately 19 g $cm^{-1}$. The TLA-227 climbs without bound through the STP because the normal stresses have essentially no gravity forces to oppose them. Time between frames is approximately 7.5 sec.

(From "Experiments on Free Surface Phenomena" by G. S. Beavers and D. D. Joseph, Journal of Non-Newtonian Fluid Mechanics, 5, 1979, pp. 323-352.)





(a), (b) Top views of the four-petal configurations bifurcating from an axisymmetric time-periodic flow. The two views are photographs at two different instants during a cycle: $\omega$ = 9.5 cycles/s, $\Theta$ = 200°. (c) Top view and (d) side view of the three-petal configuration bifurcating from an axisymmetric time-periodic flow: $\omega$ = 9.2 cycles/s, $\Theta$ = 235°.

(From "Novel Weissenberg Effects", by G. S. Beavers and D. D. Joseph, Journal of Fluid Mechanics, 81, 1977, pp. 265-272.)