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PERTURBATION OF STATES OF REST AND RIGID MOTION OF SIMPLE FLUIDS AND SOLIDS *

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Summary

In the lecture I advocate perturbing states of rest and rigid motion with arbitrary motions. This procedure leads to general expressions for the relation between stress and deformation and defines the parameters which must be measured in order to distinguish one material from another.

I. Introduction

Rheologists do not agree on what should be measured in order to differentiate one simple material from another. The reason is that so many rheologists have pet constitutive equations. So the definition of characterizing quantities differs from rheologist to rheologist. This is not a very salubrious condition for the development of good rheometrical science.

We (Beavers and I) have been trying to avoid the problems which arise from using pet constitutive equations [1,2]. To avoid making overspecialized assumptions we consider general constitutive equations but we consider them only in a restricted class of deformations. This simplifies the constitutive problem because only a restricted set of responses are possible when the deformations are restricted. Everybody knows this, and mostly everybody agrees that it is a good way to proceed, but not so many rheologists actually do proceed in this way.

We have our best results for the class of deformations which perturb states

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in which the extra stress associated with straining motion vanishes. This means perturbations of states of rigid motion in fluids and perturbations of rest states (of elastic deformations) in solids. The equations which govern the motions of a viscoelastic solid perturbing rigid motions are probably like the ones I am going to describe for fluids, after the effect of elastic (that is, static) deformations are taken into account.

In the lecture I will try to describe the application of the principles just described to problems of viscoelastic fluids and solids. The theory of fluids is taken from my 1977 paper on "Rotating Simple Fluids" [3]. We shall be interested in drawing the consequences of the equations; the details of the mathematical derivations can be found in the original paper and will not be repeated here. The theory of motions of viscoelastic solids which perturb the state of rest is joint work of P. Dixit and myself [4].

II. Rotating simple fluids

The stress in an incompressible simple fluid is given by

$$T = -p\mathbf{1} + S\,, \tag{II.1}$$

where $-p\mathbf{1}$ is a constitutively indeterminate part of T and S is the extra stress given by

$$S = \mathcal{F}_{s=0}^{\infty}[G(s)], \qquad 0 \leqslant s = t - \tau < \infty \tag{II.2}$$

where, for short,

$$\mathcal{F} = \mathcal{F}[G(s)] \equiv \mathcal{F}_{s=0}^{\infty}[G(s)]$$

and $\mathcal{F}$ is a response functional whose argument functions are histories

$$\begin{aligned} &G(s) = C_t(t-s) - \mathbf{1}\,,\\ &C_t(\tau) = F_t^{\mathrm{T}}(\tau)F_t(\tau)\,,\\ &F_t(\tau) = \nabla_x \chi_t(x, \tau) \qquad (F_{ij} = \partial\chi_i/\partial x_j), \end{aligned} \tag{II.3}$$

where

$$\xi = \chi_t(x, \tau), \qquad x = \chi_t(x, t) \tag{II.4}$$

is the position of the particle which is presently at x at an earlier time τ and $s = t - \tau \in (-\infty, 0]$.

It is a generally accepted but apparently arbitrary procedure to prescribe histories up to a certain time and to require that the further evolution of the motion be determined by the dynamic equations governing motion. This is called an initial-history problem.

We suppose that each instant $t > 0$, the fluid is driven by a prescribed body force $f(x, t)$ in the region $\mathcal{V}(t)$ occupied by the fluid and by the motion of the boundary $\partial\mathcal{V}(t)$. Then

$$\rho[\partial U/\partial t + (U \cdot \nabla)U] = -\nabla p + \nabla \cdot \mathcal{F}[G(s)] + \rho f(x, t) \tag{II.5}$$

and div $U(x, t) = 0$ in $\mathcal{V}$, $t > 0$ and, on the boundary

$$U(x, t) = q(x, t), \qquad x \in \partial \mathcal{V}(t) . \tag{II.6}$$

I call the prescribed functions $f(x, t)$ and $q(x, t)$ the driving data. When $\tau \leqslant 0$, the history of the motion of the fluid $\mathcal{V}(\tau)$ and on the boundary are prescribed arbitrarily.

$$U(\chi_t(x, \tau), \tau) \text{ is prescribed in } \mathcal{V}(\tau), \tau \leqslant 0 . \tag{II.7}$$

The relative position at time $\tau < t$ of the particle at x enters into (II.5) through the argument $G(s)$ of the extra stress. It is then possible to regard (II.5), div $U = 0$ and the path equations $U(\chi_t(x, \tau), \tau) = \partial \chi_t / \partial \tau$ as seven equations for seven unknown scalars U, p and χ_t.

From a strict point of view it is not possible to prescribe (II.7). The velocity at all times, even when $\tau < 0$, is determined by the equations of motion. But we may suppose that given U we may always find the f and q which is associated with the given U. So it is actually the prescribed driving data which is prescribed when $t < 0$.

It is not possible to say much about the response of simple fluids when $\mathcal{F}$ is so generally specified. It is possible to give some special solutions on which the motion is so severely restricted that $\mathcal{F}$ reduces to a simple form. We can make progress by perturbing the data leading to these special solutions.

III. The difference between perturbing the stress and perturbing the driving data

It is well known and easy to show that $\mathcal{F}[0] = 0$, and $G(s) = 0$ when $\chi_t(x, \tau)$ is the history of the motion of a rigid body. Obviously, perturbations of $\mathcal{F}[G(s)]$ (with $s=0$ to ∞) must be in powers of G. Hence

$$\mathcal{F}\Big[\mathop{G(s)}_{s=0}^{\infty}\Big] = \mathcal{F}_1\Big[\mathop{G(s)}_{s=0}^{\infty}\Big] + \mathcal{F}_2\Big[\mathop{G(s_1)}_{s_1=0}^{\infty} \Big| \mathop{G(s_2)}_{s_2=0}^{\infty}\Big]$$

$$+ \mathcal{F}_3\Big[\mathop{G(s_1)}_{s_1=0}^{\infty} \Big| \mathop{G(s_2)}_{s_2}^{\infty} \Big| \mathop{G(s_3)}_{s_3}^{\infty}\Big] + \ldots , \tag{III.1}$$

where $\mathcal{F}_1$ is a linear operator, $\mathcal{F}_2$ is bilinear, $\mathcal{F}_3$ is trilinear and so on. Usually we express these multilinear operators in terms of integrals. Since $\mathcal{F}[G(s)]$ is the stress in an isotropic fluid, the multilinear operators must likewise be isotropic and when expressed in terms of integrals, the kernels in the integrals must be isotropic tensors [5,6].

In general, it is not possible to perturb $G(s)$ arbitrarily. The history is determined by the equations of motion, ultimately through the driving data. When the data is a small deviation, proportional to ϵ, from that giving rise to rigid motions, then the history $G(s, \epsilon)$ may be assumed to be a small deviation from $G(s, 0) = 0$. In fact, our problem is to determine how $G(s, \epsilon)$ perturbing

$G(s, 0) = 0$ is induced by the dynamics when the given data is perturbed. Our main results are canonical forms of the stress and the equations of motion, free of redundant terms, which can be solved sequentially to predict the motion and evolution of the history once the material functions which appear in the expressions for the stress are known.

There is a big difference between just saying that $G(s, \epsilon)$ is small for nearly rigid motions and actually computing the G's that arise from perturbing the data which gives rise to rigid motions. To dramatize this difference we note that even though $G(s) = 0$ when evaluated on the history of any rigid body motion, only certain rigid body motions are compatible with the equations of motion. To prove this we should imagine that the data depends on a parameter ϵ and that when $\epsilon = 0$,

$$\rho\left[\frac{\partial U^{\langle 0\rangle}}{\partial t} + U^{\langle 0\rangle}\cdot\nabla U^{\langle 0\rangle}\right] = -\nabla p^{\langle 0\rangle} + \operatorname{div}\mathcal{F}[G^{\langle 0\rangle}(s)] + \rho f^{\langle 0\rangle}(x, t) \tag{III.2}$$

and $\operatorname{div} U^{\langle 0\rangle}(x, t) = 0$ in the region of space $\mathcal{V}(t)$ occupied by the fluid. In addition

$$U^{\langle 0\rangle}(x, t) = q^{\langle 0\rangle}(x, t), \qquad x \in \partial\mathcal{V}(t)\,. \tag{III.3}$$

This problem is incompletely specified because the response functional $\mathcal{F}$ is unspecified. But if $U^{\langle 0\rangle}(\chi_t(x, \tau), \tau) = \Omega(\tau)\wedge\chi_t^{\langle 0\rangle}(x, \tau)$ then $G^{\langle 0\rangle} = 0$ and $\mathcal{F}[0] = 0$. So if $f^{\langle 0\rangle}(x, t) = 0$, then (III.2) reduces to

$$\rho\overset{\circ}{\Omega}\wedge x = -\nabla(p + \tfrac{1}{2}p|\Omega\wedge x|^2)\,, \tag{III.4}$$

showing that $\overset{\circ}{\Omega}\wedge x$ is a gradient; that is,

$$2\rho\overset{\circ}{\Omega} = 0\,. \tag{III.5}$$

So the data $f^{\langle 0\rangle} = 0$; $q_0 = \Omega\wedge x$ for $x \in \partial\mathcal{V}$ leads to rigid body motions of the fluid in $\mathcal{V}(t)$ only if Ω is a constant vector; that is, the spin of $\mathcal{V}(t)$ is constant in time and there is no precession of the axis of rotation.

IV. Canonical forms for the stress

We are interested in the stresses which arise when the data giving rise to steady rigid rotation is perturbed by arbitrary data. We call the coefficients S_n induced by the expansion of $G(s, \epsilon)$,

$$\mathcal{F}[G(s, \epsilon)] = \epsilon S_1 + \epsilon^2 S_2 + \ldots, \tag{IV.1}$$

canonical forms of the stress at order n. We are going to call attention to some of the properties of the canonical forms, emphasizing S_1 and S_2, without repeating the details [3] of the mathematical derivations.

The theory shows that derivatives of the Cauchy strain

$$\mathcal{G}(s, \epsilon) = -\frac{\mathrm{d}}{\mathrm{d}s}\dot{G}(s, \epsilon) = \nabla\tilde{U}\cdot{}^{\mathrm{T}}F_t + F_t^{\mathrm{T}}\cdot\nabla\tilde{U}\,, \tag{IV.2}$$

where $\tilde{U}(x, \tau, \epsilon) = U(\chi_t(x, \tau, \epsilon), \tau, \epsilon)) = \partial\chi_t/\partial\tau$. $\mathcal{G}(s, \epsilon)$ is a better kinematic variable than the Cauchy strain $G(s, \epsilon)$ because $\mathcal{G}(s, \epsilon)$ leads to a canonical theory framed in terms of velocity rather than displacement:

$$\mathcal{G}(s, \epsilon) = \epsilon\mathcal{G}^{\langle 1\rangle}(s) + \epsilon^2\mathcal{G}^{\langle 2\rangle}(s) + \dots, \tag{IV.3}$$

where

$$\mathcal{G}^{\langle l\rangle}(s) = Q^{\mathrm{T}}(\Omega s)[A^{\langle l\rangle}(s) + B^{\langle l\rangle}(s)]\,Q(\Omega s), \qquad l = 1, 2, \dots, \tag{IV.4}$$

$$[Q(\Omega s)] = \begin{bmatrix} \cos\Omega s & \sin\Omega s & 0 \\ -\sin\Omega s & \cos\Omega s & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\left.\begin{aligned} A^{\langle l\rangle}(s) &= \nabla_\xi U^{\langle l\rangle}(\xi, \tau) + \text{transpose}, \\ B^{\langle 1\rangle}(s) &= 0, \end{aligned}\right. \tag{IV.5}$$

$$B^{\langle 2\rangle}(s) = (\chi^{\langle 1\rangle}\cdot\nabla_\xi)A^{\langle 1\rangle}(s) + (A^{\langle 1\rangle}\cdot\nabla_\xi\chi^{\langle 1\rangle} + \text{transpose}). \tag{IV.6}$$

Equation (IV.4) shows that $\mathcal{G}^{\langle l\rangle}(s)$ is defined in terms of $U^{\langle l\rangle}$ and lower order quantities. It seems to us that $\mathcal{G}$ is the right kinematic measure for fluids because $\mathcal{G}$ leads to problems framed in terms of velocities whereas G is the right kinematic measure for solids because G leads to problems framed in terms of displacement.

An encapsulated derivation of the canonical forms of the stress S_n for perturbations of rigid body motions of fluids is given below

$$\begin{aligned}
\mathcal{F}[G(s, \epsilon)] &= \mathcal{F}_1[G(s, \epsilon)] + \mathcal{F}_2[G(s, \epsilon)G(s, \epsilon)] + \dots \\
&= \int_0^\infty \frac{\mathrm{d}G}{\mathrm{d}s}\, G(s, \epsilon)\,\mathrm{d}s + \int_0^\infty\int_0^\infty \left\{\frac{\partial^2\gamma(s_1, s_2)}{\partial s_1\,\partial s_2} G(s_1, \epsilon)G(s_2, \epsilon)\right. \\
&\quad \left. + \frac{\partial^2\hat{\alpha}(s_1, s_2)}{\partial s_1\,\partial s_2}[\operatorname{tr} G(s_1, \epsilon)]\,G(s_2, \epsilon)\right\}\mathrm{d}s_1\,\mathrm{d}s_2 + \dots \\
&= \int_0^\infty G(s)\,\mathcal{G}(s, \epsilon)\,\mathrm{d}s + \int_0^\infty\int_0^\infty \{\gamma(s_1, s_2)\,\mathcal{G}(s_1, \epsilon)\,\mathcal{G}(s_2, \epsilon) \\
&\quad + \hat{\alpha}(s_1, s_2)[\operatorname{tr}\mathcal{G}(s_1, \epsilon)]\,\mathcal{G}(s_2, \epsilon)\}\,\mathrm{d}s_1\,\mathrm{d}s_2 + \dots \\
&= \mathcal{F}_1[\mathcal{G}(s, \epsilon)] + \mathcal{F}_2[\mathcal{G}(s_1, \epsilon)\,\mathcal{G}(s_2, \epsilon)] + \dots \equiv \tilde{\mathcal{F}}[\mathcal{G}(s, \epsilon)] \\
&= \tilde{\mathcal{F}}[\epsilon\mathcal{G}^{\langle 1\rangle}(s) + \epsilon^2\mathcal{G}^{\langle 2\rangle}(s) + \dots] = \epsilon\,\tilde{\mathcal{F}}_1[\mathcal{G}^{\langle 1\rangle}(s)] \\
&\quad + \epsilon^2\{\tilde{\mathcal{F}}_1[\mathcal{G}^{\langle 2\rangle}(s)] + \tilde{\mathcal{F}}_2[\mathcal{G}^{\langle 1\rangle}(s_1), \mathcal{G}^{\langle 1\rangle}(s_2)]\} + \dots \\
&= \epsilon S_1 + \epsilon^2 S_2 + O(\epsilon^3),
\end{aligned} \tag{IV.7}$$

where

$$\tilde{\mathcal{F}}_1[\mathcal{G}^{\langle n \rangle}(s)] \equiv \int_0^\infty G(s)\mathcal{G}^{\langle n \rangle}(s)\,\mathrm{d}s \tag{IV.8}$$

and

$$\tilde{\mathcal{F}}_2[\mathcal{G}^{\langle 1 \rangle}(s_1), \mathcal{G}^{\langle 1 \rangle}(s_2)] = \int_0^\infty \int_0^\infty \gamma(s_1, s_2)\mathcal{G}(s_1)\mathcal{G}(s_2)\,\mathrm{d}s_1\,\mathrm{d}s_2 \,. \tag{IV.9}$$

The canonical stress S_1 is completely specified when the shear relaxation modulus $G(s)$ is known. The second-order stress S_2 is completely specified when $G(s)$ and $\gamma(s_1, s_2) = \gamma(s_2, s_1)$ are known. The material function $\hat{\alpha}(s_1, s_2)$ does not enter into the second-order theory because tr $\mathcal{G}^{\langle 1 \rangle} = 0$.

Many persons swear by constitutive models which depend on the shear relaxation modulus $G(s)$ alone, and not on the other moduli like $\gamma(s_1, s_2)$. Despite some demonstrations of agreement with experiments, I would guess that these $G(s)$ models are not good.

Though the fluid is isotropic the perturbation stresses S_n have a preferred direction, defined by the axis of rotation through the orthogonal matrices $Q(\Omega s)$ defining $\mathcal{G}^{\langle 1 \rangle}(s)$ (see (IV.6)).

In our derivation of the canonical form of stress perturbing rigid motion we assumed that the canonical forms of the stress could be represented by integrals. A good mathematical justification for $\mathcal{F}_1 = S_1$, given by (IV.8) can be constructed along lines laid out by Coleman and Noll [5]. This representation may be justified when $\mathcal{F}[\cdot]$ is Frechet differentiable on the zero history in Hilbert space topologies defined by weighted integrals (with weights which vanish at infinity). An integral representation for $\mathcal{F}_1$ is then implied by the Riesz representation theorem.

An integral representation for the higher order stresses may be partially justified as a "Weierstrass" approximation to the true stress [7].

V. Four perturbation equations and four unknown fields

The canonical forms of the equations of motion are the perturbation equations which arise by identification of different powers of ϵ. At zeroth order

$$\chi_0(s) = \chi_t^{\langle 0 \rangle}(x, \tau) = Q(\Omega s) \cdot x \tag{V.1}$$

and

$$U^{\langle 0 \rangle}(\chi_0, \tau) = U^{\langle 0 \rangle}(\chi_0) = \Omega \wedge \chi_0 \,. \tag{V.2}$$

When $\Omega = 0$, $U^{\langle 0 \rangle} = 0$ and $\chi_0 = x$ describes the fluid in a state of rest.

At order $n \geqslant 1$ we find that

$$\operatorname{div} U^{\langle n \rangle}(x, t) = 0 \text{ in } \mathcal{V}(t), \tag{V.3}$$

$$\rho[\partial U/\partial t + U^{\langle 0\rangle}\nabla U^{\langle n\rangle} + U^{\langle n\rangle}\nabla U^{\langle 0\rangle}] + \nabla p^{\langle n\rangle}$$

$$- \int_0^\infty G(s) Q^{\mathrm{T}}(\Omega s)\nabla^2_{\chi_0} U^{\langle n\rangle}(\chi_0(s), t-s)\,\mathrm{d}s$$

$$+ \text{lower order terms} = 0 \text{ in } \mathcal{V}(t), \qquad \text{(V.4)}$$

where $U^{\langle n\rangle}(x, t)$ takes on prescribed values on the boundary $\partial\mathcal{V}$ of $\mathcal{V}$. Equations (V.1) and (V.2) are four equations in four unknowns. As in the Navier–Stokes theory, we can find $U^{\langle n\rangle}$ and $p^{\langle n\rangle}$ by solving four equations in four unknowns. The particle paths $\chi^{\langle n\rangle}$ satisfy

$$\frac{\partial \chi^{\langle n\rangle}(\tau)}{\partial \tau} = U^{\langle n\rangle}(\chi_0(\tau), \tau) + \chi^{\langle n\rangle}\cdot\nabla_{\chi_0} U^{\langle 0\rangle} + \text{lower order terms}, \qquad \text{(V.5)}$$

$$\chi^{\langle n\rangle}(t) = 0\,.$$

$\chi^{\langle n\rangle}(\tau)$ is needed for the forcing terms in the equations which determine $U^{\langle n+1\rangle}$ and $p^{\langle n+1\rangle}$. But $U^{\langle n\rangle}$ is independent of $\chi^{\langle n\rangle}$ and $\chi^{\langle n\rangle}$ is given by quadrature involving $U^{\langle n\rangle}$ and other terms of lower order. The history of the *strain* (given in terms $\chi^{\langle n\rangle}$) is passive; that is, it enters into the equations of motion through forcing terms computed on solutions of lower order. But the history of the velocity is active; that is, it enters into the dominant term from the stress in the linear operator which needs inverting at each stage of the perturbation. The computation of the coefficient fields $U^{\langle n\rangle}(x, t)$, $p^{\langle n\rangle}(x, t)$ and $\chi^{\langle n\rangle}(x, t)$ may be carried sequentially. The perturbation equations take on a simple form in cylindrical coordinates (see Section VII).

Suppose that there are two solutions $U^{\langle n\rangle}$, $p^{\langle n\rangle}$ of (V.3) and (V.4) satisfying the same equations and boundary conditions but with different initial histories. The difference between the two solutions satisfies $\operatorname{div} V = 0$ and

$$\rho[\partial V/\partial t + U^{\langle 0\rangle}\cdot\nabla V + V\cdot\nabla U^{\langle 0\rangle}] + \nabla p$$

$$- \int_0^\infty G(s) Q^{\mathrm{T}}(\Omega s)\nabla^2_{\chi_0} V(\chi_0(s), t-s)\,\mathrm{d}s = 0 \qquad \text{(V.6)}$$

in $\mathcal{V}(t)$ and $V = 0$ on $\partial\mathcal{V}(t)$. It can be shown that $V(t) \to 0$ as $t \to \infty$ if $G(s) > 0$, $G'(s) < 0$ and $G(s \to \infty) \to 0$ exponentially. This means that solutions of (V.3) and (V.4) are ultimately uniquely determined by the data, independent of the initial history (see Refs. [3][8]). So if the theory is good we can expect a one-to-one correspondence between solutions and experiments uncomplicated by problems of stability, bifurcation and nonuniqueness.

VI. Some other approximations are special cases of this one

When $\Omega = 0$ the perturbation equations govern small, but otherwise arbitrary perturbations of the state of rest [2,9]. Then

$$U^{\langle 0\rangle} \equiv 0, \qquad \chi^{\langle 0\rangle} = x, \qquad Q(0) = 1 \qquad \text{(VI.1)}$$

at first order, $U^{\langle 1\rangle}(x, t)$ is prescribed on $\partial\mathcal{V}$, div $U^{\langle 1\rangle} = 0$ and

$$\rho\partial U^{\langle 1\rangle}/\partial t = -\nabla p^{\langle 1\rangle} + \int_0^\infty G(s)\nabla^2 U^{\langle 1\rangle}(x, t-s)\,ds\,. \tag{VI.2}$$

When $U^{\langle 1\rangle}(x, t)$ is determined from (VI.2) we may find $\chi^{\langle 1\rangle}$ by integrating

$$\partial\chi^{\langle 1\rangle}/\partial\tau = U^{\langle 1\rangle}(x, \tau), \qquad \chi^{\langle 1\rangle}(t) = 0\,. \tag{VI.3}$$

At second order, $U^{\langle 2\rangle}(x, t)$ is prescribed on $\partial\mathcal{V}$, div $U^{\langle 2\rangle} = 0$ and

$$\rho\left[\frac{\partial U^{\langle 2\rangle}}{\partial t} + U^{\langle 1\rangle}\cdot\nabla U^{\langle 1\rangle}\right] = -\nabla p^{\langle 2\rangle} + \int_0^\infty G(s)\nabla^2 U^{\langle 2\rangle}(x, t-s)\,ds$$

$$+ \operatorname{div}\Big\{\int_0^\infty G(s)[(\chi^{\langle 1\rangle}\cdot\nabla)A^{\langle 1\rangle} + A^{\langle 1\rangle}\cdot\nabla\chi^{\langle 1\rangle} + \nabla^T\chi^{\langle 1\rangle}\cdot A^{\langle 1\rangle}]\,ds$$

$$+ \int_0^\infty\int_0^\infty \gamma(s_1, s_2)A^{\langle 1\rangle}(s_1)\cdot A^{\langle 1\rangle}(s_2)\,ds_1\,ds_2\Big\}, \tag{VI.4}$$

$$A^{\langle 1\rangle}(s) = \nabla U^{\langle 1\rangle} + \nabla^T U^{\langle 1\rangle}\,.$$

When $U^{\langle 2\rangle}$ is determined from (VI.4) we may find $\chi^{\langle 2\rangle}$ by integrating

$$\frac{\partial\chi^{\langle 2\rangle}}{\partial\tau} = U^{\langle 2\rangle}(x, \tau) + (\chi^{\langle 1\rangle}\cdot\nabla)U^{\langle 1\rangle}, \qquad \chi^{\langle 2\rangle}(t) = 0\,. \tag{VI.5}$$

The so-called "slow motion" or "retarded motion" approximation arises as a perturbation of the rest state when the perturbed data is steady. If the perturbed data is steady then the perturbed motion is steady, (VI.2) becomes

$$0 = -\nabla p^{\langle 1\rangle} + \mu\nabla^2 U^{\langle 1\rangle}(x), \tag{VI.6}$$

where

$$\mu = \int_0^\infty G(s)\,ds\,.$$

We may integrate (VI.3)

$$\chi^{\langle 1\rangle} = -U^{\langle 1\rangle}(x)s\,. \tag{VI.8}$$

Using (VI.7) and (VI.8) we may write (VI.4) as

$$-U^{\langle 1\rangle}\cdot\nabla U^{\langle 1\rangle} = -\nabla p^{\langle 2\rangle} + \mu\nabla^2 U^{\langle 2\rangle}(x) + \operatorname{div}[\alpha_1 A_2 + \alpha_2 A^{\langle 1\rangle}\cdot A^{\langle 1\rangle}]\,, \tag{VI.9}$$

where

$$\alpha_1 = -\int_0^\infty sG(s)\,ds \tag{VI.10}$$

and

$$\alpha_2 = \int_0^\infty \int_0^\infty \gamma(s_1, s_2)\, ds_1\, ds_2 . \tag{VI.11}$$

The same relations (VI.7), (VI.10) and (VI.11) arise for slightly unsteady (retarded) motions [10]. The higher-order approximations in the theory of slow steady motion of viscoelastic fluids arise from the higher order theory of perturbations of the rest state in a similar way. The stresses S_n which arise in the theory of slow steady motions of viscoelastic fluids are called the stress tensors for "fluids of grade n". For example,

$$S_1 = \mu A^{\langle 1 \rangle},$$
$$S_2 = \mu A^{\langle 1 \rangle} + \alpha_1 A_2 + \alpha_2 A^{\langle 1 \rangle} \cdot A^{\langle 1 \rangle},$$
$$A_2 = (U^{\langle 1 \rangle} \cdot \nabla) A^{\langle 1 \rangle} + A^{\langle 1 \rangle} \cdot \nabla U^{\langle 1 \rangle} \cdot A^{\langle 1 \rangle} .$$

It is useful to call attention here to a criticism of constitutive equations which are assumed to depend on $G(s)$ alone. Such constitutive equations should logically be independent of material functions like $\gamma(s_1, s_2)$, among others, which are presumably independent of $G(s)$. In the steady case this means that constitutive equations which depend on $G(s)$ alone cannot depend on α_2.

VII. The perturbed constitutive equation is not isotropic but is axially symmetric. Equations of motion in cylindrical coordinates

Of course, the fluid is isotropic and the constitutive equation is isotropic. But the rigid motion of the fluid introduces a preferred direction in the perturbed constitutive equation which is most conveniently expressed by introducing coordinates with axial symmetry, say cylindrical coordinates (r, θ, z). In these coordinates it becomes apparent that there are two effects, an inertial and a viscoelastic effect, of the steady rigid rotation. The inertial effects are very well understood but the viscoelastic effects are new and best understood by examining the dynamic equations in cylindrical coordinates

$$\rho \begin{bmatrix} \partial_t + \Omega\partial_\theta & -2\,\Omega & 0 \\ 2\,\Omega & \partial_t + \Omega\partial_\theta & 0 \\ 0 & 0 & \partial_t + \Omega\partial_\theta \end{bmatrix} \begin{bmatrix} U_r^{\langle n \rangle}(r, \theta, z, t) \\ U_\theta^{\langle n \rangle}(r, \theta, z, t) \\ U_z^{\langle n \rangle}(r, \theta, z, t) \end{bmatrix}$$
$$+ \begin{bmatrix} \partial_r \\ \dfrac{1}{r}\partial_\theta \\ \partial_z \end{bmatrix} P^{\langle n \rangle}(r, \theta, z, t)$$

$$-\int_0^\infty G(s)\begin{bmatrix}\nabla^2-\frac{1}{r^2} & -\frac{2}{r^2}\partial_\theta & 0\\ \frac{2}{r^2}\partial_\theta & \nabla^2-\frac{1}{r^2} & 0\\ 0 & 0 & \nabla^2\end{bmatrix}\begin{bmatrix}U_r^{\langle n\rangle}(r,\theta-\Omega s,z,t-s)\\ U_\theta^{\langle n\rangle}(r,\theta-\Omega s,z,t-s)\\ U_z^{\langle n\rangle}(r,\theta-\Omega s,z,t-s)\end{bmatrix}ds$$

$$=\begin{bmatrix}f_r^{\langle n\rangle}(r,\theta,z,t)\\ f_\theta^{\langle n\rangle}(r,\theta,z,t)\\ f_z^{\langle n\rangle}(r,\theta,z,t)\end{bmatrix}. \qquad \text{(VII.1)}$$

The $f^{\langle n\rangle}$ are inhomogeneous terms which depend on lower orders, $f^{\langle 1\rangle}=0$, $f^{\langle 2\rangle}$ is quadratic in $U^{\langle 1\rangle}$ and the hereditary integrals in $f^{\langle n\rangle}$, $n>1$ have integrands with arguments $(r, \theta-\Omega s, z, t-s)$. The main viscoelastic effect is the appearance of the argument $\theta-\Omega s$ in the hereditary integrals. This is because the history of the Cauchy strain is computed on a particle and is evaluated at the positions that the particle occupied in the rigid rotation, that is on the circles of fixed radius $r=r^{\langle 0\rangle}$ height $z=z^{\langle 0\rangle}$ at the variable angular position $\theta^{\langle 0\rangle}=\theta-\Omega s$.

VIII. Reduction to "fluids of grade *n*" is possible only when the perturbation of steady rigid rotation of fluid is steady and axisymmetric

Fluids of "grade n" are the form of the stresses which perturb the rest state with slow steady motions (see Section VI). Since we may write our equations relative to an observer rotating with an angular velocity $\boldsymbol{\Omega}=e_z\Omega$ and a steady motion would appear to be "slow" relative to this rotating observer it is plausible, but not correct, to take fluids of "grade n" for approximations of steady perturbations of steady rigid body motions of fluids.

We can take fluids of "grade n" as the correct perturbation stresses of steady rigid body motions of fluids when the perturbation is *steady* and *axisymmetric*. Then (VII.1) becomes

$$\begin{aligned}&2\rho\Omega\;\; U^{\langle n\rangle}(x)+\nabla p^{\langle n\rangle}(x)-\mu\nabla^2U^{\langle n\rangle}(x)=f^{\langle n\rangle}, \qquad f^{\langle 1\rangle}=0\\ &f^{\langle 2\rangle}=-\rho U^{\langle 1\rangle}\cdot\nabla U^{\langle 1\rangle}+\operatorname{div}(\alpha_1A_2+\alpha_2A^{\langle 1\rangle 2}), \qquad \text{etc.}\end{aligned} \qquad \text{(VIII.1)}$$

But when perturbed motion is steady and not axisymmetric, the integrands are a Fourier series in $e^{il\theta}$ which reduce to a single term $l=0$ in the axisymmetric case. For example, since $u(r,\theta,z)$ is periodic in θ with period 2π, we have

$$\int_0^\infty G(s)u(r,\theta-\Omega s,z)\,ds=\sum_{-\infty}^{\infty}u_l(r,z)\,e^{il\theta}\eta^*(l\Omega), \qquad \text{(VIII.2)}$$

$$\eta^*(l\Omega)=\int_0^\infty G(s)\,e^{-il\Omega s}\,ds. \qquad \text{(VIII.3)}$$

IX. The complex viscosity

Rheologists frequently work with the "complex viscosity" $\eta^*(\Omega)$ defined by (VIII.3) when $l = 1$. The function $\eta^*(\cdot)$ arises from (VII.1) when solutions are decomposed into Fourier series in θ and t

$$\begin{bmatrix} U^{\langle n\rangle}(r,\theta,z,t) \\ p^{\langle n\rangle}(r,\theta,z,t) \\ f^{\langle n\rangle}(r,\theta,z,t) \end{bmatrix} = \sum_{m=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} e^{i(l\theta+m\omega t)} \begin{bmatrix} U_{lm}(r,z) \\ p_{lm}(r,z) \\ f_{lm}(r,z) \end{bmatrix}. \tag{IX.1}$$

We are assuming that solutions are $2\pi/\omega$ periodic in t. The values $m = 0$ and $l = 0$ are included in the summation and the series are real-valued.

If all the coefficients with $l \neq 0$ are zero, then the flow is axisymmetric. If all the coefficients are zero then the flow is steady. The Fourier coefficients of (IX.1) satisfy

$$\rho \begin{bmatrix} i(\omega m + \Omega l) & -2\Omega & 0 \\ 2\Omega & i(\omega m + \Omega l) & 0 \\ 0 & 0 & i(\omega m + \Omega l) \end{bmatrix} \begin{bmatrix} u_{lm}(r,z) \\ v_{lm}(r,z) \\ w_{lm}(r,z) \end{bmatrix} + \begin{bmatrix} \partial_r \\ i\frac{l}{r} \\ \partial_z \end{bmatrix} p_{lm}(r,z)$$

$$-\eta^*(\omega m + \Omega l) \begin{bmatrix} \nabla^2 - \frac{1}{r^2}(l^2+1) & -2i\frac{l}{r^2} & 0 \\ 2i\frac{l}{r^2} & \nabla^2 - \frac{1}{r^2}(l^2+1) & 0 \\ 0 & 0 & \nabla^2 - \frac{l^2}{r^2} \end{bmatrix} \begin{bmatrix} u_{lm}(r,z) \\ v_{lm}(r,z) \\ w_{lm}(r,z) \end{bmatrix}$$

$$= \begin{bmatrix} f_{rlm}(r,z) \\ f_{\theta lm}(r,z) \\ f_{zlm}(r,z) \end{bmatrix}, \tag{IX.2}$$

where

$$\eta^*(\omega m + \Omega l) = \int_0^\infty G(s)\, e^{-i(\omega m + \Omega l)s}\, ds\,^\dagger .$$

It is clear that the complex viscosity $\eta^*(\cdot)$ is an important quantity which always arises when time-periodic motions perturb steady rigid ones. The argument of η^*, however, depends on the prescribed forcing data for the prob-

---

* Abbott and Walters [11] derived the equations for $m = 0, l = 1$ by a different method and applied the method to the theory of rheometers [11,12].

lem which selects the values of l and m. So $\eta^*(\cdot)$ is a special type of material function arising for periodic perturbation of steady rigid motions. The values which η^* takes in a certain problem do not depend on the material alone but they also depend on the prescribed data of the problem. So $\eta^*(\cdot)$ is not a material function like the shear relaxation modulus which depends on the material alone.

Finally we note that $\eta^*(0) = \mu$ is the viscosity of the fluid. When Ω is large the asymptotic expansion

$$\eta^*(\Omega) = \int_0^\infty G(s)\, e^{-i\Omega s}\, dx = \frac{G(0)}{i\Omega} + \frac{G'(0)}{\Omega^2} + O(\Omega^{-3})$$

is useful, for example, in the study of boundary layers of Ekmans type which can occur in viscoelastic fluids when Ω is large.

X. Perturbation of the rest state of a viscoelastic solid

My purpose is to show how the approach we use to characterize fluids can be applied to viscoelastic solids. The theory for solids has been worked by P. Dixit and me and the details of the analysis will re reported elsewhere [4]. Here we shall confine our attention to certain aspects of the theory of incompressible solids.

The constitutive equation for an initially isotropic incompressible simple solid was found by Green and Rivlin [13]. It may be written as (see Ref. [14]).

$$\boldsymbol{T} = -p\mathbf{1} + \mathcal{F}[\boldsymbol{B}(t), \underset{s=0}{\overset{\infty}{\boldsymbol{G}(s)}}]. \tag{X.1}$$

$\boldsymbol{T}$ is the stress, $\boldsymbol{G}(s)$ is the history of right-relative Cauchy—Green tensor (the same $\boldsymbol{G}(s)$ which we used for fluids) and

$$\boldsymbol{B}(t) = \boldsymbol{F}(t) \cdot \boldsymbol{F}^{\mathrm{T}}(t)$$

is the left Cauchy—Green tensor defined in terms of the deformation gradient

$$\boldsymbol{F} = \nabla x(X, t) \qquad (F_{ij} = \partial x_i/\partial X_j),$$

where $x = x(X, t)$ designates points of the body in its deformed configuration and X designates material points. Our convention is that X is a coordinate point of the reference configuration *. p is a reaction pressure associated with incompressibility

$$\det \boldsymbol{F} = 1\,, \tag{X.2}$$

and p is constitutively indeterminate. We determine p from the solution of

* The tensor $\boldsymbol{G}(s)$ is based on the relative deformation tensor $\boldsymbol{F}_t(x, t)$, see (II.3).

the equations governing the deformation. We shall define the zero level of stress by requiring that the body be free of stresses in its undeformed or "natural" state,

$$F = 1 = B(t), \qquad G(s) = 0\,, \qquad \mathcal{F}[1, 0] = 0, \qquad T = 0\,. \tag{X.3}$$

Now I shall make some categorical statements which hold strictly for simple solids. *All static deformations are elastic.* In a static deformation $G(s) = 0$ and

$$T = -p1 + \mathcal{F}[B(t), 0] \tag{X.4}$$

defines the relation between stress and deformation in an elastic material. Static deformations are the special case of dynamic responses corresponding to states of rest. So perturbing states of rest.is the same as perturbing elasticity. If the material has not come to rest $G(s) \neq 0$ and the constitutive response is not elastic but instead depends on history. *There is no such thing as an elastic material. A viscoelastic material will undergo an elastic response when the deformations are independent of time.* Of course, it is always possible to imagine that the constitutive equation is independent of $G(s) \neq 0$. Then we get an elastic material, which can be studied in its own right. It may be useful to study "elastic" materials because elastic responses and nearly elastic responses of viscoelastic materials are important. But it is not so good to define "elastic materials" in the context of material science because elastic materials arise only in static deformations of viscoelastic ones. For dynamic responses we need information about the dependence of $\mathcal{F}$ on $G(s)$. Now I am going to show how to get some of the kinds of information we need.

XI. Perturbations of the rest state

The rest state is the class of elastic responses of a viscoelastic material. We shall now perturb the rest state with small time-dependent data and derive the perturbed constitutive equations and the perturbed equations of motion.

In the rest state $G(s) = 0$. As in fluids, we perturb $G(s) = 0$ with powers of G. Then

$$\begin{aligned}\mathcal{F}[B(t), G(s)] &= \mathcal{F}[B(t), 0] + \mathcal{F}_1[B, 0 \mid G(s)] \\ &\quad + \mathcal{F}_2[B, 0 \mid G(s_1) \mid G(s_2)] + O(\|G\|^3)\,,\end{aligned} \tag{XI.1}$$

where the dependence of $\mathcal{F}_n$ on the history of $G(s)$ is understood, and not explicit.

(XI.1) reminds us of the fluids equation (III.1) except here we have to worry about the tensor $B(t)$. We can identify $\mathfrak{f}(B) = \mathcal{F}[B, 0]$ as the elasticity part of $\mathcal{F}[B, G(s)]$; the other terms in the Fréchet expansion vanish when $G(s)$ does. Now we follow Green and Rivlin [13] and Coleman and Noll [5] and assume, as in the case of fluid, that the Fréchet derivatives can be expressed in terms of integrals.

$$\mathcal{F}[B(t), G(s)] = \mathfrak{f}(B) + \int_0^\infty K(B(t), s)G(s)\,ds$$
$$+ \int_0^\infty \int_0^\infty \Gamma(B(t), s_1, s_2)G(s_1)G(s_2)\,ds_1\,ds_2 + \ldots \tag{XI.2}$$

where K is a fourth and Γ is a sixth-order tensor. Since we are assuming that the material is isotropic, $\mathfrak{f}(B)$ is an isotropic tensor function and so are the integrands. Methods for finding the most general gorms for isotropic tensor functions can be found in the paper of Wineman and Pikin [15] and are reviewed in Section 11 of the treatise by Truesdell and Noll [14] and in the paper of Rivlin [16]. The reduction to isotropic form is like the Hamilton—Cayley reduction of a tensor polynomial of degree $m > 2$ to $m = 2$. We find that, modulo terms proportional to 1,

$$\mathfrak{f}(B) = \mathfrak{f}_1 B + \mathfrak{f}_2 B^2, \tag{XI.3}$$

$$\begin{aligned} K(B, s)G(s) &= B\,\mathrm{tr}[(\phi_{00}1 + \phi_{01}B + \phi_{02}B^2)G(s)] \\ &+ B^2\,\mathrm{tr}[(\phi_{10}1 + \phi_{11}B + \phi_{12}B^2)G(s)] \\ &+ (\phi_{20}1 + \phi_{21}B + \phi_{22}B^2)G(s) \\ &+ G(s)(\phi_{20}1 + \phi_{21}B + \phi_{22}B^2)\,, \end{aligned} \tag{XI.4}$$

where the $\mathfrak{f}$'s and ϕ's depend on I_B, II_B and III_B, the invariants of B, and the ϕ's also depend on s. The terms proportional to 1 have been absorbed into p.

The expressions (XI.2—4) are general constitutive equations for the stress in motions which perturb data giving rise to the rest state of viscoelastic solids. Suppose $F = F_0$, $B = B_0$ on the rest state and the data perturbing this state is proportional to ϵ. Then we set $B = B_0 + \epsilon B^{\langle 1 \rangle} + \ldots$, $G = \epsilon G^{\langle 1 \rangle}(s) + \ldots$ and find the canonical forms of the stresses and governing equations by identification. The first-order theory, proportional to ϵ, is dependent on the unknown functions of $\mathfrak{f}_1$, $\mathfrak{f}_2$, ϕ_{00}, ϕ_{10}, ϕ_{01}, ϕ_{02}, ϕ_{11}, ϕ_{12}, ϕ_{20}, ϕ_{21}, ϕ_{22} of the strain invariants and appear already in the first-order theory. In fact, there are too many unknown functions of many variables to have much hope of finding them all for even one single solid. This same type of difficulty occurs in theory of motions of viscoelastic fluids perturbing viscometric flows.

XII. Perturbation of the natural state of a viscoelastic solid

A more satisfactory theory for the purposes of material science can be obtained by restricting the theory of perturbations of the rest state to the case in which the rest state is the natural state. *The natural state is the unstressed and undeformed state of the body*. It is the most simple of the dynamical states of the body. By perturbing the natural state we derive the simplest possible dynamical theory for general classes of simple solids. In the incompressible case, we get the material characterized for small amplitude motions through second order when we know two constants and three material functions.

Suppose that ϵ is a measure of the data perturbing the natural state. To be definite let us suppose that the displacement $\epsilon U(x, t)$ the boundary $\partial \mathcal{V}(t, \epsilon)$ of the body $\mathcal{V}(t, \epsilon)$ is prescribed:

$$
\begin{aligned}
&x = X + \epsilon U(x, t)\,, \\
&x \in \partial \mathcal{V}(t, \epsilon)\,, \\
&X \in \partial \mathcal{V}(t, 0) = \partial \mathcal{V}_0
\end{aligned}
\qquad \text{(XII 1)}
$$

where $\mathcal{V}_0$ is the boundary of the body in the natural state. We can do mixed problems with tractions prescribed on parts of the body. But to keep things simple here we stay with (XII.1).

Now everything in sight is expanded in powers of:

$$x - X = \epsilon u^{\langle 1\rangle}(X, t) + \epsilon^2 u^{\langle 2\rangle}(X, t) + O(\epsilon^3) = u(X, t, \epsilon), \qquad \text{(XII.2)}$$

$$F(X, t, \epsilon) = 1 + \nabla u = 1 + \epsilon F^{\langle 1\rangle}(X, t) + \epsilon^2 F^{\langle 2\rangle}(X, t) + O(\epsilon^3)\,, \qquad \text{(XII.3)}$$

where

$$F^{\langle n\rangle}(X, t, \epsilon) = \nabla u^{\langle n\rangle}, \qquad (F_{ij}^{\langle n\rangle} = \partial u_i^{\langle n\rangle}/\partial X_j)\,,$$

$$G(s, \epsilon) = F_t^{\mathrm{T}}(\tau, \epsilon) F_t(\tau, \epsilon) - 1 = \epsilon G^{\langle 1\rangle}(s) + \epsilon^2 G^{\langle 2\rangle}(s) + O(\epsilon^3)\,,$$

where

$$
\begin{aligned}
&G^{\langle 1\rangle}(s) = 2\{E^{\langle 1\rangle}(t-s) - E^{\langle 1\rangle}(t)\}\,, \\
&G^{\langle 2\rangle}(s) = 2\{E^{\langle 2\rangle}(t-s) - E^{\langle 2\rangle}(t)\} + \mathcal{E}^{\langle 2\rangle}(t, s)\,, \\
&E^{\langle n\rangle} = \tfrac{1}{2}\,(F^{\langle n\rangle} + F^{\mathrm{T}\langle n\rangle})
\end{aligned}
\qquad \text{(XII.5)}
$$

and

$$
\begin{aligned}
\mathcal{E}^{\langle 2\rangle}(t, s) = {} & F^{\mathrm{T}\langle 1\rangle}(t-s) \cdot F^{\langle 1\rangle}(t-s) + F^{\mathrm{T}\langle 1\rangle}(t) \cdot F^{\langle 1\rangle}(t) \\
& - 2\, F^{\mathrm{T}\langle 1\rangle} \cdot E^{\langle 1\rangle}(t-s) + F^{\langle 1\rangle}(t) \cdot F^{\langle 1\rangle}(t) \\
& + F^{\mathrm{T}\langle 1\rangle}(t) \cdot F^{\mathrm{T}\langle 1\rangle}(t) - 2\, E^{\langle 1\rangle}(t-s) \cdot F^{\langle 1\rangle}(t)\,.
\end{aligned}
$$

$$
\begin{aligned}
B(t, \epsilon) = F(t, \epsilon) \cdot F^{\mathrm{T}}(t, \epsilon) = 1 + 2\,\epsilon E^{\langle 1\rangle}(t) + \epsilon^2\{2\, E^{\langle 2\rangle}(t) \\
+ F^{\langle 1\rangle}(t) \cdot F^{\mathrm{T}\langle 1\rangle}(t)\} + O(\epsilon^3)\,.
\end{aligned}
\qquad \text{(XII.6)}
$$

Since $\det F(X, t, \epsilon) = 1$ we find, using (XII.3) that

$$\mathrm{tr}\, F^{\langle 1\rangle} = 0\,.$$

Hence

$$\mathrm{tr}\, E^{\langle 1\rangle} = \mathrm{tr}\, G^{\langle 1\rangle} = 0\,. \qquad \text{(XII.7)}$$

We turn next to the expansion of the stress. Inserting (XII.4) and (XII.6) into (XI.2) and expanding $\mathfrak{f}$, K, Γ in powers of ϵ, we find that $\mathcal{T}$ is given in terms of multinomials in the tensors $B^{\langle n\rangle}(t)$, $G^{\langle m\rangle}(s)$ with tensor-valued coefficients which are independent of B and G. These tensor-valued constant coefficients are selected so that the expression for the stress is isotropic. After some work we come up with the following formulas for the stresses.

$$T = \epsilon T^{\langle 1\rangle} + \epsilon^2 T^{\langle 2\rangle} + O(\epsilon^3)\,, \tag{XII.8}$$

$$T^{\langle 1\rangle}(t) = -p^{\langle 1\rangle}\mathbf{1} + 2\,\gamma E^{\langle 1\rangle}(t) + 2\int_0^\infty \zeta(s)E^{\langle 1\rangle}(t-s)\,\mathrm{d}s\,, \tag{XII.9}$$

where

$$\gamma = \beta - \int_0^\infty \zeta(s)\,\mathrm{d}s\,.$$

$$\begin{aligned} T^{\langle 2\rangle}(t) = {} & -p^{\langle 2\rangle}\mathbf{1} + 2\,\gamma E^{\langle 2\rangle} + 2\int_0^\infty \zeta(s)E^{\langle 2\rangle}(t-s)\,\mathrm{d}s \\ & + \beta F^{\langle 1\rangle}\cdot F^{\mathrm{T}\langle 1\rangle} + \beta^{[2]}B^{\langle 1\rangle}\cdot B^{\langle 1\rangle} + \int_0^\infty \zeta(s)\,\mathcal{E}^{\langle 2\rangle}(t,s)\,\mathrm{d}s \\ & + \int_0^\infty \zeta^{[2]}(s)[B^{\langle 1\rangle}(t)\cdot G^{\langle 1\rangle}(s) + G^{\langle 1\rangle}(s)\cdot B^{\langle 1\rangle}(t)]\,\mathrm{d}s \\ & + \int_0^\infty\int_0^\infty \alpha(s_1, s_2)G^{\langle 1\rangle}(s_1)\cdot G^{\langle 1\rangle}(s_2)\,\mathrm{d}s,\,\mathrm{d}s_2\,. \end{aligned} \tag{XII.10}$$

The functions $p^{\langle 1\rangle}$ and $p^{\langle 2\rangle}$ are to be determined from the solution of the equations at first and second order.

The first-order theory is complete when one elastic constant β and one material function, the relaxation modulus, $\zeta(s)$ are known. The second-order theory is complete when two elastic constants β and $\beta^{[2]}$ and three material functions $\zeta(s)$, $\zeta^{[2]}(s)$ and $\alpha(s_1, s_2)$ are known.

The coefficients in the expansions (XII.2—6) are to be obtained sequentially from the solution of perturbation equations which arise equally from perturbing either the Cauchy or the Piola—Kirchoff equations of motion. There is a characteristic linear operator

$$\mathcal{G}(\cdot) = \rho_0\,\frac{\partial^2}{\partial t^2} - \gamma\nabla^2(\cdot) - \nabla^2\int_0^\infty \zeta(s)(\cdot)(t-s)\,\mathrm{d}s \tag{XII.11}$$

to invert at each order in the perturbation sequence. For example, when displacements are prescribed

$$\left.\begin{aligned}\mathcal{G}u^{\langle 1\rangle} + \nabla p^{\langle 1\rangle} &= 0,\\ \operatorname{div} u^{\langle 1\rangle} &= 0,\end{aligned}\right\} \text{ in } \mathcal{V}_0 \tag{XII.12}$$

$u^{\langle 1\rangle}(X, t)$ is prescribed for $X \in \partial\mathcal{V}_0$,

$$\left.\begin{aligned}\mathcal{G}U^{\langle 2\rangle} + \nabla \Pi^{\langle 2\rangle} &= M_2,\\ \operatorname{div} U^{\langle 2\rangle} &= \theta_2,\end{aligned}\right\} \tag{XII.13}$$

$U^{\langle 2\rangle}(X, t)$ is prescribed for $X \in \partial\mathcal{V}_0$,

Initial history of $u^{\langle 2\rangle}(X, t)$, $t \leqslant 0$ is prescribed in $\mathcal{V}_0$, where

$$M_2 = F^{\mathrm{T}\langle 1\rangle} \cdot \nabla p^{\langle 1\rangle} + \operatorname{div}\{\beta^{[2]} B^{\langle 1\rangle} \cdot B^{\langle 1\rangle}$$

$$+ \int_0^\infty \zeta(s)[-G^{\langle 1\rangle}(s) \cdot B^{\langle 1\rangle} + 2(F^{\mathrm{T}\langle 1\rangle}(t-s) - F^{\mathrm{T}\langle 1\rangle}(t)) \cdot E^{\langle 1\rangle}(t-s)]\,\mathrm{d}s$$

$$+ \int_0^\infty \zeta^{[2]}(s)[B^{\langle 1\rangle} \cdot G^{\langle 1\rangle}(s) + G^{\langle 1\rangle}(s) \cdot B^{\langle 1\rangle}]\,\mathrm{d}s$$

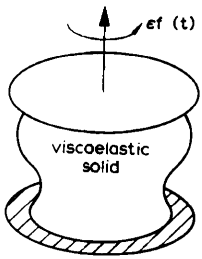
$$+ \int_0^\infty \int_0^\infty \alpha(s_1, s_2) G^{\langle 1\rangle}(s_1) \cdot G^{\langle 1\rangle}(s_2)\,\mathrm{d}s_1\,\mathrm{d}s_2\}$$

and

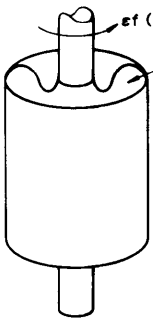
$$\theta_2 = \tfrac{1}{2}\operatorname{tr}[F^{\langle 1\rangle} \cdot F^{\langle 1\rangle}].$$

The higher-order problems have a similar structure and higher-order problems may be solved sequentially.

The theory just sketched presumably governs all small amplitude motions perturbing the natural state of viscoelastic solids satisfying (XI.2). However, the theory is almost completely useless without procedures to determine the values of the material parameters β, $\beta^{[2]}$, $\zeta(s)$, $\zeta^{[2]}(s)$ and $\alpha(s_1, s_2)$. As in the theory of fluids, we need solutions of the perturbation equations leading to the design of experiments to determine the values of the material parameters. And, again as in fluids, we are presently studying the possibility of using free surface problems for rheometrical measurements of solids. In such problems, as in the examples sketched below, the shape of the free surface depends on the material parameters and by measuring the shape we can determine the values of some of the parameters.



The top and bottom plates are rigid and are rotated in their own plane



Rotation of an annulus of viscoelastic material bonded to a stationary rigid cylinder (a Weissenberg effect in solids).

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