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D. D. Joseph

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FACTORIZATION THEOREMS AND REPEATED BRANCHING OF SOLUTIONS AT A SIMPLE EIGENVALUE*

D. D. Joseph

Department of Aerospace Engineering and Mechanics
University of Minnesota
Minneapolis, Minnesota 55455

STABILITY AND BIFURCATION IN $\mathbb{R}_1$†

In this paper I prove factorization theorems which show that, under certain typical hypotheses, the stability of steady and time-periodic solutions can change only at a turning point or at a point of bifurcation.

It is best to start in $\mathbb{R}_1$ where the hypotheses require C_2 smoothness plus a transversality (strict crossing) condition of Hopf's type. I show that these hypotheses imply that points of bifurcation are just double points of plane curves, that other types of bifurcation (double points with higher-order contact, triple points, and higher multiple points) are arbitrarily excluded. The same conclusions hold for general problems in Banach space when the eigenvalue of the Fréchet derivative on the solution branch is algebraically simple. The conclusion applies to symmetry-breaking steady bifurcation of steady solutions of partial differential equations of evolution type and to symmetry breaking T-periodic bifurcations of T-periodic solutions of partial differential equations of evolution type.

We consider an evolution equation in $\mathbb{R}_1$ of the form

$$u_t + F(\mu, u) = 0 \tag{1}$$

where $F(\cdot, \cdot)$ has two continuous derivatives in $\mathbb{R}_1 \times \mathbb{R}_1$. It is conventional in the study of stability of bifurcation to arrange things so that

$$F(\mu, 0) = 0 \quad \text{for all} \quad \mu \in \mathbb{R}_1. \tag{2}$$

But we shall not require (**2**). Instead we require that equilibrium solutions of (1) have $u = \epsilon, \epsilon_t = 0$, and

$$F(\mu, \epsilon) = 0. \tag{3}$$

The study of bifurcation of equilibrium solutions of the autonomous problem (**1**) is equivalent to the study of singular points of the plane curves (**3**).

In our study of equilibrium solutions (**3**) it is desirable to introduce the following classification of points:

(i) A *regular point* of $F(\mu, \epsilon) = 0$ is one for which the implicit function theorem works,

$$F_\mu \neq 0 \quad \text{or} \quad F_\epsilon \neq 0. \tag{4}$$

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†This section elaborates on joint work initiated by S. Rosenblat[8] and D. D. Joseph.[5]

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If (4) holds, then, we can find a unique curve $\mu = \mu(\epsilon)$ or $\epsilon = \epsilon(\mu)$ through the point.

(ii) A *regular turning point* is a point at which $\mu_\epsilon(\epsilon)$ changes sign and $F_\mu(\mu, \epsilon) \neq 0$.

(iii) A *singular point* of the curve $F(\mu, \epsilon) = 0$ is a point at which

$$F_\mu = F_\epsilon = 0 \tag{5}$$

(iv) A *double point* of the curve $F(\mu, \epsilon) = 0$ is a singular point through which pass two and only two branches of $F(\mu, \epsilon) = 0$ possessing distinct tangents.

(v) A *singular turning (double) point* of the curve $F(\mu, \epsilon)$ is a double point at which μ_ϵ changes sign.

(vi) A *cusp point* of the curve $F(\mu, \epsilon) = 0$ is a point of higher order contact between two branches of the curve. The two branches of the curve have the same tangent at a cusp point.

(vii) A *higher order singular point* of the curve $F(\mu, \epsilon) = 0$ is a singular point at which all three second derivatives of $F(\mu, \epsilon)$ are null.

I am going to connect the study of stability to the study of bifurcation under the "strict crossing" hypothesis introduced by Hopf and used in almost all studies of bifurcation and stability. I will explain what is meant by "strict crossing" in due course; for now it will suffice to remark that this hypothesis restricts the study of bifurcation to *double points;* cusp points and higher order singular points are excluded.

It is necessary to be precise about double points. Suppose (μ_0, ϵ_0) is a singular point. Then equilibrium curves passing through the singular points satisfy

$$2F(\mu, \epsilon) = F_{\mu\mu}\delta\mu^2 + 2F_{\epsilon\mu}\delta\epsilon\delta\mu + F_{\epsilon\epsilon}\delta\epsilon^2 + o(\delta\mu^2 + \delta\epsilon\delta\mu + \delta\epsilon^2) = 0 \tag{6}$$

where $\delta\mu = \mu - \mu_0$, $\delta\epsilon = \epsilon - \epsilon_0$, $F_{\mu\mu} = F_{\mu\mu}(\mu_0, \epsilon_0)$, and so forth. In the limit, as $(\mu, \epsilon) \to (\mu_0, \epsilon_0)$ the Equation 6 for the curves $F(\mu, \epsilon) = 0$ reduce to the quadratic equation.

$$F_{\mu\mu}\mathrm{d}\mu^2 + 2F_{\epsilon\mu}\mathrm{d}\epsilon\mathrm{d}\mu + F_{\epsilon\epsilon}\mathrm{d}\epsilon^2 = 0 \tag{7}$$

for the tangents to the curve. We find that

$$\begin{bmatrix} \mu_\epsilon^{(1)} & (\epsilon_0) \\ \mu_\epsilon^{(2)} & (\epsilon_0) \end{bmatrix} = -\frac{F_{\epsilon\mu}}{F_{\mu\mu}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \sqrt{\frac{D}{F_{\mu\mu}^2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} \tag{8}$$

or

$$\begin{bmatrix} \epsilon_\mu^{(1)} & (\mu_0) \\ \epsilon_\mu^{(2)} & (\mu_0) \end{bmatrix} = -\frac{F_{\epsilon\mu}}{F_{\epsilon\epsilon}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \sqrt{\frac{D}{F_{\epsilon\epsilon}^2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} \tag{9}$$

where

$$D = F_{\epsilon\mu}^2 - F_{\mu\mu}F_{\epsilon\epsilon}. \tag{10}$$

If $D < 0$ there are no real tangents through (μ_0, ϵ_0) and the point (μ_0, ϵ_0) is an isolated point solution of $F(\mu, \epsilon) = 0$.

We shall consider the case when (μ_0, ϵ_0) is *not* a higher order singular point. Then (μ_0, ϵ_0) is a double point if and only if $D > 0$. If $D = 0$ then the slope at the singular point of higher order contact is given by (**3**) or (**4**). If $D > 0$ and $F_{\mu\mu} \neq 0$, then there are two tangents with slopes $\mu_\epsilon^{(1)}(\epsilon_0)$ and $\mu_\epsilon^{(2)}(\epsilon_0)$ given by (**8**). If $D > 0$ and $F_{\mu\mu} = 0$, then $F_{\epsilon\mu} \neq 0$ and

$$d\epsilon[2d\mu F_{\epsilon\mu} + d\epsilon F_{\epsilon\epsilon}] = 0 \tag{11}$$

and there are two tangents $\epsilon_\mu(\mu_0) = 0$ and $\mu_\epsilon(\epsilon_0) = -F_{\epsilon\epsilon}/2F_{\epsilon\mu}$. If $\epsilon_\mu(\mu_0) = 0$ then $F_{\mu\mu}(\mu_0, \epsilon_0) = 0$. So all possibilities are covered in the following two cases:

(i) $D > 0$, $F_{\mu\mu} \neq 0$ with tangents $\mu_\epsilon^{(1)}(\epsilon_0)$ and $\mu_\epsilon^{(2)}(\epsilon_0)$.
(ii) $D > 0$, $F_{\mu\mu} = 0$ with tangents $\epsilon_\mu(\mu_0) = 0$ and $\mu_\epsilon(\epsilon_0) = -F_{\epsilon\epsilon}/2F_{\epsilon\mu}$.

Now I am going to connect stability and bifurcation. To study the stability of the solution $u = \epsilon$, we study the linearized equation

$$Z_t + F_\epsilon(\mu, \epsilon) Z = 0$$

by the spectral method

$$Z = e^{-\gamma t} Z'$$

where

$$\gamma = F_\epsilon(\mu, \epsilon). \tag{12}$$

The solution $u = \epsilon$ is stable when $\gamma > 0$ and is unstable when $\gamma < 0$.

THEOREM 1 (Factorization Theorem). For every equilibrium solution $F(\mu, \epsilon) = 0$ for which $\mu = \mu(\epsilon)$ we have

$$\gamma(\epsilon) = F_\epsilon(\mu(\epsilon), \epsilon) = -\mu_\epsilon(\epsilon) F_\mu(\mu(\epsilon), \epsilon) \equiv -\mu_\epsilon \hat{\gamma}(\epsilon). \tag{13}$$

The proof of Theorem 1 follows from (**12**) and the equation

$$\frac{dF(\mu(\epsilon), \epsilon)}{d\epsilon} = F_\epsilon(\mu(\epsilon), \epsilon) + \mu_\epsilon(\epsilon) F_\mu(v(\epsilon), \epsilon) = 0. \tag{14}$$

This type of factorization may be proved for the stability of bifurcation solutions in Banach spaces more complicated than $\mathbb{R}_1$.[3,5,6,7] But the theorem is most easily understood in $\mathbb{R}_1$. One of the main implications of the factorization theorem is that $\gamma(\epsilon)$ *must change sign as* ϵ *is varied across a regular turning point.* This implies that the solution $u = \epsilon$, $\mu = \mu(\epsilon)$ is stable on one side of a regular turning point and is unstable on the other side.

THEOREM 2. (A) Any point (μ_0, ϵ_0) of the curve $\mu = \mu(\epsilon)$ for which $\hat{\gamma}(\epsilon_0) = 0$ is a singular point. (B) Any point (μ_0, ϵ_0) of the curve $\epsilon(\mu)$ for which $\gamma(\mu_0) = 0$ is a singular point.

The proof of (A) follows from (**13**) and the proof of (B) from

$$\gamma(\mu) = F_\epsilon(\mu, \epsilon(\mu)), \frac{dF}{d\mu} = F_\mu + \epsilon_\mu F_\epsilon = 0. \tag{15}$$

The next theorem connects the hypothesis of strict loss of stability to bifurcation into double points.

THEOREM 3. Suppose that (μ_0, ϵ_0) is a singular point and (A) $\gamma_\epsilon(\epsilon_0) \neq 0$ or (B) $\gamma_\mu(\mu_0) \neq 0$. Then (μ_0, ϵ_0) is a double point.

In case (A) we find from (**13**) that at the singular point $(\mu(\epsilon_0), \epsilon_0)$

$$\gamma_\epsilon(\epsilon_0) = F_{\epsilon\epsilon} + \mu_\epsilon F_{\epsilon\mu} = -\mu_\epsilon^2 F_{\mu\mu} - \mu_\epsilon F_{\epsilon\mu} \neq 0 \qquad (\mathbf{16})$$

Equation 16 shows that the characteristic quadratic (**7**) holds at $(\mu(\epsilon_0), \epsilon_0)$. Since there is a curve through this point, $D \geq 0$ and we need to show that $D \neq 0$. We shall assume that $D = F_{\epsilon\mu}^2 - F_{\mu\mu}F_{\epsilon\epsilon} = 0$ and show that this assumption contradicts (**16**). We first note that (**16**) implies that not all three of the second derivatives of F are null at $(\mu(\epsilon_0), \epsilon_0)$. If $F_{\mu\mu}F_{\epsilon\epsilon} \neq 0$ and $D = 0$ then (**8**) becomes $\mu_\epsilon(\epsilon_0) = -F_{\epsilon\mu}/F_{\mu\mu}$ and (**16**), may be written as $F_{\epsilon\epsilon} - F_{\epsilon\mu}^2/F_{\mu\mu} = -D/F_{\mu\mu} \neq 0$. So $D \neq 0$ after all. If $F_{\mu\mu}F_{\epsilon\epsilon} = 0$ and $D = 0$ then $F_{\epsilon\mu} = 0$ and (**16**) may be written as $\gamma_\epsilon = F_{\epsilon\epsilon} = -\mu_\epsilon^2 F_{\mu\mu} \neq 0$. So $D \neq 0$ after all.

In case (B) we solve $F(\mu, \epsilon) = 0$ for $\epsilon(\mu)$. At the singular point (μ_0, ϵ_0), we have strict loss of stability.

$$\gamma_\mu = F_{\epsilon\mu}(\mu_0, \epsilon(\mu_0) + \epsilon_\mu(\mu_0) F_{\epsilon\epsilon}(\mu_0, \epsilon(\mu_0)) \neq 0 \qquad (\mathbf{17})$$

and $D \geq 0$. Assuming $D = 0$ we find, using (**9**), that

$$\epsilon_\mu = -F_{\epsilon\mu}/F_{\epsilon\epsilon} \qquad (\mathbf{18})$$

if $F_{\epsilon\epsilon} \neq 0$. Then (**17**) and (**18**) imply that $\gamma_\mu = 0$, and if $D = 0$ and $F_{\epsilon\epsilon} = 0$, then $F_{\epsilon\mu} = 0$ so that $\gamma_\mu = 0$. So $D = 0$ is inconsistent with $\gamma_\mu \neq 0$ and $D > 0$ after all.

The analysis of bifurcation in $\mathbb{R}_1$ just given shows that double point bifurcation is implied by a strict crossing hypothesis of the Hopf type. The situation is more complicated when these hypotheses are relaxed. If $\gamma_\epsilon = 0$ when $\gamma = 0$ we may get cusp bifurcation; or if all three second derivatives vanish, then the cubic equation can give a triple point (three real roots for the slopes) or no bifurcation (two complex conjugate roots). If third derivatives also vanish we face the problem of classifying the roots of a quartic. For example, we may get four bifurcating branches.

It is possible to make precise statements about the stability of solutions near double points of bifurcation. All of the possibilities for the stability of double point bifurcation can be described by the cases (A) and (B), which were fully specified under (**11**). In case (A) two curves $\mu^{(1)}(\epsilon)$ and $\mu^{(2)}(\epsilon)$ pass through the double point (μ_0, ϵ_0). In case (B) two curves, $\epsilon^{(1)}(\mu)$ (with $\epsilon_\mu^{(1)}(\mu_0) = 0$) and $\mu_\epsilon^{(2)}$, pass through the double point. The eigenvalue $\gamma^{(1)}$ belongs to the curve with superscript (**1**) and $\gamma^{(2)}$ to the curve with superscript (**2**).

THEOREM 4. Suppose (μ_0, ϵ_0) is a double point. Then, in case (A),

$$\gamma^{(1)}(\epsilon) = -\mu_\epsilon^{(1)}(\epsilon)\{\hat{s}\sqrt{D}(\epsilon - \epsilon_0) + o(\epsilon - \epsilon_0)\}, \qquad (\mathbf{19})$$

and

$$\gamma^{(2)}(\epsilon) = \mu_\epsilon^{(2)}(\epsilon)\{\hat{s}\sqrt{D}(\epsilon - \epsilon_0) + o(\epsilon - \epsilon_0)\}, \qquad (\mathbf{20})$$

where $\hat{s} = F_{\mu\mu}/|F_{\mu\mu}|$ and D and $F_{\mu\mu}$ are evaluated at $\epsilon = \epsilon_0$. And in case (B),

$$\gamma^{(1)}(\mu) = s\sqrt{D}(\mu - \mu_0) + o(\mu - \mu_0) \qquad (\mathbf{21})$$

and

$$\gamma^{(2)}(\epsilon) = -s\mu_\epsilon^{(2)}(\epsilon)\{\sqrt{D}(\epsilon - \epsilon_0) + o(\epsilon - \epsilon_0)\} \tag{22}$$

where $s = F_{\epsilon\mu}/|F_{\epsilon\mu}|$

Proof. If $\mu = \mu((\epsilon)$ we have (**13**) in the form,

$$\begin{aligned}\gamma(\epsilon) &= -\mu_\epsilon(\epsilon)\,F_\mu(\mu(\epsilon),\epsilon)\\ &= -\mu_\epsilon(\epsilon)\{F_{\mu\mu}(\mu_0,\epsilon_0)\,\mu_\epsilon(\epsilon_0) + F_{\epsilon\mu}(\mu_0,\epsilon_0)(\epsilon - \epsilon_0) + o(\epsilon - \epsilon_0)\}\end{aligned} \tag{23}$$

The formulas (**19**) and (**20**) arise from (**23**) when $\mu_\epsilon(\epsilon_0)$ is replaced with the values given by (**8**). If $\epsilon = \epsilon(\mu)$ with $\epsilon_\mu(\mu_0) = 0$ then $F_{\mu\mu}(\mu_0,\epsilon_0) = 0$, $F_{\epsilon\mu}{}^2(\mu_0,\epsilon_0) = D$, and

$$\begin{aligned}\gamma(\mu) &= F_\epsilon(\mu,\epsilon(\mu)) = F_{\epsilon\mu}(\mu_0,\epsilon_0)(\mu - \mu_0) + o(\mu - \mu_0)\\ &= s\sqrt{D}(\mu - \mu_0) + o(\mu - \mu_0).\end{aligned}$$

Theorem 4 gives an exhaustive classification relating the stability of solutions near a double point to the slope of the bifurcation curves near that point. The result may be summarized as follows. Suppose $|\epsilon - \epsilon_0| > 0$ is small. Then (**19**) and (**20**) show that $\gamma^{(1)}(\epsilon)$ and $\gamma^{(2)}(\epsilon)$ have the same (different) sign if $\mu_\epsilon^{(1)}(\epsilon)$ and $\mu_\epsilon^{(2)}(\epsilon)$ have different (the same) sign. A similar conclusion can be drawn from (**21**) and (**22**). The possible distributions of stability of solutions is sketched in FIGURE 1.

Almost all the work in the theory of bifurcation is restricted to problems satisfying hypothesis (**2**). Then $F(\mu,0) = F_\mu(0,0) = F_{\mu\mu}(0,0) = 0$. The strict crossing hypothesis, introduced by Hopf, states that $\gamma_\mu^{(1)}(0) = F_{\mu\epsilon}(0,0) < 0$. Then we get $D > 0$ and $\gamma^{(2)}(\epsilon) = -\mu_\epsilon^{(2)}(\epsilon)\,\gamma_\mu^{(1)}(0)\,\{(\epsilon - \epsilon_0) + o(\epsilon - \epsilon_0)\}$.

We now suppose that $F(\cdot,\cdot)$ has four continuous partial derivatives and show what happens to the stability of solutions bifurcating at a cusp point of second-order contact and at a triple point. When $\mu = \mu(\epsilon)$ all derivatives $\mathfrak{F}(\epsilon) \equiv F(\mu(\epsilon),\epsilon) = 0$ vanish. Then we have (**14**).

$$\frac{d^2\mathfrak{F}}{d\epsilon^3} = F_{\epsilon\epsilon} + 2\mu_\epsilon F_{\epsilon\mu} + \mu_\epsilon{}^2 F_{\mu\mu} + \mu_{\epsilon\epsilon}F_\mu = 0, \tag{24}$$

$$\begin{aligned}\frac{d^3\mathfrak{F}}{d\epsilon^3} &= F_{\epsilon\epsilon\epsilon} + 3\mu_\epsilon F_{\epsilon\epsilon\mu} + 3\mu_\epsilon{}^2F_{\epsilon\mu\mu} + \mu_\epsilon{}^3F_{\mu\mu\mu} + 3\mu_{\epsilon\epsilon}F_{\epsilon\mu}\\ &\quad + 3\mu_{\epsilon\epsilon}\mu_\epsilon F_{\mu\mu} + \mu_{\epsilon\epsilon\epsilon}F_\mu = 0,\end{aligned} \tag{25}$$

$$\begin{aligned}\frac{d^4\mathfrak{F}}{d\epsilon^4} &= F_{\epsilon\epsilon\epsilon\epsilon} + 4\mu_\epsilon F_{\epsilon\epsilon\epsilon\mu} + 6\mu_\epsilon{}^2F_{\epsilon\epsilon\mu\mu} + 4\mu_\epsilon{}^3F_{\epsilon\mu\mu\mu} + \mu_\epsilon{}^4F_{\mu\mu\mu\mu} + 4\mu_{\epsilon\epsilon\epsilon}F_{\epsilon\mu}\\ &\quad + 4\mu_{\epsilon\epsilon\epsilon}\mu_\epsilon F_{\mu\mu} + 3\mu_{\epsilon\epsilon}{}^2F_{\mu\mu} + 6\mu_{\epsilon\epsilon}F_{\epsilon\epsilon\mu} + 12\mu_\epsilon\mu_{\epsilon\epsilon}F_{\epsilon\mu\mu}\\ &\quad + 6\mu_{\epsilon\epsilon}\mu_\epsilon{}^2F_{\mu\mu\mu} + \mu_{\epsilon\epsilon\epsilon\epsilon}F_\mu = 0\end{aligned} \tag{26}$$

When $\epsilon = \epsilon(\mu)$, $\mathfrak{f}(\mu) = F(\mu,\epsilon(\mu)) = 0$, and

$$\frac{d^2\mathfrak{f}}{d\mu^2} = F_{\mu\mu} + 2\epsilon_\mu F_{\epsilon\mu} + \epsilon_\mu{}^2F_{\epsilon\epsilon} + \epsilon_{\mu\mu}F_\epsilon = 0,$$

then

$$\frac{d^3 \ell}{d\mu^3} = F_{\mu\mu\mu} + 3\epsilon_\mu F_{\epsilon\mu\mu} + 3\epsilon_\mu{}^2 F_{\epsilon\epsilon\mu} + \epsilon_\mu{}^3 F_{\epsilon\epsilon\epsilon} + 3\epsilon_{\mu\mu} F_{\epsilon\mu} + 3\epsilon_{\mu\mu}\epsilon_\mu F_{\epsilon\epsilon} + \epsilon_{\mu\mu\mu} F_\epsilon = 0, \tag{27}$$

$$\frac{d^4 \ell}{d\mu^4} = F_{\mu\mu\mu\mu} - 4\epsilon_\mu F_{\epsilon\mu\mu\mu} + 6\epsilon_\mu{}^2 F_{\epsilon\epsilon\mu\mu} + 4\epsilon_\mu{}^3 F_{\epsilon\epsilon\epsilon\mu} + \epsilon_\mu{}^4 F_{\epsilon\epsilon\epsilon\epsilon} + 4\epsilon_{\mu\mu\mu} F_{\epsilon\mu} + 4\epsilon_{\mu\mu\mu}\epsilon_\mu F_{\epsilon\epsilon} + 3\epsilon_{\mu\mu}{}^2 F_{\epsilon\epsilon} + 6\epsilon_{\mu\mu} F_{\epsilon\mu\mu} + 12\epsilon_\mu\epsilon_{\mu\mu} F_{\epsilon\epsilon\mu} + 6\epsilon_{\mu\mu}\epsilon_\mu{}^2 F_{\epsilon\epsilon\epsilon} + \epsilon_{\mu\mu\mu\mu} F_\epsilon = 0.$$

At a cusp point $F = F_\epsilon = F_\mu = D = 0$. In case (A), $F_{\mu\mu} \neq 0$ $\mu_\epsilon(\epsilon_0) = -F_{\epsilon\mu}/F_{\mu\mu}$, (**24**) is satisfied identically, (**25**) becomes

$$F_{\epsilon\epsilon\epsilon} + 3\mu_\epsilon(\epsilon_0) F_{\epsilon\epsilon\mu} + 3\mu_\epsilon{}^2(\epsilon_0) F_{\epsilon\mu\mu} + \mu_\epsilon{}^3 F_{\mu\mu\mu} = 0,$$

and the coefficient of $\mu_{\epsilon\epsilon\epsilon}$ in (**26**) vanishes, leaving a quadratic equation for the

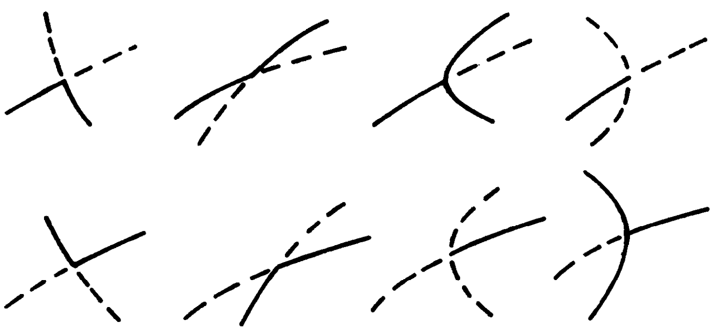

FIGURE 1. Stability of solutions in the neighborhood of double point bifurcation.

curvature $\mu_{\epsilon\epsilon}$

$$\mu_{\epsilon\epsilon}{}^2 + 2\mu_{\epsilon\epsilon}\xi/F_{\mu\mu} + \zeta/F_{\mu\mu} = 0 \tag{29}$$

where

$$\xi = F_{\epsilon\epsilon\mu} + 2\mu_\epsilon F_{\epsilon\mu\mu} + \mu_\epsilon{}^2 F_{\mu\mu\mu}$$

and

$$3\zeta = F_{\epsilon\epsilon\epsilon\epsilon} + 4\mu_\epsilon F_{\epsilon\epsilon\epsilon\mu} + 6\mu_\epsilon{}^2 F_{\epsilon\epsilon\mu\mu} + 4\mu_\epsilon{}^3 F_{\epsilon\mu\mu\mu} + \mu_\epsilon{}^4 F_{\mu\mu\mu\mu}.$$

Equation (**29**) has two roots

$$\begin{pmatrix} \mu_{\epsilon\epsilon}{}^{(1)} & (\epsilon_0) \\ \mu_{\epsilon\epsilon}{}^{(2)} & (\epsilon_0) \end{pmatrix} = -\frac{\xi}{F_{\mu\mu}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \sqrt{\frac{\mathfrak{D}_1}{F_{\mu\mu}{}^2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \tag{30}$$

where

$$\mathfrak{D}_1 = \xi^2 - F_{\mu\mu}\zeta.$$

In case (B), $F_{\mu\mu} = 0$ and $F_{\epsilon\epsilon} \neq 0$; and since $D = 0$, $F_{\epsilon\mu} = 0$ and $\epsilon_\mu(\mu_0) = 0$. Equation 27 then shows that $F_{\mu\mu\mu} = 0$ and (**28**) reduces to a quadratic equation for the curvature $\epsilon_{\mu\mu}(\mu_0)$:

$$\epsilon_{\mu\mu}{}^2 + 2\epsilon_{\mu\mu}F_{\epsilon\mu\mu}/F_{\epsilon\epsilon} + F_{\mu\mu\mu\mu}/3F_{\epsilon\epsilon} = 0 \tag{31}$$

Equation 31 has two roots

$$\begin{pmatrix} \epsilon_{\mu\mu}{}^{(1)} & (\mu_0) \\ \epsilon_{\mu\mu}{}^{(2)} & (\mu_0) \end{pmatrix} = -\frac{F_{\epsilon\mu\mu}}{F_{\epsilon\epsilon}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \sqrt{\frac{\mathfrak{D}_2}{F_{\epsilon\epsilon}{}^2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \tag{32}$$

where

$$\mathfrak{D}_2 = F_{\epsilon\mu\mu}{}^2 - F_{\epsilon\epsilon}F_{\mu\mu\mu\mu}/3F_{\epsilon\epsilon}.$$

At a point of second-order contact the two curves have common tangents and different real-valued curvatures. It follows that $\mathfrak{D}_1 > 0$ or $\mathfrak{D}_2 > 0$ at a point of second-order contact.

Restricting our attention to a point of second-order contact, we expand the factor $F_\mu(\mu(\epsilon), \epsilon)$ into a series of powers of $(\epsilon - \epsilon_0)$ and find that in case (A)

$$\begin{pmatrix} \gamma^{(1)} & (\epsilon) \\ \gamma^{(2)} & (\epsilon) \end{pmatrix} = -\tfrac{1}{2}\hat{s}\sqrt{\mathfrak{D}_1}\begin{pmatrix} \mu_\epsilon{}^{(1)} & (\epsilon) \\ -\mu_\epsilon{}^{(2)} & (\epsilon) \end{pmatrix}(\epsilon - \epsilon_0)^2 + O(\epsilon - \epsilon_0)^3. \tag{33}$$

In case (B), we expand $\gamma(\mu) = F_\epsilon(\mu, \epsilon(\mu))$ into a series of powers in $(\mu - \mu_0)$ and find that

$$\begin{pmatrix} \gamma^{(1)} & (\mu) \\ \gamma^{(2)} & (\mu) \end{pmatrix} = -\tfrac{1}{2}s\sqrt{\mathfrak{D}_2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}(\mu - \mu_0)^2 + O(\mu - \mu_0)^3. \tag{34}$$

It follows from (**32**) and (**34**) that the stability of any branch passing through a cusp point of second order changes sign if and only if $\mu_\epsilon(\epsilon)$ does. The possible distributions of stability at a cusp point are exhibited in FIGURE 2.

We turn next to the case in which all second order derivatives of $F(\mu, \epsilon) = 0$ are null at a singular point. Confining our attention to the case in which $F_{\mu\mu\mu} \neq 0$ we may write (**25**) as

$$(\mu_\epsilon - \mu_\epsilon{}^{(1)})(\mu_\epsilon - \mu_\epsilon{}^{(2)})(\mu_\epsilon - \mu_\epsilon{}^{(3)}) = \mu_\epsilon{}^3 + 3\mu_\epsilon{}^2\frac{F_{\epsilon\mu\mu}}{F_{\mu\mu\mu}} + 3\mu_\epsilon\frac{F_{\epsilon\epsilon\mu}}{F_{\mu\mu\mu}} + \frac{F_{\epsilon\epsilon\epsilon}}{F_{\mu\mu\mu}} = 0 \tag{35}$$

where $\mu_\epsilon{}^{(1)}$, $\mu_\epsilon{}^{(2)}$ and $\mu_\epsilon{}^{(3)}$ are values of $\mu_\epsilon(\epsilon)$ at $\epsilon = \epsilon_0$. It follows from (**35**) that

$$\frac{F_{\epsilon\epsilon\mu}}{F_{\mu\mu\mu}} = \frac{1}{3}(\mu_\epsilon{}^{(1)}\mu_\epsilon{}^{(2)} + \mu_\epsilon{}^{(1)}\mu_\epsilon{}^{(3)} + \mu_\epsilon{}^{(2)}\mu_\epsilon{}^{(3)}) \tag{36}$$

and

$$\frac{F_{\epsilon\mu\mu}}{F_{\mu\mu\mu}} = -\frac{1}{3}(\mu_\epsilon^{(1)} + \mu_\epsilon^{(2)} + \mu_\epsilon^{(3)}). \tag{37}$$

If the three roots of (35) are real and distinct three bifurcating solutions pass through the singular point (μ_0, ϵ_0). The stability of these branches may be determined from the sign of

$$\begin{aligned}\gamma(\epsilon) &= -\mu_\epsilon(\epsilon)F_\mu(\mu(\epsilon),\epsilon)\\ &= -\tfrac{1}{2}\mu_\epsilon(\epsilon)\{\xi(\epsilon-\epsilon_0)^2 + O(\epsilon-\epsilon_0)^3\}\\ &= -\tfrac{1}{2}\mu_\epsilon(\epsilon)F_{\mu\mu\mu}\{\tfrac{1}{3}(\mu_\epsilon^{(1)}\mu_\epsilon^{(2)} + \mu_\epsilon^{(1)}\mu_\epsilon^{(3)} + \mu_\epsilon^{(2)}\mu_\epsilon^{(3)})\\ &\quad - \tfrac{2}{3}\mu_\epsilon(\epsilon_0)(\mu_\epsilon^{(1)} + \mu_\epsilon^{(2)} + \mu_\epsilon^{(3)}) + \mu_\epsilon^2(\epsilon_0)\}(\epsilon-\epsilon_0)^2\\ &\quad + O(\epsilon-\epsilon_0)^3.\end{aligned}$$

We find that

$$\begin{bmatrix}\gamma(\epsilon)^{(1)}\\ \gamma(\epsilon)^{(2)}\\ \gamma(\epsilon)^{(3)}\end{bmatrix} = -\tfrac{1}{6}F_{\mu\mu\mu}\begin{bmatrix}\mu_\epsilon^{(1)}(\epsilon)(\mu_\epsilon^{(1)} - \mu_\epsilon^{(2)})(\mu_\epsilon^{(1)} - \mu_\epsilon^{(3)})\\ \mu_\epsilon^{(2)}(\epsilon)(\mu_\epsilon^{(1)} - \mu_\epsilon^{(2)})(\mu_\epsilon^{(3)} - \mu_\epsilon^{(2)})\\ \mu_\epsilon^{(3)}(\epsilon)(\mu_\epsilon^{(1)} - \mu_\epsilon^{(3)})(\mu_\epsilon^{(2)} - \mu_\epsilon^{(3)})\end{bmatrix}(\epsilon-\epsilon_0)^2 + O(\epsilon-\epsilon_0)^3 \tag{38}$$

where it may be assumed, without loss of generality, that $\mu_\epsilon^{(1)} > \mu_\epsilon^{(2)} > \mu_\epsilon^{(3)}$. The

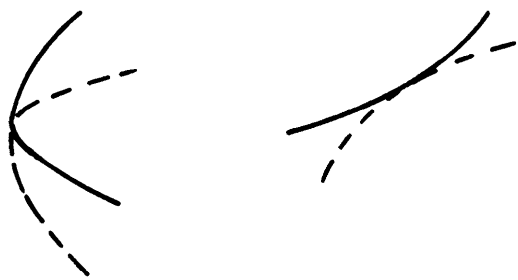

FIGURE 2. Stability of solutions bifurcating at a cusp point of second order.

distribution of stability of the three distinct branches is easily determined from (38). We leave further deductions about bifurcation and stability at a singular point where the second derivatives are all null as an exercise for the interested reader. It will suffice here to remark that the stability of a branch passing through such a point can change if and only if $\mu_\epsilon(\epsilon)$ changes sign there.

In the next two theorems, I give a global characterization of the stability of equilibrium solutions. I first note that $\gamma \neq 0$ at regular points of the curve $F(\mu, \epsilon) = 0$ at which $\mu_\epsilon \neq 0$. The factorization theorem $\gamma = -\mu_\epsilon(\epsilon)F_\mu(\mu(\epsilon),\epsilon)$ shows that γ can change sign at a stationary regular point ($\mu_\epsilon = 0$, $F_\mu \neq 0$) only if it be a turning point.

THEOREM 5. Assume that all singular points of solutions of $F(\mu, \epsilon) = 0$ are double points. The stability of such solutions must change at each regular turning point and at each singular point (which is not a turning point) and only at such points.

Theorem 5 gives a fairly complete catalogue of the stability of solution on connected branches of $F(\mu, \epsilon) = 0$. But solutions of $F(\mu, \epsilon) = 0$ need not be connected (see FIGURE 3 for a typical example). It is, however, possible to relate the stability of equilibrium solutions of isolated branches that pierce the line $\mu =$ const with solutions of $F(\mu, \epsilon) = 0$. Label these points in an increasing sequence $\epsilon_1 < \epsilon_2 < \cdots < \epsilon_n$. Then $F(\mu, \epsilon_1) = F(\mu, \epsilon_2) = \cdots = F(\mu, \epsilon_n) = 0$. Of course,

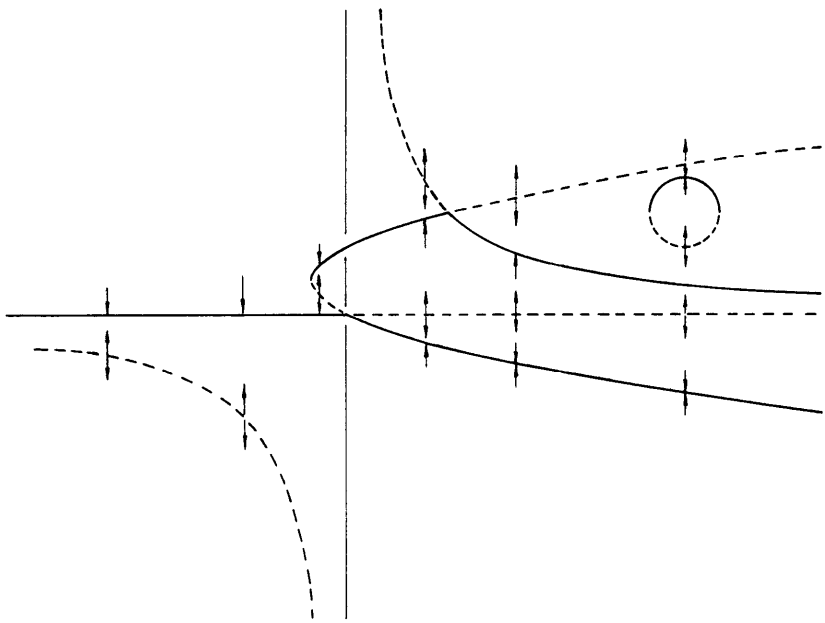

FIGURE 3. Bifurcation, stability, and domains of attraction of equilibrium solutions of

$$\frac{du}{dt} = u(9 - \mu u)(\mu + 2u - u^2)([\mu - 10]^2 + [u - 3]^2 - 1).$$

The equilibrium solution $\mu = 9/u$ in the third quadrant and the circle are isolated solutions that cannot be obtained by bifurcation analysis.

$F(\mu, \epsilon)$ is of one sign between any two successive zeros. It follows that if $F_\epsilon(\mu, \epsilon_l) < 0$, then $F_\epsilon(\mu, \epsilon_{l+1}) \geq 0$ and $F_\epsilon(\mu, \epsilon_{l-1}) \geq 0$. Suppose, for example, that $F_\epsilon(\mu, \epsilon_{l-1}) = 0$, then $\gamma(\epsilon_{l-1}) = 0$ so that (μ, ϵ_{l-1}) is a singular point or a regular stationary point, $\mu_\epsilon(\epsilon_{l-1}) = 0$. The stability of solutions at regular stationary points is completely described by Theorem 1 and at singular double points by Theorem 5. Isolated (conjugate) singular points may be ignored because $D < 0$ implies that $F(\mu_0, \epsilon_0) = 0$ is an extreme value of $F(\mu, u)$, which does not change sign as ϵ is varied across ϵ_0 at a fixed $\mu = \mu_0$. In any event, it will usually be possible to shift μ slightly so that $F_\epsilon(\mu, \epsilon_l) \neq 0$ at each and every piercing point.

THEOREM 6. If $F_\epsilon(\mu, \epsilon_l) \neq 0$ on each and every piercing point of solutions of $F(\mu, \epsilon) = 0$ and the line $\mu = $ const, then the sign of $\gamma(\epsilon)$ at such points is a sequence of alternating sign. If the solution (μ, ϵ_l) of $F(\mu, \epsilon) = 0$ is stable (unstable) then the solutions (μ, ϵ_{l-1}) and (μ, ϵ_{l+1}) are unstable (stable).

Theorem 6 is an obvious global extension in $\mathbb{R}_1$ of a local theorem of H. Weinberger,[9] which holds in a general Banach space. Similar conclusions have been derived by Benjamin[1,2] in interesting applications of the theory of Leray-Schauder Degree to steady solutions of the Navier Stokes Equations.

FACTORIZATION THEOREMS[3,5,6,7]

Consider the evolution equation

$$\frac{dV}{dt} + \underline{F}(t, \mu, \underline{V}) = 0, \tag{39}$$

where $\underline{V}$ is an element of a Banach space $\underline{X}$ and $\underline{F}(t, \mu, \underline{V})$ is analytic map from $\mathbb{R}_1 \times \mathbb{C} \times \underline{X}$ to $\underline{Y} \supset \underline{X}$. Two types of problems will be considered. In the first the external data is steady and $\underline{F}(t, \mu, \underline{V}) = \underline{F}(\mu, \underline{V})$ is independent t. In this case (**39**) is an autonomous problem. In the second case the external data is T-periodic and $\underline{F}(t, \cdot, \cdot) = \underline{F}(t + T, \cdot, \cdot)$.

In the applications I have in mind $\underline{V} \in \underline{X}$ corresponds to a solutions $\underline{V}(\underline{x}, t)$ of a partial differential equation defined over a region $\mathcal{V}$ of space. It is convenient to introduce a scalar product

$$(u, v) = (\overline{v, u}) = \int_{\mathcal{V}} u(x)\bar{v}(x)\, d\mathcal{V}, \tag{40}$$

where the overbar designates complex conjugate. I shall discuss three different types of equilibrium solutions of (**39**):

(a) Steady solutions of autonomous problems

$$\underline{V} = \underline{u}(\epsilon), \quad \mu = \mu(\epsilon),$$

(b) Limit cycle solutions of autonomous problems (Hopf's type)

$$\underline{V} = \underline{u}(s, \epsilon) = \underline{u}(s + 2\pi, \epsilon), \quad s = \omega(\epsilon)t, \quad \mu = \mu(\epsilon),$$

(c) nT-periodic solutions of T-periodic problems[4]

$$\underline{V} = \underline{u}(t, \epsilon) = \underline{u}(t + nT, \epsilon), \quad \mu = \mu(\epsilon), \quad n = 1, 2, 3, 4.$$

I have assumed that the equilibrium solutions can be defined parametrically through a parameter $\epsilon = f(\underline{u})$ where f is a linear functional that will distinguish different solutions. For example, f can be taken as a projection of the solution into a certain null-space, ϵ is then an amplitude and the quantities $\underline{u}$, μ, ω are known to be analytic in ϵ when $\underline{F}$ is.

The equilibrium solutions satisfy

$$\underline{F}(\mu(\epsilon), \ \underline{u}(\epsilon)) = 0, \tag{41a}$$

$$\omega(\epsilon)\dot{\underline{u}}(s, \epsilon) + \underline{F}[\mu(\epsilon), \ u(s, \epsilon)] = 0, \tag{41b}$$

$$\dot{\underline{u}}(t, \epsilon) + \underline{F}[t, \mu(\epsilon), \ \underline{u}(t, \epsilon)] = 0. \tag{41c}$$

Differentiating (**41**) with respect to ϵ we get

$$\mu_\epsilon \underline{F}_\mu[\mu(\epsilon), \underline{u}(\epsilon)] + \underline{F}_u[\mu(\epsilon), \underline{u}(\epsilon) | u_\epsilon] = 0, \qquad \textbf{(42a)}$$

$$\omega_\epsilon \underline{\dot{u}} + \mu_\epsilon \underline{F}_\mu[\mu(\epsilon), \underline{u}(s,\epsilon)] + \mathcal{J}\, \underline{u}_\epsilon = 0, \qquad \textbf{(42b)}$$

$$\mu_\epsilon \underline{F}_\mu[\mu(\epsilon), \underline{u}(t,\epsilon)] + \mathbb{J}\, \underline{u}_e = 0. \qquad \textbf{(42c)}$$

where

$$\mathcal{J}(\cdot) = \omega(\epsilon)(\dot{\cdot})(s,\epsilon) + \underline{F}_u[\mu(\epsilon), \underline{u}(s,\epsilon) | (\cdot)],$$

$$\mathbb{J}(\cdot) = (\dot{\cdot})(t,\epsilon) + F_u[t, \mu(\epsilon), u(t,\epsilon) | (\cdot)],$$

and

$$\operatorname{dom} \underline{F}_u[\mu(\epsilon), \underline{u}(\epsilon) | \cdot] = \underline{X}$$

$$\operatorname{dom} \mathcal{J} = \underline{X}_{2\pi} = \underline{X} \cap 2\pi\text{-periodic functions of } s,$$

$$\operatorname{dom} \mathbb{J} = \underline{X}_{nT} = \underline{X} \cap nT\text{-periodic functions of } t.$$

We define scalar products on $X_{2\pi}$, and X_{nT}

$$\langle \underline{u}, \underline{v} \rangle_{2\pi} \equiv \frac{1}{2\pi} \int_0^{2\pi} (\underline{u}(s,\epsilon), \underline{v}(\underline{s},\epsilon))\, d\underline{s},$$

$$\langle \underline{u}, \underline{v} \rangle_{nT} \frac{1}{nT} \int_0^{nT} [\underline{u}(t,\epsilon), \underline{v}(t,\epsilon)]\, ds.$$

The linearized equations for small disturbances Z of equilibrium solution u are

$$\underline{\dot{Z}} + \underline{F}_u[\mu(\epsilon), \underline{u}(\epsilon) | \underline{Z}] = 0, \qquad \textbf{(43a)}$$

$$\underline{\dot{Z}} + \underline{F}_u[\mu(\epsilon), \underline{u}(s,\epsilon) | \underline{Z}] = 0, \qquad \textbf{(43b)}$$

$$\underline{\dot{Z}} + \underline{F}_u(t, \mu(\epsilon), \underline{u}(t,\epsilon) | Z] = 0. \qquad \textbf{(43c)}$$

Spectral problems for these equations may be formed using the method of Floquet. Setting $\underline{Z} = e^{-\gamma t} \underline{\zeta}$ we get

$$-\gamma \underline{\zeta} + \underline{F}_u[\mu(\epsilon), \underline{u}(\epsilon) | \underline{\zeta}] = 0, \qquad \textbf{(44a)}$$

$$-\gamma \underline{\zeta} + \mathcal{J}\, \underline{\zeta} = 0, \underline{\zeta}(s) = \underline{\zeta}(s + 2\pi), \qquad \textbf{(44b)}$$

$$-\gamma \underline{\zeta} + \mathbb{J}\, \underline{\zeta} = 0, \underline{\zeta}(t) = \underline{\zeta}(t + nT). \qquad \textbf{(44c)}$$

The adjoint spectral problems are

$$-\overline{\gamma} \underline{\zeta}^* + \underline{F}_u^*(\mu(\epsilon), \underline{u}(\epsilon) | \underline{\zeta}^*) = 0, \qquad \textbf{(45a)}$$

$$-\overline{\gamma} \underline{\zeta}^* + \mathcal{J}^* \underline{\zeta}^* = 0, \qquad \underline{\zeta}^*(s) = \underline{\zeta}^*(s + 2\pi), \qquad \textbf{(45b)}$$

$$-\overline{\gamma} \underline{\zeta}^* + \mathbb{J}^* \underline{\zeta}^* = 0, \qquad \underline{\zeta}^*(t) = \underline{\zeta}^*(t + nT). \qquad \textbf{(45c)}$$

Now we can start to work out the factorization. We form scalar products of (**42**) with adjoint eigenfunctions and find that Equations 42 are solvable only if

$$\gamma(\underline{u}_\epsilon, \underline{\zeta}^*) + \mu_\epsilon(\underline{F}_\mu, \underline{\zeta}^*) = 0, \tag{46a}$$

$$\gamma < \underline{u}_\epsilon, \underline{\zeta}^* >_{2\pi} + \mu_\epsilon < \underline{F}_\mu, \underline{\zeta}^* >_{2\pi} = 0, \tag{46b}$$

$$\gamma < \underline{u}_\epsilon, \underline{\zeta}^* >_{nT} + \mu_\epsilon < \underline{F}_\mu, \underline{\zeta}^* >_{nT} = 0. \tag{46c}$$

In deriving (**46b**) we made use of the equation

$$\mathcal{J}\, \mathring{u}(s, \epsilon) = 0, \tag{47}$$

which follows, by differentiation with respect to s, from (**41b**). Comparing (**47**) with (**45b**), we find that

$$\gamma \; < \mathring{\underline{u}}, \underline{\zeta}^* >_{2\pi} = 0 \tag{48}$$

The first hypothesis (H.1) for the factorization

$$\gamma(\epsilon) = -\mu_\epsilon(\epsilon)\,\hat{\gamma}(\epsilon) \tag{49}$$

is that the inner products in the first term of each of Equations (**46**) does not vanish. Then using analyticity with respect to ϵ we conclude that coefficient of μ_ϵ in the equations that arise when (**46**) and (**49**) are combined must vanish.‡ Hence

$$\hat{\gamma}(\epsilon) = (\underline{F}_\mu, \underline{\zeta}^*)/(\underline{u}_\epsilon, \underline{\zeta}^*) \tag{50a}$$

$$\hat{\gamma}(\epsilon) = < \underline{F}_\mu, \underline{\zeta}^* >_{2\pi} / < \underline{u}_\epsilon, \underline{\zeta}^* >_{2\pi}, \tag{50b}$$

$$\hat{\gamma}(\epsilon) = < \underline{F}_\mu, \underline{\zeta}^* >_{nT} / < \underline{u}_\epsilon, \underline{\zeta}^* >_{nT} \tag{50c}$$

The second hypothesis (H.2) is that γ is an algebraically simple eigenvalue of the operators $F_u(\mu, \underline{u}|\cdot)$, $\mathcal{J}$ and $\mathbf{J}$ and these operators are Fredholm with compact resolvents from $\underline{Y}$, $\underline{Y}_{2\pi}$, $\underline{Y}_{nT}$ to $\underline{X}$, $\underline{X}_{2\pi}$, $\underline{X}_{nT}$.

THEOREM 7. (Factorization Theorem). If (H.1) and (H.2) hold then (**49**) and (**50**) are valid along with the decompositions

$$\underline{\zeta} = b(\epsilon)\{\underline{u}_\epsilon(\epsilon) + \mu_\epsilon \underline{q}(\epsilon)\} \tag{51a}$$

$$\underline{\zeta} = b(\epsilon)\left\{-\frac{\omega_\epsilon(\epsilon)}{\gamma(\epsilon)}\mathring{\underline{u}}(s,\epsilon) + \underline{u}_\epsilon + \mu_\epsilon \underline{q}(s,\epsilon)\right\} \tag{51b}$$

$$\zeta = b(\epsilon)\{\underline{u}_\epsilon(t,\epsilon) + \mu_\epsilon \underline{q}(t,\epsilon)\} \tag{51c}$$

where $\underline{q}(\epsilon)$, $\underline{q}(s, \epsilon)$ and $\underline{q}(t, \epsilon)$ satisfy

$$-\hat{\gamma}\underline{u}_\epsilon + \underline{F}_\mu[\mu(\epsilon), \underline{u}(\epsilon)] + (\gamma \underline{q} - \underline{F}_u[\mu(\epsilon), \underline{u}(\epsilon)|\underline{q}] = 0 \text{ and } (\underline{q}, \underline{\zeta}^*) = 0, \tag{52a}$$

$$-\hat{\gamma}\underline{u}_\epsilon(s,\epsilon) + \underline{F}_\mu[\mu(\epsilon), \underline{u}(s,\epsilon)] + (\gamma - \mathcal{J})\underline{q} = 0,$$
$$< \underline{q}, \underline{\zeta}^* >_{2\pi} = 0, \text{ and } \underline{q}(s,\epsilon) = \underline{q}(s + 2\pi, \epsilon), \tag{52b}$$

$$-\hat{\gamma}\underline{u}_\epsilon + \underline{F}_\mu[\mu(\epsilon), \underline{u}(t,\epsilon)] + (\gamma - \mathbf{J})\underline{q} = 0,$$
$$< \underline{q}, \underline{\zeta}^* >_{nT} = 0, \text{ and } \underline{q}(t,\epsilon) = \underline{q}(t + nT, \epsilon). \tag{52c}$$

‡Assuming $\mu(\epsilon)$ is not constant on a branch.

Proof. (H.2) asserts that $\underline{F}_u$, $\mathcal{J}$, and $\mathbf{J}$ are Fredholm operators with compact resolvents. Since γ is to be an algebraically simple eigenvalue, each of the three Equations 52 is uniquely solvable for $\underline{q}$ on the complement of the null space associated with γ when the Equations 50 hold. The orthogonality of $\underline{q}$ and $\underline{\zeta}^*$ and $\underline{u}$ and $\underline{\zeta}^*$ Equation 48 imply that the normalizing factors $b(\epsilon)$ in (51) may be expressed as

$$\underline{\zeta} = \frac{(\underline{\zeta}, \underline{\zeta}^*)}{(\underline{u}_\epsilon, \underline{\zeta}^*)} [\underline{u}_\epsilon(\epsilon) + \mu_\epsilon \underline{q}(\epsilon)], \tag{53a}$$

$$\underline{\zeta} = \frac{<\underline{\zeta}, \underline{\zeta}^*>_{2\pi}}{<\underline{u}_\epsilon, \underline{\zeta}^*>_{2\pi}} \left[- \frac{\omega_\epsilon(\epsilon)}{\gamma(\epsilon)} \mathring{\underline{u}}(s, \epsilon) + \underline{u}_\epsilon(s, \epsilon) + \mu_\epsilon \underline{q}(s, \epsilon)\right], \tag{53b}$$

$$\underline{\zeta} = \frac{<\underline{\zeta}, \underline{\zeta}^*>_{nT}}{<\underline{u}_\epsilon, \underline{\zeta}^*>_{nT}} [\underline{u}_\epsilon(t, \epsilon) + \mu_\epsilon \underline{q}(t, \epsilon)]. \tag{53c}$$

This completes the proof of the factorization under the hypotheses (H.1) and (H.2).

The hypothesis (H.2) is insufficient for the discussion of turning points on the bifurcation curve for periodic solutions (Case II). At such points the factorization, (**49**) shows that $\gamma(\epsilon) = 0$ and a difficulty is already apparent in (**52b**). This difficulty may be traced to the fact that $\gamma(\epsilon) = 0$ cannot be an algebraically simple eigenvalue of $\gamma(\epsilon)$. The following theorem was proved by Joseph[6]:

The algebraic multiplicity of the eigenvalue $\gamma(\epsilon) = 0$ of $\mathcal{J}(\epsilon)$ is at least two. Relative to such an eigenvalue, we have at least a two-link Jordan chain

$$\mathcal{J}\, \mathring{\underline{u}}(s, \epsilon) = 0,$$

$$\mathcal{J}\{-(\underline{u}_\epsilon(s, \epsilon) + \mu_\epsilon \underline{q}(s, \epsilon)/\omega_\epsilon(\epsilon)\} = \mathring{\underline{u}}(s, \epsilon)$$

whenever $\omega_\epsilon \neq 0$ when $\gamma(\epsilon) = 0$. If $\omega_\epsilon = 0$ when $\gamma = 0$, then the geometric multiplicity of $\gamma(\epsilon) = 0$ is at least two, and $\mathring{u}$ and $\underline{u}_\epsilon + \mu_\epsilon \underline{q}$ are both eigenfunction on the null space of $\mathcal{J}(\epsilon)$.

If we suppose that $\omega_\epsilon \neq 0$ when $\gamma(\epsilon) = 0$ then the algebraic multiplicity of $\gamma(\epsilon) = 0$ is two and the geometric multiplicity is one. In this case the Riesz-Schauder Theory[6] implies that $<\underline{\zeta}, \underline{\zeta}^*>_{2\pi} = 0$. At such points the factorization theorem still holds but the normalizing factor *cannot* be taken as $<\underline{\zeta}, \underline{\zeta}^*>_{2\pi}/ <\underline{u}_\epsilon, \underline{\zeta}^*>_{2\pi}$.

When $\mu_\epsilon(\epsilon) = 0$ we find that $\gamma(\epsilon) = \mu_\epsilon(\epsilon)\hat{\gamma}(\epsilon) = 0$ and

$$\underline{\zeta} = b\underline{u}_\epsilon(\epsilon), \tag{54a}$$

$$\underline{\zeta} = b\left\{\frac{-\omega_\epsilon(\epsilon)}{\mu_\epsilon(\epsilon)\hat{\gamma}(\epsilon)} \mathring{\underline{u}}(s, \epsilon) + \underline{u}_\epsilon(s, \epsilon)\right\}, \tag{54b}$$

$$\underline{\zeta} = b\underline{u}_\epsilon(t, \epsilon). \tag{54c}$$

If $\omega_\epsilon \neq 0$, then in case II, $\underline{\zeta} \propto \mathring{\underline{u}}(s, \epsilon)$

We may now define, as in $\mathbb{R}_1$, a regular turning point as a point at which $\mu_\epsilon(\epsilon)$ changes sign and $\hat{\gamma}(\epsilon) \neq 0$ where $\hat{\gamma}(\epsilon)$ is given by (**50**).

STABILITY AND SYMMETRY-BREAKING BIFURCATION

I shall now specify the assumptions which insure that problems (a) and (c) under (**40**) are essentially problems in $\mathbb{R}_1$. What we need to insure is that γ is real-valued and algebraically simple at points at which $\gamma = 0$. If we assume that $\gamma = 0$ at singular points, then we cannot break the temporal symmetry pattern of a solution through bifurcation; instead we get new steady solutions as a bifurcation of old steady solutions and new T-periodic solutions from old T-periodic solutions. When $\gamma = 0$ at criticality, bifurcation from steady solutions to T-periodic solutions or from T-periodic solutions to nT-periodic solutions ($n = 3, 4$) or tori is impossible. In fluid mechanics or, more generally, in problems governed by partial differential equations, such bifurcations will lead to new patterns of spatial symmetry.

I shall consider the T-periodic problem (III) and require conditions that allow only double-point, T-periodic bifurcation. For this we need a strict crossing condition under the hypothesis:

(H.3) Let d be an open rectangle in the (μ, ϵ) plane and suppose that $\underline{u}(t) = \underline{u}(t + T)$ for some point in d and dom $\mathbb{J} = \underline{X}_T$ in d. Let γ satisfying (H.1) and (H.2) be real-valued and suppose that all of the other eigenvalues of $\mathbb{J}$ have positive real parts in d.

I have already proved the factorization theorem (Theorem 7) under the assumptions (H.1) and (H.2). This theorem is the equivalent of Theorem 1, which gives the form of the factorization $\mathbb{R}_1$. My aim now is to show that under the hypothesis (H.3) we get the equivalent of Theorems 2 through 5. These theorems concern the behavior of solutions near a singular point (μ_0, ϵ_0). Various definitions of the amplitude ϵ are possible and all the good ones are equivalent. In the present context, we use (H.1) to justify the following definitions: Let $\underline{u}[t, \mu(\epsilon), \epsilon] = \underline{u}[t + T, \mu(\epsilon), \epsilon] \equiv \underline{u}(\epsilon)$ and, in the case where $\mu_\epsilon(\epsilon) = \infty$, $\underline{u}[t, \mu, \epsilon(\mu)] = \underline{u}[t + T, \mu, \epsilon(\mu)] \equiv u(\mu)$. By $\underline{u}_0$ I mean $u(\epsilon_0) = \underline{u}(\mu_0)$ where, in a loose notation which confuses the values of a function with the function, I write $\underline{u}(\epsilon) = \underline{u}(\mu)$, $\underline{\zeta}(\epsilon)$, and $\underline{\zeta}^*(\epsilon) = \underline{\zeta}^*(\mu)$. The amplitude ϵ may be defined in any convenient way.

It is useful in the analysis to introduce the T-periodic operators from $\underline{X}_T$ to $\underline{Y}_T$:

$$\underline{G}(\mu, \underline{u}) = \dot{\underline{u}} + \underline{F}(t, \mu, \underline{u}) = 0, \tag{55}$$

$$\underline{G}_u(\mu, \underline{u} \mid \underline{v}) = \mathbb{J}\underline{v}, \tag{56}$$

$$\underline{G}_\mu(\mu, \underline{u}) = \underline{F}_\mu(t, \mu, \underline{u}), \tag{57}$$

$$\begin{aligned} \underline{G}_{uu}(\mu, \underline{u} \mid \underline{v}_1 \mid \underline{v}_2) &= \underline{F}_{uu}(t, \mu, \underline{u} \mid \underline{v}_1 \mid \underline{v}_2) \\ &= G_{uu}(\mu, \underline{u} \mid \underline{v}_2 \mid \underline{v}_1), \end{aligned} \tag{58}$$

$$\underline{G}_{\mu\mu}(\mu, \underline{u}) = \underline{F}_{\mu\mu}(t, \mu, \underline{u}), \tag{59}$$

$$\underline{G}_{u\mu}(\mu, \underline{u} \mid \underline{v}) = \underline{F}_{u\mu}(t, \mu, \underline{u} \mid \underline{v}). \tag{60}$$

A singular point is a point where

$$\mathcal{G}_\mu \equiv \langle \underline{G}_\mu(\mu_0, \underline{u}_0), \underline{\zeta}_0^* \rangle_T = 0 \tag{61}$$

and

$$\mathcal{G}_u \equiv \langle \underline{G}_{\dot{u}}[\mu_0, \underline{u}_0 |(\cdot)], \underline{\zeta}_0^* \rangle_T = 0. \tag{62}$$

Consider T-periodic solutions $\underline{u}(\epsilon)$ of

$$\underline{G}[\mu(\epsilon), \underline{u}(\epsilon)] = 0. \tag{63}$$

Differentiating with respect to ϵ, we get

$$\mu_\epsilon \underline{G}_\mu[\mu(\epsilon), \underline{u}(\epsilon)] + \underline{G}_u[\mu(\epsilon), \underline{u}(\epsilon) \,|\, \underline{u}_\epsilon] = 0 \tag{64}$$

and

$$\underline{G}_u[\mu(\epsilon), \underline{u}(\epsilon) \,|\underline{u}_{\epsilon\epsilon}] + \mu_{\epsilon\epsilon}\underline{G}_\mu[\mu(\epsilon), \underline{u}(\epsilon)] + \mu_\epsilon^2 \underline{G}_{\mu\mu}[\mu(\epsilon), \underline{u}(\epsilon)] + 2\mu_\epsilon \underline{G}_{u\mu}[\mu(\epsilon), \underline{u}(\epsilon) \,|\, \underline{u}_\epsilon] + \underline{G}_{uu}[\mu(\epsilon), \underline{u}(\epsilon) \,|\, \underline{u}_\epsilon \,|\, \underline{u}_\epsilon] = 0. \tag{65}$$

Consider T-periodic solutions $\underline{u}(\mu)$ of

$$\underline{G}[\mu, \underline{u}(\mu)] = 0. \tag{66}$$

Differentiating with respect to μ, we get

$$\underline{G}_\mu[\mu, \underline{u}(\mu)] + \underline{G}_u[\mu, \underline{u}(\mu) \,|\, \underline{u}_\mu] = 0 \tag{67}$$

and

$$\underline{G}_u[\mu, \underline{u}(\mu) \,|\, u_{\mu\mu}] + \underline{G}_{\mu\mu}[\mu, \underline{u}(\mu)] + 2\underline{G}_{u\mu}[\mu, \underline{u}(\mu) \,|\, \underline{u}_\mu] + \underline{G}_{uu}[\mu, \underline{u}(\mu) \,|\, \underline{u}_\mu \,|\, \underline{u}_\mu] = 0. \tag{68}$$

At the singular point (μ_0, ϵ_0) we find from (67) and (68) that

$$\mu_\epsilon^2(\epsilon_0) \langle \underline{G}_{\mu\mu}(\mu_0, \underline{u}_0), \underline{\zeta}^*_0 \rangle_T + 2\mu_\epsilon(\epsilon_0) \langle \underline{G}_{\mu\mu}[\mu_0, \underline{u}_0 \,|\, \underline{u}_\epsilon(\epsilon_0)], \underline{\zeta}^*_0 \rangle_T + \langle \underline{G}_{uu}(\mu_0, \underline{u}_0 \,|\, \underline{u}_\epsilon(\epsilon_0) \,|\, \underline{u}_\epsilon(\epsilon_0), \underline{\zeta}_0^* \rangle_T = 0 \tag{69}$$

and

$$\langle \underline{G}_{\mu\mu}(\mu_0, \underline{u}_0), \underline{\zeta}^*_0 \rangle_T + 2 \langle G_{u\mu}[v_0, \underline{u}_0 \,|\, \underline{u}_\mu(\mu_0)], \underline{\zeta}_0^* \rangle_T + \langle \underline{G}_{uu}[\mu_0, \underline{u}_0 \,|\, \underline{u}_\mu(\mu_0) \,|\, \underline{u}_\mu(\mu_0)], \underline{\zeta}^*_0 \rangle_T = 0. \tag{70}$$

Equations 69 and 70 are the starting place for the derivation of a characteristic quadratic equation, like (7), for determining the tangents to the curves intersecting at a double point. They are not suitable for this purpose as yet because the quantities $\underline{u}_\epsilon(\epsilon_0)$ and $\underline{u}_\mu(\mu_0)$ depend on the branch on which they are evaluated. But we may use the factorization theorem to derive characteristic quadratic equations of the appropriate form. With this aim in mind we turn to (53c), which may be written as

$$\begin{aligned}
\underline{\zeta}(\epsilon) &= \frac{\langle \underline{\zeta}, \underline{\zeta}^* \rangle_T}{\langle \underline{u}_\epsilon, \underline{\zeta}^* \rangle_T} \{\underline{u}_\epsilon(\epsilon) + \mu_\epsilon(\epsilon) \underline{q}(\epsilon)\} \\
&= \frac{\langle \underline{\zeta}, \underline{\zeta}^* \rangle_T}{\epsilon_\mu \langle \underline{u}_\epsilon, \underline{\zeta}^* \rangle_T} \epsilon_\mu \{\underline{u}_\epsilon + \mu_\epsilon(\epsilon) \underline{q}(\epsilon)\} \\
&= \frac{\langle \underline{\zeta}, \underline{\zeta}^* \rangle_T}{\langle \underline{u}_\mu, \underline{\zeta}^* \rangle_T} \{\underline{u}_\mu(\mu) + \underline{q}(\mu)\} = \underline{\zeta}(\mu),
\end{aligned} \tag{71}$$

where, in keeping with our notational convention, we have suppressed the T-periodic dependence of $\underline{\zeta}, \underline{\zeta}^*, \underline{u}, \underline{q}$ on t. At the singular point (μ_0, ϵ_0), $\underline{\zeta}_0$ satisfying $\underline{G}_u(\mu_0, \underline{u}_0 \mid \underline{\zeta}_0) = 0$ and $\underline{q}_0$ satisfying

$$F_\mu(\mu_0, \underline{u}_0) - \underline{G}_u(\mu_0, \underline{u}_0 \mid \underline{q}_0) = 0, \ <\underline{q}_0, \underline{\zeta}_0^*>_T = 0 \tag{72}$$

are determined by the point (μ_0, ϵ_0) and are independent of the branch passing through the point. We next introduce slope parameters

$$\tilde{\mu}_\epsilon(\epsilon) = \mu_\epsilon(\epsilon) \frac{<\underline{\zeta}(\epsilon), \underline{\zeta}^*(\epsilon)>_T}{<\underline{u}_\mu(\mu), \underline{\zeta}^*(\mu)>_T} \tag{73a}$$

and

$$\tilde{\epsilon}_\mu(\mu) = \frac{<\underline{\zeta}(\mu), \underline{\zeta}^*(\mu)>_T}{<\underline{u}_\mu(\mu), \underline{\zeta}^*(\mu)>_T}. \tag{73b}$$

Combining (71) evaluated at $\epsilon = \epsilon_0$ with (69) we find that

$$\tilde{\mu}_\epsilon^2(\epsilon_0)\mathcal{G}_{\mu\mu} + 2\tilde{\mu}_\epsilon(\epsilon_0)\mathcal{G}_{u\mu} + \mathcal{G}_{uu} = 0, \tag{74}$$

where

$$\mathcal{G}_{\mu\mu} = <\underline{G}_{\mu\mu}(\mu_0, \underline{u}_0), \underline{\zeta}^*_0>_T + <\underline{G}_{uu}(\mu_0, \underline{u}_0 \mid \underline{q}_0 \mid \underline{q}_0), \underline{\zeta}^*_0>_T$$
$$- 2<\underline{G}_{\mu u}(\mu_0, \underline{u}_0 \mid \underline{q}_0), \underline{\zeta}^*_0>_T,$$
$$\mathcal{G}_{u\mu} = <\underline{G}_{u\mu}(\mu_0, \underline{u}_0 \mid \underline{\zeta}_0), \underline{\zeta}^*_0>_T - <\underline{G}_{uu}(\mu_0, \underline{u}_0 \mid \underline{q}_0 \mid \underline{\zeta}_0), \underline{\zeta}^*_0>_T$$

and

$$\mathcal{G}_{uu} = <\underline{G}_{uu}(\mu_0, \underline{u}_0 \mid \underline{\zeta}_0 \mid \underline{\zeta}_0), \underline{\zeta}^*_0>_T$$

depend only on (μ_0, ϵ_0) and not on the branch passing through (μ_0, ϵ_0). Combining (71) with (70) we find that

$$\mathcal{G}_{\mu\mu} + 2\tilde{\epsilon}_\mu(\mu_0)\mathcal{G}_{u\mu} + \tilde{\epsilon}_\mu^2(\mu_0)\mathcal{G}_{uu} = 0. \tag{75}$$

To complete the transformation of the problem of T-periodic bifurcation into the framework of the analysis in $\mathbb{R}_1$ we need to obtain formulas expressing strict crossing. The following perturbation formulas hold at (μ_0, ϵ_0):

$$\gamma_\mu(\mu_0)\underline{\zeta}_0 = \underline{G}_u(\mu_0, \underline{u}_0 \mid \underline{\zeta}_\nu) + \underline{G}_{\mu\mu}(\mu_0, \underline{u}_0 \mid \underline{\zeta}_0)$$
$$+ \underline{G}_{uu}(\mu_0, \underline{u}_0 \mid \underline{u}_\mu(\mu_0) \mid \underline{\zeta}_0)$$

and

$$-\hat{\gamma}_\epsilon \underline{u}_\epsilon + \mu_\epsilon \underline{G}_{\mu\mu}(\mu_0, \underline{u}_0) + \underline{G}_{u\mu}(\mu_0, \underline{u}_0 \mid \underline{u}_\epsilon)$$
$$+ (\gamma_\epsilon \underline{q}_0 - \mu_\epsilon \underline{G}_{u\mu}(\mu_0, \underline{u}_0 \mid \underline{q}_0) - \underline{G}_{uu}(\mu_0, \underline{u}_0 \mid \underline{u}_\epsilon \mid \underline{q}_0))$$
$$- \underline{G}_u(\mu_0, \underline{u}_0 \mid \underline{q}_\epsilon) = 0.$$

Hence

$$\hat{\gamma}_\epsilon <\underline{u}_\epsilon, \underline{\zeta}^*_0>_T + \mu_\epsilon <\underline{G}_{\mu\mu}, \underline{\zeta}^*_0>_T + <\underline{G}_{u\mu}(\mu_0, \underline{u}_0 \mid \underline{u}_\epsilon), \underline{\zeta}_0^*>_T$$
$$- \mu_\epsilon <\underline{G}_{u\mu}(\mu_0, \underline{u}_0 \mid \underline{q}_0), \underline{\zeta}^*_0>_T - <\underline{G}_{uu}(\mu_0, \underline{u}_0 \mid \underline{u}_\epsilon \mid \underline{q}_0), \underline{\zeta}^*_0>_T = 0 \tag{76}$$

and

$$\gamma_\mu <\underline{\zeta}_0, \underline{\zeta}^*_0> = <\underline{G}_{u\mu}(\mu_0, \underline{u}_0 | \underline{\zeta}_0), \underline{\zeta}^*_0>_T + <\underline{G}_{uu}(\mu_0, \underline{u}_0 | \underline{u}_\mu | \underline{\zeta}_0), \underline{\zeta}^*_0>_T. \quad (77)$$

Introducing (**71**) and (**73a**) into (**76**), we find that

$$\hat{\gamma}_\epsilon(\epsilon_0) = \tilde{\mu}_\epsilon(\epsilon_0) g_{\mu\mu} - g_{u\mu}. \quad (78)$$

Introducing (**71**) and (**73b**) into (**77**) we find that

$$\gamma_\mu(\mu_0) = g_{u\mu} + \tilde{\epsilon}_\mu(\mu_0) g_{uu}. \quad (79)$$

We have now completed the reduction to $\mathbb{R}_1$ and can prove Theorems 1 through 5 for T-periodic solutions. The proof of these theorems, summarized as Theorem 8 follows, word for word, the proof given for $\mathbb{R}_1$ in the first section when we replace F_ϵ, F_μ, $F_{\mu\mu}$, $F_{\epsilon\mu}$, $F_{\epsilon\epsilon}$, and $D = F_{\epsilon\mu}^2 - F_{\epsilon\epsilon}F_{\mu\mu}$ with $\mathcal{G}_u$, $\mathcal{G}_\mu$, $\mathcal{G}_{\mu\mu}$, $\mathcal{G}_{u\mu}$, $\mathcal{G}_{uu}$, and $D = \mathcal{G}^2_{u\mu} - \mathcal{G}_{uu}\mathcal{G}_{\mu\mu}$.

THEOREM 8. Under the assumptions (H.1) and (H.2) the following factorization holds:

$$\gamma(\epsilon) = -\mu_\epsilon(\epsilon)\hat{\gamma}(\epsilon)$$

where $\hat{\gamma}(\epsilon)$ is given by (**50c**). The term $\gamma(\epsilon)$ must change sign as ϵ is varied across a regular turning point. Any point (μ_0, ϵ_0) of the curve $\mu = \mu(\epsilon)$ for which $\gamma(\epsilon_0) = 0$ is a singular point. If (H.3) holds and $\gamma_\epsilon(\epsilon_0) \neq 0$) or $\gamma_\mu(\mu_0) \neq 0$, then (μ_0, ϵ_0) is a double point. The stability of the two solutions bifurcating at the double point is determined in the linearized approximation by

$$\gamma^{(1)}(\epsilon) = -\mu_\epsilon^{(1)}(\epsilon)\{\hat{s}\sqrt{\tilde{D}}(\epsilon - \epsilon_0) + o(\epsilon - \epsilon_0)\}$$

and

$$\gamma^{(2)}(\epsilon) = \mu_\epsilon^{(2)}(\epsilon)\{\hat{s}\sqrt{\tilde{D}}(\epsilon - \epsilon_0) + o(\epsilon - \epsilon_0)\}$$

where $\hat{s} = \mathcal{G}_{\mu\mu}/|\mathcal{G}_{\mu u}|$, or by

$$\gamma^{(1)}(\mu) = s\sqrt{\tilde{D}}(\mu - \mu_0) + O(\mu - \mu_0)$$

and

$$\gamma^{(2)}(\epsilon) = -s\mu_\epsilon^{(2)}(\epsilon)\{\sqrt{\tilde{D}}(\epsilon - \epsilon_0) + o(\epsilon - \epsilon_0),$$

where $s = \mathcal{G}_{u\mu}/|\mathcal{G}_{u\mu}|$. Assume that all singular points of $\underline{G}(\mu, \underline{u}) = 0$ are double points. The stability of such T-periodic solutions must change at each regular turning point and at each singular point (which is not a turning point) and only at such points.

Theorem 8 holds for steady solutions. I think that theorem 8 implies that the results on stability of steady solutions which have been given by Benjamin,[1] using topological degree, hold also for steady and T-periodic solutions under (H.1), (H.2), and (H.3). For the index of a solution I get

$$i = \text{sign}\,\frac{<\underline{G}_u(\mu, \underline{u}|\underline{\zeta}), \underline{\zeta}^*>_T}{<\underline{\zeta}, \underline{\zeta}^*>_T} = \text{sign}\left\{-\mu_\epsilon(\epsilon)\,\frac{<F_\mu(\mu, \underline{u}), \underline{\zeta}^*>_T}{<\underline{\zeta}, \underline{\zeta}^*>_T}\right\},$$

so that i = sign γ when $\gamma \neq 0$ and, more generally, $i = [i(+) + i(-)]/2$ where i is the index of a solution at a point of the bifurcation curve and $i(+)$ are limiting values on one and the other side of the point. Then $i = 0$ at double points that are not turning points, and $i = \pm 1$ at turning points that are double points.

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