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A new separation of variables theory for problems of Stokes flow and elasticity

Summary

Some classes of fourth-order boundary-value problems arising in the theory of Stokes flow and elasticity are solved by the method of biorthogonal series. The eigenfunctions are formed from separable solutions when the separation constants (eigenvalues) are chosen to make the solution and its normal derivative vanish at the side walls. A general and unified algorithm is presented for solving such problems on strips, in wedges, between disks, in cylinders and between cylinders and in cones. The method applies to many different equations and it always leads to the same general algorithms, the same type of biorthogonal expansion, the same type of reduction to ordinary differential equations, the same biorthogonality condition and the same type of formulas for the biorthogonal coefficients. The method leads to a new theory of biorthogonal 'Fourier' series of two-component vector-valued functions. Convergence of the series is proved for sufficiently smooth but otherwise arbitrary data. Completeness of the representations is established in a smaller, but still large, class of functions. Questions of summability by Féjers method, Gibbs phenomenon, representation of functions in weak classes and other points of analysis in the theory of trigonometric series are important open questions in this new theory.

1. Introduction

The theory described here can be regarded as the extension to fourth-order problems of the method of generalized 'Fourier series' used in the study of second-order problems. This type of extension was first introduced by R. C. T. Smith [1] in his study of stresses in a semi-infinite strip clamped at its side and loaded at its top edge. Smith's ideas were used by Joseph and Fosdick [2] to study a narrow-gap approximation for secondary motions generated by the Weissenberg effect. A more complete analysis, including numerical computations, was given by Joseph and Sturges [3] in their study of the free surface on a liquid filling a rectangular trench heated from its side. In that paper, it is shown that Smith's biorthogonal series are formally analogous to complex Fourier series and, though the biorthogonal eigenfunctions are much more complicated than trigonometric functions, the biorthogonal 'Fourier' coefficients may be computed from simple algorithms. Joseph and Sturges [3] also showed how the eigenfunction expansions should be used to compute solutions for other boundary conditions and in strips of finite height. The same type of biorthogonal expansions were used by Joseph [4] in a study of the free surface on the round edge of a flowing liquid filling a torsion flow viscometer. This is the first case where this type of eigenfunction expansion arises for a problem which is not biharmonic. Similar eigenfunction expansions are required for the axisymmetric problems of Stokes flow between concentric cylinders studies by Yoo and Joseph [5] and for the problem of axisymmetric flow in a cone studied by Liu and Joseph [6]. Yoo's thorough study [7] of secondary motions indiced by the Weissenberg effect is a notable achievement of this method of analysis. Liu and Joseph [8] showed how the corner eigenfunctions of Dean and Montagnon [9] and Moffat [10] may be used to generate biorthogonal series solution of Stokes-flow problems in wedge-shaped circular sectors. In their example, a Stokes flow is generated by buoyancy which is induced by density differences associated with heating one side wall. The secondary motion associated with the motion of a free surface on a viscoelastic fluid between oscillatory planes (Sturges and Joseph [11]) falls within the domain of application of the method of biorthogonal series. This problem may be reduced to the study of $\nabla^4\psi+\lambda^2\nabla^2\psi=0$ (λ^2 is complex) where ψ and its normal derivative vanish on the side wall. Such problems are also common in the linear theory of dynamic elasticity and in the linearized theory of buckling.

The list of problems given in the last paragraph are a small sample of those which can be solved by biorthogonal eigenfunction expansions. The eigenfunctions required in these different problems depend on the given data and the shape of the boundary; though these differ from problem to problem, the expansions for different problems share common properties which appear to be intrinsic to Stokes flow in cavities and to problems of elasticity with built-in side walls.

The method of solution in biorthogonal series requires the expansion of two-component vector-valued functions into a series of vector-valued biorthogonal eigenfunctions. The representation of arbitrary vectors with biorthogonal series is an independent problem of pure analysis with only weak connections to boundary-value problems. Smith [1] established conditions on the data sufficient to gaurantee the convergence of the biorthogonal series. But Smith's conditions eliminate most applications.

Joseph [12] and Joseph and Sturges [13] showed that the biorthogonal series will converge for arbitrary, sufficiently smooth, data. The rate of convergence depends on the data to be expanded. For 'bad' data like step functions and 'ramp' functions, convergence is conditional; better data gives absolute and even uniform convergence, as in the elementary theory of trigonometric series. And, as in the elementary theory, Féjer's method of computing Cesaro sums seems to greatly improve the rate of convergence.

The aim of this paper is to bring the recent results, just reviewed, into one place so as to emphasize their common features and to create propaganda for the method. I believe the method is very important, it bears the same relation to fourth-order problems as generalized Fourier series do to second-order problems. There are also some new results in this paper; in particular, the results stated in section 5 on the completeness of the eigenfunction expansions appear not to have noticed before.

In section 2, we consider the canonical problem first posed and solved formally by Smith [1]. In section 3, we review various extensions and generalizations of the biorthogonal series expansion of boundary-value problems. In section 4, we prove convergence of the series for arbitrary smooth data and establish rates of convergence. In section 5, we consider the problem of completeness in the sense of justifications for the expansions.

2 The canonical edge problem in a semi-infinite strip

The canonical problem is defined as follows. We seek a bounded biharmonic function $\hat{\Psi}(t, y)$ in the semi-infinite strip $\mathcal{V}=[t, y:-1\leq t\leq 1, y\leq 0]$ satisfying

$$(\tilde{\Psi}(1, y), \tilde{\Psi}(-1, y), \tilde{\Psi}_{,t}(1, y), \tilde{\Psi}_{,t}(-1, y)]=[c_1, c_2, c_3, c_4], \tag{2.1}$$

where c_1, c_2, c_3 and c_4 are constants and $\tilde{\Psi}_{,tt}(t, 0)$ are $\tilde{\Psi}_{,yy}(t, 0)$ are arbitrary, prescribed, sufficiently-smooth functions.

The function

$$\Psi(t, y)=\tilde{\Psi}(t, y)-\frac{c_3-c_4}{4}(t^2-1)+\frac{c_1-c_2-c_3-c_4}{4}(3t-t^3)+\frac{c_3+c_4}{2}t+\frac{c_1+c_2}{2} \tag{2.2}$$

satisfies

$$\nabla^4\Psi=0 \quad \text{in} \quad \mathcal{V}. \tag{2.3a}$$

$$\Psi=\Psi_{,t}=0 \quad \text{when} \quad t=\pm 1, \tag{2.3b}$$

$$\{\Psi_{,yy}(t, 0), \Psi_{,tt}(t, 0)\}=\{f(t), g(t)\}, \tag{2.3c}$$

where $f(t)$ and $g(t)$ are arbitrary, prescribed, sufficiently-smooth functions and Ψ is bounded as $y \to -\infty$. The prescribed edge function $g(t)$ is compatible with the side-wall boundary condition (2.3b) if and only if $g(t)$ satisfies the compatibility

$$\langle g \rangle = \langle tg \rangle = 0, \tag{2.4}$$

where

$$\langle \cdot \rangle = \int_{-1}^{1} \cdot \, dt.$$

R. C. T. Smith [1] gave a formal solution of (2.3) and justified the solution under the conditions that $f(\pm 1) = f'(\pm 1) = g(\pm 1) = g'(\pm 1) = 0$. These restrictions on the data rule out most of the applications. They also make the theory uninteresting from the point of view of pure 'Fourier' analysis. Fortunately, Smith's restrictions are not intrinsic and they may be dropped (*see* section 4 and 5). For the moment, we pursue the formal theory.

Separable solutions

$$\Psi^{(n)} = \phi_1^{(n)}(t) \exp (S_n y)$$

satisfy (2.3a) and (2.3b) if

$$\phi_{1,tttt}^{(n)} + 2S_n^2 \phi_{1,tt}^{(n)} + S_n^4 \phi_1^{(n)} = 0, \tag{2.5a}$$

$$\phi_1^{(n)}(\pm 1) = \phi_{,t}^{(n)}(\pm 1) = 0. \tag{2.5b}$$

The complex constants S_n are eigenvalues which arise as follows. We may always decompose the solution of (2.5) into even and odd sets. The even solutions may be written as

$$\phi_1^{(n)}(t) = S_n \sin S_n \cos S_n t - S_n t \cos S_n \sin S_n t. \tag{2.6}$$

Evidently, $\phi_1^{(n)}(\pm 1) = 0$ and we may verify that $\phi_{1,t}^{(n)}(\pm 1) = 0$ if and only if

$$\sin 2S_n + 2S_n = 0. \tag{2.7}$$

The odd solutions may be written as

$$\phi_1^{(n)}(t) = S_n \cos S_n \sin S_n t - S_n t \sin S_n \cos S_n t, \tag{2.8}$$

where

$$\sin 2S_n = 2S_n. \tag{2.9}$$

The eigenfunctions (2.6) and (2.8) are sometimes called the Papkovich–Fadle [14, 15] functions in honor of the two gentlemen who first introduced them in the study of problems of elasticity.

We note that there are no real-valued solutions of the eigenvalue equations (2.7) and (2.9) other than $S_0 = 0$. There is no eigenfunction belonging to $S_0 = 0$ satisfying (2.5*a*) and the *four* conditions (2.5*b*). It follows that the eigenvalues are all complex constants. *If* S_n *is an eigenvalue, then so is* $-S_n$ *and so is* $\bar{S}_n$, *the complex conjugate of* S_n. All of the eigenvalues are known if all be known in the first quadrant of the complex S plane. Following Joseph and Sturges [3], we shall index the eigenvalues to accentuate the analogy between the biorthogonal series (2.13) and (2.14), below, and the complex form of Fourier's trigonometric series. We first identify the roots of, say, (2.7) in the first quadrant and order them in a sequence according to the size of their real parts; that is, $0 < \text{Re}\, S_1 < \text{Re}\, S_2 < \text{Re}\, S_3$, etc. We then identify the roots of (2.7) in the fourth quadrant as

$$S_{-n} = \bar{S}_n. \tag{2.10}$$

Roots in the second quadrant are given by $-S_n$ and in the third quadrant by $-S_{-n}$. Identical conventions are adopted for the roots of (2.9). It follows from these conventions that

$$\phi_1^{(-n)}(t, S_{-n}) = \phi_1^{(n)}(t, \bar{S}_n) = \bar{\phi}_1^{(n)}(t, S_n) \tag{2.11}$$

and

$$\phi_1^{(n)}(t, -S_n) = \phi_1^{(n)}(t, S_n). \tag{2.12}$$

We now seek the solution of (2.3) as a series of separable solutions in the form $(n \neq 0)$

$$\Psi(t, y) = \sum_{-\infty}^{\infty} C_n \phi_1^{(n)}(t) \exp(S_n y)/S_n^2. \tag{2.13}$$

This series is a formal solution of (2.3) if the constants C_n can be selected so that

$$\begin{bmatrix} f(t) \\ g(t) \end{bmatrix} \equiv \mathbf{f}(t) = \sum_{-\infty}^{\infty} C_n \boldsymbol{\phi}^{(n)}(t) = \sum_{-\infty}^{\infty} C_n \begin{bmatrix} \phi_1^{(n)}(t) \\ \phi_2^{(n)}(t) \end{bmatrix}, \tag{2.14}$$

where

$$\phi_2^{(n)}(t) = \phi_{1,tt}^{(n)}/S_n^2. \tag{2.15}$$

One of Smith's basic contributions is an algorithm for computing the constants C_n. Combining (2.15) and (2.4) we find that

$$\phi_{2,tt}^{(n)} + S_2^n(2\phi_2^{(n)} + \phi_1^{(n)}) = 0. \tag{2.16}$$

Equations (2.15) and (2.16) may be written as

$$\boldsymbol{\phi}_{,tt}^{(n)} + S_n^2 \mathbf{A} \cdot \boldsymbol{\phi} = 0, \tag{2.17}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}. \tag{2.18}$$

The matrix $\mathbf{A}$ has an important place in the theory. We call it the biorthogonality matrix.

An adjoint (row) vector

$$\mathbf{\Psi}^{(r)}=[\Psi_1, \Psi_2]$$

belonging to S_n satisfies the differential system

$$\mathbf{\Psi}^{(n)}_{,tt}+S_n^2\mathbf{\Psi}\cdot\mathbf{A}=0 \tag{2.19}$$

and

$$\Psi_2^{(n)}(\pm 1)=\Psi_{2,t}^{(n)}(\pm 1)=0. \tag{2.20}$$

Equations (2.19) and (2.20) are derived in the usual way. We introduce a scalar product

$$\mathbf{\Psi}\cdot\mathbf{\phi}=[\Psi_1, \Psi_2]\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix}=\Psi_1\phi_1+\Psi_2\phi_2$$

and an inner product identity

$$\langle\mathbf{\Psi}\cdot(\mathbf{\phi}_{,tt}+S_n^2\mathbf{A}\cdot\mathbf{\phi})\rangle=[\mathbf{\Psi}\cdot\mathbf{\phi}_{,t}-\mathbf{\Psi}_{,t}\cdot\mathbf{\phi}]_{-1}^{1}+\langle(\mathbf{\Psi}_{,n}+S_n^2\mathbf{A})\cdot\mathbf{\phi}\rangle.$$

which holds for all $\mathbf{\Psi}$, $\mathbf{\phi}\in C_2\,[1,-1]$. If $\mathbf{\phi}=\mathbf{\phi}^{(n)}$, then

$$[\Psi_2\phi_{2,t}-\Psi_{2,t}\phi_2]_{-1}^{1}+\langle(\mathbf{\Psi}_{,tt}+S_n^2\mathbf{\Psi}\cdot\mathbf{A})\cdot\mathbf{\phi}\rangle=0. \tag{2.21}$$

The adjoint problem is the subset of $\Psi\in C_2[-1, 1]$, for which (2.21) holds when $\mathbf{\phi}$ is allowed to range over $C_2[-1, 1]$.

The vectors $\mathbf{\phi}^{(n)}$ and $\mathbf{\Psi}^{(n)}$ form a biorthogonal set. Using (2.14), (2.15), (2.19) and (2.20) we find that

$$\langle\mathbf{\Psi}^{(m)}\cdot\mathbf{A}\cdot\mathbf{\phi}^{(n)}\rangle=\delta_{nm}k_m, \tag{2.22}$$

where

$$\langle\mathbf{\Psi}^{(m)}\cdot\mathbf{A}\cdot\mathbf{\phi}^{(m)}\rangle=k_m, \tag{2.23}$$

The 'Fourier' coefficients C_n in the biorthogonal series (2.14) may be computed using (2.23):

$$C_n=\frac{1}{k_n}\langle\mathbf{\Psi}^{(n)}\cdot\mathbf{A}\cdot\mathbf{f}\rangle. \tag{2.24}$$

If the data vector $\mathbf{f}(t)$ is real-valued, then

$$C_{-n}=\bar{C}_n \tag{2.25}$$

and the series (2.13) giving $\Psi(t, y)$ is real-valued.

It is useful to maintain a distinction between the eigenfunction $\phi_1^{(n)}$ of (2.5) and the first component of the eigenvector $\mathbf{\phi}^{(n)}$ satisfying (2.17) and (2.15b). These sets are identical except that $S_0=0$ is not an eigenvalue of

(2.5). When $S_0=0$, we find a nontrivial eigenvector of (2.17) and (2.5*b*) and an adjoint eigenvector:

$$\begin{bmatrix}\phi_1^{(0)}\\ \phi_2^{(0)}\end{bmatrix}=\begin{bmatrix}0\\ 1\end{bmatrix},\qquad [\Psi_1^{(0)}; \Psi_2^{(0)}]=[1, 0],\qquad k_0=-2 \tag{2.26}$$

and

$$\begin{bmatrix}\phi_1^{(0)}\\ \phi_2^{(0)}\end{bmatrix}=\begin{bmatrix}0\\ t\end{bmatrix},\qquad [\Psi_1^{(0)}, \Psi_2^{(0)}]=[t, 0],\qquad k_0=-2. \tag{2.27}$$

These eigenvectors do not enter into the solution of (2.3), because (2.4) follows from (2.3) and rules them out.

Explicit formulae for the eigenvectors are listed below. For $n=0, \pm 1, \pm 2, \ldots$

$$\phi_1^{(n)}=\Psi_2^{(n)},\qquad \Psi_1^{(n)}-\phi_2^{(n)}=2\phi_1^{(n)}. \tag{2.28}$$

When $n=\pm 1, \pm 2, \ldots$ and S_n are the roots of $\sin 2S+2S=0$, the eigenfunctions are even functions of t; $\phi_1^{(n)}(t)$ is given by (2.6) and

$$\phi_2^{(n)}=-\phi_1^{(n)}-2\cos S_n \cos S_n t,\qquad k_n=-4\cos^4 S_n. \tag{2.29}$$

When $n=\pm 1, \pm 2, \ldots$ and S_n are the roots of $\sin 2s-2s=0$, the eigenfunctions are odd functions of t; $\phi_1^{(n)}(t)$ is given by (2.8) and

$$\phi_2^{(n)}=-\phi_1^{(n)}+2\sin S_n \sin S_n t,\qquad k_n=-4\sin^4 S_n.$$

Whenever the formal solution of (2.3) is justified, we have an immediate and precise mathematical realization of St Venant's principle. Stating this principle in an informal way, we note that the solutions of (2.3) in the strip decay very rapidly to

$$\Psi(t, y)\sim C_1\phi_1^{(1)}(t)\exp S_1 y. \tag{2.30}$$

That the rapidity of this decay is exponential with a large decay constant (~ 3.2) can be inferred from the numerical values of the first quadrant roots of (2.7) and (2.9). The first three roots of (2.7) are given by Robbins and Smith [16] as

$$S_1=2.106\,196+i\,1.125\,365,$$
$$S_2=5.356\,269+i\,1.551\,575,$$
$$S_3=8.536\,683+i\,1.775\,544.$$

The first three roots of (2.9) are given by Hillman and Salzer [17] as

$$S_1=3.748\,838+i\,1.384\,339,$$
$$S_2=6.949\,980+i\,1.676\,105,$$
$$S_3=10.119\,259+i\,1.858\,384.$$

It follows that the interior form of the solution is independent of the precise details of the edge loads when these loads are self-equilibrated; that is, when (2.4) holds. The edge loads enter only through the constant C_1 and this constant is determined by a projection $\langle \mathbf{\Psi}^{(1)} \cdot \mathbf{A} \cdot \mathbf{f} \rangle$ of the data vector. The interior solution is a decaying system of closed eddies with a fixed spatial period (*see* Fig. 1).

3 Extensions of the theory

In this section we will describe the extensions of the theory presented in Section 2 to other problems. These extensions show that the method of biorthogonal series should have a distinguished place in the education of mathematically-minded scientists. The method of biorthogonal series is the method of 'Fourier series', which is appropriate to many problems of fourth order. Such problems cannot be avoided, in any event, in the study of mechanics.

3.1 Other boundary conditions (Joseph and Sturges [13]).

Suppose the data vector is given by

$$\begin{bmatrix} \Psi_{,y}(t,0) \\ \Psi_{,tt}(t,0) \end{bmatrix} = \begin{bmatrix} f(t) \\ g(t) \end{bmatrix} \equiv \mathbf{f}(t). \tag{3.1}$$

This is the form which the data takes in fluid mechanics, when the velocity is given by a stream function in $\mathcal{V}$ and is prescribed on $y=0$. Then, $\Psi(t, y)$ is given by (2.13) if we can find the C_n for which

$$\mathbf{f}(t) = \begin{bmatrix} f(t) \\ g(t) \end{bmatrix} = \lim_{N\to\infty} \sum_{-N}^{N} C_n \left\{ \begin{bmatrix} \phi_1^{(n)}(t) \\ \phi_2^{(n)}(t) \end{bmatrix} + \left(\frac{1}{S_n} - 1\right) \begin{bmatrix} \phi_1^{(n)} \\ 0 \end{bmatrix} \right\}, \tag{3.2}$$

where $C_0 = 0$. Using the biorthogonality condition (2.22), we get an infinite number ($N\to\infty$) of inhomogeneous equations for the coefficient C_n:

$$\langle \mathbf{\Psi}^{(n)} \cdot \mathbf{A} \cdot \mathbf{f} \rangle = C_n k_n + \sum_{-N}^{N} C_m B_{mn}, \tag{3.3}$$

where

$$B_{nm} = \left(\frac{1}{S_m} - 1\right) \langle \phi_1^{(m)} \cdot \phi_1^{(n)} \rangle.$$

We solve the infinite system in the usual way–by truncation. There are $2N$ equations for the real and imaginary parts of C_n. Proofs of convergence

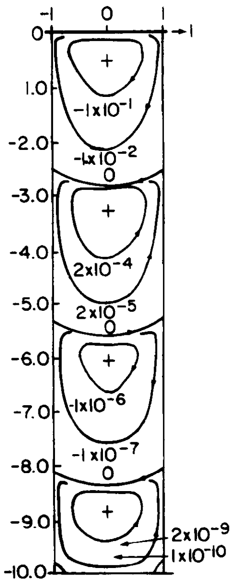

Fig. 1. Level lines of the streamlines for flow in a cavity with depth/width ratio of 5. $\Psi(t, y)$ is biharmonic of Ψ and Ψ_t vanish at the $t=\pm 1$. The normal component of velocity vanishes on the top and bottom, the tangential component vanishes at the bottom and is equal to one on the top: i.e.,

$$\Psi_y(t,0)-1=\Psi(t,0)=\Psi_y(t,-H)=\Psi(t,-H)=0.$$

This problem is of the type solved in section 3.1. The data is 'bad' and convergence of the series for $\Psi_y(t,0)$ and $\Psi(t,0)$ is conditional (*see* Joseph and Sturges [13]). Away from the edges, the flow is given by (2.30) with $S_1 = 2.106\,196 + i\,1.253\,65$.

are presently unknown, but the numerical tests of convergence always work out (*see* Fig. 1).

If the trench has a bottom at $y=-d$, then a data vector $\mathbf{f}(t)$ is prescribed at the top $(y=0)$ and a data vector $\mathbf{g}(t)$ is also prescribed at the bottom. It is necessary to retain the second and third quadrant eigenvalues, which now correspond to *bounded* eigensolutions. We get a solution in the form

$$\Psi(t,y)=\sum_{-\infty}^{\infty}(C_n \exp(S_n y)+D_n \exp(-S_n y))\phi_1^{(n)}(t). \tag{3.5}$$

where $C_0 = D_0 = 0$. The coefficients C_n and D_n may be computed by applying the biorthogonality condition (2.22) to the data vector at the bottom and top of the strip (*see* Fig. 1).

3.2 Vibration problems for elastic plates and viscoelastic fluids (Sturges and Joseph [11]).

Let $\mathscr{V}$ be the semi-infinite strip defined above (2.1). We want to solve the following problem:

$$\left.\begin{aligned} &\nabla^4\Psi-\lambda^2\nabla^2\Psi=0 \text{ in } \mathscr{V} \ (\lambda^2 \text{ is complex}),\\ &\Psi(\pm 1, y)=\Psi_{,t}(\pm 1, y)=0,\\ &\mathbf{f}(t)=\begin{bmatrix} f(t)\\ g(t)\end{bmatrix}=\begin{bmatrix}\Psi_{,yy}(t,0)\\ \Psi_{,tt}(t,0)\end{bmatrix},\\ &\langle g\rangle=\langle tg\rangle=0,\\ &\Psi(t,y) \text{ is bounded in } \mathscr{V}.\end{aligned}\right\} \tag{3.6}$$

This problem, like that treated in section 2, is in canonical form and it may be solved by the methods used in section 2. We get $\Psi(t, y)$ in the series form given by (2.13). However, now we get the even eigenfunctions

$$\phi_1^{(n)}(t)=\cos\sqrt{(S_n^2-\lambda^2)}\cos S_n t-\cos S_n \cos t\sqrt{(S_n^2-\lambda^2)}, \tag{3.7}$$

where the eigenvalues S_n are determined by

$$S_n \tan S_n=\sqrt{(S_n^2-\lambda^2)}\tan\sqrt{(S_n^2-\lambda^2)}. \tag{3.8}$$

Since λ is a prescribed complex constant, it is no longer true that $S_{-n}=\bar{S}_n$. Now we order the fourth quadrant eigenvalues in a sequence $\operatorname{Re} S_{-1}<\operatorname{Re} S_{-2}<\operatorname{Re} S_{-3}\ldots$. If S_n is a eigenvalue, so is $-S_n$.

The computation of C_n follows along the lines laid out in section 2. We get

$$\mathbf{f}(t)=\sum_{-\infty}^{\infty} C_n\boldsymbol{\phi}^{(n)}(t), \qquad \boldsymbol{\phi}^{(n)}=\begin{bmatrix}\phi_1^{(n)}(t)\\ \phi_2^{(n)}(t)\end{bmatrix}, \tag{3.9}$$

where $C_0=0$.

$$\boldsymbol{\phi}_{,tt}+S_n^2\mathbf{B}_n\cdot\boldsymbol{\phi}=0,\ \phi_1^{(n)}(\pm 1)=\phi_{1,t}^{(n)}(\pm 1)=0,$$

$$\mathbf{B}_n\begin{bmatrix} 0 & -1\\ 1-\dfrac{\lambda^2}{S_n^2} & 2-\dfrac{\lambda^2}{S_n^2}\end{bmatrix}, \tag{3.10}$$

$$C_n=\frac{1}{k_n}\langle\boldsymbol{\Psi}^{(n)}\cdot\mathbf{A}\cdot\mathbf{f}\rangle \tag{3.11}$$

($\mathbf{A}$ = biorthogonality matrix),

$$\boldsymbol{\Psi}_{,tt}+S_n^2\boldsymbol{\Psi}^{(n)}\cdot\mathbf{B}_n=0, \qquad \boldsymbol{\Psi}^{(n)}=[\Psi_1^{(n)}, \phi_1^{(n)}]. \tag{3.12}$$

The expression (3.11) for C_n and the definition of k_n arise from the biorthogonality condition

$$\langle\boldsymbol{\Psi}^{(m)}\cdot\mathbf{A}\cdot\boldsymbol{\phi}^{(n)}\rangle=k_n\delta_{mn}. \tag{3.13}$$

We can solve (3.6) for other boundary conditions, as in section 3.1 above, or in finite domains. The same basic method of solution will work for many variations in the problem (3.6).

3.3 Biharmonic problems in circular sectors (Liu and Joseph [81])

The pie-shaped planar region $\mathcal{V}$ is specified in polar coordinates (r, θ) as

$$\mathcal{V}=[r, \theta:\ 0\leq\eta\leq r<1,\ -\beta\leq\theta\leq\beta].$$

We consider the following biharmonic problem:

$$\begin{aligned}&\nabla^4\Psi(r,\theta)=0 \text{ in } \mathcal{V}.\\ &\Psi(r, \pm\beta)=\Psi_{,\theta}(r, \pm\beta)=0, \qquad (3.14)\\ &\mathbf{f}(\theta)=\begin{bmatrix} r\left(\frac{1}{r}\Psi_{,r}\right)_{,r} \\ \frac{1}{r^2}\Psi_{,\theta\theta}\end{bmatrix}=\begin{bmatrix} f(\theta)\\ g(\theta)\end{bmatrix} \text{ is prescribed on } r=\eta, 1,\end{aligned}$$

and

$$\langle g\rangle=\langle\theta g(\theta)\rangle=0,$$

where

$$\langle\cdot\rangle\equiv\int_{-\beta}^{\beta}\cdot\, d\theta.$$

This problem comes up, for example, in the study of the motion under a free surface on a liquid in a pie-shaped trench heated from its side (*see* Fig. 2 and Table 1) or in the study of the normal displacements of a thin elastic strip clamped at $\theta=\pm\beta$ with displacements and couples prescribed on the radial boundaries.

The solution of this problem can be found by the method of section 2. We get

$$\Psi=\sum_{-\infty}^{\infty}[C_n t^{\lambda_n}+D_n t^{-\lambda_n+2}]\phi_1{}^{(n)}(\theta)/\lambda_n(\lambda_n-2), \qquad (3.15)$$

where $C_0=D_0=0$. The solutions split into even and odd sets:

$$\left.\begin{aligned}&\phi_1^{(n)}(\theta)=\cos(\lambda_n-2)\beta\cos\lambda_n\theta-\cos\lambda_n\beta\cos(\lambda_n-2)\theta,\\ &\sin[2\beta(\lambda_n-1)]+(\lambda_n-1)\sin 2\beta=0;\end{aligned}\right\} \qquad (3.16)$$

$$\left.\begin{aligned}&\phi_1^{(n)}(\theta)=\sin(\lambda_n-2)\beta\sin\lambda_n\theta-\sin\lambda_n\beta\sin(\lambda_n-2)\theta,\\ &\sin[2\beta(\lambda_n-1)]-(\lambda_n-1)\sin 2\beta=0.\end{aligned}\right\} \qquad (3.17)$$

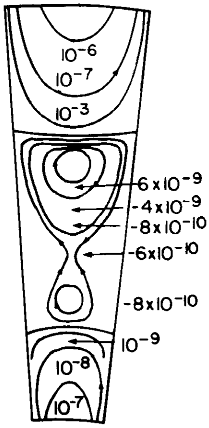

Fig. 2. (After Liu and Joseph [8].) Stokes flow in a wedge. $2\beta = 10°$, $a/b = 0.5$. Level lines of

$$\Psi(r,\theta) = \sum_{-\infty}^{\infty} C_n t^{\lambda_n - 3} + D_n t^{-\lambda_n - 1} \frac{\phi_1(n)}{\lambda_n(\lambda_n - 2)}.$$

Table 1 *Convergence of the top edge series and the bottom edge series when* $2\beta = 10°$ *and* $a/b = 0.5$ *in Fig.* 2

$$\frac{\Psi}{t^3} \times 10^6 = \frac{10^6 \times f(\theta,\beta)}{16} = 10^6 \times \sum_{-N}^{N} (C_n t^{\lambda_n - 3} + D_n t^{-\lambda_n - 1}) \frac{\phi_1^{(n)}}{\lambda_n(\lambda_n - 2)}$$

θ (degrees)	$\frac{10^6 \times f(\theta,\beta)}{16}$	$N=1$	$N=3$	$N=5$	$N=9$	$N=10$
On the top: $t = 1$						
0	2.433 72	2.447 76	2.433 52	2.433 63	2.433 68	2.433 69
1	2.242 67	2.253 77	2.242 92	2 242 68	2.242 63	2.242 64
2	1.716 47	1.718 89	1.716 09	1.716 42	1.716 42	1.716 44
3	0.995 85	0.987 91	0.995 74	0.995 79	0.995 81	0.995 83
4	0.314 85	0.306 05	0.315 50	0.314 92	0.314 81	0.314 82
5	0	0	0	0	0	0
On the bottom: $t = 0.5$						
0	2.433 72	2.324 07	2.445 25	2.433 88	2.433 40	2.433 69
1	2.242 67	2.180 42	2.242 55	2.242 50	2.242 35	2.242 63
2	1.716 47	1.753 39	1.718 76	1.716 39	1.716 17	1.716 44
3	0.995 85	1.090 52	0.994 26	0.996 20	0.995 60	0.995 83
4	0.314 85	0.371 74	0.312 85	0.314 07	0.314 68	0.314 82
5	0	0	0	0	0	0

The eigenvalues $\mu_n = \lambda_n - 1$ are symmetrically distributed in the complex μ-plane; if λ_n is an eigenvalue, then $\bar{\lambda}_n$, $2-\lambda_n$ and $2-\bar{\lambda}_n$ are also eigenvalues. Moreover, $\phi_1^{(n)}(\theta, \lambda_n) = -\phi_1^{(n)}(\theta, 2-\lambda_n)$.

The coefficients C_n and D_n are determined from the edge data using the biorthogonality condition (3.20). This condition may be obtained after the eigenvector problem is reduced to ordinary differential equations by methods like those discussed in section 2. We get

$$\mathbf{f}(\theta) = \sum_{-\infty}^{\infty} C_n \boldsymbol{\phi}^{(n)}(\theta), \qquad \boldsymbol{\phi}^{(n)} = \begin{bmatrix} \phi_1^{(n)}(\theta) \\ \phi_2^{(n)}(\theta) \end{bmatrix}, \tag{3.18}$$

$$\boldsymbol{\phi}_{,\theta\theta}^{(n)} + \mathbf{A}_n \cdot \boldsymbol{\phi}^{(n)} = 0, \qquad \phi_1^{(n)}(\pm\beta) = \phi_{1,\theta}^{(n)}(\pm\beta) = 0$$

$$\mathbf{A}_n = \begin{bmatrix} 0 & -\lambda_n(\lambda_n - 2) \\ \lambda_n(\lambda_n - 2) & (\lambda_n - 2)^2 + \lambda_n^2 \end{bmatrix}, \tag{3.19}$$

$$C_n = \frac{1}{k_n} \langle \boldsymbol{\Psi}^{(n)} \cdot \mathbf{A} \cdot \mathbf{f} \rangle \tag{3.20}$$

($\mathbf{A}$ = biorthogonality matrix),

$$\boldsymbol{\Psi}_{,\theta\theta}^{(n)} + \boldsymbol{\Psi}^{(n)} \cdot \mathbf{A}_n = 0, \qquad \boldsymbol{\Psi}^{(n)} = [\Psi_1^{(n)}(\theta), \phi_1^{(n)}(\theta)]. \tag{3.21}$$

The expression (3.11) for C_n and the definition of k_n arise from the biorthogonality condition

$$\langle \boldsymbol{\Psi}^{(m)} \cdot \mathbf{A} \cdot \boldsymbol{\phi}^{(n)} \rangle = k_n \delta_{mn}. \tag{3.22}$$

Many variations of the canonical problem (3.14) are tractable to analysis by the method of separation of variables and biorthogonal series.

3.4 Stokes-flow problems between parallel disks (Joseph [4])

We now consider an axisymmetric problem of Stokes flow between parallel disks. The problem statement is given in the caption to Fig. 3.

The solution of this Stokes-flow problem is given by

$$\Psi(r, t) = \sum_{-\infty}^{\infty} C_n \phi_1^{(n)}(t) F(S_n, r)/S_n^2,$$

where

$$F(S_n, r) = \frac{r}{R} \frac{I_1(S_n r)}{I_1(S_n R)}$$

and $I_1(S_n r)$ is a modified Bessel function of the first kind. We find that

$$\mathbf{f}(t) = \sum_{-\infty}^{\infty} C_n \boldsymbol{\phi}^{(n)}(t),$$

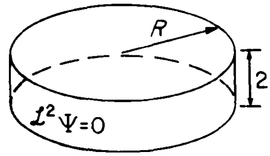

Fig. 3. Stokes flow between parallel disks. The stream function $\Psi(r, t)$ satisfies the Stokes–Beltrami equation $\mathcal{L}\Psi = 0$, where

$$\mathcal{L}=\frac{\partial^2}{\partial r^2}-\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial t^2}.$$

On the disks $\Psi(r, \pm 1) = \Psi_{,t}(r \pm 1) = 0$. The data vector

$$\mathbf{f}(t)=\begin{bmatrix} f(t) \\ g(t) \end{bmatrix} \equiv \begin{bmatrix} R\left(\frac{1}{r}\Psi_{,r}\right)_{,r} \\ \Psi_{,tt} \end{bmatrix}$$

is prescribed on the round edge. $g(t)$ satisfies the compatibility condition (2.4).

$$\mathcal{V}=[r, t: 0 \leqslant r \leqslant R, -1 \leqslant t \leqslant 1].$$

where $\boldsymbol{\phi}^{(n)}(t)$, S_n and C_n are exactly the eigenvectors, eigenvalues and biorthogonal coefficients given in section 2.

3.5 Stokes-flow problems between coaxial cylinders (Yoo and Joseph [5])

This problem arises, for example, in the study of the secondary motions associated with thermally induced convective currents or in the study of secondary motions induced by the Weissenberg effect in viscoelastic fluids (Yoo [7]).

The problem statement is as follows. A stream function $\Psi(r, y)$ for axisymmetric flow in cylindrical coordinates is defined in the region $\mathcal{V}$ between coaxial cylinders,

$$\mathcal{V}=[r, y: 0<a \leqslant r \leqslant b,\ b-a=2,\ y \leqslant 0].$$

The streamfunction satisfies

$$\mathcal{L}^2\Psi=0 \text{ in } \mathcal{V}, \tag{3.23}$$

where

$$\mathcal{L}=\left(\frac{\partial^2}{\partial r^2}-\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial y^2}\right).$$

$$\Psi(a, y)=\Psi(b, y)=\Psi_{,r}(a, y)=\Psi_{r}(b, y)=0, \tag{3.24}$$

and

$$\Psi(r, y) \text{ is bounded in } \mathscr{V}. \tag{3.25}$$

At the edge $y=0$, the values of the normal component of velocity and the shear stress are prescribed in terms of the data vector

$$\mathbf{f}(r)=\begin{bmatrix} f(r) \\ g(r) \end{bmatrix}=\begin{bmatrix} \Psi_{,yy} \\ r\left(\frac{1}{r}\Psi_{,r}\right)_{,r} \end{bmatrix}. \tag{3.26}$$

The prescription of $g(r)$ on $y=0$ is compatible with the wall boundary conditions (3.24) if and only if

$$\langle g(r)\rangle=\langle r^2 g(r)\rangle=0, \tag{3.27}$$

where

$$\langle\cdot\rangle=\int_a^b \frac{1}{r}\cdot dr.$$

The solution of the Stokes-flow problem (3.23)–(3.27) is given by

$$\Psi(r, y)=\sum_{-\infty}^{\infty} C_n \phi_1^{(n)}(r)/p_n^2,$$

where $C_0=0$ and

$$\phi_1^{(n)}(r)=A_1^{(n)} rJ_1(p_n r)+A_2^{(n)} rY_1(p_n r) + A_3^{(n)} r^2 J_0(p_n r)+A_4^{(n)} r^2 Y_0(p_n r),$$

where $J_l(p_n r)$ and $Y_l(p_n r)$ are Bessel functions. The constants $A_l^{(n)}$ and the eigenvalues p_n are selected to satisfy the side-wall boundary conditions (3.24) and will not be given here (*see* Yoo and Joseph [5]). The eigenvalues are symmetrically located in the four quadrants of the complex p_n plane and the numbering convention used for S_n in Section 2 applies also to p_n.

The coefficients C_n may be obtained by methods like those used in section 2. We find that

$$r\left(\frac{1}{r}\boldsymbol{\phi}_{,r}^{(n)}\right)_{,r}+p_n^2 \mathbf{A}\cdot\boldsymbol{\phi}^{(n)}=0, \qquad \boldsymbol{\phi}^{(n)}=\begin{bmatrix}\phi_1^{(n)} \\ \phi_2^{(n)}\end{bmatrix},$$

where $\phi_1^{(n)}(r)$ and $\phi_{1,r}^{(n)}(r)$ vanish at $r=a$ and $r=b$,

$$r\left(\frac{1}{r}\boldsymbol{\Psi}_{,r}^{(n)}\right)_{,r}+p_n^2 \boldsymbol{\Psi}^{(n)}\cdot\mathbf{A}=0, \qquad \boldsymbol{\Psi}^{(n)}=[\Psi_1^{(n)}, \phi_1^{(n)}]$$

and $\mathbf{A}$ is the biorthogonality matrix. From these equations, we find the biorthogonality condition;

$$\langle\boldsymbol{\Psi}^{(m)}\cdot\mathbf{A}\cdot\boldsymbol{\phi}^{(n)}\rangle=k_n\delta_{mn}.$$

The C_n are then given by

$$C_n = \frac{1}{k_n}\langle \mathbf{\Psi}^{(n)} \cdot \mathbf{A} \cdot \mathbf{f} \rangle.$$

Stokes-flow problems between coaxial cylinders can be worked when the cylinder is bounded and not semi-infinite and for other boundary conditions on the top and bottom. The same type of analysis can be carried out for Stokes-flow edge problems in circular cylinders ($a = 0$).

3.6 Stokes flow in cones (Liu and Joseph [6])

Let $\mathcal{V}$ be the right circular cone of polar radius $r = 1$ and polar angle $\theta = \theta_0$ shown in Fig. 4. The motion inside $\mathcal{V}$ is governed by the Stokes–Beltrami equation

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1-\xi^2}{r^2}\frac{\partial^2}{\partial \xi^2}\right)^2 \Psi(r, \xi) = 0, \tag{3.28}$$

where $\xi = \cos\theta$. On the conical side walls, $\xi = \xi_0$,

$$\Psi(r, \xi_0) = \Psi_{,\xi}(r, \xi_0) = 0. \tag{3.29}$$

The data vector is prescribed at the spherical edge $r = 1$:

$$\mathbf{f}(\xi) \equiv \begin{bmatrix} f(\xi) \\ g(\xi) \end{bmatrix} = \begin{bmatrix} r^4\left(\frac{1}{r}\Psi_{,r}\right)_{,r} \\ (1-\xi^2)\Psi_{,\xi\xi} \end{bmatrix}. \tag{3.30}$$

This data prescription is equivalent to the statement that the normal component of velocity and the shear stress $S_{r\theta}$ are prescribed on the spherical cap at $r = 1$. The prescription (3.30) of $g(\xi)$ is compatible with

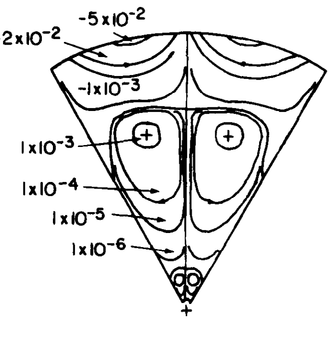

Fig. 4. (After Liu and Joseph [6].) Stokes flow in a cone. Level lines of $r^{3/2+\mu_1}\phi_1(\xi)$ for $2\theta_0 = 60°$, $\mu_1 = 5.888\,24 + i\,2.030\,12$.

the side-wall boundary condition if and only if

$$\int_{\xi_0}^{1} \frac{g(\xi)}{1-\xi^2}\,d\xi = \int_{\xi_0}^{1} \frac{\xi g(\xi)}{1-\xi^2}\,d\xi = 0. \tag{3.31}$$

The solution of the Stokes-flow problem (3.28)–(3.31) is given by

$$\Psi(r,\xi) = \sum_{-\infty}^{\infty} \frac{C_n}{\mu_n^2 - \frac{9}{4}} r^{3/2+\mu_n} \phi_1^{(n)}(\xi) \tag{3.32}$$

where $C_0 = 0$ and

$$\phi_1^{(n)}(\xi) = (1-\xi^2)^{1/2}\{P^1_{\mu_n-3/2}(\xi_0)P^1_{\mu_n-1/2}(\xi) - P^1_{\mu_n-1/2}(\xi_0)P^1_{\mu_n-3/2}(\xi)\}, \tag{3.33}$$

where P^1_μ (cos θ) is an associated Legendre function. It is obvious that $\phi_1^{(n)}(\xi_0) = 0$. $\phi_{1,\xi}^{(n)}(\xi_0)$ also vanishes if $\mu = \mu_n$ are selected as eigenvalues; that is, as roots $\mu = \mu_n$ of the characteristic equation

$$\begin{aligned}\{(\mu+3/2)P^1_{\mu+3/2}(\xi_0) + (\mu-3/2)P^1_{\mu+1/2}(\xi_0)\}&\\ \times\{(\mu-1/2)P^1_{\mu+1/2}(\xi_0) + (\mu+1/2)P^1_{\mu-3/2}(\xi_0)\}&\\ = 4\mu^2\xi_0^2 P^1_{\mu+1/2}(\xi_0)P^1_{\mu-3/2}(\kappa_0).&\end{aligned} \tag{3.34}$$

In deriving (3.34) we used recursion relations.

The eigenvalues μ_n are all complex-valued. It is easy to verify, using the fourth-order differential equation satisfied by $\phi_1^{(n)}(\xi)$, that if μ_n is an eigenvalue, so is $\bar{\mu}_n$ and $-\mu_n$. The μ_n in the first quadrant of the complex μ-plane are arranged in an increasing sequence according to the size of their real parts.

The coefficients C_n may be obtained by methods like those used in section 2. We find that

$$(1-\xi^2)\boldsymbol{\phi}^{(n)}_{,\xi\xi} + \mathbf{A}_n \cdot \boldsymbol{\phi}^{(n)} = 0, \qquad \boldsymbol{\phi}^{(n)} = \begin{bmatrix}\phi_1^{(n)}\\ \phi_2^{(n)}\end{bmatrix},$$

where

$$\phi_1^{(n)}(\xi_0) = \phi_{1,\xi}^{(n)}(\xi_0) = 0,$$
$$(1-\xi^2)\boldsymbol{\Psi}^{(n)}_{,\xi\xi} + \boldsymbol{\Psi}^{(n)} \cdot \mathbf{A}_n = 0 \qquad \boldsymbol{\Psi}^{(n)} = [\Psi_1^{(n)}, \phi_1^{(n)}]$$

and

$$\mathbf{A}_n = \begin{bmatrix} 0 & -(\mu_n^2 - 9/4)\\ \mu_n^2 - 1/4 & 2(\mu_n^2 - 5/4)\end{bmatrix}.$$

From these equations, we find the biorthogonality condition

$$\langle \boldsymbol{\Psi}^{(m)} \cdot \mathbf{A} \cdot \boldsymbol{\phi}^{(n)} \rangle = k_n \delta_{mn},$$

where $\mathbf{A}$ is the biorthogonality matrix and

$$\langle\cdot\rangle=\int_{\xi_0}^{1}\left(\frac{1}{1-\xi^2}\right)\mathrm{d}\xi.$$

The C_n are then given by

$$C_n=\frac{1}{k_n}\langle\boldsymbol{\Psi}^{(n)}\cdot\mathbf{A}\cdot\mathbf{f}\rangle.$$

In dealing with fourth-order problems in cones, it is necessary to compute associated Legendre functions of arbitrary complex order. Though representations of these functions in terms of integrals and hypergeometric series have been given by mathematicians of antiquity, the computational value of such representations is unknown. Liu and Joseph [6] worked successfully with

$$P_\nu^1(\cos\theta)=\frac{i(\nu+1)}{2\pi}\int_0^{2\pi}(\cos\theta+i\sin\theta\cos t)^\nu\cos t\,\mathrm{d}t$$

for complex and unrestricted values of ν.

The methods used here can be applied to problems in truncated cones and problems with other boundary conditions. It can also be applied to problems of Stokes flow between cones. The analysis of flow between cones is important in understanding the flow in cone and plate rheometers used for rheological measurements. In the analysis of flow between cones it is necessary to introduce the other kind of Legendre functions $Q_\mu^1(\xi)$ which are singular at $\xi=1$. There are some ancient representations of $Q_\mu^1(\xi)$ which are supposed to be good for complex, unrestricted values of μ.

4 Mathematical theory of biorthogonal expansions of vector-valued functions

In sections 2 and 3, we studied different boundary-value problems governed by partial differential equations of the fourth order. These different problems were solved formally. The formal solutions can be justified provided only that in each and every problem a biorthogonal series representation

$$\mathbf{f}(t)=\sum_{-\infty}^{\infty}C_n\boldsymbol{\phi}^{(n)}(t)=\sum_{-\infty}^{\infty}\frac{C_n}{k_n}\langle\boldsymbol{\psi}^{(n)}\cdot\mathbf{A}\cdot\mathbf{f}\rangle\boldsymbol{\phi}^{(n)}(t) \tag{4.1}$$

can be justified.

The representation (4.1) asserts that a certain large class of functions $f(t)$ and $g(t)$ may be expanded as a two-component vector

$$\mathbf{f}(t)=\begin{bmatrix}f(t)\\g(t)\end{bmatrix} \tag{4.2}$$

in a series of characteristic vectors

$$\boldsymbol{\phi}^{(n)}(t) = \begin{bmatrix} \phi_1^{(n)}(t) \\ \phi_2^{(n)}(t) \end{bmatrix}$$

generated as eigensolutions of a second-order ordinary differential equation. The biorthogonal coefficients C_n are expressed through a scalar product

$$C_n = \frac{1}{k_n} \langle \boldsymbol{\Psi}^{(n)} \cdot \mathbf{A} \cdot \mathbf{f} \rangle \tag{4.3}$$

involving the data vector $\mathbf{f}$, the biorthogonal matrix $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}$, and adjoint characteristic vectors

$$\boldsymbol{\Psi}^{(n)} = [\Psi_1^{(n)}, \phi_1^{(n)}]$$

generated as eigensolutions of the second-order problem adjoint to the one for $\boldsymbol{\phi}^{(n)}$. The formula (4.3) always arises from a biorthogonality condition of the form

$$\langle \boldsymbol{\Psi}^{(m)} \cdot \mathbf{A} \cdot \boldsymbol{\phi}^{(n)} \rangle = k_m \delta_{nm}. \tag{4.4}$$

The representation (4.1) is notable because of the following.

1. The form of the expansion, the form of the biorthogonality condition (4.4) and the value of the biorthogonality matrix are always the same, though the eigenvector bases $\boldsymbol{\phi}^{(n)}$, $\boldsymbol{\Psi}^{(n)}$ and the eigenvalues differ from problem to problem.
2. Though two functions $f(t)$ and $g(t)$ are expanded, only one set of biorthogonal coefficients are needed. In general, we expect to have nontrivial representations of the function $g(t) = 0$ or $f(t) = 0$ with $C_n = 0 \forall n$ if and only if $f(t) = g(t) = 0$.
3. Since each different boundary-value problem generates a different basis $\boldsymbol{\phi}^{(n)}(t)$, we may have many different representations of the vector $\mathbf{f}(t)$ in terms of biorthogonal series.

The problem of representing $\mathbf{f}(t)$ in biorthogonal series can be studied as an independent mathematical problem with only incidental connections to the separation of variables theory for boundary-value problems satisfying partial differential equations of order four. This independent problem is not more strongly tied to differential equations than trigonometric series are tied to theory of differential equations for harmonic oscillators.

Now we are going to consider results which establish the conditions on $f(t)$ which justify the expansion (4.1). There are two main types of results which are required.

1. The conditions under which the series (4.1) converges.
2. The conditions under which the series (4.1) converges to $\mathbf{f}(t)$: i.e., the conditions under which the bases $\boldsymbol{\phi}^{(n)}(t)$ are complete.

From now on we will confine our attention to the basis $\boldsymbol{\phi}^{(n)}(t)$ and eigenvalues S_n generated by the canonical edge problem in the semi-infinite strip of width two. This is the problem considered by Smith [1] and the convergence proofs of Joseph [12] and Joseph and Sturges [13] apply to it. The restriction of the analysis to the basis $\boldsymbol{\phi}^{(n)}(t)$ of section 2 has greater generality than might first be supposed. To prove convergence, we need to establish the asymptotic distribution of the eigenvalues and the asymptotic forms of the eigenfunctions. It turns out that these asymptotic forms are practically the same for the $\boldsymbol{\phi}^{(n)}$ of section 2 and for the many different $\boldsymbol{\phi}^{(n)}(t)$ of section 3. Hence, all of our convergence proofs may be expected to carry over with only slight and fairly obvious modifications. The method we use in section 5 to establish completeness for the basis $\boldsymbol{\phi}^{(n)}$ of section 2 is Smith's extension of the residue method of Titchmarsh [18]. This method carries over immediately to the analysis of completeness of the bases generated by the problems treated in section 3.

In the study of the separation of variables solutions of fourth-order problems, we required that the edge data and side-wall boundary values should be compatible. This compatibility condition is expressed by (2.4) and it eliminates the eigenvalue $S_0 = 0$ and the eigenfunctions (2.26) and (2.27). The restriction (2.4) on the functions $g(t)$ which may be expanded in biorthogonal series is apparent and not real. For if (2.4) does not hold for $g(t) = g_e(t) + g_0(t)$, where $g_e = \frac{1}{2}[g(t) + g(-t)]$ and $g_0 = \frac{1}{2}[g(t) - g(-t)]$, then (2.4) does hold for $\hat{g}(t) = f(t) - \langle g_e \rangle - t\langle tg_0 \rangle$. Hence, *in the sequel, we shall* ***assume****, without loss of generality, that* $\langle g \rangle = \langle tg \rangle = 0$.

We can obtain an expansion formula for arbitrary $\mathbf{f}(t)$ by superposition from the expansion formula

$$\mathbf{f}(t) = \mathbf{f}(-t) = \sum_{-\infty}^{\infty} C_n \boldsymbol{\phi}^{(n)}(t), \qquad C_0 = 0, \tag{4.5}$$

for even data ($\boldsymbol{\phi}^{(n)}$ belong to S_n satisfying $\sin 2S_n + 2S_n = 0$) and the expansion formula

$$\mathbf{f}(t) = -\mathbf{f}(-t) = \sum_{-\infty}^{\infty} C_n \boldsymbol{\phi}^{(n)}(t), \qquad C_0 = 0 \tag{4.6}$$

for odd data ($\boldsymbol{\phi}^{(n)}$ belonging to S_n satisfying $\sin 2S_n - 2S_n = 0$). It is sufficient to consider, say, the even data. The results and proofs for odd data are essentially the same.

Smith [1] has justified (4.5) and (4.6) under the restrictions that

$$f(\pm 1) = g(\pm 1) = f'(\pm 1) = g'(\pm 1) = 0 \tag{4.7}$$

and

$$(f''(t), g''(t)) \text{ are of bounded variation.} \tag{4.8}$$

He notes that if $f(t)$ and $g(t)$ are merely of bounded variation, then the series on the left of (4.5) and (4.6) may diverge. For example, we have the following asymptotic results for the eigenvalues S_n satisfying $\sin 2S_n + 2S_n$ and the associated (even) eigenvectors (2.6), (2.28) and (2.29)

$$2S_n \to (2n - \tfrac{1}{2})\pi + i \log (4n-1)\pi, \tag{4.9}$$

$$\begin{bmatrix} \sin S_n t \\ \cos S_n t \end{bmatrix} = \frac{1}{2} \begin{bmatrix} i \\ 1 \end{bmatrix} [(4n-1)\pi]^{|t|/2} e^{-i(n-1/4)\pi|t|} + O(n^{-|t|/2}) \tag{4.10}$$

and

$$k_n = -4 \cos^4 S_n = -\frac{(4n-1)^2\pi^2}{4} + O(n). \tag{4.11}$$

When n is very large,

$$S_n = O(n), \quad k_n = O(n^2) \tag{4.12}$$

and

$$\mathbf{\Phi}^{(n)}, \mathbf{\Psi}^{(n)} = O(n^{(3+|t|)/2}). \tag{4.13}$$

Smith notes that, if it is assumed that $f(t)$ and $g(t)$ are of bounded variation, then

$$C_n = O(n^{-1}), \qquad C_n \mathbf{\Phi}^{(n)} = O(n)$$

and the series (4.5) will diverge. Smith argued that the divergence of (4.5) and (4.6) need not necessarily effect the practical value of the solution since (2.13) converges rapidly for any $y > 0$, however small (*see* Smith [1], p. 23).

Smith's restrictions (4.7) are too severe. They rule out the applications in which the values on $y = 0$ are important. And they rule out the possibility of a mathematical theory of biorthogonal expansions of the form (4.5) and (4.6) except in the very restricted class satisfying (4.7) and (4.8). He says (p. 237) that the 'Details of the calculation make it unlikely that the conditions imposed on $(f(t), g(t))$ can be much relaxed.' Fortunately, Smith's statements are incorrect; we get uniform absolute convergence to $\mathbf{f}(t)$ even when $g(\pm 1) \neq 0$ and $g'(\pm 1) \neq 0$ and we get conditional and not uniform convergence with no conditions on the boundary values of $f(t)$ or $g(t)$.

The most important type of hypothesis on $f(t)$ and $g(t)$ are, like (4.7), associated with the boundary values at $t = \pm 1$. We consider three cases. In the first case,

$$f(\pm 1) = f'(\pm 1) = 0. \tag{*}$$

Uniform convergence of (4.5) and (4.6) is possible under the hypothesis

(*) because

$$f(t)=\sum_{-\infty}^{\infty} C_n\phi_1^{(n)}(t) \tag{4.14}$$

is compatible with $\phi_1^{(n)}(\pm 1)=\phi_{1,t}^{(n)}(\pm 1)=0$. From the point of view of differential equations, (*) is normal for nice problems because $f=\Psi_{,yy}(t,0)$ and $\Psi(y,t)=\Psi_{,t}(y,t)$ vanish on $t\pm 1$. In the second case,

$$f(\pm 1)=0. \tag{**}$$

Uniform convergence to $f'(t)$ is impossible because the left and right side of the first derivative of (4.14) do not match when $t=\pm 1$. In the third case,

$$f(\pm 1),\ f'(\pm 1),\ g(\pm 1),\ g'(\pm 1) \text{ are unrestricted.} \tag{***}$$

Uniform convergence to $f(t)$ is impossible. In all three cases, we get interior, point-wise convergence but the convergence is uniform only under the hypothesis (*).

The proof of convergence under hypotheses (*) and (**) and certain smoothness assumptions follow from easy estimates of C_n and majorization by numerical series (Joseph [12]). In the case (***), the convergence is conditional; the proof is delicate and will not be given here (*see* Joseph and Sturges [13]).

To prove convergence under (*), we assume that $f(t)$ and $g(t)$ are in the class $C_2\,[-1,1]\cap C_4^p\,[-1,1]$ with a finite number of jumps, at most, in the third and fourth derivative. Then, integrating by parts, we find that

$$\begin{aligned}
k_nC_n &= \langle \mathbf{\Psi}^{(n)}\cdot\mathbf{A}\cdot\mathbf{f}\rangle = \langle (2\phi_1^{(n)}(t)-\Psi_1^{(n)}(t)g(t)+\phi_1^{(n)}(t)f(t)\rangle \\
&= -\frac{1}{S_n^2}\langle \phi_{1,tt}^{(n)}(t)g(t)+\Psi_{1,tt}^{(n)}(t)f(t)\rangle \\
&= -\frac{1}{S_n^2}[\phi_{1,t}^{(n)}(t)g(t)+\Psi_{1,t}^{(n)}(t)f(t)]_{-1}^{1} \\
&\quad +\frac{1}{S_n^2}[\phi_1^{(n)}(t)g'(t)+\Psi_1^{(n)}(t)f'(t)]_{-1}^{1} \\
&\quad -\frac{1}{S_n^2}\langle \phi_1(t)g''(t)+\Psi_1(t)f''(t)\rangle.
\end{aligned} \tag{4.15}$$

Since $f(\pm 1)=f'(\pm 1)=\phi_1^{(n)}(\pm 1)=\phi_{1,t}^{(n)}(\pm 1)=0$ and (from (2.28) and (2.29)) $\Psi_1^{(n)}=\phi_1^{(n)}-2\cos S_n \cos S_nt$, we have

$$-k_nS_n^2C_n=\langle (g+f)''\phi_1^{(n)}(t)-2\cos S_n \cos S_ntf''\rangle. \tag{4.16}$$

The largest term in the integrand is $\phi_1^{(n)}(t)=O(n^{(3+|t|)/2})$ and

$$\begin{aligned}
\langle (g+f)''\phi_1^{(n)}(t)\rangle &= S_n[\sin S_n\langle (g+f)''\cos S_nt\rangle-\cos S_n\langle t(g+f)''\sin S_nt\rangle] \\
&= 2(g+f)''-\sin S_n\langle (g+f)''\sin S_nt\rangle-\cos S_n\langle (t(g+f)'')'\cos S_nt\rangle,
\end{aligned} \tag{4.17}$$

where we have used the fact that f and g are even functions of t. A similar reduction holds when f and g are odd. The last term of (4.16) and the last two terms of (4.17) can be integrated once more by parts over the intervals where the integrands are continuous. The integration by parts introduces S_n into the denominator so that the remaining integrals are of $O(1)$ when n is large. We therefore have proved that

If $f(\pm 1) = f'(\pm 1) = 0$ *and* $f(t)$ *and* $g(t) \in C_2[-1, 1] \cap C_4^p[-1, 1]$, *then, when* n *is large,*

$$C_n \to \frac{1}{k_n S_n^2} O(1) \to O\left(\frac{1}{n^4}\right) \tag{4.18}$$

and, for each t, $-1 \leq t \leq 1$, *the series* (4.5) *may be majorized by a convergent numerical series*

$$C \sum_{n=1} 1/n^{(5-|t|)/2}, \ -1 \leq t \leq 1. \tag{4.19}$$

So, in case (*), we get uniform absolute convergence without conditions on the boundary values of $g(t)$.

In case (**), we find (Joseph [12]) that $C_n = O(1/n^3)$ and the series (4.5) may be majorized by the convergent numerical series

$$C \sum_{n=1} 1/n^{(3-|t|)/2}, \qquad 1 < t < 1.$$

The convergence is absolute but it need not be uniform. As we have already noted, the differentiated series (4.5) *cannot converge to* f' (± 1), but it may converge conditionally.

If all conditions on the boundary values of $f(t)$ are discarded, we get $C_n = O(1/n^2)$. The series converges, but not absolutely and not uniformly (*see* Joseph and Sturges [13]).

It is necessary to maintain a distinction between convergence and convergence to $\mathbf{f}(t)$. Smith proved convergence to $\mathbf{f}(t)$ under the hypotheses (4.7) and (4.8). If one checks through Smith's computation in the appendix to his paper, one finds that the conditions $g(\pm 1) = g'(\pm 1) = 0$ which he required are not used in his demonstration (*see* sections 4 and 5 of this paper). We therefore have convergence to $\mathbf{f}(t)$ under hypothesis (*) when $\mathbf{f}''(t)$ is of bounded variation or for $\mathbf{f}(t)$ in the class $C_2[-1, 1] \cap C_4^p[-1, 1]$. We shall call these conditions normal for nice problems, and for such problems we get uniform pointwise convergence and the formal solution of the boundary-value problem is justified.

There is, at present, no theorem of completeness when $\mathbf{f}(t)$ is in a bigger class. Indeed, we have already noted that if no boundary conditions are prescribed for $\mathbf{f}(t)$ at $t = \pm 1$, or if (**) is prescribed, we cannot have uniform convergence since, at least, $f'(\pm 1) \neq 0$ cannot be represented by the series (4.5) or (4.6). The interior convergence of (4.5) and (4.6) can, however, be guaranteed in the cases (**) and (***) provided that $\mathbf{f}(t) \in C_1[-1, 1] \cap C_3^{\mathrm{P}}[-1, 1]$ in case (**) and to $C_0[-1, 1] \cap C_2^{\mathrm{P}}[-1, 1]$ in case (**). There is as yet no theorem guaranteeing interior pointwise convergence to $\mathbf{f}(t)$ for case (**) or case (***), but our numerical work shows that we do get such convergence.

The situation which prevails in case (***), when $f'(\pm 1) \neq 0$ and $f(\pm 1) \neq 0$, is analogous to Fourier series, where, for example, the representation

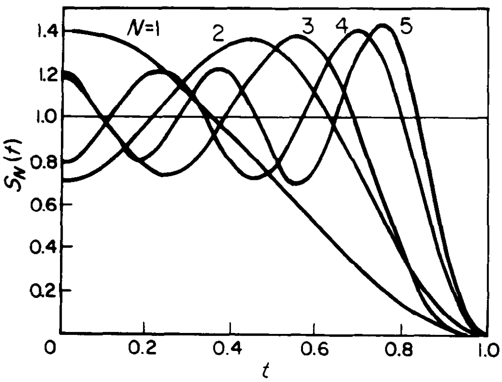

Fig. 5. Convergence of the partial sums

$$S_N(t) = \sum_{-N}^{N} \frac{-1}{\cos^4 S_n} \phi_1^{(n)}(t)$$

of the biorthogonal series $\lim_{N\to\infty} S_N(t)$ representing the unit step function $f(t)=1,\ -1<t<1$.

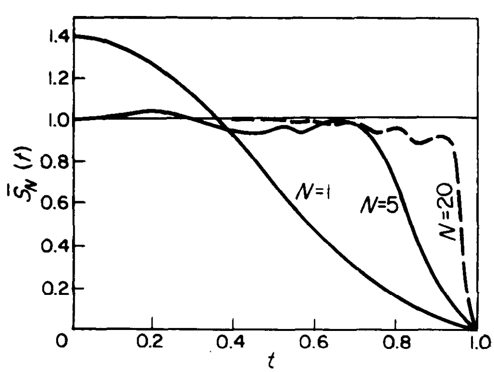

Fig. 6. Convergence of the Cesaro sums

$$\bar{S}_N(t) = \frac{1}{N} \sum_{M=1}^{N} S_M(t) \text{ to the unit step function } f(t) = 1,\ -1<t<1.$$

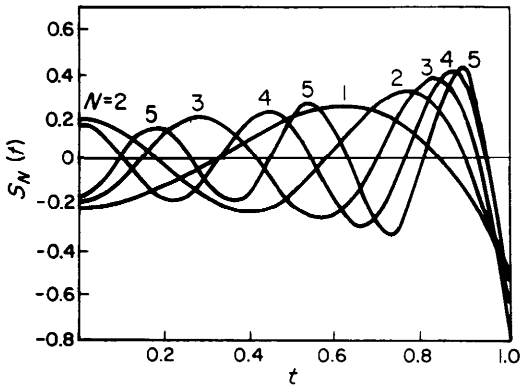

Fig. 7. Convergence of the partial sums

$$S_N(t)=\sum_{-N}^{N}\frac{-1}{\cos^4 S_n}\,\phi_2^{(n)}(t)$$

of the biorthogonal series $\lim_{N\to\infty} S_N(t)$ representing the zero function $g(t)=0$, $-1<t<1$.

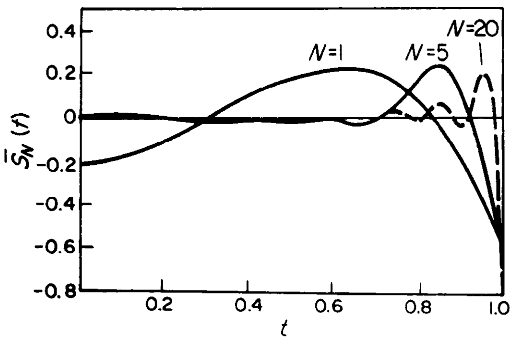

Fig. 8. Convergence of the Cesaro sums

$\bar{S}_N(t)=\dfrac{1}{N}\sum_{M=1}^{N} S_M(t)$ to the zero function $g(t)=0$, $-1<t<1$.

$$1=\sum_{n=1}^{\infty}\frac{2-2\cos n\pi}{n\pi}\sin n\pi t, \qquad 0<t<1, \tag{4.20}$$

must fail when $t=\pm 1$. We get conditional interior convergence to 1 at each interior point and the convergence is not uniform.

The step function may also be represented by a biorthogonal series of even eigenvectors

$$\mathbf{f}(t)=\begin{bmatrix}1\\0\end{bmatrix}=\sum_{-\infty}^{\infty} C_n\begin{bmatrix}\phi_1^{(n)}(t)\\ \phi_2^{(n)}(t)\end{bmatrix}, \qquad C_0=0, \tag{4.21}$$

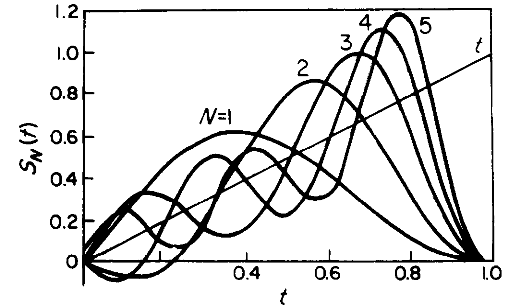

Fig. 9. Convergence of the partial sums

$$S_N(t)=\sum_{-N}^{N}-\frac{1}{S_n^2}\phi_1^{(n)}(t)$$

of the biorthogonal series $\lim_{N\to\infty} S_N(t)$ representing the unit ramp function $f(t)=t,\ -1<t<1$.

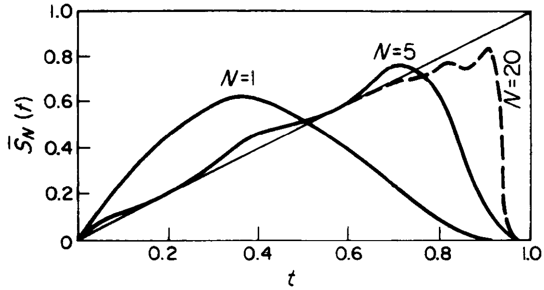

Fig. 10. Convergence of the Cesaro sums

$$\bar{S}_N(t)=\frac{1}{N}\sum_{M=1}^{N} S_M(t)$$

to the unit ramp function $f(t)=t,\ -1,<t<1$.

where $C_n=-1/\cos^4 S_n$. Similarly, we can expand the ramp function

$$\mathbf{f}(t)=\begin{bmatrix} t \\ 0 \end{bmatrix}=\sum_{-\infty}^{\infty}\begin{bmatrix}\phi_1^{(n)}(t)\\ \phi_2^{(n)}(t)\end{bmatrix}, \qquad C_0=0, \tag{4.22}$$

in the odd set of eigenfunctions. We get different nontrivial representations of $g(t)=0$ from (4.21) and (4.22).

In Figs 5 through 8 (taken from Joseph and Sturges [13]), we have exhibited the convergence of the partial sums

$$\mathbf{S}_N(t)=\sum_{-N}^{N}\frac{1}{\cos^4 S_n}\begin{bmatrix}\phi_1^{(n)}(t)\\ \phi_2^{(n)}(t)\end{bmatrix}\to\begin{bmatrix}1\\0\end{bmatrix}$$

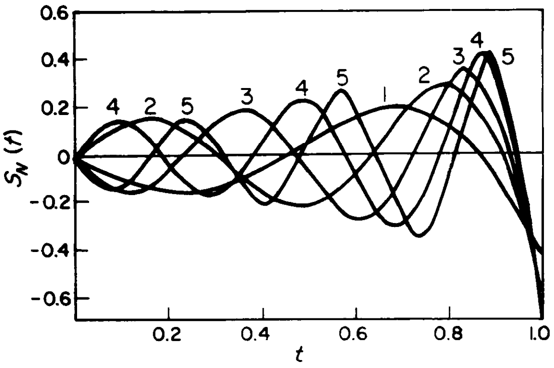

Fig. 11. Convergence of the partial sums

$$S_N(t)=\sum_{-N}^{N}-\frac{1}{S_n^2}\phi_2^{(n)}(t)$$

of the biorthogonal series $\lim_{N\to\infty} S_N(t)$ representing the zero function $g(t)=0$, $-1<t<1$.

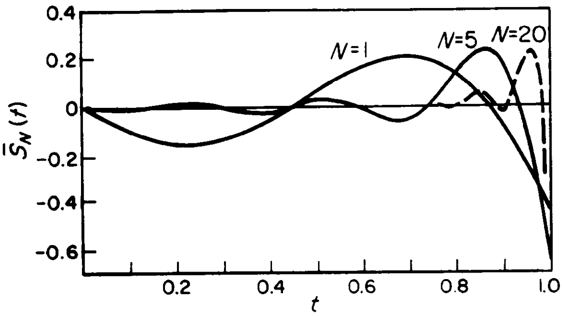

Fig. 12. Convergence of the Cesaro sums

$$\bar{S}_N(t)=\frac{1}{N}\sum_{N=1}^{N} S_M(t)$$

to the zero function $g(t)=0$, $-1<t<1$.

and the Cesaro sum

$$\mathbf{S}_M(t)=\frac{1}{M}\sum_{N=1}^{M}\mathbf{S}_N(t)\to\begin{bmatrix}1\\0\end{bmatrix}$$

of even eigenfunction to the step-function vector. In Figs 9 through 12 (taken from Joseph and Sturges [13]), we have exhibited the convergence of the partial sums

$$\mathbf{S}_N(t) = -\sum_{-N}^{N} \frac{1}{S_n^2} \begin{bmatrix} \phi_1^{(n)}(t) \\ \phi_2^{(n)}(t) \end{bmatrix} \to \begin{bmatrix} t \\ 0 \end{bmatrix}$$

and the Cesaro sum

$$\mathbf{S}_M(t) = \frac{1}{M} \sum_{M=1}^{M} \mathbf{S}_N(t) \to \begin{bmatrix} t \\ 0 \end{bmatrix}$$

of odd eigenvectors to the ramp-function vector.

The oscillatory character of conditional convergence is exhibited in Fig. 13 (taken from Joseph and Sturges [13]).

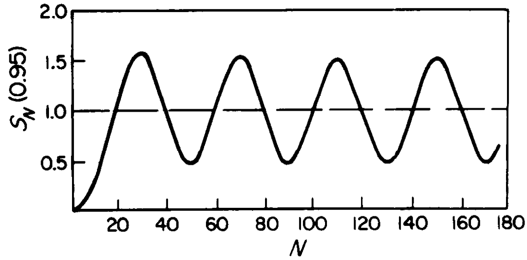

Fig. 13. Oscillatory character of the convergence to $f(t)=1$ (at $t=0.95$) of the partial sums

$$S_N(t) = \sum_{-N}^{N} \frac{-1}{\cos^4 S_n} \phi_1^{(n)}(t).$$

As in the case of trigonometric series, Féjer's method of summing Cesaro sums leads to greatly improved convergence. The numerical work also suggests the appearance of a Gibbs phenomenon.

5. Justification of the expansion formula

To justify the expansion (4.5), we shall follow the method of residues used by Smith [1]. The method is a generalization of one used by Titchmarsh [18] to study eigenfunction expansions associated with second-order differential equations. In fact, both authors make perfectly explicit the calculations used in the general spectral theory for linear operators. Of course, explicit computation leads to explicit results without a lot of hard-to-verify hypotheses. In the general theory, one considers an operator $\mathbf{T}(=\mathbf{A}^{-1}\,\mathrm{d}^2/\mathrm{d}t^2)$ with eigenvalues S^2 and domain $\mathcal{D}$, mapping $\mathcal{D} \to \mathcal{R} \supset \mathcal{D}$. The resolvent $\mathbf{R}(S)$ of $\mathbf{T} - S^2\mathbf{1}$ is the operator from $\mathcal{R} \to \mathcal{D}$ which inverts the problem $(\mathbf{T} - S^2\mathbf{1})\mathbf{x} = \mathbf{f}$, $\forall \mathbf{f} \in \mathcal{R}$ when S^2 is in the resolvent set; that is when S^2 is not an eigenvalue of $\mathbf{T}$. The expansion theorem can be obtained computing residues in the complex S plane at the singularities of the resolvent, provided that (a) there is a sequence of closed contours C_N not passing through eigenvalues of $\mathbf{T}$, (b) the minimum distance from the origin of the S-plane to C_N tends to infinity with N and (c)

$$\lim_{N\to\infty} \{\sup_{S\in C_N} \|\mathbf{R}(S)\mathbf{f}\|\} = 0, \qquad \forall \mathbf{f} \in \mathcal{R}.$$

In Smith's analysis, the class $\mathcal{R}$ of functions are the vector-valued fields satisfying (4.7) and (4.8). We will now show that the conditions $q(\pm 1) = g'(\pm 1) = 0$, which Smith assumed, are not required in his proof. It follows that we may take $\mathcal{R}$ as the larger class of vector fields satisfying (4.8) and (*); that is,

$$\mathbf{f}(t) = \begin{bmatrix} f(t) \\ g(t) \end{bmatrix}, \qquad f(\pm 1) = f'(\pm 1) = \langle g \rangle = \langle tg \rangle = 0.$$

$$\mathbf{f}(t) \in C_1[-1, 1], \qquad f''(t) \text{ of bounded variation.}$$

Smith solves the nonhomogeneous equation

$$\mathbf{M}''(t) + S^2\mathbf{A} \cdot \mathbf{M}(t) = \mathbf{A} \cdot \mathbf{f}(t), \tag{5.1}$$

where $\mathbf{A}$ is the biorthogonality matrix; $\mathbf{M}(t)$ is a two-component column vector such that

$$\mathbf{E} \cdot \mathbf{M}(\pm 1) + \hat{\mathbf{E}} \cdot \mathbf{M}'(\pm 1) = 0, \qquad \mathbf{E} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad \hat{\mathbf{E}} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}. \tag{5.2}$$

A Green-function solution of this problem may be obtained by the method of variation of parameters. We seek $\mathbf{M}$ in the form

$$\mathbf{M}(t) = \mathbf{X}_1(t) \cdot \mathbf{F}_1(t) + \mathbf{X}_2(t) \cdot \mathbf{F}_2(t), \tag{5.3}$$

where $\mathbf{X}_1(t) = \mathbf{X}(t-1)$, $\mathbf{X}_2(t) = \mathbf{X}(t+1)$ and the 2×2 matrix $\mathbf{X}(t)$ satisfies

$$\mathbf{X}''(t) + S^2\mathbf{A} \cdot \mathbf{X} = 0, \qquad \mathbf{E} \cdot \mathbf{X}(0) + \hat{\mathbf{E}} \cdot \mathbf{X}'(0) = 0 \tag{5.4}$$

and $\mathbf{F}_1(t)$ and $\mathbf{F}_2(t)$ are such that

$$\mathbf{X}_1 \cdot \mathbf{F}_1' + \mathbf{X}_2 \cdot \mathbf{F}_2' = 0 \tag{5.5}$$

with $\mathbf{F}_1(-1) = \mathbf{F}_2(1) = 0$. Equations (5.1), (5.3) and (5.5) imply that

$$\mathbf{X}_1' \cdot \mathbf{F}_1' + \mathbf{X}_2' \cdot \mathbf{F}_2' = \mathbf{A} \cdot \mathbf{f}. \tag{5.6}$$

We next introduce the 2×2 matrices $\mathbf{Y}_1(t) = \mathbf{Y}(t-1)$ and $\mathbf{Y}_2(t) = \mathbf{Y}(t+1)$ satisfying

$$\mathbf{Y}''(t) + S^2\mathbf{Y} \cdot \mathbf{A} = 0, \qquad \mathbf{Y}[0] \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \mathbf{Y}'(0) \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = 0 \tag{5.7}$$

and define the Wronskians

$$\mathbf{W}_{ij}(t) = \mathbf{Y}_j \cdot \mathbf{X}_i' - \mathbf{Y}_j' \cdot \mathbf{X}_i \qquad (i, j = 1, 2). \tag{5.8}$$

Clearly, $\mathbf{W}'_{ij}=0$ and $\mathbf{W}_{ij}$ is a constant matrix. The boundary conditions imply that $\mathbf{W}_{11}=\mathbf{W}_{22}=0$. Premultiplying (5.6) by $\mathbf{Y}_2$ and (5.5) by Y_2, we find that

$$\mathbf{W}_{12}\mathbf{F}'_1=\mathbf{Y}_2\cdot\mathbf{A}\cdot\mathbf{f}$$

and similarly,

$$\mathbf{W}_{21}\mathbf{F}'_2=\mathbf{Y}_1\cdot\mathbf{A}\cdot\mathbf{f}.$$

Hence, after solving for $\mathbf{F}'_1$ and $\mathbf{F}'_2$ and integrating, we get

$$\begin{aligned}\mathbf{F}_1&=\mathbf{W}_{12}^{-1}\int_{-1}^{t}\mathbf{Y}_2(u)\cdot\mathbf{A}\cdot\mathbf{f}(u)\,du,\\ \mathbf{F}_2&=-\mathbf{W}_{21}^{-1}\int_{t}^{1}\mathbf{Y}_1(u)\cdot\mathbf{A}\cdot\mathbf{f}(u)\,du.\end{aligned}\tag{5.9}$$

Equations (5.3) and (5.9) solve (5.1) and (5.2). Moreover,

$$\mathbf{X}(t)=\begin{bmatrix}\sin St-St\cos St & St\sin St\\ \sin St+St\cos St & 2\cos St-St\sin St\end{bmatrix},$$

$$\mathbf{Y}(t)=\begin{bmatrix}2\cos St+St\sin St & St\sin St\\ 3\sin St-St\cos St & \operatorname{Sin} St-St\cos St\end{bmatrix}.$$

$$\mathbf{W}_{12}^{-1}=\frac{1}{2S(\sin^2 2S-4S^2)}\begin{bmatrix}-2S\sin 2s & \sin 2S+2S\cos 2S\\ -\sin 2S+2S\cos 2S & 2S\sin 2S\end{bmatrix}.\tag{5.10}$$

and

$$\mathbf{W}_{21}^{-1}=\frac{1}{2S(\sin^2 2S-4S^2)}\begin{bmatrix}-2S\sin 2s & -\sin 2S-2S\cos 2S\\ \sin 2S-2S\cos 2S & 2S\sin 2S\end{bmatrix}.\tag{5.11}$$

Smith notes that $S\mathbf{M}(t,S)$, $-1\leq t\leq 1$, is a meromorphic function of S with poles at the zeros S_n of

$$\sin^2 2S-4S^2=(\sin 2S+2S)(\sin 2S-2S)=0.$$

The zeros of $\sin 2S+2S$ and $\sin 2S-2S$ are symmetric in the four quadrants of the S-plane. Smith shows that the residue at a zero of $\sin 2S+2S$ is

$$\tfrac{1}{2}C_n\begin{bmatrix}\phi_1^{(n)}(t)\\ \phi_2^{(n)}(t)\end{bmatrix},\qquad n=\pm 1,\ \pm 2,\ldots,$$

where $\phi_1^{(n)}(t)$, $\phi_2^{(n)}(t)$ are the even eigenfunctions (2.6) and (2.29) and C_n is computed from (2.24) on the even adjoint eigenfunctions $\Psi_1^{(n)}(t)$ and $\Psi_2^{(n)}(t)$. Similarly, the residue of $\mathbf{M}S$ at a zero of $\sin 2S-2S$ is given by (5.12) computed on the odd eigenfunction and adjoint eigenfunctions. $S\mathbf{M}(S,t)$ has no residue at $S=0$.

To verify (5.12), for example at $S=S_n$, where $\sin 2S_n=-2S_n$, note that

$$(S-S_n)\mathbf{W}_{12}^{-1}=-\frac{1}{8S\cos^2 S_n}\begin{bmatrix} S_n & -\sin^2 S_n \\ \cos^2 S_n & -S_n \end{bmatrix},$$

$$(S-S_n)\mathbf{W}_{21}^{-1}=-\frac{1}{8S_n\cos^2 S_n}\begin{bmatrix} S_n & \sin^2 S_n \\ -\cos^2 S_n & -S_n \end{bmatrix}$$

and

$$(S-S_n)S_n\mathbf{M}(S_n)=-\frac{1}{8\cos^2 S_n}\left\{\begin{bmatrix} -\phi_1^{(n)}(t) & \dfrac{\sin^2 S}{S}\phi_1^{(n)}(t) \\ -\phi_2^{(n)}(t) & \dfrac{\sin^2 S}{S}\phi_2^{(n)}(t) \end{bmatrix}\right.$$

$$\times\int_{-1}^{t}\mathbf{Y}_2(u)\cdot\mathbf{A}\cdot\mathbf{f}(u)\,\mathrm{d}u$$

$$\left.-\begin{bmatrix} \phi_1^{(n)}(t) & \dfrac{\sin^2 S}{S}\phi_1^{(n)}(t) \\ \phi_2^{(n)}(t) & \dfrac{\sin^2 S}{S}\phi_2^{(n)}(t) \end{bmatrix}\int_t^1 \mathbf{Y}_1(u)\cdot\mathbf{A}\cdot\mathbf{f}(u)\,\mathrm{d}u\right\}$$

$$=\tfrac{1}{2}C_n\begin{bmatrix}\phi_1^{(n)}(t)\\ \phi_2^{(n)}(t)\end{bmatrix}.$$

Let us assume that

$$\mathbf{M}=\frac{1}{S_2}\begin{bmatrix} f(t)\\ g(t)\end{bmatrix}-\frac{1}{S^2}\mathbf{X}_1(t)\cdot\mathbf{W}_{12}^{-1}\cdot\int_{-1}^{t}\mathbf{Y}_2(u)\cdot\mathbf{f}''(u)\,\mathrm{d}u$$

$$+\frac{1}{S^2}\mathbf{X}_2(t)\cdot\mathbf{W}_{21}^{-1}\cdot\int_t^1\mathbf{Y}_1(u)\cdot\mathbf{f}''(u)\,\mathrm{d}u$$

$$=\frac{1}{S^2}\begin{bmatrix} f(t)\\ g(t)\end{bmatrix}+\mathbf{G}(s,t) \tag{5.13}$$

and

$$\lim_{N\to\infty}\frac{1}{2\pi i}\int_{C_N} S\mathbf{G}(S,t)\,\mathrm{d}s=0,\qquad 1\leqslant t\leqslant 1. \tag{5.14}$$

C_N is the square with vertices $2N\pi\ (\pm1\pm i)$ and the contour is described in the anticlockwise sense. Smith proved that (5.14) holds provided only that $\mathbf{f}''(t)$ is of bounded variation. Taking account of the numbering convention established in section 2, $(S_n,\ \boldsymbol{\phi}^{(n)}(S_n),\ C_n(S_n))=(-S_n,\boldsymbol{\phi}^{(n)}(-S_n),\ C_n(-S_n))$, we find, using (5.3), (5.9) and (5.12), that

$$\lim_{N\to\infty}\frac{1}{2\pi i}\int_{C_N} S\mathbf{M}(t,S)\,\mathrm{d}s=\sum_{-\infty}^{\infty}C_n\begin{bmatrix}\phi_1^{(n)}(t)\\ \phi_2^{(n)}(t)\end{bmatrix} \tag{5.15}$$

and, using (5.13) and (5.14), that

$$\lim_{N\to\infty}\frac{1}{2\pi i}\int_{C_N} S\mathbf{M}(t,S)=\begin{bmatrix} f(t)\\ g(t)\end{bmatrix}. \tag{5.16}$$

It follows that, if (5.13) holds, then (4.5) and (4.6) are valid and the series on the right converges pointwise, absolutely and uniformly to the data vector on the left.

It was evidently the derivation of (5.13) which induced Smith to introduce the restrictions $g(\pm1)=g'(\pm1)=0$. But his demonstration does not require these restrictions because his P satisfies the equation under his (X) without requiring $g(\pm1)=g'(\pm1)$. More directly, we shall now show that (5.3) and (5.9) imply (5.13) provided only that $f(\pm1)=f'(\pm1)=0$. Noting first that, by (5.7), $\mathbf{Y}_i(u)\cdot\mathbf{A}=-\mathbf{Y}_i''/S^2$ $(i=1,2)$, we integrate by parts:

$$\begin{aligned} S^2\mathbf{M}&=\mathbf{X}_1(t)\cdot\mathbf{W}_{12}^{-1}\cdot\int_{-1}^{t}\mathbf{Y}_2(u)\cdot\mathbf{A}\cdot\mathbf{f}(u)\,du\\ &\quad-\mathbf{X}_2(t)\cdot\mathbf{W}_{21}^{-1}\cdot\int_{t}^{1}\mathbf{Y}_1(u)\cdot\mathbf{A}\cdot\mathbf{f}(u)\,du\\ &=-\mathbf{X}_1(t)\cdot\mathbf{W}_{21}^{-1}\cdot\{[\mathbf{Y}_2'\cdot\mathbf{f}]_{-1}^{t}-[\mathbf{Y}_2\cdot\mathbf{f}']_{-1}^{t}\}\\ &\quad+\mathbf{X}_2(t)\cdot\mathbf{W}_{21}^{-1}\cdot\{[\mathbf{Y}_i'\cdot\mathbf{f}]_t^{1}-[\mathbf{Y}_i\cdot\mathbf{f}]_t^{1}\}\\ &\quad+S^2\mathbf{G}(S,t). \end{aligned}\tag{5.17}$$

We next observe that

$$\mathbf{Y}_2(-1)=\mathbf{Y}_1(1)=\mathbf{Y}(0)=\begin{bmatrix}2&0\\0&0\end{bmatrix}$$

and

$$\mathbf{Y}_2'(-1)=\mathbf{Y}'(1)=\mathbf{Y}'(0)=S\begin{bmatrix}0&0\\2&0\end{bmatrix}.$$

Since, by hypothesis,

$$\mathbf{f}(\pm1)=\begin{bmatrix}0\\ g(\pm1)\end{bmatrix},\qquad f'(\pm1)=\begin{bmatrix}0\\ g'(\pm1)\end{bmatrix},$$

we calculate

$$\begin{aligned} S^2\mathbf{M}=&-\{\mathbf{X}_1(t)\cdot\mathbf{W}_{21}^{-1}\cdot\mathbf{Y}_2'(t)+\mathbf{X}_2(t)\cdot\mathbf{W}_{21}^{-1}\cdot\mathbf{Y}_1(t)\}\cdot\mathbf{f}(t)\\ &+(\mathbf{X}_1(t)\cdot\mathbf{W}_{21}^{-1}\cdot\mathbf{Y}_2(t)+\mathbf{X}_2(t)\cdot\mathbf{W}_{21}^{-1}\cdot\mathbf{Y}(t)\}\cdot\mathbf{f}(t)\\ &+S^2\mathbf{G}(S,t). \end{aligned}\tag{5.18}$$

But the second bracket in (5.18) vanishes by (5.5) and, differentiating the second bracket, we get (5.18) in the form

$$S^2\mathbf{M}=\{\mathbf{X}_1'(t)\cdot\mathbf{W}_{12}^{-1}\cdot\mathbf{Y}_2(t)+\mathbf{X}_2'(t)\cdot\mathbf{W}_{21}^{-1}\cdot\mathbf{Y}_1(t)\}\cdot\mathbf{f}(t)+S^2\mathbf{G}(S,t) \tag{5.19}$$

To prove that the bracket in (5.19) is $\mathbf{1}$, we note that $\mathbf{W}_{22}=0$, $\mathbf{Y}_2(t)$ is not singular and $\mathbf{X}_2\cdot\mathbf{W}_{21}^{-1}\cdot\mathbf{Y}_1=-\mathbf{X}_1\cdot\mathbf{W}_{21}^{-1}\cdot\mathbf{Y}_2$. Then

$$\begin{aligned}0&=\mathbf{Y}_2^{-1}\cdot(\mathbf{Y}_2\cdot\mathbf{X}_2'-\mathbf{Y}_2'\cdot\mathbf{X}_2)\cdot\mathbf{W}_{21}^{-1}\cdot\mathbf{Y}_1\\&=\mathbf{X}_2'\cdot\mathbf{W}_{21}^{-1}\cdot\mathbf{Y}_1+\mathbf{Y}_2^{-1}\cdot(\mathbf{Y}_2'\cdot\mathbf{X}_1)\cdot\mathbf{W}_{21}^{-1}\cdot\mathbf{Y}_2\\&=\mathbf{X}_2'\cdot\mathbf{W}_{21}^{-1}\cdot\mathbf{Y}_1+\mathbf{Y}_2^{-1}(-\mathbf{W}_{21}+\mathbf{Y}_2\cdot\mathbf{X}_1')\cdot\mathbf{W}_{12}^{-1}\cdot\mathbf{Y}_2\\&=\mathbf{X}_2'\cdot\mathbf{W}_{21}^{-1}\cdot\mathbf{Y}_1'+\mathbf{X}_1\cdot\mathbf{W}_{12}^{-1}\cdot\mathbf{Y}_2-\mathbf{1}.\end{aligned}$$

Hence,

$$S^2\mathbf{M}=\mathbf{f}(t)+S^2\mathbf{G}(S,t);$$

i.e., (5.13) holds.

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