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Hydrodynamic Stability and Bifurcation

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Our understanding of hydrodynamic stability has been greatly enriched by recent developments in the mathematical theory of bifurcation. Bifurcation theory brings the theory of stability closer to physics and leads to simple criteria by which one can judge when linear analysis will describe observed motions. It also leads to an understanding of the formation of symmetry patterns and the destruction of these patterns in turbulent flow. The notion of bifurcating solutions of permanent form leads to concepts which give rise to classifications of motions which can be observed and to their qualitative properties.

1. Introduction

The goal of hydrodynamics is to describe and predict the motions of fluids under applied forces. For incompressible Navier-Stokes fluids, in many circumstances, these forces scale with the Reynolds number. When the Reynolds number is small, hydrodynamics is not so difficult because there is a unique correspondance between the given data and the predicted motions. But when the Reynolds number is larger hydrodynamics is complicated; there are many solutions; non-uniqueness is the rule; sets of solutions must be described and stable and observable subsets must be separated from the others.

To study these hard problems we use stability and bifurcation theory. These two theories are related but not the same. They are both old theories. Some of the best results in the general theory of bifurcation, and nearly all of the hydrodynamical results are relatively new.

In bifurcation theory we try to find the solutions of permanent form which arise from the instability of other flows. The bifurcating flows may be stable or unstable; only the stable ones would be observed as solutions of permanent forms.

The simplest example of a solution of permanent form is a steady flow. The next most complicated is a time-periodic flow, then a quasi-periodic flow with two frequencies. In more complicated situations, like turbulence, the objects which bifurcate are not solutions, but sets of solutions which do not have a permanent form. These solutions are evidently confined to some lower dimensional region of state space. This lower dimensional region is called an attractor if solutions which are not on it are attracted to it. The point to emphasize here is that the solutions on such an attractor need not have a permanent form. The solutions on the attractor need not be stable but the attractor is.

Turbulent attractors seem to occur in some flows after three or four bifurcations. For example, the experiments of Gollub and Swinney (1975) and Swinney, Fenstermacher and Gollub (1977) indicate that a turbulent attractor appears after a few bifurcations of flow between rotating cylinders in the sequence: Taylor vortices, undulating Taylor vortices with one frequency (Hopf bifurcation), undulating Taylor vortices with two frequencies (quasiperiodic bifurcation), diminished undulation with a broad band spectrum (turbulence). A similar sequence of bifurcations leading to turbulence has been observed in the flow between rotating spheres studied by Munson and Menguturk (1975). (Their results are discussed by Joseph, 1976, Vol. I, p. 214). A theory for turbulent attractors which appear after a small number of bifurcations has been given by Ruelle and Takens, 1971; McLaughlin and Martin, 1975; Chenciner and Iooss, 1977; Iooss, 1977; and Ruelle, 1977, among others. In other flows, like flow through a pipe, turbulence appears immediately, just as the undirectional laminar shear flow loses stability and its appearance is still a big mystery.

In my lecture I am going to stick to simple attractors. I want to show that a big light is cast on hydrodynamics by the theory of bifurcation and how the theory of hydrodynamic stability is illuminated therein. I will consider symmetry-breaking steady bifurcation of steady solutions, time-periodic bifurcation of steady solutions and subharmonic bifurcation of time-periodic solutions.

I intend this lecture to be a survey from a personal point of view. I am not going to attempt to give an account of the many deserving papers that have been written in this field and I have made no attempt to provide a good bibliography. In fact a very good bibliography, up to 1976 is available in my books, "Stability of Fluid Motions I & II which is published as a Springer-Tract in Natural Philosophy. I will confine my citations to a few references.

2. Uniqueness of solutions of permanent form when the Reynolds number is low

The proposition to be proved here is that all solutions of the Navier-Stokes equations which are close to rigid body motions are stable and the solutions of permanent form are unique. This proposition deserves more explanation because it is very important and, in fact, is a special result which holds for Navier-Stokes but not generally. I will be able to give better explanations after sketching the proof.

Let $\mathcal{V}(t)$ be the region of space occupied by the fluid. We want to discuss the stability of solutions $(\underline{V}(\underline{x},t); \underline{P}(\underline{x},t)$ = (velocity, pressure) of permanent form which satisfy the Navier-Stokes equations when the body force field $\underline{F}(\underline{x},t)$ and the velocity $\underline{V} = \underline{V}(\underline{x},t)$, $\underline{x} \in \partial\mathcal{V}$ are prescribed. In $\mathcal{V}(t)$, div $\underline{V} = 0$ and

$$\underline{V}_t + (\underline{V}\cdot\nabla)\underline{V} = -\nabla P + \lambda\nabla^2\underline{V} + \underline{F}(\underline{x},t). \qquad (2.1)$$

Equation (2.1) is dimensionless and $\lambda = 1/R$ and $R = m\ell^2/\nu$ is a Reynolds number based on m which we make take as some typical value of the symmetric part of the velocity gradient, ℓ is a typical length and ν is the kinematic viscosity. A perturbed solution $(\underline{V}^*, P^*)$ satisfies the same equations and boundary conditions but differs from $(\underline{V},P)$ initially. The difference $(\underline{V} - \underline{V}^*, P - P^*) = (\underline{u},p)$ satisfies the following initial-boundary-value-problem:

$$\underline{u}_t + (\underline{V}\cdot\nabla)\underline{u} + (\underline{u}\cdot\nabla)\underline{V} + (\underline{u}\cdot\nabla)\underline{u} = -\nabla p + \lambda\nabla^2\underline{u} \qquad (2.2)$$

where

$$\underline{u} \in H = \{\underline{v}: \ \text{div}\ \underline{v} = 0,\ \underline{v}|_{\partial\mathcal{V}} = 0,\ \langle|\nabla\underline{v}|^2\rangle < \infty\} \qquad (2.3)$$

and

$$\langle\cdot\rangle = \int_{\mathcal{V}} \cdot \, d\mathcal{V}$$

Of course, $\underline{u}(\underline{x},0)$ is prescribed.

After multiplying (2.2) by $\underline{u}$ and integrating over $\mathcal{V}(t)$ we get

$$\frac{1}{2}\frac{d}{dt}\langle|\underline{u}|^2\rangle = -\langle\underline{u}\cdot\underline{D}(\underline{V})\cdot\underline{u}\rangle - \lambda\langle|\nabla\underline{u}|^2\rangle \tag{2.4}$$

where $\underline{D}(\underline{V})$ is the symmetric part of $\nabla\underline{V}$. The term $-\langle\underline{u}\cdot\underline{D}(\underline{V})\cdot\underline{u}\rangle$ arises from the third term on the left of (2.2). Since $\nabla\underline{V} = \underline{D} + \underline{\Omega}$, where $\underline{\Omega}$ is antisymmetric

$$\underline{u}\cdot\nabla\underline{V}\cdot\underline{u} = \underline{u}\cdot\underline{D}(\underline{V})\cdot\underline{u}$$

The second and last term on the left of $\underline{u}\cdot(2.2)$ and the pressure on the right vanish because they can be cast into divergence forms and integrated to the boundary where they vanish. When $\underline{D}/\lambda = R\underline{D}$ is small we always get stability. For rigid body motions $R\underline{D} = 0$. <u>Hence flows which perturb rigid motions are always stable</u>.

It is good to draw attention to the fact that the only nonlinear term in (2.2), the term $(\underline{u}\cdot\nabla)\underline{u}$, has no "energy":

$$\langle\underline{u}\cdot(\underline{u}\cdot\nabla)\underline{u}\rangle = \langle\underline{u}\cdot\nabla\frac{|\underline{u}|^2}{2}\rangle = \langle\operatorname{div}(\underline{u}\frac{|\underline{u}|^2}{2})\rangle$$

$$= \int_{\partial\mathcal{V}}(\underline{n}\cdot\underline{u})\frac{|\underline{u}|^2}{2} = 0 \tag{2.5}$$

where $\underline{n}$ is the outward normal on $\partial\mathcal{V}$.

The vanishing of (2.5) is a big event in hydrodynamics. It allows us to get (2.4) in a form independent of the amplitude of $\underline{u}$. Each term of (2.4) is homogeneous of degree two in $\underline{u}$ and the form of (2.4) is unchanged when $\underline{u}$ is replaced by $\hat{a}\underline{u}$ for any non-zero number $\hat{a}$. In general problems of continuum mechanics there are other nonlinear terms, like material nonlinearities, and these other kinds of nonlinear terms do appear in the energy equation. In these general problems you cannot get a global stability result of the type I am now going to derive from the analysis of energy.

First I rewrite (2.4)

$$\mathcal{E} = \frac{1}{2} <|\underline{u}|^2>.$$

$$\mathcal{D} = <|\nabla \underline{u}^2|>,$$

$$\mathcal{U} = - <\underline{u} \cdot \underline{D}(\underline{V}) \cdot \underline{u}>$$

and

$$\frac{d\mathcal{E}}{dt} = \mathcal{D}\left[\frac{\mathcal{U}}{\mathcal{D}} - \lambda\right] \leq \mathcal{D}[\tilde{\lambda}(t,\lambda) - \lambda]$$

where

$$\tilde{\lambda}(\lambda,t) = \max_{H} \frac{\mathcal{U}(\lambda,t)}{\mathcal{D}} > 0. \qquad (2.6)$$

Now it is well known and easy to prove that there is $\Lambda(\mathcal{V})$ depending on $\mathcal{V}$ such that

$$D \geq \Lambda \mathcal{E} .$$

So, if $\lambda > \tilde{\lambda}(\lambda,t)$ then

$$\frac{d}{dt} \ln \mathcal{E} = - \Lambda(t) \; [\lambda - \tilde{\lambda}(\lambda,t)]$$

and

$$\mathcal{E}(t) < \mathcal{E}(0) \exp \{- \int_0^t \Lambda(\lambda - \tilde{\lambda}) d\tau \, . \qquad (2.7)$$

It follows that if λ is large enough (or R is small enough) the energy of any disturbance of $\underline{V}$ tends asymptotically to zero. On the other hand, the maximizing $\underline{u} = \tilde{\underline{u}} \in H$ for (2.6) serves admirably as the disturbance whose energy will increase initially at the **smallest** R. Of course, such initially increasing disturbances may eventually decay.

The uniqueness of solutions of permanent form at small Reynolds numbers follows almost immediately from (2.5). Suppose that $\underline{V}$ and $\underline{V}^*$ are almost periodic functions of t. Then $\underline{u}(\underline{x},t)$ and $\mathcal{E}(t)$ are almost periodic and

$$\frac{1}{T}\int_0^T \mathcal{E}^2(t)dt \le \mathcal{E}^2(0)\,\frac{1}{T}\int_0^T \exp\{-\int_0^t \Lambda(\lambda-\tilde{\lambda})d\tau\}dt \;.$$

Clearly, if

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \mathcal{E}^2(t)dt \to 0$$

and, since $\mathcal{E}(t)$ is almost periodic, $\mathcal{E}(t) = 0$ and $\underline{V} = \underline{V}^*$, almost everywhere. It is an immediate corollary of this result that when the Reynolds number is small, time-periodic motions and steady solutions of the Navier-Stokes equations are unique.

What does this type of uniqueness mean? It means that when R is small, solutions of the Navier-Stokes are uniquely determined by the data. If the prescribed motions of the boundary of $\mathcal{V}$ are steady relative to some fixed frame and if the applied forces $\underline{F}$ are also steady and R is small then there is only one steady solution of the Navier-Stokes equations. If the data is time-periodic then there is only one time-periodic solution. If the data is almost periodic, there is only one almost periodic solution.

Moreover (2.7) shows that when R is small the unique solution corresponding to the given data is globally stable. So we can't get any other solution of permanent form when R is small, all other solutions decay exponetially to the unique one.

It follows that the unique solution at small R is a window to the outside. There is a one to one correspondance between the given data and this unique solution. You don't see boundary data or prescribed forces in (2.2). The data, is there all right but you don't see it because it is buried in the solution $\underline{V}$. So when we talk about stability and bifurcation of $\underline{V}$ we can just as well talk about the stability and bifurcation of the <u>null solution</u> $\underline{u} = 0$ of (2.2). This solution, the null solution, is globally stable when R is small, when the motion $\underline{V}$ is nearly that of a rigid body and, through $\underline{V}$, it is uniquely determined by the data.

We are going to assume that the motion $\underline{V}(\underline{x},t) = \underline{V}(\underline{x},t,R)$ exists for all $R \in \mathbb{R}^+$ even though it may not be unique when R is large. Whether or not $\underline{V}(\underline{x},t,R)$ is stable or unique it always gives a unique specification of the data for (2.1) in the problem (2.2).

Note: the material in this section is found in my book "Stability of Fluid Motion-I". The theorems are inspired by works of Orr, Thomas, Hopf and Serrin which are cited in my book.

3. Evolution equations governing disturbances of the null solution

We want to know the value of R for which an initial disturbance of $\underline{u} = 0$ will lead eventually to permanent motions other than $\underline{u} = 0$. Since we aren't clever enough to analyze (2.2) in full generality, we study the special class of indefinitely small disturbances for which (2.2) may be linearized (set $\underline{u} = \delta\underline{z}$, $p = \delta\pi$, $\delta\to 0$). Then we seek a critical value $R = R_c$ such that when $R > R_c$ indefinitely small disturbances will grow. To find the value R_c we study the linearized equations. New we can write (2.2) as

$$\frac{\partial \underline{u}}{\partial t} + (\underline{V}\cdot\nabla)\underline{u} + (\underline{u}\cdot\nabla)\underline{V} + (\underline{u}\cdot\nabla)\underline{u} + (\mu-\lambda_c)\nabla^2\underline{u} = -\nabla p \quad (3.1)$$

where $\lambda_c = 1/R_c$ and $\mu = -\lambda + \lambda_c$. This rewriting doesn't really change anything except to map critical value λ_c into the critical value $\mu = 0$. The linearized problem associated with (3.1) is

$$\frac{\partial \underline{z}}{\partial t} + (\underline{V}\cdot\nabla)\underline{z} + (\underline{z}\cdot\nabla)\underline{V} + (\mu-\lambda_c)\nabla^2\underline{z} = -\nabla\pi \quad (3.2)$$

where $\underline{z} \in H$. The linearized problem (3.2) can be very tough even when $\underline{V}$ is very simple. For example, we get the very tough Orr-Sommerfeld problem when $\underline{V}$ is a parallel steady flow between parallel planes. But we can say a lot about bifurcation and stability when $\underline{V}(\underline{x},t) = \underline{V}(\underline{x})$ is steady or when $\underline{V}(\underline{x},t) = \underline{V}(\underline{x},t + T)$ is T-periodic.

The pressure p in (3.1) is a reaction pressure which must be determined so that $\underline{u}$ will satisfy div $\underline{u} = 0$. We can get rid of p by taking the curl of (3.1). Another way to get rid of p is to form the scalar product of (3.1) with any element $\underline{v} \in H$. These $\underline{v}$'s all have div $\underline{v} = 0$ and they depend on t only through $\underline{x}$. Hence

$$\frac{d}{dt} \langle \underline{u} \cdot \underline{v} \rangle + \langle [\underline{V} \cdot \nabla \underline{u} + \underline{u} \cdot \nabla \underline{V} + \underline{u} \cdot \nabla \underline{u} + (\mu - \lambda_c) \nabla^2 \underline{u}] \cdot \underline{v} \rangle = 0 \tag{3.3}$$

Equation (3.3) is to hold for every $\underline{v} \in H$. If we say the correct things about H, then (3.3) is equivalent to (3.1). Now we rewrite (3.3) as an evolution equation on a Banach space

$$\underline{u}_t + F(t,\mu, \underline{u}) = 0. \tag{3.4}$$

Eq. (3.4) is equivalent to (3.1) and, if you like, you can think of (3.4) as a symbolic way of writing (3.1). In fact you can do everything I am going to do with (3.4) to (3.1), but you then have to do more writing.

Our understanding about F in (3.4) is that the t in the first argument comes from $\underline{V}(\underline{x},t)$. So if $\underline{V} = \underline{V}(\underline{x})$ is steady then

$$F(t,\mu, \underline{u}) = F(\mu, \underline{u}) \tag{3.5}$$

is independent of t and (3.4) is an autonomous problem. And, if $\underline{V}(\underline{x},t) = \underline{V}(\underline{x}, t + \tau)$ then

$$F(t,\mu, \underline{u}) = F(t + T,\mu, \underline{u}) \tag{3.6}$$

is T-periodic and, of course, (3.4) is not autonomous. The t-dependence of F is very important because it specifies the data for the original problem (2.1) through the unique correspondance of this data and $\underline{V}(\underline{x},t)$.

The linearized problem corresponding to (3.4) may be obtained from (3.3) by putting $\underline{u} = \delta \underline{z}$, $\delta \to 0$. We get

$$\underline{z}_t + F_u(t,\mu,0|\underline{z}) = 0 \tag{3.7}$$

for $\underline{z} \in H$. $F_u(t,\mu,0|\underline{z}) \equiv F_u(t,\mu|\underline{z}) = F_u(t + T, \mu|\underline{z})$ is the derivative of the nonlinear operator at the point $\underline{u} = 0$.

4. Spectral problems and spectral hypotheses

Suppose that $\underline{V}$ is steady. Then we use method of exponential disturbances

$$z = e^{-\sigma t}\,\underline{\zeta}(\underline{x}) \tag{4.1}$$

to study stability. We get

$$-\sigma\,\underline{\zeta} + F_u\,(\mu,\ 0|\underline{\zeta}) = 0 \tag{4.2}$$

$\underline{\zeta} \in H$ as an eigenvalue problem for σ.

Suppose that $\underline{V}$ is T-periodic. Then we use the method of Floquet

$$z = e^{-\sigma t}\,\underline{\zeta}(\underline{x},t) \qquad \underline{\zeta}(\underline{x},t) = \underline{\zeta}(\underline{x},t+\tau) \tag{4.3}$$

to study stability. We get

$$-\sigma\underline{\zeta} + \underline{\zeta}_t + F_u(t,\mu,\ 0|\underline{\zeta}) = 0 \tag{4.4}$$

where $\underline{\zeta} \in H_T = H \cap$T-perodic functions.

The complex numbers

$$\sigma(\mu) = \xi(\mu) + i\omega(\mu)$$

are eigenvalues. For Navier-Stokes problems and many other problems they form a discrete set of finite multiplicity. We have introduced μ so that when μ is negative

$$\xi(\mu) > 0 \tag{4.5}$$

for all eigenvalues σ. When $\mu = 0$ (the critical condition) there are some eigenvalues for which

$$\xi(0) = 0 \tag{4.6}$$

and all the rest have $\xi(0) > 0$. For slightly larger values of μ some of the eigenvalues have negative real parts and the null solution is unstable.

Two methods are used to study problems of bifurcation and stability: the method of nonlinear stability theory and the method of bifurcation theory. The difference between these two methods is in the way the problem to be studied is specified. Nonlinear stability theory starts with statements about $\underline{V}$; bifurcation theory starts with statements about $\sigma(0) = i\omega(0)$. In nonlinear stability theory it is necessary to restrict ones attention to the special class of $\underline{V}$'s which are known in sufficient detail to make possible explicit analysis by perturbation methods or numerical methods. Very detailed hydrodynamical results follow from nonlinear stability analysis but the results lack generality because in general $\underline{V}$ is not known in sufficient detail. Bifurcation analysis, on the other hand, does not require that $\underline{V}$ be specified in detail; instead, it is necessary to make assumptions about the spectrum of the spectral problem for $\underline{V}$. Bifurcation theory leads to a fairly general classification of the qualitative structure of solutions which arise out of successive loss of stability. We don't get detailed hydrodynamics from bifurcation theory and we don't get a general qualitative theory from nonlinear stability theory. So the two methods are complementary where they are genuinely different.

The methods of nonlinear stability theory are perhaps better known to fluid mechanicians. Many papers using this method are found, for example, in the Journal of Fluid Mechanics. The methods of bifurcation theory are perhaps better known to mathematicians. Many papers using this method are found, for example, in the Archive for Rational Mechanics and Analysis. I am going to discuss bifurcation theory in this lecture. The most complete results known in bifurcation theory are based on

Hopf's hypotheses:

(i) $\sigma(0) = i\omega(0)$ is an isolated, algebraically simple eigenvalue

(ii) $\xi_\mu(0) < 0$. The loss of stability of the null solution is strict. These hypotheses are fairly typical in applications though they are violated in special cases which are frequently associated with problems of symmetry breaking bifurcations in spatial regions with a high degree of symmetry.

5. Symmetry breaking steady bifurcation of steady solutions and Hopf bifurcation of steady solutions

When $\underline{V}(\underline{x})$ is steady and $\omega(0) = 0$ we get an infinitesimal steady solution at criticality and a real nonlinear solution of (3.1) bifurcates. Physically such bifurcation typically occurs as a break in the symmetry of $\underline{V}(\underline{x})$. For example, uniform Couette flow between cylinders breaks into Taylor vortices or the conduction state of a fluid layer heated from below breaks into cellular convection.

In the study of bifurcation it is convenient to introduce an amplitude ε. This amplitude can be defined in various ways. For example, we can define the amplitude ε as the projection of $\underline{u}$ with some fixed vector, say $\varepsilon = \langle \underline{u} \cdot \underline{\zeta}^* \rangle$. A bifurcation diagram is a curve in the (μ, ε) plane. Bifurcating solutions exist for each pair (μ, ε) on the curve. In the applications in fluid dynamics we always get a single-valued curve $\mu = \mu(\varepsilon)$.

Now I am going to explain the bifurcation results which follow from Hopf's hypotheses. The main mathematical consequences of these hypotheses are summarized in the bifurcation diagrams 5.1 and 5.2. The following features of these diagrams deserve emphasis.

(i) The solution $\underline{u} = 0$ ($\varepsilon = 0$) is stable to small disturbances when $\mu = -1/R + 1/R_c < 0$. It is stable to all disturbances when, $\mu < \mu_E = -1/R_E + 1/R_c$ where

$$\frac{1}{R_E} = \max_H \frac{\mathcal{I}(1/R)}{\mathcal{D}} > 0$$

(see §2).

(ii) An analytic in ε family of solutions bifurcates at $\mu = 0$. In the typical case (5.1) we get a non-zero slope at the origin, $\mu_\varepsilon(0) \neq 0$. This is called two-sided bifurcation because the bifurcating solution goes off to both sides near $\varepsilon = 0$. If $\mu_\varepsilon(0) = 0$ and $\mu_{\varepsilon\varepsilon}(0) \neq 0$ we get one-sided bifurcation, either to the right of $\mu = 0$ as in (5.2a) or to the left as in (5.2b).

(iii) Local results are results for small values of ε. A bifurcating solution is called locally supercritical if it exists for values of μ for which $\underline{u} = 0$ is unstable. Here supercritical solutions are those which branch off to the side $\mu > 0$. Subcritical solutions go the other way.

(iv) Supercritical bifurcating solutions are stable, subcritical bifurcating solutions are unstable when ε is small.

(v) Subcritical bifurcating solutions cannot exist in the shaded regions on the left of Figs. 5.1 and 5.2. These shaded regions are regions of uniqueness in which only one solution of permanent form, $\underline{u} \equiv 0$, is possible. It follows that subcritical solutions cannot exist for $R < R_E$ and that bounded analytic bifurcation must turn around and come back. I shall call points at which μ_ε changes sign turning points. Such points are sometimes called critical points.

(vi) Factorization theorems (Joseph & Nield, 1975 Joseph, 1977) may be used to extend the local (small $|\varepsilon|$) results of Hopf to unrestricted values of $|\varepsilon|$. The local result says that supercritical bifurcation is stable and subcritical bifurcation is unstable. The global result states, that, given conditions like those assumed by Hopf, bifurcating branches having the good direction $\varepsilon\mu_\varepsilon(\varepsilon) > 0$ are stable and bifurcating branches having the bad direction $\varepsilon\mu_\varepsilon(\varepsilon) < 0$ are unstable.

(vii) The bifurcation diagrams show that the solution picture is very badly represented by the linear theory of stability of $\underline{u} = 0$ whenever there is subcritical bifurcation. When there is subcritical bifurcation there are stable subcritical bifurcating solutions for $\mu < 0$. We get these as a snap-through instability in which the solution $\underline{V}(\underline{x})$ loses stability for $\mu < 0$ to a large disturbance. Such a disturbance cannot be attracted by the unstable bifurcating solution so it goes to the stable branch with large values of ε.

In the next section I will give simple arguments to establish the bifurcation pictures and to explain the significance (and some of the limitations) of Hopf's hypotheses. To get simple arguments I cheat in the following way:

First I regard (3.4) as an equation in a Banach space. Then I forget about hydrodynamics and treat (3.4) in the simplest realization of a Banach space, $\mathbb{R}_1$. In $\mathbb{R}_1$ I can prove everything and the bifurcation results which are obtained are global and complete. It is necessary to add that though Hopf bifurcation into periodic solutions cannot occur in $\mathbb{R}_1$ we can have Hopf bifurcation already in $\mathbb{R}_2$ and bifurcation into nonperiodic attractors in $\mathbb{R}_3$. However, even in these more complicated cases the bifurcation pictures shown in (5.1) hold. Surprisingly, $\mathbb{R}_1$ has already much that we need to know about the general properties of bifurcating solutions.

Before turning to analysis of bifurcation and stability in $\mathbb{R}_1$ I should like to make more precise the relation between Hopf bifurcation and the diagrams shown in Fig. 5.1 and 5.2. We now suppose that Hopf's hypotheses hold and $\omega(0) = \omega_0 \neq 0$. There are two eigenvectors $\underline{\zeta}$ and $\overline{\underline{\zeta}}$ (the complex conjugate of $\underline{\zeta}$) of (4.2) belonging to the eigenvalues $i\omega_0$ and $-i\omega_0$. This means there are two solutions $\underline{z} = e^{-i\omega_0 t}\underline{\zeta}$ and $\overline{\underline{z}}$ of the linearized stability problem

$$\text{(5.1)} \qquad J\,\underline{z} = \underline{z}_t + F_u(\mu,0|\underline{z}) = 0 \ .$$

Another way to say this is that the null space of the operator J, whose domain is of $2\pi/\omega_0$ periodic functions, is two-dimensional (spanned by $\underline{z}$ and $\overline{\underline{z}}$) or zero is a double, semi-simple eigenvalue of J.

In studying Hopf bifurcation we seek a solution $(\mu(\varepsilon), \underline{u}(t,\varepsilon))$ of the nonlinear evolution problem with amplitude ε such that

$$\text{(5.2)} \qquad \lim_{\varepsilon\to 0} (\underline{u}(t,\varepsilon)/\varepsilon) \to b\underline{z} + \overline{b}\overline{\underline{z}} = \underline{u}_\varepsilon(t) \ .$$

In the general theory of perturbation it is necessary to determine the value of the complex constant b and the other initiating ε-dependent parameters. However, the Hopf bifurcation is degenerate because (5.1) is autonomous. This means that (5.1) is invariant to a shift in the origin of time. It follows from this that we may take b as real-valued because if b is complex say, $b = \pm |b|e^{ic}$, then we may shift $t \to t - c = \tau$

so that $u_\varepsilon(\tau) = \pm |b|(\underline{z}(\tau) + \overline{\underline{z}}(\tau))$. So we have to find a family of solutions which turn out to be unique within a time shift. When framed in terms of the phase plane for two dimensional problems, the Hopf bifurcation is the bifurcation of an equilibrium point into a limit cycle. We may regard any point on the limit cycle as the origin of time.

The analysis of Hopf bifurcation can be found in many places; for example in my book (Vol. I) or in Sattingers (1972) notes. The Hopf solution consists of three analytic functions of $\varepsilon[\mu(\varepsilon), \underline{u}(t) = \underline{u}(t + \frac{2\pi}{\omega(\varepsilon)}), \omega(\varepsilon)]$ = [bifurcation curve, bifurcating time-periodic solutions, amplitude dependent frequency]. Analysis shows that bifurcating time-periodic solutions are always one sided; that is,

$$\mu(\varepsilon) = \mu(-\varepsilon) \tag{5.3}$$

and

$$\omega(\varepsilon) = \omega(-\varepsilon) \ . \tag{5.4}$$

It follows from (5.3) that the bifurcation diagram (5.1) can never occur when a time-periodic solution bifurcates from a steady solution. On the other hand, the bifurcation diagrams (5.2) are expected for Hopf bifurcation with the understanding that these curves are symmetric in the sense of (5.3). The stability properties of bifurcating time-periodic solutions are just those shown in the diagrams (5.2) (see Joseph and Nield (1975) and Joseph (1977)).

It is perhaps useful to add that the bifurcation picture (5.1) is perhaps typical of most problems involving steady symmetry breaking bifurcation. We make this statement plausible by noting that in $\mathbb{R}_1$ we cannot get one-sided bifurcation unless a certain derivative vanishes (see the analysis following (6.11)). On the other hand, in regions with a high degree of spatial symmetry, like circular or hexagonal cylinders, this derivative (which for partial differential equations is a projection of a functional derivative) may vanish and, in addition, the spectrum may degenerate so that several eigenfunctions share the same eigenvalue (see Joseph, 1976, Vol. II, Chap. X). In this case it usually suffices to study bifurcation at a semi-simple (multiple) eigenvalue and the bifurcation pictures which result can have a different structure, as shown in Figs. 8.2 and 8.3.

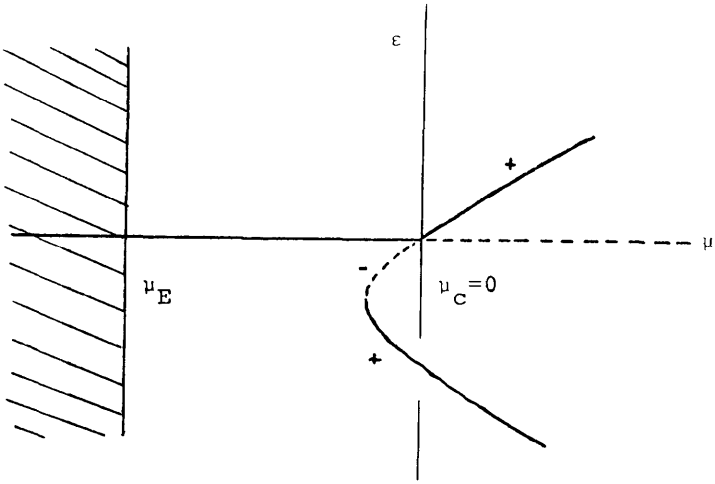

Fig. 5.1. **Two-sided** bifurcation. This type of bifurcation diagram is typical for steady bifurcations. It occurs, for example, in the problem of Bénard convection when the material properties depend on temperature or the conduction solution ($\varepsilon = 0$) is not linear (see Chapter X of Joseph, Vol. II, 1976). It also appears to characterize the problem of bifurcation of Couette flow between spheres (see Chapter VII of Joseph, Vol. I, 1976). The energy method shows that in the shaded region the null solution (say, heat conduction or Couette flow) is globally stable and unique. In fact the energy method frequently gives overly conservative estimates of the region of global stability of the null solution. When $\mu > \mu_c$ the null solution is unstable. The factorization theorem shows that bifurcating solutions with plus signs ($\varepsilon\mu_\varepsilon > 0$) are stable and those minus signs ($\varepsilon\mu_\varepsilon < 0$) are unphysical and also unstable. For example, when $\varepsilon\mu_\varepsilon < 0$, $\mu(\varepsilon)$ increases when $|\varepsilon|$ decreases. For the Benard problem such solutions transport <u>less</u> heat when the temperature difference is <u>increased</u>. Or in the problem of rotating spheres, the unphysical solutions with $\varepsilon\mu_\varepsilon < 0$ associate uncreased rotational speed with decreasing torques.

The same type of bifurcation diagram holds for the problem of symmetry breaking T-periodic bifurcation of T-periodic flows. This is the case n = 1 of nT-periodic (subharmonic) bifurcation of T-periodic flows discussed in §8.

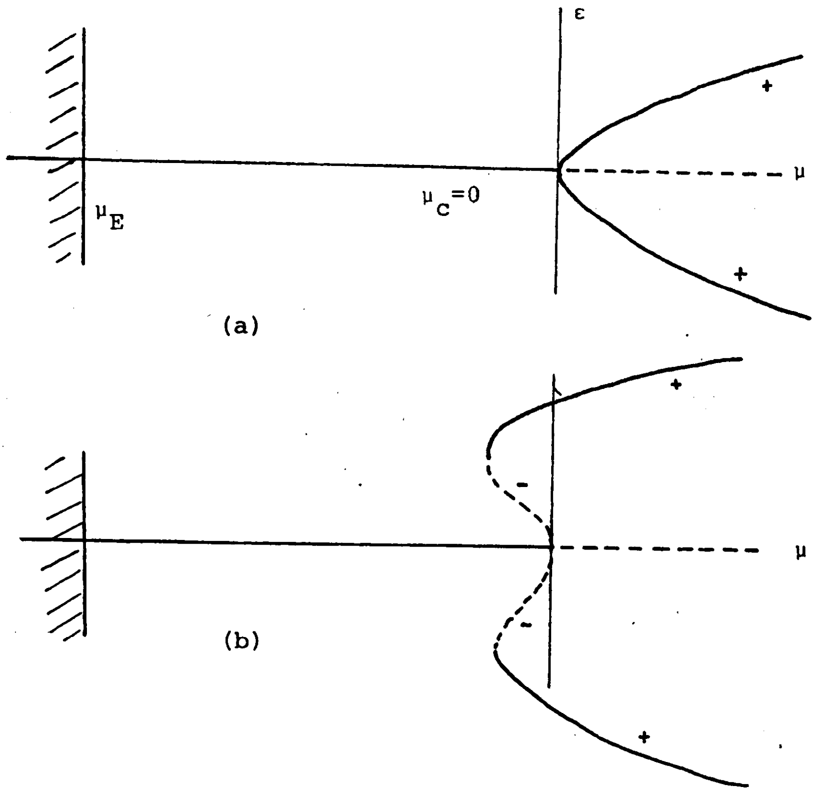

Fig. 5.2. One-sided bifurcation: (a) is locally supercritica and (b) is locally subcritical. Bifurcation diagrams of type (a) hold in idealized problems of Bénard convection and in the problem of bifurcation of Couette flow into Taylor vortices between infinitely long cylinders. Subcritical one-sided bifurcation (b) of steady solutions into steady solutions with different patterns of symmetry have not yet been given in the hydrodynamical context. The bifurcation of steady solutions into time-periodic solutions (Hopf bifurcation) is always one-sided because $\mu(\varepsilon) = \mu(-\varepsilon)$. Fig. 5.2(b) is appropriate in the description of Hopf bifurcation of parallel shear flows. For example, Fig. (5.2b) describes the bifurcation of plane Poiseuille flow. The subcritical time-periodic bifurcating solution ($\varepsilon\mu_\varepsilon < 0$) is unphysical and unstable. On such solutions, the mass flux is decreased as the pressure gradient is increased. We get instability in the

form of snap-through to turbulent slugs when $\mu < \mu_c$ (Reynolds numbers less than the critical value of linearized theory).

Snap through instabilities are not possible for the one-sided supercritical bifurcating solutions shown in (a). These solutions bifurcate to the "good" side ($\varepsilon\mu_\varepsilon > 0$) and are stable.

If you didn't know bifurcation theory and just studied the linearized theory of stability you would conclude that the linearized theory was good if your problem was like that shown in (a) or that the linearized theory was bad if your problem was like that shown in (b). In problems (a), like the idealized Bénard problem and Taylor problem would find that the null solution was stable for $\mu < \mu_c$ and unstable for $\mu > \mu_c$. Moreover calculations of the flow replacing the null solution for μ slightly larger than μ_c could be well approximated by the eigenfunctions of the linearized theory of stability. On the other hand, in cases like (b) you would observe flows on the stable branches with large values of ε when $\mu < \mu_c$. Such flows would ordinarily not be well-described by the linearized eigenfunctions.

Bifurcation theory explains why it is true that the predictions of linearized theories of stability agree with experiments in some cases and not in others.

The same type of bifurcation diagram holds for the problem of 2T-periodic bifurcation of T-periodic flows (see §8).

6. Stability and bifurcation in $\mathbb{R}_1$ †

We consider an evolution equation in $\mathbb{R}_1$ of the same form as (3.4):

$$(6.1)\qquad u_t + F(\mu,u) = 0 .$$

where $F(\cdot,\cdot)$ has two continuous derivatives in $\mathbb{R}_1 \times \mathbb{R}_1$. It is conventional in the study of stability of bifurcation to arrange things so that

$$(6.2)\qquad F(\mu,0) = 0 \quad \forall \mu \in \mathbb{R}_1 .$$

But we shall not require (6.2). Instead we require that equilibrium solutions of (6.1) have $u = \varepsilon$, $\varepsilon_t = 0$ and

$$(6.3)\qquad F(\mu,\varepsilon) = 0 .$$

The study of bifurcation of equilibrium solutions of the autonomous problem (6.1) is equivalent to the study of singular points of the plane curves (6.3).

In our study of equilibrium solutions (6.3) it is desirable to introduce the following classification of points:

(i) A <u>regular point</u> of $F(\mu,\varepsilon) = 0$ is one for which the implicit function theorem works,

$$(6.4)\qquad F_\mu \neq 0 \quad \text{or} \quad F_\varepsilon \neq 0 .$$

If $(6.4)_1$ holds then we can find a unique curve $\mu = \mu(\varepsilon)$ through the point. If $(6.4)_2$ holds then we can find a unique

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† The results given in this section grow out of joint work of D. D. Joseph (1977) and S. Rosenblat (1977).

curve $\varepsilon = \varepsilon(\mu)$ through the point.

(ii) A <u>regular turning point</u> is a point at which $\mu_\varepsilon(\varepsilon)$ changes sign and $F_\mu(\mu,\varepsilon) \neq 0$.

(iii) A <u>singular point</u> of the curve $F(\mu,\varepsilon) = 0$ is a point at which

$$(6.5) \qquad F_\mu = F_\varepsilon = 0 .$$

(iv) A <u>double point</u> of the curve $F(\mu,\varepsilon) = 0$ is a singular point through which pass two and only two branches of $F(\mu,\varepsilon) = 0$ possessing distinct tangents.

(v) A <u>singular turning (double) point</u> of the curve $F(\mu,\varepsilon) = 0$ is a double point at which μ_ε changes sign.

(vi) A <u>cusp point</u> of the curve $F(\mu,\varepsilon) = 0$ is a point of higher order contact between two branches of the curve. The two branches of the curve have the same tangent at a cusp point.

(vii) A <u>higher order singular point</u> of the curve $F(\mu,\varepsilon) = 0$ is a singular point at which all three second derivatives of $F(\mu,\varepsilon) = 0$ are null.

I am going to connect the study of stability to the study of bifurcation under the "strict crossing" hypothesis introduced by Hopf and used in almost all studies of bifurcation and stability. I will explain what is meant by "strict crossing" in due course; for now it will suffice to remark that this hypothesis restricts the study of bifurcation to <u>double points</u>; cusp points and higher order singular points are excluded.

It is necessary to be precise about double points. Suppose (μ_0, ε_0) is a singular point. The equilibrium curves passing through the singular points satisfy

$$(6.6)\qquad 2F(\mu,\varepsilon) = F_{\mu\mu}\ \delta\mu^2 + 2F_{\varepsilon\mu}\ \delta\varepsilon\ \delta\mu + F_{\varepsilon\varepsilon}\ \delta\varepsilon^2 + o\ (\delta\mu^2 + \delta\varepsilon\ \delta\mu + \delta\varepsilon^2) = 0$$

where $\delta\mu = \mu - \mu_0$, $\delta\varepsilon = \varepsilon - \varepsilon_0$ and $F_{\mu\mu} = F_{\mu\mu}(\mu_0,\varepsilon_0)$, etc. In the limit, as $(\mu, \varepsilon) \to (\mu_0, \varepsilon_0)$ the equation (6.6) for the curves $F(\mu,\varepsilon) = 0$ reduce to the quadratic equation

$$(6.7)\qquad F_{\mu\mu}\ d\mu^2 + 2F_{\varepsilon\mu}\ d\varepsilon\ d\mu + F_{\varepsilon\varepsilon}\ d\varepsilon^2 = 0$$

for the tangents to the curve. We find that

$$(6.8)\qquad \begin{bmatrix} \mu_\varepsilon^{(1)}(\varepsilon_0) \\ \mu_\varepsilon^{(2)}(\varepsilon_0) \end{bmatrix} = -\frac{F_{\varepsilon\mu}}{F_{\mu\mu}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \sqrt{\frac{D}{F_{\mu\mu}^2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

or

$$(6.9)\qquad \begin{bmatrix} \varepsilon_\mu^{(1)}(\mu_0) \\ \varepsilon_\mu^{(2)}(\mu_0) \end{bmatrix} = -\frac{F_{\varepsilon\mu}}{F_{\varepsilon\varepsilon}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \sqrt{\frac{D}{F_{\varepsilon\varepsilon}^2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

where

$$(6.10)\qquad D = F^2_{\varepsilon\mu} - F_{\mu\mu}\ F_{\varepsilon\varepsilon}\ .$$

If $D < 0$ there are no real tangents through (μ_0, ε_0) and the point (μ_0, ε_0) is an isolated point solution of $F(\mu,\varepsilon) = 0$.

We shall consider the case when (μ_0, ε_0) is <u>not</u> a higher order singular point. Then (μ_0, ε_0) is a double point if and only if $D > 0$. If $D = 0$ then the slope at the singular point of higher order contact is given by (6.3) or (6.4). If $D > 0$ and $F_{\mu\mu} \neq 0$, then there are two tangents with slopes $\mu_\varepsilon^{(1)}(\varepsilon_0)$ and $\mu_\varepsilon^{(2)}(\varepsilon_0)$ given by (6.8). If $D > 0$ and $F_{\mu\mu} = 0$, then $F_{\varepsilon\mu} \neq 0$ and

$$(6.11) \qquad d\varepsilon\,[2\, d\mu\, F_{\varepsilon\mu} + d\varepsilon\, F_{\varepsilon\varepsilon}] = 0$$

and there are two tangents $\varepsilon_\mu(\mu_0) = 0$ and $\mu_\varepsilon(\varepsilon_0) = -F_{\varepsilon\varepsilon}/2F_{\varepsilon\mu}$. If $\varepsilon_\mu(\mu_0) = 0$ then $F_{\mu\mu}(\mu_0,\varepsilon_0) = 0$. So all possibilities are covered in the following two cases:

(A) $D > 0$, $F_{\mu\mu} \neq 0$ with tangents $\mu_\varepsilon^{(1)}(\varepsilon_0)$ and $\mu_\varepsilon^{(2)}(\varepsilon_0)$

(B) $D > 0$, $F_{\mu\mu} = 0$ with tangents $\varepsilon_\mu(\mu_0) = 0$ and $\mu_\varepsilon(\varepsilon_0) = -F_{\varepsilon\varepsilon}/2F_{\varepsilon\mu}$.

Now I am going to connect stability and bifurcation. To study the stability of the solution $u = \varepsilon$ we study the linearized equation

$$z_t + F_\varepsilon(\mu,\varepsilon)z = 0$$

by the spectral method

$$z = e^{-\gamma t} z'$$

where

(6.12) $\gamma = F_\varepsilon(\mu,\varepsilon)$.

The solution $u = \varepsilon$ is stable when $\gamma > 0$ and is unstable when $\gamma < 0$.

Theorem 1 (Factorization theorem): For every equilibrium solution $F(\mu,\varepsilon) = 0$ for which $\mu = \mu(\varepsilon)$ we have

(6.13) $\gamma = F_\varepsilon(\mu(\varepsilon),\varepsilon) = -\mu_\varepsilon(\varepsilon)\, F_\mu(\mu(\varepsilon),\varepsilon) \equiv \mu_\varepsilon\, \hat{\gamma}(\varepsilon)$

The proof of theorem 1 follows from (6.12) and the equation

(6.14) $\frac{dF(\mu(\varepsilon),\varepsilon)}{d\varepsilon} = F_\varepsilon(\mu(\varepsilon),\varepsilon) + \mu_\varepsilon(\varepsilon) F_\mu(\mu(\varepsilon),\varepsilon) = 0$.

This type of factorization may be proved for the stability of bifurcating solutions in Banach spaces more complicated than $\mathbb{R}_1$ (Joseph, 1977). But the theorem is most easily understood in $\mathbb{R}_1$. One of the main implications of the factorization theorem is that $\gamma(\varepsilon)$ must change sign as ε is varied across a regular turning point. This implies that the solution $u = \varepsilon$, $\mu = \mu(\varepsilon)$ is stable on one side of a regular turning point and is unstable on the other side.

The next theorem connects stability and singular points.

Theorem 2: (A) Any point (μ_0, ε_0) of the curve $\mu = \mu(\varepsilon)$ for which $\hat{\gamma}(\varepsilon_0) = 0$ is a singular point. (B) Any point (μ_0, ε_0) of the curve $\varepsilon(\mu)$ for which $\gamma(\mu_0) = 0$ is a singular point.

The proof of (A) follows from (6.13) and the proof of (B) from

$$(6.15) \qquad \gamma(\mu) = F_{\varepsilon}(\mu,\varepsilon(\mu)), \ \frac{dF}{d\mu} = F_{\mu} + \varepsilon_{\mu}F_{\varepsilon} = 0 \ .$$

The next theorem connects the hypothesis of strict loss of stability into bifurcation into double points.

Theorem 3. Suppose that (μ_0,ε_0) is a singular point and (A) $\gamma_{\varepsilon}(\varepsilon_0) \neq 0$ or (B) $\gamma_{\mu}(\mu_0) \neq 0$. Then (μ_0,ε_0) is a double point. In case (A) we find from (6.13) that at the singular point $(\mu(\varepsilon_0),\varepsilon_0)$

$$(6.16) \qquad \gamma_{\varepsilon}(\varepsilon_0) = F_{\varepsilon\varepsilon} + \mu_{\varepsilon} F_{\varepsilon\mu} = - \mu_{\varepsilon}^2 F_{\mu\mu} - \mu_{\varepsilon} F_{\varepsilon\mu} \neq 0$$

Eq. $(6.16)_2$ shows that the characteristic quadratic (6.7) holds at $(\mu(\varepsilon_0),\varepsilon_0)$. Since there is a curve through this point $D \geq 0$ and we need to show that $D \neq 0$. We shall assume that $D = F_{\varepsilon\mu}^2 - F_{\mu\mu} F_{\varepsilon\varepsilon} = 0$ and show that this assumption contradicts (6.16). We first note that (6.16) implies that not all three of the second derivatives of F are null at $(\mu(\varepsilon_0),\varepsilon_0)$. If $F_{\mu\mu} F_{\varepsilon\varepsilon} \neq 0$ and $D = 0$ then (6.8) becomes $\mu_{\varepsilon}(\varepsilon_0) = - F_{\varepsilon\mu}/F_{\mu\mu}$ and $(6.16)_1$ may be written as $F_{\varepsilon\varepsilon} - F_{\varepsilon\mu}^2/F_{\mu\mu} = - D/F_{\mu\mu} \neq 0$. So $D \neq 0$ after all. If $F_{\mu\mu} F_{\varepsilon\varepsilon} = 0$ and $D = 0$ then $F_{\varepsilon\mu} = 0$ and (6.16) may be written as $\gamma_{\varepsilon} = F_{\varepsilon\varepsilon} = - \mu_{\varepsilon}^2 F_{\mu\mu} \neq 0$. So $D \neq 0$ after all.

In case (B) we solve $F(\mu,\varepsilon) = 0$ for $\varepsilon(\mu)$. At the singular point (μ_0,ε_0), we have strict loss of stability

(6.17) $\gamma_\mu = F_{\varepsilon\mu}(\mu_0, \varepsilon(\mu_0)) + \varepsilon_\mu(\mu_0) F_{\varepsilon\varepsilon}(\mu_0, \varepsilon(\mu_0)) \neq 0$

and $D \geq 0$. Assuming $D = 0$ we find, using (6.9) that

(6.18) $\varepsilon_\mu = - F_{\varepsilon\mu}/F_{\varepsilon\varepsilon}$

if $F_{\varepsilon\varepsilon} \neq 0$. Then (6.17) and (6.18) imply that $\gamma_\mu = 0$, and if $D = 0$ and $F_{\varepsilon\varepsilon} = 0$, then $F_{\varepsilon\mu} = 0$ so that $\gamma_\mu = 0$. So $D = 0$ is inconsistent with $\gamma_\mu \neq 0$ and $D > 0$ after all.

The analysis of bifurcation in $\mathbb{R}_1$ just given shows that double point bifurcation is implied by a strict crossing hypothesis of the Hopf type. The situation is more complicated when these hypotheses are relaxed. If $\gamma_\varepsilon = 0$ when $\gamma = 0$ we may get cusp bifurcation; or if all three second derivatives vanish, then the cubic equation can give a triple point (three real roots for the slopes) or no bifurcation (two complex conjugate roots). If third derivatives also vanish we face the problem of classifying the roots of a quartic. For example, we may get four bifurcating branches.

It is possible to make precise statements about the stability of solutions near double points of bifurcation. All of the possibilities for the stability of double point bifurcation be described by the cases (A) and (B) which were fully specified under (6.11). In case (A) two curves $\mu^{(1)}(\varepsilon)$ and $\mu^{(2)}(\varepsilon)$ pass through the double point (μ_0, ε_0). In case (B) two curves, $\varepsilon^{(1)}(\mu)$ (with $\varepsilon_\mu^{(1)}(\mu_0) = 0$) and $\mu_\varepsilon^{(2)}$, pass through the double point. The eigenvalue $\gamma^{(1)}$ belongs to the curve with superscript (1) and $\gamma^{(2)}$ to the curve with superscript (2).

Theorem 4. Suppose (μ_0, ε_0) is a double point. Then, in case (A)

$$(6.19) \qquad \gamma^{(1)}(\varepsilon) = -\mu_\varepsilon^{(1)}(\varepsilon)\{\hat{s}\sqrt{D}(\varepsilon-\varepsilon_0) + o(\varepsilon-\varepsilon_0)\},$$

and

$$(6.20) \qquad \gamma^{(2)}(\varepsilon) = \mu_\varepsilon^{(2)}(\varepsilon)\{\hat{s}\sqrt{D}(\varepsilon-\varepsilon_0) + o(\varepsilon-\varepsilon_0)\}$$

where $\hat{s} = F_{\mu\mu}/|F_{\mu\mu}|$ and $F_{\mu\mu}$ and D are evaluated at $\varepsilon = \varepsilon_0$. And in case (B)

$$(6.21) \qquad \gamma^{(1)}(\mu) = s\sqrt{D}(\mu-\mu_0) + o(\mu-\mu_0)$$

and

$$(6.22) \qquad \gamma^{(2)}(\varepsilon) = -s\mu_\varepsilon^{(2)}(\varepsilon)\{\sqrt{D}\,(\varepsilon-\varepsilon_0) + o(\varepsilon-\varepsilon_0)$$

where $s = F_{\varepsilon\mu}/|F_{\varepsilon\mu}|$ and $F_{\varepsilon\mu}$ is evaluated at (μ_0, ε_0).

Proofs: If $\mu = \mu(\varepsilon)$ we have (6.13) in the form

$$(6.23) \qquad \gamma(\varepsilon) = -\mu_\varepsilon(\varepsilon)\, F_\mu(\mu(\varepsilon),\varepsilon)$$

$$= -\mu_\varepsilon(\varepsilon)\{[F_{\mu\mu}(\mu_0,\varepsilon_0)\mu_\varepsilon(\varepsilon_0) + F_{\varepsilon\mu}(\mu_0,\varepsilon_0)](\varepsilon-\varepsilon_0)$$

$$+ o\,(\varepsilon-\varepsilon_0)\}$$

The formulas (6.19) and (6.20) arise from (6.23) when $\mu_\varepsilon(\varepsilon_0)$

is replaced with the values given by (6.8). If $\varepsilon = \varepsilon(\mu)$ with $\varepsilon_{\mu}(\mu_0) = 0$ then $F_{\mu\mu}(\mu_0,\varepsilon_0) = 0$, $F^2_{\varepsilon\mu}(\mu_0,\varepsilon_0) = D$ and

$$\gamma(\mu) = F_{\varepsilon}(\mu,\varepsilon(\mu)) = F_{\varepsilon\mu}(\mu_0,\varepsilon_0)(\mu-\mu_0) + o(\mu-\mu_0)$$

$$= s\sqrt{D}(\mu-\mu_0) + o(\mu-\mu_0) .$$

Theorem 4 gives an exhaustive classification relating the stability of solutions near a double point to the slope of the bifurcation curves near that point. The result may be summarized as follows. Suppose $|\varepsilon - \varepsilon_0| > 0$ is small. Then (6.19) and (6.20) show that $\gamma^{(1)}(\varepsilon)$ and $\gamma^{(2)}(\varepsilon)$ have the same (different) sign if $\mu_{\varepsilon}^{(1)}(\varepsilon)$ and $\mu_{\varepsilon}^{(2)}(\varepsilon)$ have different (the same) sign. A similar conclusion can be drawn from (6.21) and (6.22). The possible distributions of stability of solutions is sketched in Fig. 6.1.

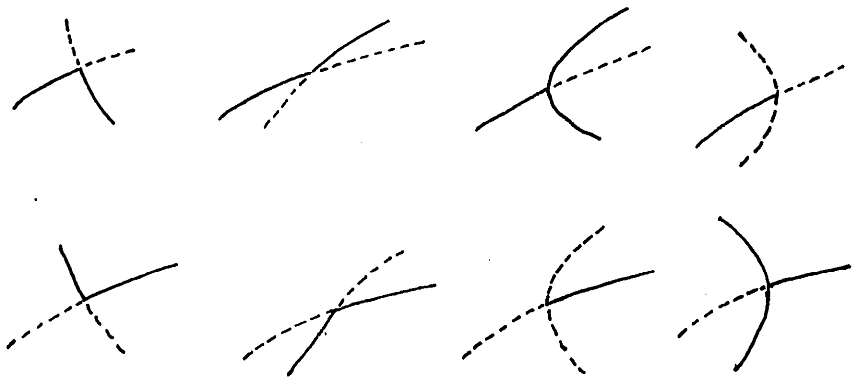


Fig. 6.1. Stability of solutions in the neighborhood of double point bifurcation.

Almost all the work in the theory of bifurcation is restricted to problems satisfying hypothesis (6.2). Then $F(\mu,0) = F_{\mu}(0,0) = F_{\mu\mu}(0,0) = 0$. The strict crossing hypothesis, introduced by Hopf, states that $\gamma_{\mu}(0) = F_{\mu\varepsilon}(0,0) < 0$. Then we get $D > 0$ and $\gamma^{(2)}(\varepsilon) = -\mu_{\varepsilon}^{(2)}(\varepsilon)\{\gamma_{\mu}^{(1)}(0)(\varepsilon-\varepsilon_0)+O(\varepsilon-\varepsilon_0)\}$.

Though we have excluded the analysis of cusp point bifurcation it is possible to deduce some definite results about the stability of solutions which bifurcate at a cusp point. Assuming now that $F(\mu,\varepsilon)$ has four continuous partial derivatives we note that at a cusp point $D = 0$. Then we may write (6.19)-(6.22) as

$$\gamma^{(1)}(\varepsilon) = -\mu_{\varepsilon}^{(1)}(\varepsilon)\ \{a\ (\varepsilon-\varepsilon_0)^2 + O(\varepsilon-\varepsilon_0)^3\}$$

and

$$\gamma^{(2)}(\varepsilon) = \mu_{\varepsilon}^{(2)}(\varepsilon)\ \{-a\ (\varepsilon-\varepsilon_0)^2 + O(\varepsilon-\varepsilon_0)^3\}$$

or

$$\gamma^{(1)}(\mu) = c(\mu-\mu_0)^2 + O(\mu-\mu_0)^3$$

and

$$\gamma^{(2)}(\mu) = -c(\mu-\mu_0)^2 + O(\mu-\mu_0)^3$$

where a and c are constants formed from third and fourth derivatives of F. It follows from these formulas that the stability of any branch passing through a cusp point changes sign if and only if $\mu_{\varepsilon}(\varepsilon)$ changes sign. Possible distributions of stability of solutions bifurcating at a cusp point are shown in Fig. 6.2 (see Theorem 6).

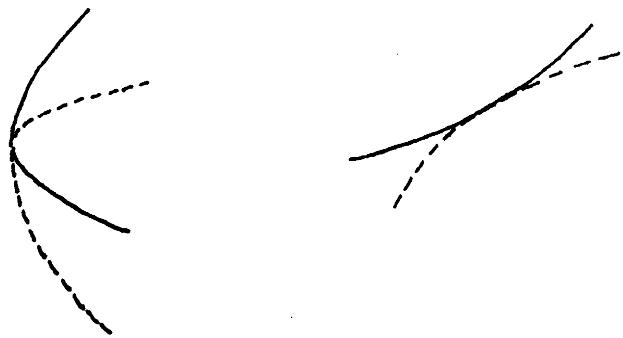


Fig. 6.2. **Stability of solutions bifurcating at a cusp point.**

In the next two theorems I give a global characterization of the stability of equilibrium solutions. I first note that $\gamma \neq 0$ at regular points of the curve $F(\mu,\varepsilon) = 0$ at which $\mu_\varepsilon \neq 0$. The factorization theorem $\gamma = -\mu_\varepsilon(\varepsilon)F_\mu(\mu(\varepsilon),\varepsilon)$ shows that γ can change sign at a stationary regular point ($\mu_\varepsilon = 0$, $F_\mu \neq 0$) only if it be a turning point.

Theorem 5: Assume that all singular points of solutions of $F(\mu,\varepsilon) = 0$ are double points. The stability of such solutions must change at each regular turning point and at each singular point (which is not a turning point) and only at such points.

Theorem 5 gives a fairly complete catalogue of the stability of solutions on connected branches of $F(\mu,\varepsilon) = 0$. But solutions of $F(\mu,\varepsilon) = 0$ need not be connected (see Fig. 6.3 for a typical example). It is however possible to relate the stability of equilibrium solutions of disconnected branches which pierce the line μ = const. Consider the points of intersection of the line μ = const. with solutions of $F(\mu,\varepsilon)=0$.

Label these points in an increasing sequence $\varepsilon_1 < \varepsilon_2 < \ldots < \varepsilon_n$. $F(\mu,\varepsilon_1) = F(\mu,\varepsilon_2) = \ldots = F(\mu,\varepsilon_n) = 0$. Of course, $F(\mu,\varepsilon)$ is of one sign between any two successive zeros. It follows that if $F_\varepsilon(\mu,\varepsilon_\ell) < 0$, then $F_\varepsilon(\mu,\varepsilon_{\ell+1}) \geq 0$ and $F_\varepsilon(\mu,\varepsilon_{\ell-1}) \geq 0$. Suppose, for example, that $F_\varepsilon(\mu,\varepsilon_{\ell-1}) = 0$, then $\gamma(\varepsilon_{\ell-1}) = 0$ so that $(\mu, \varepsilon_{\ell-1})$ is a singular point or a regular stationary point, $\mu_\varepsilon(\varepsilon_{\ell-1}) = 0$. The stability of solutions at regular stationary points is completely described by theorem 1 and at singular double points by theorem 5. Isolated (conjugate) singular points may be ignored because $D < 0$ implies that $F(\mu_0,\varepsilon_0) = 0$ is an extreme value of $F(\mu,\varepsilon)$ which does not change sign as ε is varied across ε_0 at a fixed $\mu = \mu_0$. In any event it will always be possible to shift μ slightly so that $F_\varepsilon(\mu,\varepsilon_\ell) \neq 0$ at each and every piercing point.

Theorem 6: If $F_\varepsilon(\mu,\varepsilon_\ell) \neq 0$ on each and every piercing point of solutions of $F(\mu,\varepsilon) = 0$ and the line μ = const, then the sign of $\gamma(\varepsilon)$ at such points is a sequence of alternating sign. If the solution (μ,ε_ℓ) of $F(\mu,\varepsilon) = 0$ is stable (unstable) then the solutions $(\mu,\varepsilon_{\ell-1})$ and $(\mu,\varepsilon_{\ell+1})$ are unstable (stable).

Theorem 6 is an obvious global extension in $\mathbb{R}_1$ of a local theorem of H. Weinberger (1976) which holds in a general Banach space. Similar conclusions have been derived by Benjamin (1976, 1977) in interesting applications of the theory of Leray-Schauder degree to steady solutions of the Navier Stokes equations.

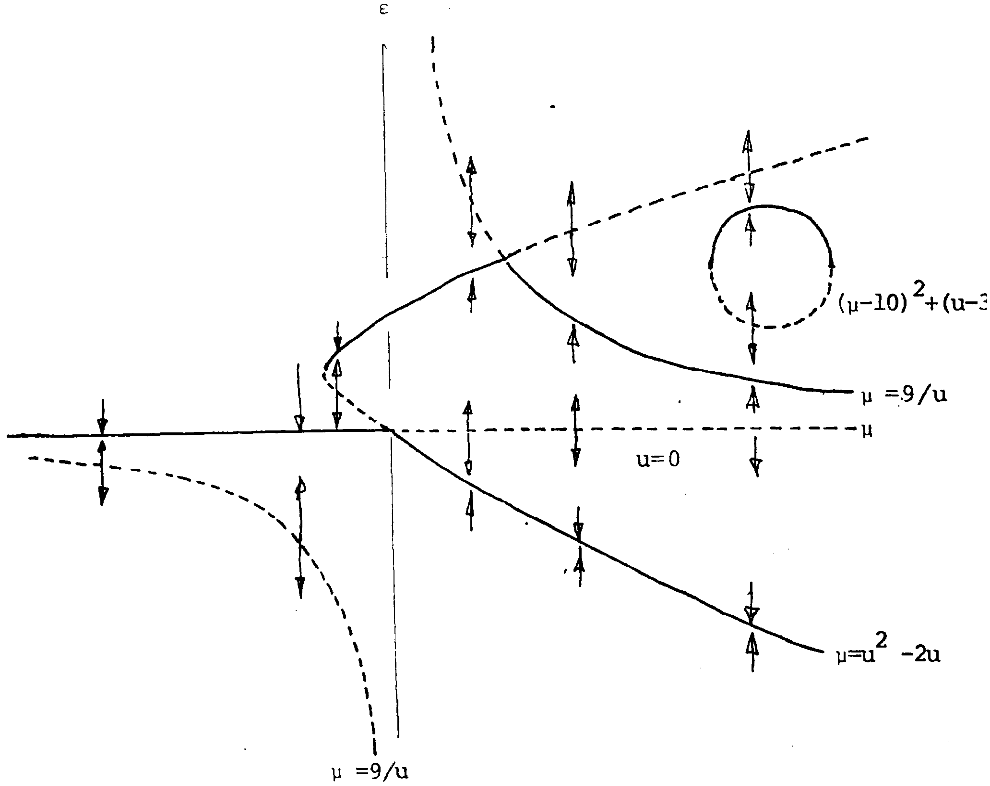


Fig. 6.3. Bifurcation, stability and domains of attraction of equilibrium solutions of

$$\frac{du}{dt} = u(9 - \mu u)(\mu + 2u - u^2)([\mu - 10]^2 + [u - 3]^2 - 1) .$$

The equilibrium solution $\mu = 9/u$ in the third quadrant and the circle are isolated solutions which cannot be obtained by bifurcation analysis.

7. Hydrodynamical interpretations of the bifurcation diagrams

I want now to probe more deeply into the hydrodynamical implications of our bifurcation analysis. I have already stressed that the direction of bifurcation is extremely important. In hydrodynamical problems the amplitude $|\varepsilon|$ can usually be defined in a physical way; for example, $|\varepsilon|$ can be interpreted as the heat transported by convection and μ as the temperature difference driving the convection. Or we could take $|\varepsilon|$ as the mass flux and μ as the pressure gradient driving the flow through a pipe. Then the good direction of bifurcation is the one in which the heat transported increases with the temperature difference or the one in which the mass flux increases with the pressure gradient. In the bad direction the relations go the other way.

I have argued that, as a consequence of the factorization theorem, the good direction on the primary bifurcation coming off $\underline{u} = 0$ is stable and the bad direction, it is comforting to say, is unstable. But in bifurcation studies things are frequently different than what they seem and if we inspect Fig. 6.3 we find a stable branch of secondary bifurcation in the bad direction ($\mu = 9/u$ in the 1st quadrant) and on the isolated solution on the circle. Of course the evolution equation of Fig. 6.3 is not really a model for the Navier-Stokes equations which are nonlinear precisely in the term $(\underline{u}\cdot\nabla)\underline{u}$ and not in an arbitrary way. Nevertheless, it is almost certainly true that the Navier-Stokes equations support stable isolated solutions which describe the observed hydrodynamics.

One type of isolated solution which appears to be relevant in hydrodynamics is shown in Fig. 7.1. This figure is

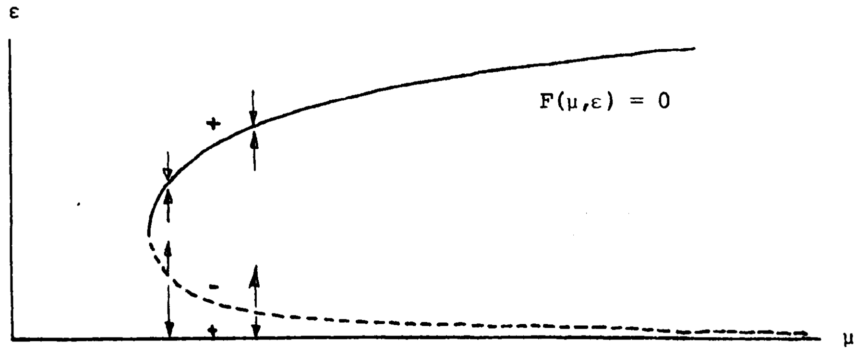


Fig. 7.1. $u = \varepsilon = 0$ is stable for all $\mu \in \mathbb{R}_1$. But for $\mu > \mu_0$, $u = 0$ has a small domain of attraction and the stable bifurcating solution on the upper branch has a large domain of attraction.

very much like the bifurcation diagrams which may be expected in the problems of stability and bifurcation of Plane Couette flow, of Couette flow between rotating cylinders when the inner one is at rest and of Hagen-Poiseuille flow down a round pipe. In all these flows the basic flow (the null solution) is stable to infinitesimally small disturbances at all Reynolds numbers. This "stability" is achieved in experiments; if great care is taken to suppress disturbances, the basic flow can be maintained at very high Reynolds numbers. But larger disturbances of the basic flow are unstable and are attracted to some stable solution or set of solutions with large values of ε.

The bifurcation diagrams which probably apply to the problem of Poiseuille flow through the annulus between two concentric cylinders are sketched in Fig. 7.2. The bifurcation of Poiseuille flow is known to be time-periodic so that $R = R(\varepsilon)$ is an even function of ε. The critical Reynolds number $(R = R_c)$ of the linearized theory of stability depends very stongly on $\eta = a/b$ and $R_c(\eta) \to \infty$ as $\eta \to 0$. But the observed limit of stability to natural disturbances is not very sensitive to η and seems to be fixed somewhere between 2000 and 4000. Unstable time-periodic bifurcating solutions on the lower branch of bifurcation curve have been observed in the experiments of Nishioka, Iida and Ichikawa (1975).

When $a = 0$ we have flow through a round pipe, Hagen-Poiseuille flow. It is not known if the limiting flow $a/b \to 0$ is the same as $a = 0$ but in both cases it appears that $R_c \to \infty$ and we have an isolated bifurcating solution of the type described in Fig. 7.1 and 7.2c.

We could claim perfect agreement between bifurcation theory and experiments if the observed solutions with large values of ε were time-periodic with values of $R = R(\varepsilon)$ on the upper stable branch of the bifurcation curve. However, the upper branch is known only in a very coarse approximation (Zahn, _et_. _al_., 1974). Moreover, stable periodic motions are not observed; instead we get a direct transition from laminar Poiseuille flow to turbulence without intermediate bifurcations. This direct transition is obviously one of the big events which needs to be explained.

But a good explanation, which is certainly way outside the range of understanding we can get by studying bifurcation in $\mathbb{R}_1$, still eludes us.

The following remarks about the snap-through instability to turbulence may be helpful. In finite pipes there seems to be something akin to the two stable solutions u = 0 when $R < R_c$ and the stable time-periodic solution on the upper branch (Wygnanski and Champagne, 1973; Wygnanski, Sokolov, Friedman, 1975). The flow is spatially segregated into distinct packets of laminar and turbulent flow (turbulent "puffs when R is near to R_0, and "slugs" at higher values of R). The transition from laminar to turbulent flow at a fixed place occurs suddenly as a puff or slug sweeps over the place, and the reverse transition occurs just as suddenly when it leaves the place. These observations suggest a sort of cycling in phase space between two weakly attracting solutions, one of which is laminar.

Other types of isolated solutions appear to occur in the problem of Taylor vortices and undulating Taylor vortices between rotating cylinders of finite length (Coles, 1965; T. B. Benjamin, 1977) and between rotating concentric spheres. Coles' study draws attention to the marked degree of non-uniqueness and hysteresis which characterize the spatial structure of these flows. For supercritical speeds of a rotating inner cylinder up to about ten times critical, Coles finds that in one and the same apparatus the number of vortices and the number of waves traveling around these vortices are not uniquely determined by the speed. The number of

Taylor cells in his apparatus range from 18 to 32 and the number of waves which travel along the axis of the cells range from 3 to 7. Moreover, "as many as 20 or 25 different states (each state being defined by the number of Taylor cells and the number of tangential waves) have been observed at a given speed".

Benjamin's experiments were designed to establish physically theoretical ideas, developed by him (Benjamin, 1976, 1977), which stem from bifurcation theory set in the context of Thom's castastrophe theory and stability theory set in the context of Riesz-Schauder index theory. This powerful combination of ideas have much in common with the theory of bifurcation and stability in $\mathbb{R}_1$. The index theory leads to essentially the same results as the factorization theorem. To this theory, catastrophe theory adds more control parameters which allows one to investigate how isolated solutions may be developed out of connected solutions as the extra control parameter is varied. In Benjamins experiments there are two control parameters, the Reynolds number and the height ℓ of the rotating cylinders. His apparatus is a short annulus which accomodates from two to four cells. (In such experiments, cell cross-sections never deviate by very much from squares.) He finds that there are certain states, characterized by the number of cells and the sense of circulation in the cells, which can be reached only through sudden acceleration from rest. Such cells seem to be isolated from the basic cellular solution for most values of ℓ.

Results similar to, and even more varied than those reported by Benjamin, were described to me in an informal conversation I have had with Professor Zierep of Karlsruhe at this symposium. Zierep says that he finds four or five distinct flows in a wide gap between rotating concentric cylinders which may be reached through a variety of programs for acceleration to a constant value of rotation of the inner sphere.

In the experiments which I have just described there are an integral number of cells and an integral number of undulations around the cells. The different solutions correspond to different integers and the change from one to another set of integers appears always to be discontinuous. This suggests that we are dealing here with the problem of isolated solutions. This problem is outside of the scope of the traditional bifurcation theory.

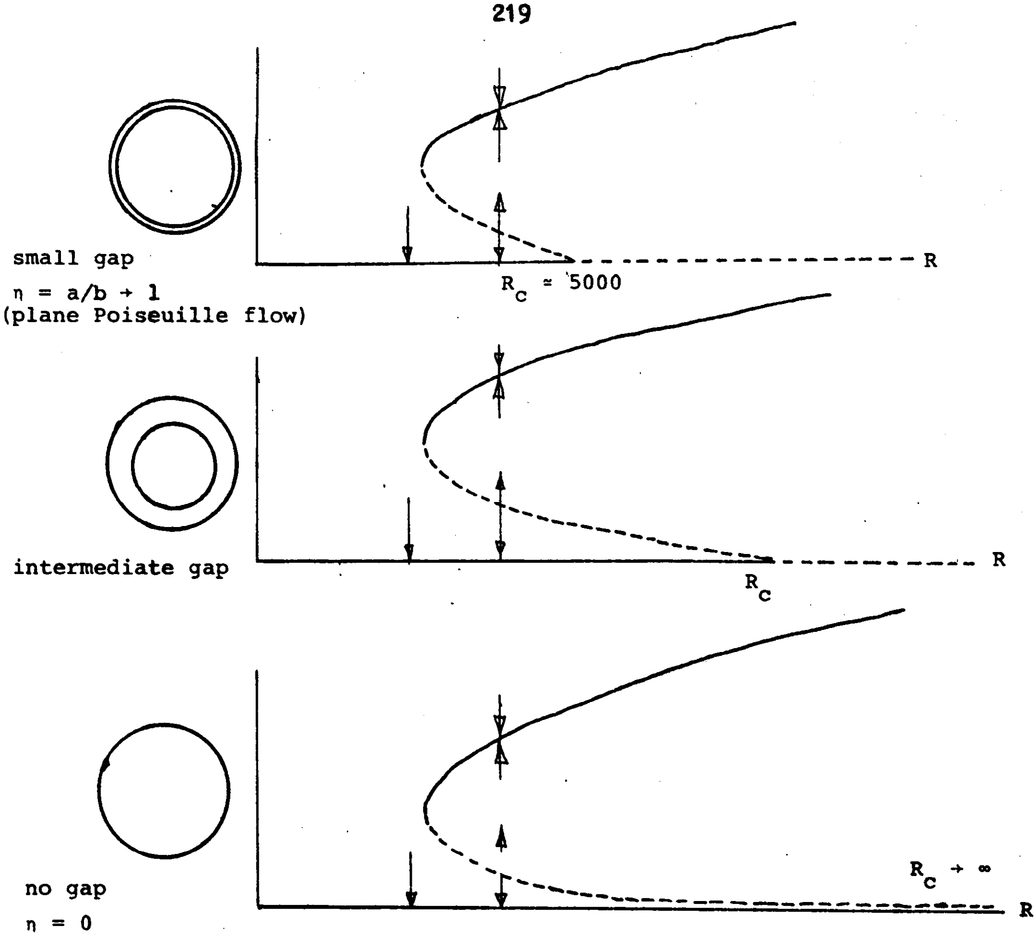


Fig. 7.2. Bifurcation of Poiseuille flow through an annular pipe. For all non-zero values of $1 \geq \eta = a/b > 0$ there is a finite critical value R_c for the linearized stability of Poiseuille flow and a time-periodic solution bifurcates subcritically at this point. $R_c \to \infty$ as $\eta \to 0$; hence (c) is an isolated solution (of the type which bifurcates from infinity). The slope of the bifurcating curves at $(R,\varepsilon)=(R_c,0)$ is known from exact computation. The form of the bifurcating curves for $\varepsilon > 0$ is a guess based on approximate analysis,

experiments, and the factorization theorem (see Joseph, 1976, Chap. IV). These are the curves which should arise from Hopf bifurcation when the analysis of bifurcation and stability is carried out in the too restricted set of functions allowed under the term "bifurcation at a simple eigenvalue."

8. Subharmonic bifurcation of forced T-periodic solutions

In the previous sections we considered some phenomena which could arise from the bifurcation of steady solutions at a simple eigenvalue. A similar analysis, at a good level of generality, can be given for the analogous problem which a arises when the basic motion is T-periodic (Iooss and Joseph, 1977). There are relatively many more hydrodynamical examples of bifurcation of steady flow because the problem of stability of time-periodic flows has only a relatively short history (see Davis, 1976). However, since the main results of bifurcation theory follow from spectral hypotheses, the relative paucity of explicit examples of loss of stability of $\underline{V}(\underline{x},t) = \underline{V}(\underline{x},t+T)$ does not interfere with the analysis of bifurcation. It is likely that in years to come many such explicit hydrodynamical examples will be forthcoming.

A disturbance $\underline{u}(\underline{x},t)$ of $\underline{V}(\underline{x},t) = \underline{V}(\underline{x},t+T)$ satisfies (3.4) and (3.6). For short, we suppress $\underline{x}$ and write $\underline{u}(t)$ and $\underline{V}(t)$. The linearization of (3.4) is given by (3.7) and the spectral problem by (4.4). We assume that, at criticality, $\xi(0) = 0$ and $\sigma(0) = i\omega(0)$ is an isolated simple eigenvalue of

(8.1) $$J_0(\cdot) = (\cdot)_t + F_u(t,0|(\cdot)) \;, \quad \text{dom } J_0 = H_T \;,$$

and that a strict crossing condition holds, $\xi_\mu(0) < 0$. If $i\omega_0$ is an eigenvalue is $-i\omega_0$. Then

(8.2) $$\underline{z} = e^{-i\omega_0 t}\,\underline{\zeta}(t) \quad \text{and} \quad \overline{\underline{z}}$$

both satisfy (3.7).

We look for subharmonic solutions, nT-periodic solutions where $n \geq 1$ is an integer. So we demand that $\underline{z}(t) = \underline{z}(t + nT)$; that is,

$$\underline{z}(t + nT) = e^{-i\omega_0 (t + nT)} \underline{\zeta}(t + nT) = e^{-i\omega_0 nT} \underline{z}(t) .$$

Hence

$$e^{-i\omega_0 nT} = \lambda_0^n = 1 \tag{8.3}$$

where $\lambda_0 = e^{-i\omega_0 T}$ is the Floquet multiplier. It follows that

$$\omega_0 = \frac{2\pi}{T} (r+k)$$

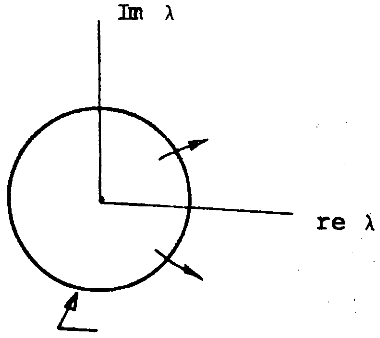
where r is an irreducible fraction

$$0 \leq r = \frac{m}{n} < 1 .$$

Repeated points ($k \in \mathbb{Z}$) in the complex σ plane map into unique points

$$\lambda_0 = e^{-i\omega_0 T} = e^{-2\pi i r} \tag{8.4}$$

in the complex λ plane. These points are dense on the unit circle in the complex λ plane (Fig. 8.1). The strict crossing condition means that, in general, as μ is increased past zero a pair of complex conjugate values $\lambda(\mu)$ pass out of the



$$\lambda_0 = e^{-2\pi i r}, \quad 0 \leq r = \frac{m}{n} < 1 .$$

Fig. 8.1. Floquet multipliers at criticality

unit circle in the λ plane. In two cases, $r = m/n = 0$, $r = 1/2$, λ_0 is real and $z = \bar{z}$.

Summarizing these results, we have stated that at criticality

$$\underline{z} = \exp\{-2\pi imt/nT\}\, \underline{\zeta}(t)$$

so that $J_0\underline{z} = J_0\bar{\underline{z}} = 0$. We now introduce the operator $\mathbb{J}_0$ which is the same as J_0 except that $\mathbb{J}_0$ acts on nT-periodic functions; $H_{nT} = \text{dom}\ \mathbb{J}_0 \subset \text{dom}\ J_0 = H_T$. In general $\underline{z} \neq \bar{\underline{z}}$, $\mathbb{J}_0\,\underline{z} = \mathbb{J}_0\bar{\underline{z}} = 0$ and zero is a double semi-simple eigenvalue of $\mathbb{J}_0$. In the special cases $(m/n) = (0/1), (1/2)$ where $\underline{z} = \bar{\underline{z}}$, zero is a simple eigenvalue of $\mathbb{J}_0$. In the two special cases $n = 1$ and $n = 2$ we have bifurcation at a simple eigenvalue within the frame of the classical theory. In the semi-simple case we find a solution which may be decomposed into a part on the null space of $\mathbb{J}_0$ and a part on the natural supplementary space

$$(8.5) \qquad \underline{u}(t,\varepsilon) = b(\varepsilon)\underline{z} + \overline{b}(\varepsilon)\overline{\underline{z}} + \underline{w}(t,\varepsilon)$$

where $\underline{u}$, $\underline{z}$ and $\underline{w}$ are nT-periodic and

$$\langle \underline{w}, \underline{z}^* \rangle_{nT} = \langle \underline{w}, \overline{z}^* \rangle_{nT} = 0 .$$

In contrasting (8.5) to (5.2) it is instructive to note that (8.5) is not invariant to arbitrary translations of the origin of time and we must find the projection $b(\varepsilon) = \langle \underline{u}(t,\varepsilon), z^* \rangle_{nT}$ from the perturbation theory at a double semi-simple eigenvalue. Iooss and Joseph (1977) find that nT-periodic is in general, possible only when

$$r = \frac{m}{n} = \frac{0}{1};\ \frac{1}{2};\ \frac{1}{3};\ \frac{2}{3};\ \frac{1}{4};\ \frac{3}{4},$$

that is when n = 1, 2, 3, 4. For all other values of r, rational or irrational we get bifurcation into a torus. One feature which distinguishes the solutions on the torus from the nT-periodic (subharmonic) solutions is the fact that the period $\tau = nT$ of subharmonic is fixed and independent of the amplitude ε.

The bifurcation diagrams for subharmonic bifurcation of T-periodic solutions is shown in Figs. 5.1, 5.2, 8.2 and 8.3.

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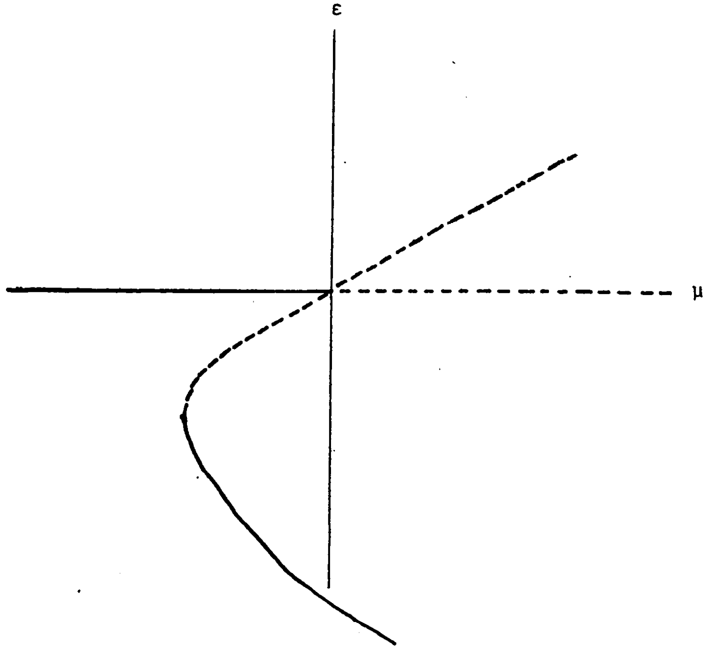

Fig. 8.2. Bifurcation of a 3T-periodic solution from a T-periodic one. The bifurcation is two-sided and is unstable on both sides of criticality when $|\varepsilon|$ is small. The factorization theorem (Joseph, 1977) shows the solution will regain stability at a regular turning point.

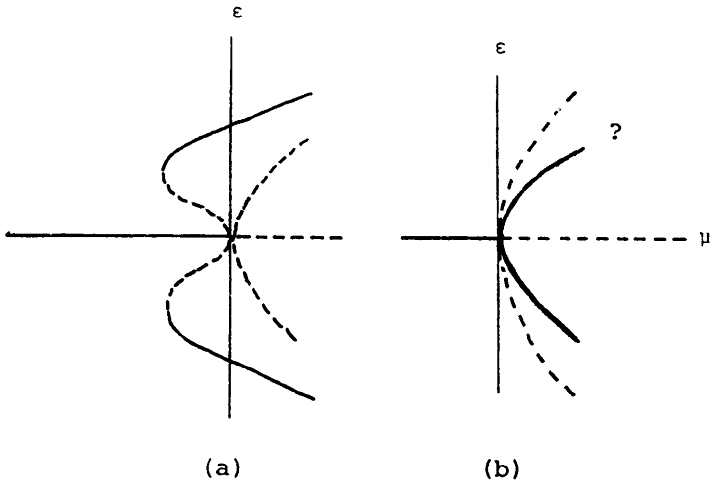

Fig. 8.3. Bifurcation of a 4T-periodic solution from a T-periodic one. There are three possible types of bifurcation from the semi-simple double eigenvalue: (a) Two solutions, both unstable bifurcate on each side of criticality. (b) Two solutions bifurcate on the same side of criticality and one of them is unstable. the stability of the other is determined by the details of the problem. (c) No 4T-periodic solutions (it is not known if there is a torus in this case).

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