

## STOKES FLOW IN CONICAL TRENCHES\*

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**Abstract.** In this paper we develop a separation of variables theory for solving problems of Stokes flow in cone-shaped trenches formed as the intersection of a cone of circular cross-section and a spherical shell centered at the vertex of the cone.<sup>1</sup> The theory leads to a new set of Stokes flow eigenfunctions which describe axisymmetric motions in the vertex of a cone. Asymptotic formulas for the distribution of eigenvalues are derived; an adjoint system is defined and is used to develop an algorithm for the computation of the coefficients in an eigenfunction expansion of edge data prescribed on the spherical boundaries.

**Introduction.** The results to be achieved here are analogous to results known to hold in two-dimensional wedges (Liu and Joseph (1977)). The existence of edge eigenfunctions (eddies) in the wedge was discovered by Dean and Montagnon (1949) and their results were extended by Moffatt (1964); however, these authors did not give asymptotic formulas for the eigenvalues or show how to use the eigenfunctions to solve boundary-value problems.

We turn directly to the problem described in the abstract. A motivating introduction to the problem is given in companion paper (in this volume) by Yoo and Joseph (1978) and will not be repeated here.

**1. The cone eigenfunctions.** The Stokes flow equation for steady axisymmetric flow in spherical coordinates is

$$(1.1) \quad \left( \frac{\partial^2}{\partial r^2} + \frac{1-\xi^2}{r^2} \frac{\partial^2}{\partial \xi^2} \right)^2 \Psi = 0,$$

where  $\xi = \cos \theta$  is the polar angle and  $\phi$  is the azimuthal angle around the axis of symmetry. The velocity vanishes on the cone  $\xi = \xi_0$ . Hence

$$(1.2) \quad \Psi(r, \xi_0) = \partial_\xi \Psi(r, \xi_0) = 0.$$

Separable solutions of (1.1) and (1.2) in the form

$$(1.3) \quad \Psi \sim r^{3/2 \pm \mu} \phi(\xi)$$

exist when  $\phi(\xi)$  satisfies

$$(1.4) \quad (1-\xi^2)\phi^{iv} - 4\xi(1-\xi^2)\phi''' + 2(\mu + \frac{1}{2})(\mu - \frac{1}{2})(1-\xi^2)\phi'' + (\mu^2 - \frac{9}{4})(\mu^2 - \frac{1}{4})\phi = 0,$$

where

$$(1.5) \quad \phi(\xi_0) = \phi'(\xi_0) = 0.$$

Our convention is that  $\text{Re } \mu > 0$  so that the minus sign in (1.3) may be dropped in problems in the full cone  $r > 0$ . Solutions of (1.4) may be found in terms of the

\* Received by the editors August 12, 1976.

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<sup>1</sup> The theoretical ideas and asymptotic analysis are due to D. Joseph while most of the calculations and all of the numerical computations are due to C. H. Liu.

associated Legendre functions

$$(1 - \xi^2)^{1/2} (P_{\mu+1/2}^1(\xi), P_{\mu-3/2}^1(\xi), Q_{\mu+1/2}^1(\xi), Q_{\mu-3/2}^1(\xi)).$$

The Legendre functions  $Q_{\mu}^1(\xi)$  are unbounded on the polar axis  $\xi = 1$  and may be discarded. Bounded solutions of (1.4) which vanish on the cone are then in the form

$$(1.6) \quad \phi(\xi) = (1 - \xi^2)^{1/2} \{P_{\mu-3/2}^1(\xi_0)P_{\mu+1/2}^1(\xi) - P_{\mu+1/2}^1(\xi_0)P_{\mu-3/2}^1(\xi)\}.$$

The function  $\phi(\xi)$  is  $O(1 - \xi^2)$  when  $\xi$  is close to 1 and  $\phi(1) = 0$ . The derivatives  $\phi'(\xi_0) = 0$  if and only if the complex numbers  $\mu = \mu_n$  appear as eigenvalues of the characteristic equation

$$(1.7) \quad \begin{aligned} &((\mu + \frac{3}{2})P_{\mu-3/2}^1(\xi_0) + (\mu - \frac{3}{2})P_{\mu+1/2}^1(\xi_0)) \\ &\cdot ((\mu - \frac{1}{2})P_{\mu+1/2}^1(\xi_0) + (\mu + \frac{1}{2})P_{\mu-3/2}^1(\xi_0)) \\ &= 4\mu^2 \xi_0^2 P_{\mu+1/2}^1(\xi_0)P_{\mu-3/2}^1(\xi_0). \end{aligned}$$

To derive (1.7) we have made use of the standard recursion relations:

$$\begin{aligned} (1 - \xi^2) \frac{dP_{\mu}^{\nu}}{d\xi} &= (\mu + 1)P_{\mu}^{\nu} - (\mu - \nu + 1)P_{\mu+1}^{\nu} \\ &= -\mu\xi P_{\mu}^{\nu} + (\mu + \nu)P_{\mu-1}^{\nu}. \end{aligned}$$

It is easy to verify, using (1.4) and (1.5), that if  $\mu$  is an eigenvalue, then  $-\mu$  and the complex conjugate  $\bar{\mu}$  are also eigenvalues.

There are an infinite number of first quadrant eigenvalues  $\mu_1, \mu_2, \mu_3, \dots$  which cluster at infinity and may be arranged in the order of the size of their real parts. To obtain the asymptotic distribution of first quadrant eigenvalues we consider the simplified form which (1.7) takes when  $|\mu|$  is large. When  $l/|\mu|$  is small and  $|\mu|$  is large we may use Watson's asymptotic expression

$$\begin{aligned} P_{\mu+l}^1(\cos \theta_0) &= \sqrt{\frac{2(\mu+l)}{\pi \sin \theta_0}} \left\{ \left(1 + \frac{1}{4(\mu+l)}\right) \cos \left[ \left(\mu+l + \frac{1}{2}\right) \theta_0 + \frac{\pi}{4} \right] \right. \\ &\quad \left. - \frac{3 \cot \theta_0}{8(\mu+l)} \sin \left[ \left(\mu+l + \frac{1}{2}\right) \theta_0 + \frac{\pi}{4} \right] + O\left(\frac{1}{|\mu+l|^2}\right) \right\}. \end{aligned}$$

Retaining terms of order  $l/\mu$  we may reduce this expression to

$$(1.8) \quad \begin{aligned} P_{\mu+l}^1(\cos \theta) &= \sqrt{\frac{2\mu}{\pi \sin \theta_0}} \left\{ \left(1 + \frac{2l+1}{4\mu}\right) \cos \left[ \left(\mu+l + \frac{1}{2}\right) \theta_0 + \frac{\pi}{4} \right] \right. \\ &\quad \left. - \frac{3 \cot \theta}{8\mu} \sin \left[ \left(\mu+l + \frac{1}{2}\right) \theta_0 + \frac{\pi}{4} \right] + O\left(\frac{1}{|\mu|^2}\right) \right\}. \end{aligned}$$

Retaining the two terms of highest order in  $1/\mu$ , we may rewrite (1.7) as

$$(1.9) \quad \begin{aligned} (P_{\mu-3/2}^1 + P_{\mu+1/2}^1)^2 - 4 \cos^2 \theta P_{\mu+1/2}^1 P_{\mu-3/2}^1 + \frac{2}{\mu} \{ (P_{\mu-3/2}^1)^2 - (P_{\mu+1/2}^1)^2 \} \\ + O\left(\frac{1}{|\mu|^2}\right) = 0. \end{aligned}$$

We next combine (1.8) and (1.9), using various simple trigonometric identities to find that (1.9) may be written as

$$(1.10) \quad \sin^2(2\theta_0) + \frac{1}{\mu} \sin(2\theta_0) \sin\left(2\mu\theta_0 + \frac{\pi}{2}\right) + O\left(\frac{1}{|\mu|^2}\right) = 0.$$

It follows from (1.10) that the asymptotic distribution of eigenvalues may be determined as the roots of

$$(1.11) \quad \mu \sin(2\theta_0) + \cos(2\mu\theta_0) = 0.$$

Introducing  $z = 2\mu\theta_0 = \zeta + i\eta$  we find, from (1.11), that

$$(1.12a) \quad \zeta \frac{\sin(2\theta_0)}{2\theta_0} = -\cos \zeta \cosh \eta,$$

and

$$(1.12b) \quad \eta \frac{\sin(2\theta_0)}{2\theta_0} = \sin \zeta \sinh \eta.$$

When  $\zeta$  and  $\eta$  are both large, we find following Hardy (1902), as a first approximation, that  $\cos \zeta \rightarrow -1$ ,  $\cosh \eta \rightarrow \frac{1}{2}e^\eta$ ,

$$(1.13a) \quad \zeta \rightarrow (2n-1)\pi, \quad n > 1$$

and

$$(1.13b) \quad \eta \rightarrow \ln \left[ (4n-2)\pi \frac{\sin(2\theta_0)}{2\theta_0} \right].$$

Some important properties of the solutions may be obtained from (1.12). For example,  $\eta(\theta_0)$ , the imaginary part of the eigenvalue  $\mu$ , is a decreasing function of  $\theta_0$  which decreases to zero at a certain critical apex angle  $\theta^*$ . In the limit  $\eta \rightarrow 0$  we find from (1.12) that

$$\zeta \frac{\sin(2\theta^*)}{2\theta^*} = -\cos \zeta$$

and

$$\frac{\sin(2\theta^*)}{2\theta^*} = \sin \zeta.$$

The smallest positive root  $\zeta_1$  of  $\zeta + \cot \zeta = 0$  is  $\zeta_1/(2\theta^*) = 2.79836$ . This leads to a critical opening angle of  $\theta^* = 65.074$ . Of course, in the limit  $\eta \rightarrow 0$ ,  $\zeta_1/(2\theta^*)$  is not large; nevertheless the approximate critical angle is not in bad agreement with the exact values  $\theta^* \sim 76.95^\circ$  and  $2.98 > \zeta_1/(2\theta^*) > 2.20$  (see Table 3).

The stream functions

$$r^{3/2+\mu_n} \phi_1^{(n)}(\xi) = r^{3/2+(\zeta_n/(2\theta_0))} \left[ \cos\left(\frac{\eta_n}{2\theta_0} \ln r\right) + i \sin\left(\frac{\eta_n}{2\theta_0} \ln r\right) \right] \phi_1^{(n)}(\xi),$$

$n = 1, 2, 3, \dots$ , form a nested sequence of corner eddies which, when  $n$  is large, has the form

$$(1.14a) \quad r^{[3/2+(n-1/2)\pi/\theta_0]} e^{i(\eta_n/(2\theta_0)) \ln r},$$

where

$$(1.14b) \quad \eta_n \rightarrow \ln \left[ (4n - 2)\pi \frac{\sin(2\theta_0)}{2\theta_0} \right].$$

The intensity of all eddies decays and their frequency increases as  $r \rightarrow 0$ . The intensity of eddies belonging to the higher eigenvalues is very very weak. The most persistent eddy is the one  $n = 1$  for which  $\zeta_n$  is the smallest. This eddy decays less rapidly as  $r \rightarrow 0$  than the eddies belonging to larger values of  $n$ .

The eddy structure of a given flow depends on the global solution over the whole field of flow. The global solution manifests itself in the constants  $C_n$  which appear in (2.6). However, in deep cones the eddy structure will always be dominated by the structure of the eigensolution with  $n = 1$ .

To compute exact values from (1.6) and (1.7) it is necessary to have an accurate representation for the associated Legendre function  $P_\nu^1(\cos \theta)$  when the index  $\nu$  is complex and unrestricted. In the computations for the Tables 1, 2, 3 and Figs. 1 and 2 we evaluated the integral representation

$$(1.15) \quad P_\nu^1(\cos \theta) = \frac{i(\nu + 1)}{2\pi} \int_0^{2\pi} (\cos \theta + i \sin \theta \cos t)^\nu \cos t dt$$

TABLE I  
Twenty first-quadrant roots of (1.7)

| $n$ | Complex roots $\mu_n$ ( $2\theta_0 = 20^\circ$ ) | $n$ | Complex roots $\mu_n$ ( $2\theta_0 = 30^\circ$ ) |
|-----|--|-----|--|
| 1   | 17.72643464 + 6.76835418i                        | 1   | 11.80153377 + 4.43057853i                        |
| 2   | 35.76843180 + 9.01141743i                        | 2   | 23.83791117 + 5.93780623i                        |
| 3   | 53.79948921 + 10.24477552i                       | 3   | 35.86188920 + 6.76560916i                        |
| 4   | 71.82257062 + 11.09994955i                       | 4   | 47.87881070 + 7.33897834i                        |
| 5   | 89.84039158 + 11.75582447i                       | 5   | 59.89152907 + 7.77834816i                        |
| 6   | 107.85458785 + 12.28826311i                      | 6   | 71.90150199 + 8.13479248i                        |
| 7   | 125.86618587 + 12.73659989i                      | 7   | 83.90956731 + 8.43477982i                        |
| 8   | 143.87585792 + 13.12389348i                      | 8   | 95.91624643 + 8.69381675i                        |
| 9   | 161.88406149 + 13.46483564i                      | 9   | 107.92188285 + 8.92177695i                       |
| 10  | 179.89111852 + 13.76937024i                      | 10  | 119.92671301 + 9.12533929i                       |
| 11  | 197.89726212 + 14.04454388i                      | 11  | 131.93090551 + 9.30923409i                       |
| 12  | 215.90266537 + 14.29553366i                      | 12  | 143.93458404 + 9.47693509i                       |
| 13  | 233.90745955 + 14.52625585i                      | 13  | 155.93784163 + 9.63106888i                       |
| 14  | 251.91174614 + 14.73974511i                      | 14  | 167.94074966 + 9.77367002i                       |
| 15  | 269.91560487 + 14.93840099i                      | 15  | 179.94336390 + 9.90634671i                       |
| 16  | 287.91909930 + 15.12415385i                      | 16  | 191.94572862 + 10.03039229i                      |
| 17  | 305.92228077 + 15.29857975i                      | 17  | 203.94787943 + 10.14686248i                      |
| 18  | 323.92519121 + 15.46298234i                      | 18  | 215.94984532 + 10.25663028i                      |
| 19  | 341.92786526 + 15.61845219i                      | 19  | 227.95165018 + 10.36042583i                      |
| 20  | 359.93033177 + 15.76591093i                      | 20  | 239.95331385 + 10.45886607i                      |

TABLE 2  
Twenty first-quadrant roots of (1.7)

| $n$ | Complex roots $\mu_n$ ( $2\theta_0 = 60^\circ$ ) | $n$ | Complex roots $\mu_n$ ( $2\theta_0 = 90^\circ$ ) |
|-----|--|-----|--|
| 1   | 5.88824487 + 2.03012122i                         | 1   | 3.93397186 + 1.15358993i                         |
| 2   | 11.91035312 + 2.80189092i                        | 2   | 7.93933996 + 1.68192493i                         |
| 3   | 17.92541993 + 3.22370138i                        | 3   | 11.94878801 + 1.96849681i                        |
| 4   | 23.93555425 + 3.51497636i                        | 4   | 15.95567326 + 2.16566791i                        |
| 5   | 29.94290027 + 3.73766360i                        | 5   | 19.96076368 + 2.31605505i                        |
| 6   | 35.94851694 + 3.91800034i                        | 6   | 23.96467636 + 2.43763333i                        |
| 7   | 41.95297812 + 4.06956239i                        | 7   | 27.96778619 + 2.53967873i                        |
| 8   | 47.95662362 + 4.20028969i                        | 8   | 31.97032494 + 2.62760508i                        |
| 9   | 53.95966885 + 4.31522956i                        | 9   | 35.97244239 + 2.70484792i                        |
| 10  | 59.96225771 + 4.41779099i                        | 10  | 39.97423946 + 2.77372413i                        |
| 11  | 65.96449040 + 4.51038505i                        | 11  | 43.97578671 + 2.83587026i                        |
| 12  | 71.96643911 + 4.59478014i                        | 12  | 47.97713503 + 2.89248523i                        |
| 13  | 77.96815727 + 4.67231188i                        | 13  | 51.97832209 + 2.94447365i                        |
| 14  | 83.96968539 + 4.74401385i                        | 14  | 55.97937643 + 2.99253492i                        |
| 15  | 89.97105478 + 4.81070228i                        | 15  | 59.98032010 + 3.03722091i                        |
| 16  | 95.97229008 + 4.87303308i                        | 16  | 63.98117040 + 3.07897477i                        |
| 17  | 101.97341095 + 4.93154125i                       | 17  | 67.98194114 + 3.11815773i                        |
| 18  | 107.97443330 + 4.98666889i                       | 18  | 71.98264346 + 3.15506812i                        |
| 19  | 113.97537015 + 5.03878558i                       | 19  | 75.98328650 + 3.18995518i                        |
| 20  | 119.97623227 + 5.08820337i                       | 20  | 79.98387776 + 3.22302926i                        |

TABLE 3  
Twenty first-quadrant roots of (1.7). When  $\theta_0 = 76.95^\circ$  the first root becomes real-valued. When  $\theta_0 > 90^\circ$  all roots are real-valued.

| $n$ | Complex roots $\mu_n$ ( $2\theta_0 = 120^\circ$ ) | $n$ | Complex roots $\mu_n$ ( $2\theta_0 = 160^\circ$ ) |
|-----|---|-----|---|
| 1   | 2.98157825 + 0.63057346i                          | 1   | 2.20702432 + 0 i                                  |
| 2   | 5.96161923 + 1.03940740i                          | 2   | 4.51887333 + 0.30170797i                          |
| 3   | 8.96438651 + 1.25920576i                          | 3   | 6.74749318 + 0.48541852i                          |
| 4   | 11.96812914 + 1.40962088i                         | 4   | 8.99113920 + 0.60575831i                          |
| 5   | 14.97133320 + 1.52398330i                         | 5   | 11.23897632 + 0.69553773i                         |
| 6   | 17.97395768 + 1.61624325i                         | 6   | 13.48830924 + 0.76718365i                         |
| 7   | 20.97611656 + 1.69356298i                         | 7   | 15.73825268 + 0.82681088i                         |
| 8   | 23.97791632 + 1.76010768i                         | 8   | 17.98846153 + 0.87788330i                         |
| 9   | 26.97943830 + 1.81851348i                         | 9   | 20.23878553 + 0.92255380i                         |
| 10  | 29.98074250 + 1.87055439i                         | 10  | 22.48915446 + 0.96225228i                         |
| 11  | 32.98187331 + 1.91748137i                         | 11  | 24.73953408 + 0.99797702i                         |
| 12  | 35.98286392 + 1.96020954i                         | 12  | 26.98990747 + 1.03045266i                         |
| 13  | 38.98373960 + 1.99942857i                         | 13  | 29.24026646 + 1.06022194i                         |
| 14  | 41.98451987 + 2.03567109i                         | 14  | 31.49060741 + 1.08770194i                         |
| 15  | 44.98522002 + 2.06935701i                         | 15  | 33.74092911 + 1.11322009i                         |
| 16  | 47.98585222 + 2.10082324i                         | 16  | 35.99123160 + 1.13703820i                         |
| 17  | 50.98642625 + 2.13034422i                         | 17  | 38.24151560 + 1.15936890i                         |
| 18  | 53.98695008 + 2.15814644i                         | 18  | 40.49178213 + 1.18038724i                         |
| 19  | 56.98743027 + 2.18441903i                         | 19  | 42.74203236 + 1.20023906i                         |
| 20  | 59.98787225 + 2.20932151i                         | 20  | 44.99226747 + 1.21904714i                         |

numerically. The representations (1.15) is valid for unrestricted complex values of  $\nu$  and  $0 < \theta < \pi/2$ .

Eigenvalue tables for the first twenty eigenvalues are given for different cone angles in Tables 1, 2, 3; there is a critical cone angle  $\theta^* \approx 76.95^\circ$  such that when

$$\theta_0 < \theta^*$$

all the roots  $\mu_n$  of (1.7) are complex-valued. When the cone angle  $\theta_0$  is increased past  $\theta^*$ , the root  $\mu_1$  with the smallest real part is real-valued. For larger values of  $\theta_0$  the higher eigenvalues  $\mu_2, \mu_3, \dots$ , successively, become real-valued. All roots are real-valued when  $\theta_0 > 90^\circ$ .

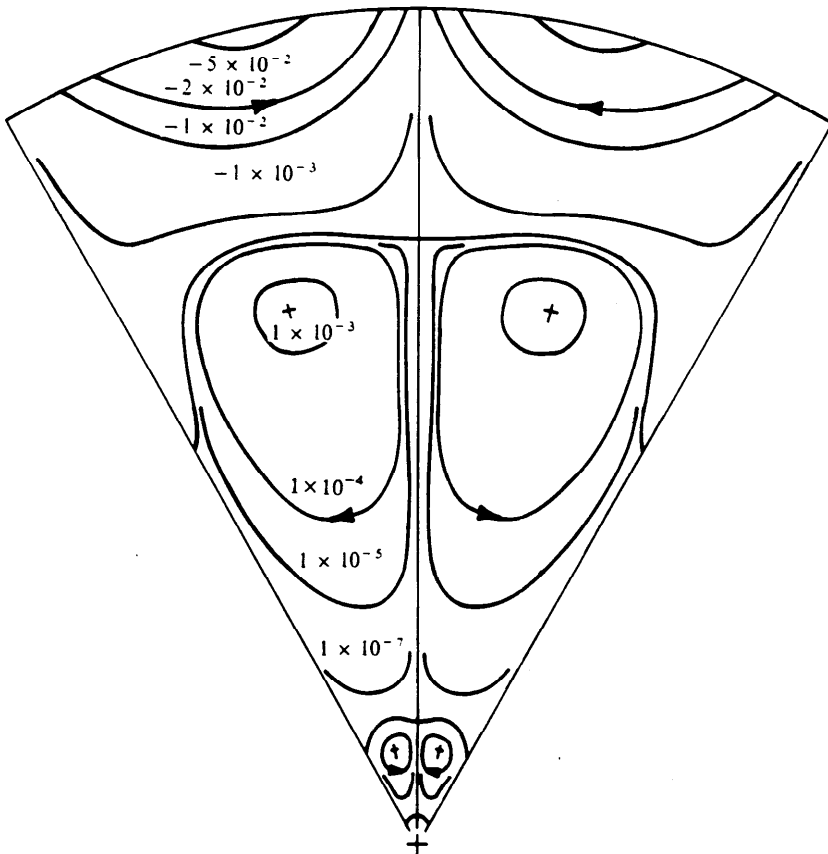


FIG. 1. Level lines of the real part of the principal eigenfunction (1.3):  $\mu = \mu_1, 2\theta_0 = 60^\circ$ ; see Table 2.

**2. The edge problem.** In the canonical form of the edge problem the values of the normal component of velocity  $u_r$ , and of the shear  $S_{r\theta}$  are prescribed on the "edge"  $r = 1$  of the cone. In  $(r, \xi)$  coordinates we have

$$u_r = \frac{-1}{r^2} \frac{\partial \Psi}{\partial \xi}, \quad u_\theta = \frac{-1}{r(1-\xi^2)^{1/2}} \frac{\partial \Psi}{\partial r}$$

and

$$S_{r\theta} = \frac{\hat{\mu}}{2} \left[ r \frac{\partial}{\partial r} \left( \frac{u_\theta}{r} \right) + \frac{1}{r} \frac{\partial u_r}{\partial \theta} \right]$$

$$= \frac{\hat{\mu}}{2} \frac{r^4}{(1-\xi^2)^{1/2}} \left[ -\frac{\partial}{\partial r} \left( \frac{1}{r^2} \frac{\partial \Psi}{\partial r} \right) + (1-\xi^2) \frac{\partial^2 \Psi}{\partial \xi^2} \right].$$

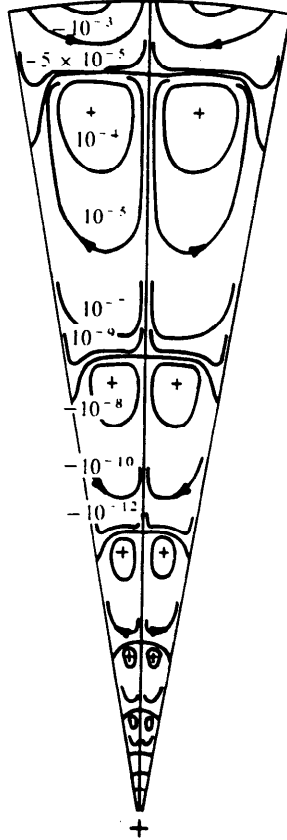


FIG. 2. Level lines of the real part of the principal eigenfunction (1.3):  $\mu = \mu_1$ ,  $2\theta_0 = 20^\circ$ ; see Table 1.

If

$$(2.1) \quad \begin{bmatrix} f(\xi) \\ g(\xi) \end{bmatrix} = \begin{bmatrix} r^4 \frac{\partial}{\partial r} \left( \frac{1}{r^2} \frac{\partial \Psi}{\partial r} \right) \\ (1-\xi^2) \frac{\partial^2 \Psi}{\partial \xi^2} \end{bmatrix}$$

are prescribed on  $r = 1$ , then  $u_r$  and  $S_{r\theta}$  are also prescribed. The prescription of edge values (2.1) are compatible with the side wall boundary condition only if

$$(2.2) \quad \int_{\xi_0}^1 \frac{g(\xi)}{1-\xi^2} d\xi = \int_{\xi_0}^1 \frac{\xi g(\xi)}{1-\xi^2} d\xi = 0.$$

We now reduce problem (1.1), (1.2) and (1.1) to ordinary differential equations. Solutions of edge problem are derived from expansions which are assumed in the form

$$(2.3) \quad r^4 \frac{\partial}{\partial r} \left( \frac{1}{r^2} \frac{\partial \Psi}{\partial r} \right) = \sum_n C_n r^{(3/2)+\mu_n} \phi_1^{(n)}(\xi)$$

and

$$(2.4) \quad (1-\xi^2) \frac{\partial^2 \Psi}{\partial \xi^2} = \sum_n C_n r^{(3/2)+\mu_n} \phi_2^{(n)}(\xi),$$

where  $\phi_1^{(n)}(\xi)$  are the eigenfunctions (1.6) belonging to the eigenvalues  $\mu_n$  and the summation is over all eigenvalues  $\mu_n$  with positive real parts. If the cone were bounded from above at  $r = 1$  and from below at  $r = r_0 < 1$  by spherical surfaces, we would replace the terms  $C_n r^{\mu_n}$  with  $C_n r^{\mu_n} + D_n r^{-\mu_n}$ . The constants  $D_n = 0$  in the full cone  $0 \leq r \leq 1$ . In the truncated cone, where  $D_n \neq 0$ , the computations proceed along lines laid out in § 4.2 of the paper by Yoo and Joseph (1978).

Now we will show how to compute the  $C_n$ . We first note that

$$\begin{aligned} & r^4 \frac{\partial}{\partial r} \left( \frac{1}{r^2} \frac{\partial}{\partial r} \right) \left[ (1-\xi^2) \frac{\partial^2 \Psi}{\partial \xi^2} \right] - (1-\xi^2) \frac{\partial^2}{\partial \xi^2} \left[ r^4 \frac{\partial}{\partial r} \left( \frac{1}{r^2} \frac{\partial \Psi}{\partial r} \right) \right] \\ &= \sum_n C_n r^{(3/2)+\mu_n} \left\{ (\mu_n^2 - \frac{9}{4}) \phi_2^{(n)} - (1-\xi^2) \phi_1^{(n)} \right\} = 0 \end{aligned}$$

and conclude that

$$(2.5) \quad (1-\xi^2) \phi_1^{(n)} = (\mu_n^2 - \frac{9}{4}) \phi_2^{(n)}.$$

Equation (8.3) implies that

$$(2.6) \quad \Psi = \sum_n \frac{C_n}{\mu_n^2 - \frac{9}{4}} r^{(3/2)+\mu_n} \phi_1^{(n)}(\xi).$$

We now expand (1.1), replacing  $\xi$  derivatives with (2.4) and forming  $r$  derivatives from (2.6), to show that

$$(2.7) \quad (1-\xi^2) \phi_2^{(n)} + 2(\mu_n^2 - \frac{5}{4}) \phi_2^{(n)} + (\mu_n^2 - \frac{1}{4}) \phi_1^{(n)} = 0.$$

We have now reduced the edge problem to the pair of ordinary differential equations (2.5) and (2.7) which are to be solved relative to the boundary conditions

$$(2.8) \quad \phi_1^{(n)}(\xi_0) = \phi_1^{(n)}(\xi_0) = 0$$

and edge conditions (2.1). Using (2.3) and (2.4) the edge conditions may be written as

$$(2.9) \quad \mathbf{f} = \begin{bmatrix} f \\ g \end{bmatrix} = \sum_n C_n \begin{bmatrix} \phi_1^{(n)}(\xi) \\ \phi_2^{(n)}(\xi) \end{bmatrix} = \sum_n C_n \Phi^{(n)}.$$

Equation (2.9) can hold for  $\xi_0 \leq \xi \leq 1$  only when  $f(1) = g(1) = f(\xi_0) = f'(\xi_0) = 0$ .



To compute the  $C_n$ , it is necessary to introduce adjoint eigenfunctions

$$[\psi_1^{(m)}, \psi_2^{(m)}] \equiv \psi^{(m)}.$$

Defining

$$\Phi^{(n)} = \begin{bmatrix} \phi_1^{(n)} \\ \phi_2^{(n)} \end{bmatrix}$$

and

$$\mathbf{A}_n = \begin{bmatrix} 0 & -(\mu_n^2 - \frac{9}{4}) \\ \mu_n^2 - \frac{1}{4} & 2(\mu_n^2 - \frac{5}{4}) \end{bmatrix},$$

we may write (2.5) and (2.7) as

$$(2.10) \quad (1 - \xi^2)\Phi''^{(n)} + \mathbf{A}_n \cdot \Phi^{(n)} = 0.$$

Equation (2.10) shows that  $\Phi^{(n)} = 0$  when  $\xi = 1$ .

To find the problem adjoint to (2.10) and (2.8) we define the scalar product

$$\langle \mathbf{a} \cdot \mathbf{b} \rangle = \int_{\xi_0}^1 \frac{\mathbf{a} \cdot \mathbf{b}}{1 - \xi^2} d\xi$$

and the linear operator

$$\mathcal{L} = (1 - \xi^2) \frac{d^2}{d\xi^2} + \mathbf{A}_n.$$

Then

$$(2.11) \quad \begin{aligned} \langle \Psi \cdot \mathcal{L}\Phi \rangle &= (\Psi \cdot \Phi')_{\xi_0}^1 - (\Psi' \cdot \Phi)_{\xi_0}^1 \\ &+ \int_{\xi_0}^1 [(1 - \xi^2)\Psi'' + \Psi \cdot \mathbf{A}_n] \frac{\Phi}{1 - \xi^2} d\xi \\ &= \psi_2(\xi_0)\phi_2'(\xi_0) - \psi_2'(\xi_0)\phi_2(\xi_0) + \langle \mathcal{L}^*\Psi \cdot \Phi \rangle, \end{aligned}$$

where  $\Phi = \phi^{(n)}$  and  $\Psi = \psi^{(n)}$  and  $\mathcal{L}^* = (1 - \xi^2)(d^2/d\xi^2) + \mathbf{A}_n$ .

It follows from (2.11) that the problem

$$(2.12) \quad 0 = \mathcal{L}^*\Psi^{(n)} = (1 - \xi^2)\Psi''^{(n)} + \Psi^{(n)} \cdot \mathbf{A}_n$$

with

$$(2.13) \quad \psi_2^{(n)}(\xi_0) = \psi_2'^{(n)}(\xi_0) = 0$$

is adjoint to (2.10) and (2.8).

Eigensolutions of (2.10) and (2.8) are in the form

$$\phi_1^{(n)} = (1 - \xi^2)^{1/2} [P_{\mu_n - 3/2}^1(\xi_0) P_{\mu_n + 1/2}^1(\xi) - P_{\mu_n + 1/2}^1(\xi_0) P_{\mu_n - 3/2}^1(\xi)]$$

and

$$\phi_2^{(n)} = (1 - \xi^2)^{1/2} \left[ \frac{\mu_n + 1/2}{\mu_n - 3/2} P_{\mu_n - 3/2}^1(\xi_0) P_{\mu_n + 1/2}^1(\xi) - \frac{\mu_n - 1/2}{\mu_n + 3/2} P_{\mu_n + 1/2}^1(\xi_0) P_{\mu_n - 3/2}^1(\xi) \right],$$

where the  $\mu_n$  are determined from (1.7). Eigensolutions of (2.12) and (2.13) are in the form

$$\psi_1^{(n)} = (1 - \xi^2)^{1/2} \left[ \frac{\mu_n - 1/2}{\mu_n - 3/2} P_{\mu_n - 3/2}^1(\xi_0) P_{\mu_n + 1/2}^1(\xi) - \frac{\mu_n + 1/2}{\mu_n - 3/2} P_{\mu_n + 1/2}^1(\xi_0) P_{\mu_n - 3/2}^1(\xi) \right]$$

and

$$\psi_2^{(n)} = \psi_1^{(n)}.$$

Since  $P_\nu^1(x) = O(1 - x^2)$  when  $x \rightarrow 1$ , the components of  $\phi^{(n)}$  and  $\psi^{(n)}$  vanish to  $O(1 - \xi^2)$  as  $\xi \rightarrow 1$ .

The eigenvectors  $\phi^{(n)}(\xi)$  and  $\psi^{(m)}(\xi)$  satisfy the following condition of biorthogonality:

$$(2.14) \quad \int_{\xi_0}^1 (1 - \xi^2)^{-1} \psi^{(m)} \cdot \mathbf{A} \cdot \phi^{(n)} d\xi = k_n \delta_{mn},$$

where  $\delta_{mn}$  is Kronecker's delta,

$$\mathbf{A} = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix},$$

and

$$k_n = \int_{\xi_0}^1 (1 - \xi^2)^{-1} \psi^{(n)} \cdot \mathbf{A} \cdot \phi^{(n)} d\xi.$$

To prove (2.14) we first multiply (2.10) by  $\psi^{(m)}$  and multiply (2.12), with  $m \neq n$  replacing  $n$ , by  $\phi^{(n)}$  and then subtract. This leads to

$$(1 - \xi^2)(\psi^{(m)} \cdot \phi^{(n)} - \psi^{(n)} \cdot \phi^{(m)}) - \psi^{(m)} \cdot (\mathbf{A}_n - \mathbf{A}_m) \cdot \phi^{(n)} = 0.$$

Since

$$\mathbf{A}_n - \mathbf{A}_m = (\mu_n^2 - \mu_m^2) \mathbf{A},$$

we find, after integrating by parts, that

$$(\mu_n^2 - \mu_m^2) \int_{\xi_0}^1 (1 - \xi^2)^{-1} \psi^{(m)} \cdot \mathbf{A} \cdot \phi^{(n)} d\xi = 0,$$

providing (2.14).

It now follows from (2.9) and (2.14) that

$$C_n = \frac{1}{k_n} \int_{\xi_0}^1 (1 - \xi^2)^{-1} \psi^{(n)} \cdot \mathbf{A} \cdot \mathbf{f} d\xi.$$

This completes the formal solution of the edge problem in the cone. We expect that Joseph's (1977) proof of convergence applies also to cone eigenfunctions. In this case the series (2.9) can be uniformly majorized by a numerical series

$$C \sum_{n=1} 1/n^{(5-\epsilon)/2}$$

whenever

$$g(1) = f(1) = f(\xi_0) = f'(\xi_0) = 0.$$

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