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THE CONVERGENCE OF BIORTHOGONAL SERIES FOR BIHARMONIC AND STOKES FLOW EDGE PROBLEMS: PART II*

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Abstract. Sufficient conditions are established for the convergence of the biorthogonal series solving edge problems which arise in elasticity and in Stokes flow in cavities. These conditions and those given in Part I, (D. D. Joseph, *The convergence of biorthogonal series for biharmonic and Stokes flow edge problems*, SIAM, J. Appl. Math., 33(1977), pp. 337–347) include all those which are likely to arise in applications. Examples of conditional convergence of the series to step functions and to ramp functions are presented. Problems previously considered to be intractable to analysis are solved by analysis.

CONTENTS

1. Introduction. In Part I (Joseph (1977)) of this two-part paper, sufficient conditions were given for the convergence of biorthogonal series which solve biharmonic and Stokes flow edge problems. The conditions of convergence depend on the compatibility of the edge data with the side-wall boundary conditions and also on the interior regularity of the edge data. In this part we show that the biorthogonal series converge even when the compatibility conditions are *not* satisfied, but in this case the convergence is conditional. The properties of conditional convergence of the biorthogonal series to the step function and to the ramp function are examined numerically. In the concluding section (§ 6) of this paper we solve the problem of slow steady flow induced in a rectangular cavity by shearing the fluid on the top with a moving plate. This problem was solved by numerical methods by Burggraf (1966) and by Pan and Acrivos (1967) and it falls in the class of conditionally convergent problems in which the side-wall compatibility conditions are not satisfied.

The canonical form of the problem to which our results apply is a biharmonic problem on a semi-infinite strip

$$\left[\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial y^2}\right]^2\Psi=0 \quad \text{in} \quad \mathcal{V}=[t, y: -1<t<1, -\infty<y<0], \tag{1.1}$$

$$\Psi(\pm 1, y)=\Psi_{,t}(\pm 1, y)=0 \tag{1.2}$$

subject to the requirement that the second derivatives of $\Psi(t, y)$ assume prescribed values

$$\begin{pmatrix} f(t) \\ g(t) \end{pmatrix}=\begin{pmatrix} \Psi_{,yy} \\ \Psi_{,tt} \end{pmatrix} \tag{1.3}$$

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on the short edge of $\mathcal{V}$, $y=0$. Using an algorithm given by R. C. T. Smith (1952), it is possible to solve (1.1)–(1.3) formally with a generalized "Fourier" series whose coefficients are selected by biorthogonality conditions. In Part I, and in this part we are concerned with the convergence of those series.

Problem (1.1)–(1.3) and allied problems which are not in canonical form, as in the problem studied in § 6, arise frequently in elasticity, in fluid mechanics and in other branches of science. The method of solution in biorthogonal series appears to be as useful for these problems as ordinary Fourier series are for simpler problems. A list of worked problems using biorthogonal series can be found in Part I.

With these preliminary remarks aside we turn now to the preparation for the statement and proof of our main theorem of convergence. For clarity, it is useful to treat even data $f(t)=f(-t)$, $g(t)=g(-t)$ and odd data $f(t)=-f(-t)$, $g(t)=-g(-t)$, separately. Arbitrary data may then be treated by superposition.

2. Convergence (even data). For real-valued even data the formal solution of (1.1)–(1.3), adapted by Joseph and Sturges (1975) from a solution first given by R. C. T. Smith (1952), is

$$\Psi(t,y)\sim\sum_{-\infty}^{\infty} C_n \exp(S_n y)\phi_1^{(n)}(t)/S_n^2 \tag{2.1}$$

where

$$\phi_1^{(n)}(t)=\bar{\phi}_1^{(-n)}(t)=S_n[\sin S_n \cos S_n t - t\cos S_n \sin S_n t]$$

are biharmonic eigenfunctions with complex-valued eigenvalues S_n determined as the first-quadrant roots of

$$2S_n=2\bar{S}_{-n}=-\sin 2S_n \tag{2.2}$$

where the overbar designates complex conjugation. It is obvious that $\phi_1^{(n)}(\pm 1)=0$ and, if (2.2) holds, $\phi_{1,t}^{(n)}(\pm 1)=0$. The eigenvalues $S_n=x_n+y_n$ are numbered $n=1,2,\ldots,\infty$ in order of increasing modulus. The coefficients C_n are given by

$$C_n=\bar{C}_{-n}=\frac{1}{k_n}\int_{-1}^{1}\{(2\psi_2^{(n)}-\psi_1^{(n)})g+\psi_2^{(n)}f\}\,dt \tag{2.3}$$

where $k_n=-4\cos^4 S_n$ and $\psi_1^{(n)}(t)$, $\psi_2^{(n)}(t)$ are adjoint eigenfunctions satisfying (see (14) of Part I)

$$\psi_{2,tt}^{(n)}+S_n^2(2\psi_2^{(n)}-\psi_1^{(n)})=0,\qquad \psi_2^{(n)}(\pm 1)=\psi_{2,t}^{(n)}(\pm 1)=0 \tag{2.4}$$

and

$$\psi_{1,tt}^{(n)}+S_n^2\psi_2^{(n)}=0. \tag{2.5}$$

The adjoint eigenfunctions are given by

$$\psi_2^{(n)}(t)=\phi_1^{(n)}(t) \tag{2.6}$$

and

$$\psi_1^{(n)}(t)=\phi_1^{(n)}-2\cos S_n \cos S_n t. \tag{2.7}$$

The formula (2.3) giving C_n is a consequence of the following property of biorthogonality satisfied by the eigenfunctions and adjoint eigenfunctions

$$\int_{-1}^{1} [\psi_1^{(n)}, \psi_2^{(n)}]\begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} \phi_1^{(m)} \\ \phi_2^{(m)} \end{pmatrix} dt = k_n\delta_{nm}. \tag{2.8}$$

Equation (2.8) is derived by standard procedures from the differential equations satisfied by the eigenfunctions and their adjoints (see Part I).

The formal solution (2.1) of (1.1)–(1.3) is the true solution if after differentiation of the series (2.1) the differentiated series converges to the prescribed edge data (1.3); that is, if

$$\begin{pmatrix} f(t) \\ g(t) \end{pmatrix} \sim \sum_{-\infty}^{\infty} C_n \begin{pmatrix} \phi_1^{(n)}(t) \\ \phi_2^{(n)}(t) \end{pmatrix} \tag{2.9}$$

where

$$\phi_2^{(n)} = \phi_{1,tt}^{(n)}/S_n^2 = -\phi_1^{(n)} - 2\cos S_n \cos S_n t.$$

We are interested in finding the conditions on $f(t)$ and $g(t)$ which will guarantee that the series on the right of (2.9) converge. In the applications we always verify by computation that the series (2.9) converge to the prescribed functions. Completeness theorems which would make it unnecessary to verify that the convergent series actually converge to the prescribed functions are presently unknown.

The properties of convergence of the series on the right of (2.9) depend on the compatibility of the data with the side-wall boundary conditions. Inspection of (2.9) shows that $f(t)$ is compatible with side-wall conditions $\phi_1^{(n)}(\pm 1) = \phi_{1,t}^{(n)}(\pm 1) = 0$ satisfied by the terms on the right of (2.9) only if

$$f(\pm 1) = f'(\pm 1) = 0. \tag{2.10}$$

In Part I we showed that the series on the right of (2.9) converges absolutely even when $f'(\pm 1) \neq 0$. Now we are going to show the series still converge when $f'(\pm 1) \neq 0$ and $f(\pm 1) \neq 0$ but, as in the Fourier representation,

$$1 \sim \sum_{n=1}^{\infty} a_n \sin n\pi t, \qquad 0 < t < 1,$$

$$a_n = 2\int_0^1 \sin n\pi t\, dt = \frac{2 - 2\cos n\pi}{n\pi},$$

the series on the right of (2.9) does not converge to $f(t)$ at $t = \pm 1$ and the convergence is conditional and not absolute.

The function $g(t)$ which is expressed in a series of terms proportional to $\phi_2^{(n)}(t)$ is compatible with the side-wall conditions if $g(t)$ satisfies the same integral conditions

$$\int_{-1}^{1} g(t)\, dt = \int_{-1}^{1} t g(t)\, dt = 0 \tag{2.11}$$

as $\phi_2^{(n)} = \phi_{,tt}^{(n)}/S_n^2$. The compatibility conditions (2.11) do not enter the proofs of convergence. It is not possible, however, to expand vectors $[f(t), g(t)]$ which violate (2.11) into a series of terms proportional to $[\phi_1^{(n)}(t), \phi_2^{(n)}(t)]$, $n > 1$, because this set is not complete when the space of functions $g(t)$ contain some which violate (2.11). To expand such functions it is necessary that the eigenvector $[\phi_1^{(0)}(t), \phi_2^{(0)}(t)] = [0, 1]$ belonging to $S_0 = 0$ be appended to the set of eigenfunctions. This eigenfunction does not satisfy the first of the conditions (2.11). The second of conditions (2.11) is automatically satisfied when $g(t)$ is an even function but restricts the class of allowed functions $g(t)$ when $g(t)$ is not even. To expand odd functions $g(t)$ it is necessary to include the eigenfunctions $[\hat{\phi}_1^{(0)}(t), \hat{\phi}_2^{(0)}] = [0, t]$ with eigenvalue $P_0 = 0$ in the set $\{\hat{\phi}_1^{(n)}, \hat{\phi}_2^{(n)}\}$. Data $[f(t), g(t)]$ violating (2.11) can be expanded formally if, instead of (2.9), we write

$$\begin{pmatrix} f(t) \\ g(t) \end{pmatrix} = C_0 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \sum_{-\infty}^{\infty} C_n \begin{pmatrix} \phi_1^{(n)}(t) \\ \phi_2^{(n)}(t) \end{pmatrix} \tag{2.12}$$

where $\Psi_2^{(0)} = 0$, $\Psi_1^{(0)} = 1$, $k_0 = -2$ and C_0 is given by (2.3). Similar formulas hold for odd data (see Part I). In the work which follows we shall assume that (2.11) holds. But (2.11) does not enter into our proof of convergence so this proof carries over directly to (2.12). We will return to expansions of the type (2.12) in our discussion of the biorthogonal expansion of the step function and the ramp function.

In Part I we noted that the conditions for convergence, $f(\pm 1) = f'(\pm 1) = g(\pm 1) = g'(\pm 1) = 0$, which were stated by Smith (1952) are too restrictive and exclude most of the problems which arise in applications.

THEOREM 3 (even data). *Suppose $f(t)$ and $g(t)$ are even, continuous, twice piecewise-continuously differentiable functions with a finite number of bounded jumps when $-1 < t < 1$. Then*

$$C_n = \frac{-1}{k_n S_n^2} \int_{-1}^{1} [\phi_{1,tt}^{(n)}(f+g) + 2fS_n^2 \cos S_n \cos S_n t]\, dt. \tag{2.13}$$

When n is large

$$C_n = O(1/n^2)$$

and for each t on the open interval $-1 < t < 1$ the series (2.9) satisfies the Leibniz criterion for conditional convergence.

Remark. Theorems 1 and 2 were proved in Part I.

The proof of this theorem will be carried out in five steps:

(i) Asymptotic formulas,
(ii) Asymptotic form of the C_n,
(iii) Dominant terms of (2.9),
(iv) Conditional convergence,
(v) Conditional convergence of dominant terms of (2.9).

(i) *Asymptotic formulas.*

LEMMA 1 (G. H. Hardy, 1902). *When n is large the first-quadrant eigenvalues $S_n = x_n + iy_n$ satisfying (2.2) are in the form*

$$2x_n = (2n - 1/2)\pi + 2\alpha_n, \tag{2.14}$$

$$2y_n = \log(4n-1)\pi + 2\beta_n \tag{2.15}$$

where α_n and β_n tend to zero as $n \to \infty$.

Hardy considered the problem $2S = \sin 2S$ but his results are trivially adapted to $2S = -\sin 2S$. Hardy's lemma is not quite good enough for our demonstrations.

LEMMA 2.

$$\alpha_n = \frac{-2\log(4n-1)\pi}{(4n-1)\pi} + o\left(\frac{\log n}{n}\right) \tag{2.16}$$

and

$$\beta_n = 2\left(\frac{\log(4n-1)\pi}{(4n-1)\pi}\right)^2 + o\left(\left[\frac{\log n}{n}\right]^2\right). \tag{2.17}$$

Proof. Define $\xi_n + i\eta_n = 2x_n + i2y_n = 2S_n$, $\xi_{n0} = (2n-\frac{1}{2})\pi$ and $\eta_{n0} = \log(4n-1)\pi$. Then with

$$\xi_n = \xi_{n0} + \varepsilon_n \qquad \text{and} \qquad \eta_n = \eta_{n0} + \delta_n \tag{2.18}$$

we may write the real and imaginary parts of $2S_n = -\sin 2S_n$ as

$$\sin(\xi_{n0} + \varepsilon_n)\cosh(\eta_{n0} + \delta_n) = -\xi_{n0} - \varepsilon_n \tag{2.19}$$

and

$$\cos(\xi_{n0} + \varepsilon_n)\sinh(\eta_{n0} + \delta_n) = -\eta_{n0} + \delta_n. \tag{2.20}$$

When n is large

$$\sin\varepsilon_n \to \varepsilon_n, \qquad \sinh\delta_n \to \delta_n$$

and

$$\cos\varepsilon_n \to 1, \qquad \cosh\delta_n \to 1.$$

Recalling that $\cos\xi_{n0} = 0$ and $\sin\xi_{n0} = -1$ we find, from (2.19), that

$$\cos\varepsilon_n\,(\cosh\eta_{n0} + \delta_n\sinh\eta_{n0}) = \xi_{n0} + \varepsilon_n \tag{2.21}$$

and, from (2.20),

$$\varepsilon_n(\sinh\eta_{n0} + \delta_n\cosh\eta_{n0}) = -\eta_{n0} - \delta_n. \tag{2.22}$$

After linearizing (2.21), (2.22) we find, using the relation $\xi_{n0} = e^{\eta_{n0}}$, that $\varepsilon_n = -2\eta_{n0}/e^{\eta_{n0}}$ and $\delta_n = \varepsilon_n^2/2$, proving (2.16) and (2.17).

Equations (2.14) and (2.15) imply that

$$\sin S_n t = \frac{i}{2}[(4n-1)\pi]^{t/2} e^{-i(n-1/4)\pi t} + O(n^{-t/2}), \tag{2.23}$$

$$\cos S_n t = \frac{1}{2}[(4n-1)\pi]^{t/2} e^{-i(n-1/4)\pi t} + O(n^{-t/2}) \tag{2.24}$$

when $0 < t < 1$. It follows that

$$k_n = -4\cos 4S_n = -\frac{(4n-1)^2\pi^2}{4} + O(n) \tag{2.25}$$

and

$$\phi_1^{(n)}, \quad \phi_2^{(n)}, \quad \psi_1^{(n)} \quad \text{and} \quad \psi_2^{(n)} = O(n^{(3+|t|)/2}) \tag{2.26}$$

when n is large.

(ii) *Asymptotic form of the* C_n. Equation (2.13) follows from (2.3), (2.6), (2.7) and (2.8). We suppose that $t=\tau$ is a point at which the derivatives of f and g are discontinuous and, to simplify the writing, that there is only one such point. Then, integrating (2.13) by parts, we find that

$$\begin{aligned} k_n S_n^2 C_n &= 4f(1)S_n^2 - \phi_1^{(n)}(\tau)((g'+f')) \\ &\quad + 2\cos S_n \cos S_n\tau((f')) - 4(\cos^2 S_n)f'(1) \\ &\quad - \int_{-1}^{1} [\phi_1^{(n)}(g''+f'') - 2f''\cos S_n \cos S_n t]\,dt \end{aligned} \tag{2.27}$$

where $((a(t))) = a(\tau_+) - a(\tau_-)$ is the jump in $a(t)$ across τ. The first term on the right of (2.27) is $O(n^2)$. The other terms on the right of (2.27) are of smaller order. To obtain the order of the terms of smaller order, it is convenient to consider two cases.

Case I. $g(t)$ and $f(t)$ have two continuous derivatives and the third derivative is piecewise continuous on $-1<t<1$. In this case the jumps in (2.27) vanish, the integral in (2.27) is $O(1)$ (see (25) of Part I) and

$$k_n S_n^2 C_n = 4f(1)S_n^2 - 4(\cos^2 S_n)f'(1) + O(1). \tag{2.28}$$

The first term on the right of (2.28) is $O(n^2)$; the second term is $O(n)$.

Case II. The hypotheses of the theorem hold. Then, instead of (2.28), we get

$$\begin{aligned} k_n S_n^2 C_n &= 4f(1)S_n^2 - \phi_1^{(n)}(\tau)((g'+f')) \\ &\quad - 2\phi_1^{(n)}(\bar\tau)(g''(\bar\tau)+f''(\bar\tau)) + O(n) \end{aligned} \tag{2.29}$$

where $\bar\tau$ is a mean value, $-1<\bar\tau<1$. The second term on the right of (2.29) is $O(n^{(3+|\tau|)/2})$ and the third term is $O(n^{(3+|\bar\tau|)/2})$ where $|\tau|$ and $|\bar\tau|$ are strictly less than one.

It is convenient to write (2.29) as follows

$$C_n = C_{n0} + C_{n1} + O(n^{-3}) \tag{2.30}$$

where

$$C_{n0} = -\frac{f(1)}{\cos^4 S_n} = O(n^{-2}) \tag{2.31}$$

and

$$C_{n1} = \frac{\text{const.}}{S_n^{(1-\zeta)/2}\cos^4 S_n} = O(n^{(-5+\zeta)/2}) \tag{2.32}$$

where $0<\zeta<1$.

(iii) *Dominant terms of* (2.9). The series on the right of (2.9) divide into dominant terms and subdominant terms. The subdominant terms are series which converge absolutely. It was shown in Part I that the series (2.9) converges absolutely when $C_n = O(n^{-3})$. It follows then from (2.30) that (2.9) converges if

$$\begin{aligned} &\sum_{-\infty}^{\infty} (C_{n0}+C_{n1})\begin{pmatrix}\phi_1^{(n)}(t)\\ \phi_2^{(n)}(t)\end{pmatrix} \\ &= \sum_{-\infty}^{\infty} (C_{n0}+C_{n1})\begin{pmatrix}\phi_1^{(n)}(t)\\ -\phi_1^{(n)}(t) - 2\cos S_n \cos S_n t\end{pmatrix} \end{aligned} \tag{2.33}$$

converges. The series

$$\sum_{-\infty}^{\infty} C_{n0} \cos S_n \cos S_n t = -f(1) \sum_{1}^{\infty} \left(\frac{\cos S_n t}{\cos^3 S_n} + \frac{\cos \bar{S}_n t}{\cos^3 \bar{S}_n} \right)$$

is absolutely convergent. To show this we use (2.24) to show that when n is large and $t \geqq 0$,

$$\frac{\cos S_n t}{\cos^3 S_n} \to \exp\{-i(n-\tfrac{1}{4})\pi(3-t)\}/\{(4n-1)\pi\}^{(3-t)/2}.$$

Hence

$$(2.34) \qquad -\frac{1}{f(1)} \sum_{-\infty}^{\infty} C_{n0} \cos S_n \cos S_n t \to \sum_{N}^{\infty} \frac{2 \cos\{(n-\frac{1}{4})\pi(3-t)\}}{[(4n-1)\pi]^{(3-t)/2}}$$

converges absolutely for all t, $0 < t < 1$.

It follows from (2.34) and (2.33) that (2.9) converges if the series

$$\sum_{-\infty}^{\infty} (C_{n0} + C_{n1}) \phi_1^{(n)}(t)$$

$$= \sum_{-\infty}^{-\infty} (C_{n0} + C_{n1}) S_n [\sin S_n \cos S_n t - t \cos S_n \sin S_n t]$$

$$= \frac{1}{2} \sum_{-\infty}^{\infty} (f_n + i g_n) \{(1-t) \sin S_n(1+t) + (1+t) \sin S_n(1-t)\}$$

$$(2.35) \qquad = \sum_{1}^{\infty} f_n \{(1-t) \sin x_n(1+t) \cosh y_n(1+t)$$

$$+ (1+t) \sin x_n(1-t) \cosh y_n(1-t)\}$$

$$- g_n \{(1-t) \cos x_n(1+t) \sinh y_n(1+t)$$

$$+ (1+t) \cos x_n(1-t) \sinh y_n(1-t)\}$$

converges, where

$$f_n + i g_n = (C_{n0} + C_{n1}) S_n = f_{-n} - i g_{-n}.$$

Since (2.35) is an even function of t the four series of (2.35) converge if the first and third series converge. The dominant terms of these series are in the form

$$(2.36) \qquad (1-t) \sum_{1}^{\infty} f_n \sqrt{[(4n-1)\pi]^{(1+t)}} \sin (n-\tfrac{1}{4})\pi(1+t)$$

and

$$(2.37) \qquad (1-t) \sum_{1}^{\infty} g_n \sqrt{[(4n-1)\pi]^{(1+t)}} \cos (n-\tfrac{1}{4})\pi(1+t).$$

To see this we note, using Lemma 1, that

$$\sin x_n(1+t) = \sin\{(n-\tfrac{1}{4})\pi(1+t) + \alpha_n\}$$

$$\sim \sin\{(n-\tfrac{1}{4})\pi(1+t)\} + \alpha_n \cos\{(n-\tfrac{1}{4})\pi(1+t)\}$$

and

$$\cosh y_n(1+t)=\sqrt{[(4n-1)\pi]^{(1+t)}}\, e^{\beta_n(1+t)}$$

$$\to\sqrt{[(4n-1)\pi]^{(1+t)}}[1+\beta_n(1+t)]$$

where α_n and β_n are given by the estimates of Lemma 2. The dominant coefficient f_n comes from the real part of

$$C_{n0}S_n\sim\frac{4f(1)}{(4n-1)\pi}+\frac{i8f(1)\ln[(4n-1)\pi]}{(4n-1)^2\pi^2}. \tag{2.38}$$

Putting these estimates together, it may be verified that the difference between the first series in (2.35) and the series (2.36) is a series of terms of $O(\ln n/n^{(3-t)/2})$ which converges absolutely on $0<t<1$. The dominant coefficient g_n arises from the imaginary part of $C_{n0}S_n$. This leads us to estimate the terms of (2.37) as $O(\ln n/n^{(3-t)/2})$. It follows that (2.37) is also subdominant and that (2.9) converges if (2.36) does.

(iv) *Conditional convergence.*

LEMMA 3. *The series*

$$f(t)=\sum n^{-\alpha}\sin[(n-\tfrac{1}{4})\pi(1+t)] \tag{2.39}$$

is conditionally convergent for each t, $-1<t<1$, *when* $0<\alpha<1$.

Proof. If (2.39) converges on a dense set $\{t_{r,s}\}\in(-1,1)$, it must converge for all $t\in(-1,1)$. Let $t_{r,s}=r/s$ where r and s are integers and $r+s$ is an odd integer. Define

$$a_n=\sin\left[\left(n-\frac{1}{4}\right)\pi\left(1+\frac{r}{s}\right)\right].$$

Then, since $r+s$ is an odd integer

$$a_{n+s}=\sin\left[\left(n-\frac{1}{4}\right)\pi\left(1+\frac{r}{s}\right)+(r+s)\pi\right]=-a_n. \tag{2.40}$$

The terms of (2.39) can be grouped into groups of s terms

$$f(t)=\sum_{m=0}^{\infty}b_m, \tag{2.41}$$

$$b_m=\sum_{ms+1}^{ms+s}a_n\Big/n^{\alpha}=\frac{a_{ms+1}}{(ms+1)^{\alpha}}+\cdots+\frac{a_{ms+l}}{(ms+l)^{\alpha}}+\cdots+\frac{a_{ms+s}}{(ms+s)^{\alpha}}.$$

Equation (2.40) implies that

$$b_{m+1}=\frac{a_{ms+1+s}}{(ms+1+s)^{\alpha}}+\cdots+\frac{a_{ms+l+s}}{(ms+l+s)^{\alpha}}+\cdots+\frac{a_{ms+2s}}{(ms+2s)^{\alpha}}$$

$$=\frac{-a_{ms+1}}{(ms+1)^{\alpha}\left(1+\dfrac{s}{ms+1}\right)^{\alpha}}-\cdots-\frac{a_{ms+l}}{(ms+l)^{\alpha}\left(1+\dfrac{s}{ms+l}\right)^{\alpha}}$$

$$-\cdots-\frac{a_{ms+s}}{(ms+s)^{\alpha}\left(1+\dfrac{s}{ms+s}\right)^{\alpha}}.$$

Now consider large values of $m \gg s$. For these

$$b_{m+1} = -b_m\left[1 - \frac{\alpha}{m} + O\left(\frac{1}{m^2}\right)\right]. \tag{2.42}$$

(2.42) shows that (2.41) or, equivalently, (2.39) is a series of alternating sign and, if $\alpha > 0$, of decreasing magnitude. It follows that (2.39) satisfies the Leibniz criterion for conditional convergence when $\alpha > 0$ and $t < 1$.

Remark. The series (2.39) diverges when $t = 1$.

(v) *Conditional convergence of dominant terms of* (2.9). The dominant terms of (2.9) are convergent if (2.36) converges. It is enough to show that

$$n^{(1+t)/2} f_n \to \frac{1}{n^{\alpha}}, \qquad \alpha > 0 \tag{2.43}$$

when n is large. Recalling that $f_n + ig_n = (C_{n0} + C_{n1})S_n$ we note, using (2.38), that $f_{n0} = re(C_{n0}S_n)$, where $f_n = f_{n0} + f_{n1}$, is of the required form with $\alpha = (1-t)/2$. We leave as an exercise for the reader the demonstration that f_{n1} is of the required form with $\alpha = 1 - (\zeta + t)/2$ (see (2.32)). This completes the proof of Theorem 3 for even data.

3. Step function. As an application of Theorem 3 for even data we consider the expansion of the step functions

$$\begin{pmatrix}1\\0\end{pmatrix} \sim \sum_{-\infty}^{\infty} C_n \begin{pmatrix}\phi_1^{(n)}(t)\\ \phi_2^{(n)}(t)\end{pmatrix}, \qquad -1 < t < 1 \tag{3.1}$$

and

$$\begin{pmatrix}0\\1\end{pmatrix} \sim C_0\begin{pmatrix}0\\1\end{pmatrix} + \sum_{-\infty}^{\infty} C_n \begin{pmatrix}\phi_1^{(n)}(t)\\ \phi_2^{(n)}(t)\end{pmatrix}, \qquad -1 < t < 1. \tag{3.2}$$

Computing C_n from (2.3), we find that

$$\text{in (3.1)} \quad C_n = -1/\cos^4 S_n, \quad n = \pm 1, \pm 2, \cdots$$

and

$$\text{in (3.2)} \quad C_0 = 1 \quad \text{and} \quad C_n = 0 \quad \text{when } n \neq 0.$$

It follows that (3.2) reduces to an identity and

$$\begin{pmatrix}1\\0\end{pmatrix} \sim \lim_{N\to\infty} \mathbf{S}_N(t) \tag{3.3}$$

where

$$\mathbf{S}_N(t) = -\sum_{-N}^{N} \frac{1}{\cos^4 S_n} \begin{pmatrix}\phi_1^{(n)}(t)\\ \phi_2^{(n)}(t)\end{pmatrix}.$$

The graph of the components of $\mathbf{S}_N(t)$ for different values of N are exhibited in Figs. 1 and 3. As in the theory of trigonometric series the introduction of Cesaro sums

$$\bar{\mathbf{S}}_N = \frac{1}{N+1} \sum_{M=1}^{N} \mathbf{S}_M$$

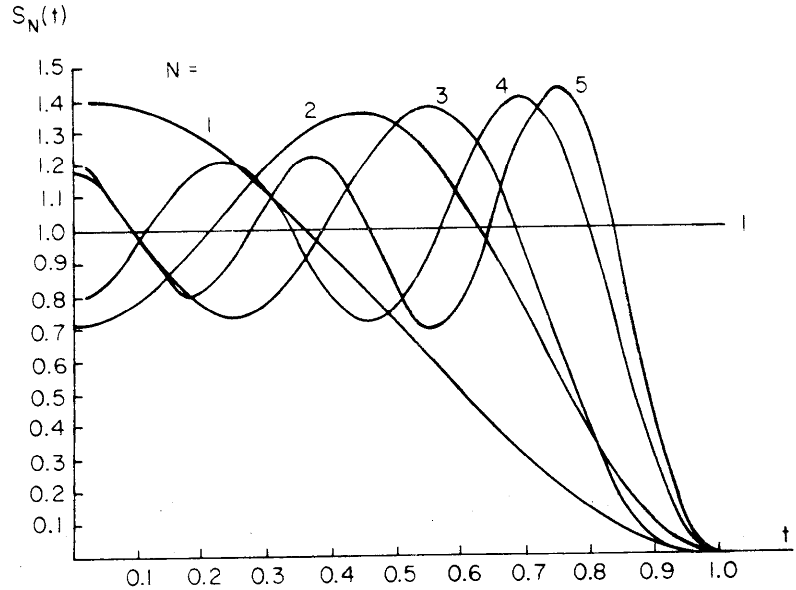

FIG. 1. *Convergence of the partial sums*

$$S_N(t) = \sum_{n=-N}^{N} \frac{-1}{\cos^4 S_n} \phi_1^{(n)}(t)$$

of the biorthogonal series $\lim_{N\to\infty} S_N(t)$ *representing the unit step function* $f(t) = 1, -1 \leqq t \leqq 1$

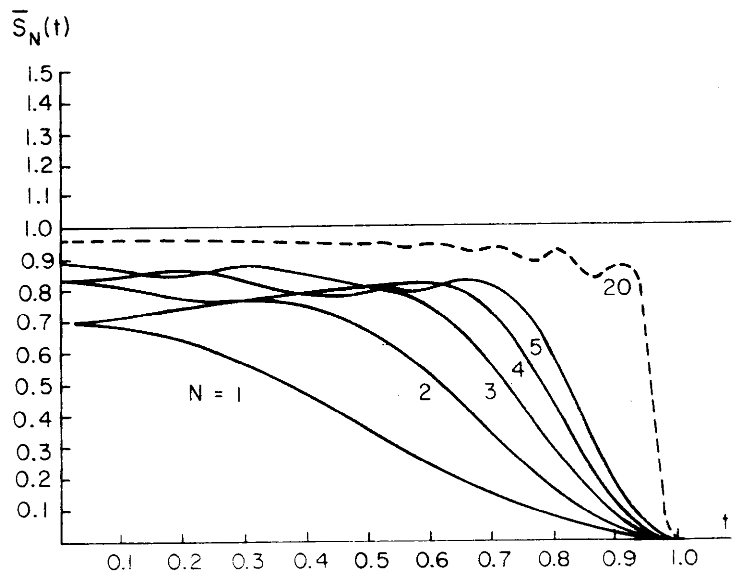

FIG. 2. *Convergence of the Cesaro sums*

$$\bar{S}_N(t) = \frac{1}{N+1} \sum_{M=1}^{N} S_M(t)$$

to the unit step function $f(t) = 1, -1 \leqq t \leqq 1$

leads to improved convergence (see Figs. 2 and 4). In Fig. 5 we have shown how the partial sums and Cesaro sums vary at a fixed point ($t = 0.95$) as a function of the truncation number N. The oscillatory character of conditional convergence is apparent from this figure.

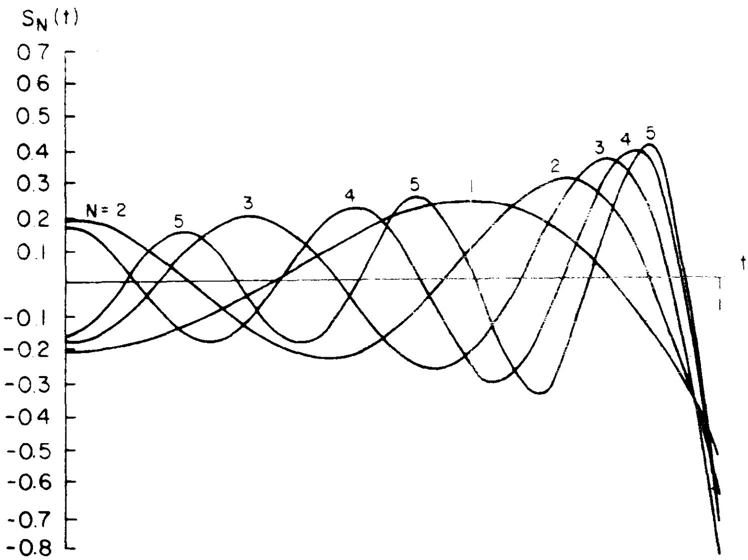

FIG. 3. *Convergence of the partial sums*

$$S_N(t) = \sum_{n=-N}^{N} \frac{-1}{\cos^4 S_n} \phi_2^{(n)}(t)$$

of the biorthogonal series $\lim_{N\to\infty} S_N(t)$ *representing the zero function* $g(t) = 0, -1 \leqq t \leqq 1$

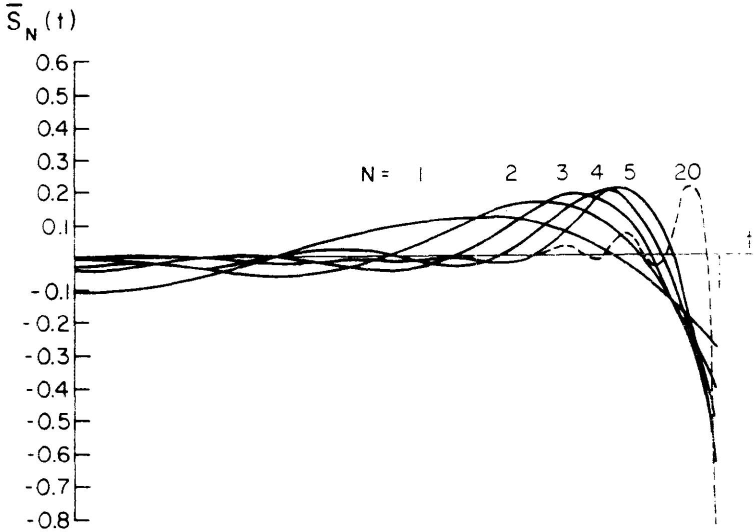

FIG. 4. *Convergence of the Cesaro sums*

$$\bar{S}_N(t) = \frac{1}{N+1} \sum_{M=1}^{N} S_M(t)$$

to the zero function $g(t) = 0, -1 \leqq t \leqq 1$

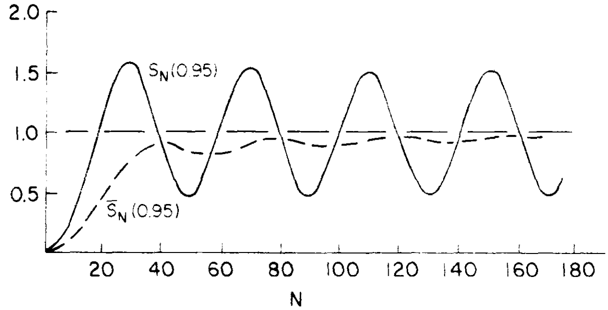

FIG. 5. *Oscillatory character of the convergence to* $f(t)=1$ *(at* $t=0.95$*) of the partial sums*

$$S_N(t)=\sum_{n=-N}^{N}\frac{-1}{\cos^4 S_n}\phi_1^{(n)}(t)$$

and Cesaro sums

$$\bar{S}_N(t)=\frac{1}{N+1}\sum_{M=1}^{N}S_M(t)$$

4. Convergence (odd data). The analysis of § 2 holds when the data is odd provided that the definitions of functions are appropriately changed. Replacing $\phi_2^{(n)}$, $\psi_2^{(n)}$ and S_n are $\hat{\phi}_2^{(n)}$, $\hat{\psi}_2^{(n)}$ and P_n where

$$\begin{aligned}
&\hat{\phi}_1(t)=P_n[\cos P_n \sin P_n t - t\sin P_n \cos P_n t],\\
&\hat{\phi}_2(t)=-\hat{\phi}_1^{(n)}+2\sin P_n \sin P_n t,\\
&\hat{\psi}_2^{(n)}=\hat{\phi}_1^{(n)},\\
&\hat{\psi}_1^{(n)}=\hat{\phi}_1^{(n)}+2\sin P_n \sin P_n t
\end{aligned}\tag{4.1}$$

and

$$2P_n=\sin 2P_n.$$

The asymptotic distribution of eigenvalues is given by

$$x_n=\left(n+\frac{1}{4}\right)\pi-\frac{\log(4n+1)\pi}{(4n+1)\pi}+o\left(\frac{\log n}{n}\right),\tag{4.2}$$

and

$$y_n=\frac{1}{2}\log(4n+1)\pi+\left(\frac{\log(4n+1)\pi}{(4n+1)\pi}\right)^2+o\left(\left[\frac{\log n}{n}\right]^2\right).\tag{4.3}$$

As in § 2, we shall suppose that the compatibility condition (2.11) holds. Then

$$C_n=\bar{C}_{-n}=\frac{1}{k_n}\int_{-1}^{1}\{(2\hat{\psi}_2^{(n)}-\hat{\psi}_1^{(n)})g+\hat{\psi}_2^{(n)}f\}\,dt,\qquad n>1.\tag{4.4}$$

When (2.11) does not hold it is necessary to include the eigenfunctions belonging to eigenvalues $P_0=0$. These are given in Part I as

$$\hat{\phi}_1^{(0)}=\hat{\psi}_2^{(0)}=0,\quad \hat{\phi}_2^{(0)}=\hat{\psi}_1^{(0)}=t,\quad k_0=-\tfrac{2}{3}\tag{4.5}$$

and, from (4.5) in (4.4),

$$C_0 = \frac{3}{2}\int_{-1}^{1} tg(t)\,dt. \tag{4.6}$$

The function ψ corresponding to the eigenfunctions with zero eigenvalue is biharmonic but it satisfies only one of the two side-wall boundary conditions $\psi = \psi' = 0$ at $t = \pm 1$.

Supposing now that (2.11) holds, we have

THEOREM 3 (odd data). *Suppose $f(t)$ and $g(t)$ are odd, continuous, twice piecewise-continuously differentiable functions with a finite number of bounded jumps when $-1 < t < 1$. Then*

$$C_n = \frac{-1}{k_n P_n^2}\int_{-1}^{1} [\hat{\phi}_{1,tt}^{(n)}(f+g) + 2fP_n^2 \sin P_n \sin P_n t]\,dt. \tag{4.7}$$

When n is large

$$C_n = O(1/n^2)$$

and for each t on the open interval $-1 < t < 1$ the series (2.9) satisfies the Leibniz criterion for conditional convergence.

The proof of this theorem follows along the path laid out in the proof given in § 2.

5. Ramp functions. As an application of the theorem for odd data we consider the expansion of the ramp function

$$\begin{pmatrix} t \\ 0 \end{pmatrix} \sim \sum_{-\infty}^{\infty} C_n \begin{pmatrix} \hat{\phi}_1^{(n)}(t) \\ \hat{\phi}_2^{(n)}(t) \end{pmatrix}, \qquad -1 < t < 1 \tag{5.1}$$

and

$$\begin{pmatrix} 0 \\ t \end{pmatrix} \sim C_0 \begin{pmatrix} 0 \\ t \end{pmatrix} + \sum_{-\infty}^{\infty} C_n \begin{pmatrix} \hat{\phi}_1^{(n)}(t) \\ \hat{\phi}_2^{(n)}(t) \end{pmatrix}, \qquad -1 < t < 1 \tag{5.2}$$

Computing C_n from (4.4) and (4.6), we find that

$$\text{in (5.1)} \quad C_n = -\frac{1}{P_n^2}, \quad n = \pm 1, \pm 2, \cdots$$

and

$$\text{in (5.2)} \quad C_0 = 1 \quad \text{and} \quad C_n = 0 \quad \text{when } n \neq 0.$$

It follows that (5.1) reduces to an identity and

$$\begin{pmatrix} t \\ 0 \end{pmatrix} \sim \lim_{N \to \infty} \mathbf{S}_N(t) \tag{5.3}$$

where

$$\mathbf{S}_N(t) = -\sum_{-N}^{N} \frac{1}{P_n^2} \begin{pmatrix} \hat{\phi}_1^{(n)}(t) \\ \hat{\phi}_2^{(n)}(t) \end{pmatrix}.$$

The graph of the components of $\mathbf{S}_N(t)$ for different values of N are exhibited in Figs. 6 and 8. As in the theory of trigonometric series the introduction of Cesaro sum

$$\bar{\mathbf{S}}_N = \frac{1}{N+1}\sum_{M=1}^{N} \mathbf{S}_M \tag{5.4}$$

leads to improved convergence (see Figs. 7 and 9). In Fig. 10 we have shown how the partial sums and Cesaro sums vary at a fixed point ($t = 0.95$) as a function of the truncation number N. The oscillatory character of conditional convergence is apparent from this figure.

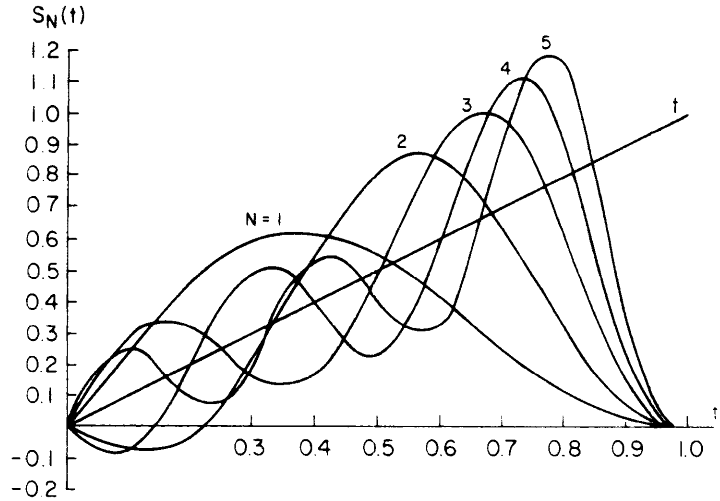

FIG. 6. *Convergence of the partial sums*

$$S_N(t) = \sum_{n=-N}^{N} \frac{-1}{P_n^2}\hat{\phi}_1^{(n)}(t)$$

of the biorthogonal series $\lim_{N\to\infty} S_N(t)$ representing the unit ramp function $f(t) = t$, $-1 \leqq t \leqq 1$

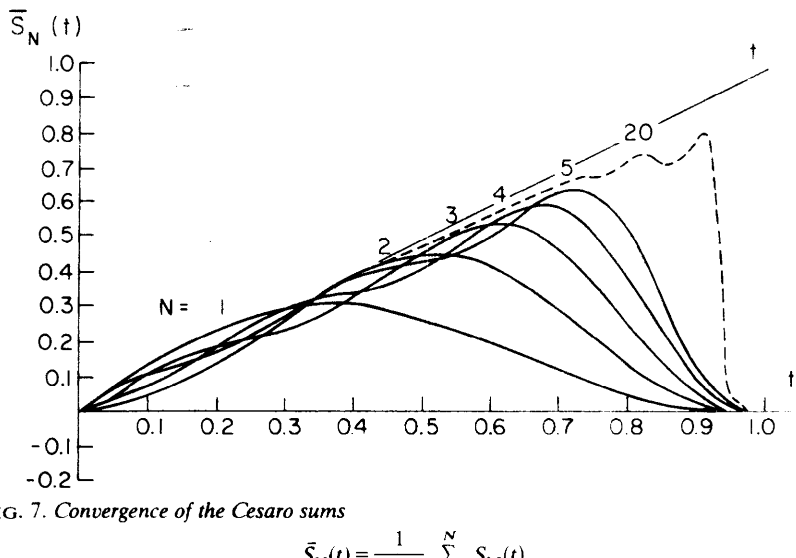

FIG. 7. *Convergence of the Cesaro sums*

$$\bar{S}_N(t) = \frac{1}{N+1}\sum_{M=1}^{N} S_M(t)$$

to the unit ramp function $f(t) = t$, $-1 \leqq t \leqq 1$

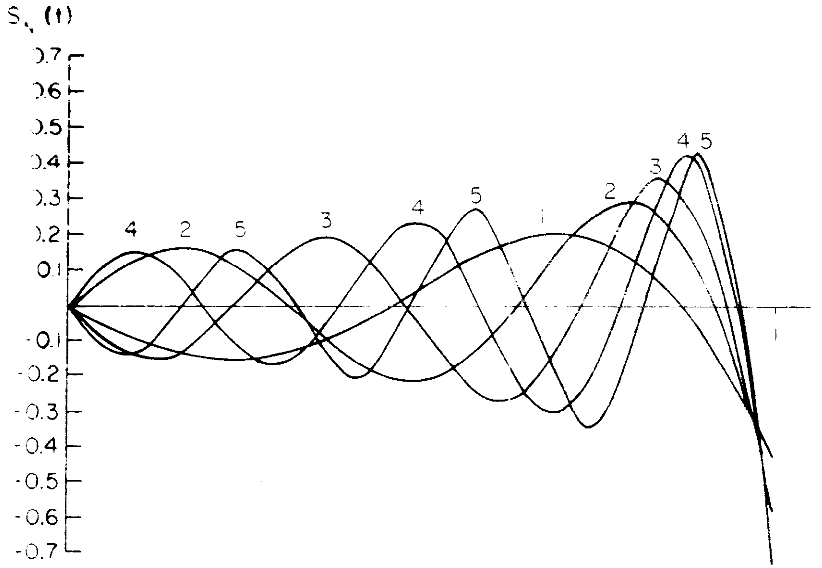

FIG. 8. *Convergence of the partial sums*

$$S_N(t)=\sum_{n=-N}^{N}\frac{-1}{P_n^2}\hat{\phi}_1^{(n)}(t)$$

of the biorthogonal series $\lim_{N\to\infty} S_N(t)$ representing the zero function $g(t)=0,\ -1\leqq t\leqq 1$

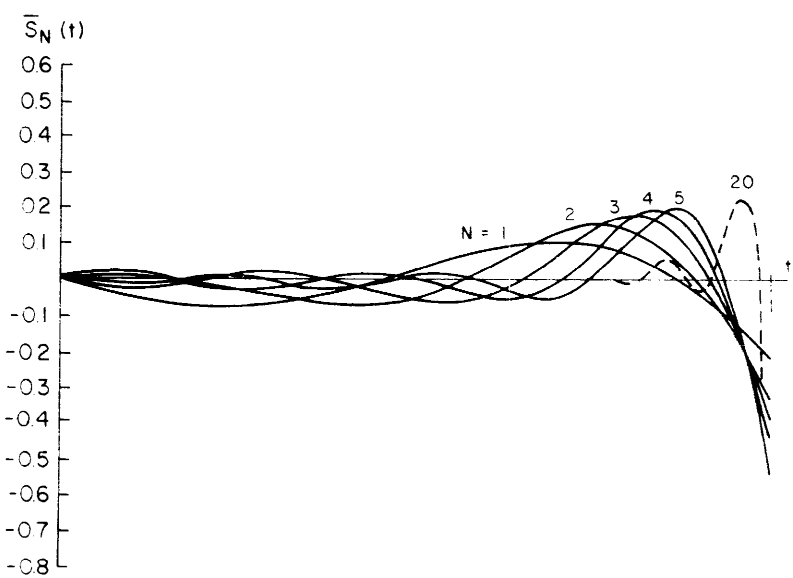

FIG. 9. *Convergence of the Cesaro sums*

$$\bar{S}_N(t)=\frac{1}{N+1}\sum_{M=1}^{N} S_M(t)$$

to the zero function $g(t)=0,\ -1\leqq t\leqq 1$

6. Stokes flow in a rectangular cavity. A two-dimensional cavity which is two units wide and H units deep is filled with liquid. The cavity is covered with a flat plate (see Fig. 11) which is drawn across the top of the cavity with a constant unit velocity in the direction of increasing t. The two-dimensional flow which is generated in the cavity can be obtained from a stream function ψ. The derivatives of ψ give the velocity components

$$(\psi_{,y}, -\psi_{,t}) = (U, V).$$

The function $\psi(t, y)$ may be obtained as the unique solution of a biharmonic problem

$$\nabla^4\psi = 0, \tag{6.1a}$$

$$\psi(\pm 1, y) = \psi_{,t}(\pm 1, y) = 0 \tag{6.1b}$$

with conditions posed on the top and bottom edge:

$$\psi_{,y}(t, 0) - 1 = \psi(t, 0) = \psi_{,y}(t, -H) = \psi(t, -H) = 0. \tag{6.2}$$

The edge conditions (6.2) cannot be put into canonical form (second, rather than first, derivatives are prescribed in the canonical form). Nevertheless, the method of biorthogonal series can be used to solve (6.1) and (6.2):[1] the series

$$\psi = \sum_{-\infty}^{\infty} \{C_n e^{S_n y} + D_n e^{-S_n(y+H)}\}\frac{\phi_1^{(n)}(t)}{S_n^2} \tag{6.3}$$

obviously satisfies (6.1a) and (6.1b). The coefficients C_n and D_n must be selected to match (6.2); that is,

$$1 = \sum_{-\infty}^{\infty} (C_n - D_n e^{-S_n H})\phi_1^{(n)}/S_n, \tag{6.4a}$$

$$0 = \sum_{-\infty}^{\infty} (C_n + D_n e^{-S_n H})\phi_1^{(n)}/S_n^2, \tag{6.4b}$$

$$0 = \sum_{-\infty}^{\infty} (C_n e^{-S_n H} - D_n)\phi_1^{(n)}/S_n^2, \tag{6.4c}$$

$$0 = \sum_{-\infty}^{\infty} (C_n e^{-S_n H} + D_n)\phi_1^{(n)}/S_n^2. \tag{6.4d}$$

[1] This problem has been studied, using numerical methods, by Burggraf (1966) for flow in a square cavity and by Pan and Acrivos (1967) for flow in deep and shallow cavities. Burggraf also studied the nonlinear problem (Reynolds numbers $R < 400$) by numerical methods; Pan and Acrivos studied the nonlinear flow in laboratory experiments ($20 \leqq R \leqq 4000$). Pan and Acrivos (p. 645) note that "As already remarked by Burggraf (1966), the form of the boundary conditions precludes an analytic solution of this system by one of the standard procedures used successfully in the field of elasticity." Burggraf (p. 133) notes that "For a rectangular boundary, the equation can be solved by elementary methods for certain types of boundary conditions common in the literature of elasticity. For the conditions of the present problem, an analytical solution in closed form is not possible by standard methods. However, the square cavity was treated by following the work of Muskhelishveli (1953), employing conformal mapping procedures. To obtain a solution, the mapping function was approximated by a finite number of terms of its infinite series, providing a polynomial approximation to the solution in the transformed variable. By comparing solutions truncated at different number of terms, it was decided that accuracy obtained by this method is worse than that of finite-difference solutions. Hence the analysis is not presented here." We have the impression that our analytic solution is as accurate or more accurate than the finite difference solutions.

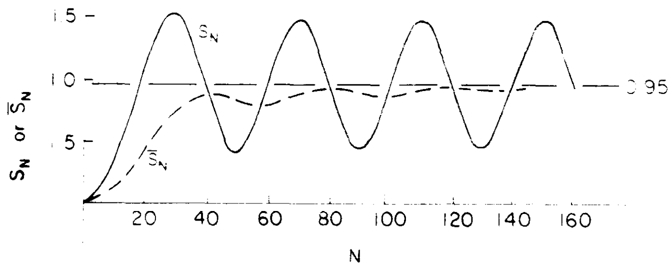

FIG. 10. *Oscillatory character of the convergence to $f(t) = t$ (at $t = 0.95$) of the partial sums*

$$S_N(t) = \sum_{n=-N}^{N} \frac{-1}{P_n^2} \hat{\phi}_1^{(n)}(t)$$

and Cesaro sums

$$\bar{S}_N(t) = \frac{1}{N+1} \sum_{M=1}^{N} S_M(t)$$

As in the canonical form of the edge problem for the biharmonic, we can determine the C_n and D_n by application of the biorthogonality condition (2.8) to the edge data (6.4). To prepare for this application we first differentiate (6.4b) and (6.4d) twice with respect to t using the relation $\phi_{1,tt}^{(n)} = S_n^2 \phi_2^{(n)}$ to eliminate $\phi_1^{(n)}$. We then write (6.4a) and the twice differentiated (6.4b) as

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \sum_{-\infty}^{\infty} (C_n + D_n e^{-S_n H}) \begin{pmatrix} \phi_1^{(n)} \\ \phi_2^{(n)} \end{pmatrix} + \sum_{-\infty}^{\infty} \left[C_n \left(\frac{1-S_n}{S_n} \right) - D_n e^{-S_n H} \left(\frac{1+S_n}{S_n} \right) \right] \begin{pmatrix} \phi_1^{(n)} \\ 0 \end{pmatrix} \tag{6.5}$$

and (6.4c) and the twice differentiated (6.4d) as

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \sum_{-\infty}^{\infty} (C_n e^{-S_n H} + D_n) \begin{pmatrix} \phi_1^{(n)} \\ \phi_2^{(n)} \end{pmatrix} + \sum_{-\infty}^{\infty} \left[C_n e^{-S_n H} \frac{1-S_n}{S_n} - D_n \frac{1+S_n}{S_n} \right] \begin{pmatrix} \phi_1^{(n)} \\ 0 \end{pmatrix}. \tag{6.6}$$

Finally, the operator

$$\int_{-1}^{1} [\psi_1^{(l)}, \psi_2^{(l)}] \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} \cdot \\ \cdot \end{pmatrix} dt$$

is applied to (6.5) and (6.6). After taking (2.8) into account, we find that

$$(C_l + D_l e^{-S_l H}) k_l + \sum_{-\infty}^{\infty} \left[C_n - D_n e^{-S_n H} \frac{1+S_n}{1-S_n} \right] A_{ln} = 4 \tag{6.7}$$

and

$$(C_l e^{-S_l H} + D_l) k_l + \sum_{-\infty}^{\infty} \left[C_n e^{-S_n H} - D_n \frac{1+S_n}{1-S_n} \right] A_{ln} = 0 \tag{6.8}$$

where

$$A_{ln} = \frac{1-S_n}{S_n} \int_{-1}^{1} \psi_2^{(l)} \phi_1^{(n)} \, dt$$

and

$$4 = \int_{-1}^{1} \psi_2^{(l)} \, dt.$$

Equations (6.7) and (6.8) form an infinite set to be solved for the coefficients C_n and D_n, $n = \pm 1, \pm 2, \cdots$. We solved this system of equations by truncation and checked the convergence of the solution of the truncated equations numerically. For $N = 20$ the solution matches the edge data quite well at the top and very well at the bottom of the cavity (see Table 1). The solution in the interior of the cavity consists of a series of edge eddies; level lines of the stream function are plotted in Fig. 12. In Fig. 13 we have enlarged the corner region so as to emphasize the existence of corner eddies of the type discussed by Moffatt (1964), and more completely by Liu and Joseph (1977). These features of the solution are also apparent from graphs computed numerically by finite differences by Pan and Acrivos (1967).

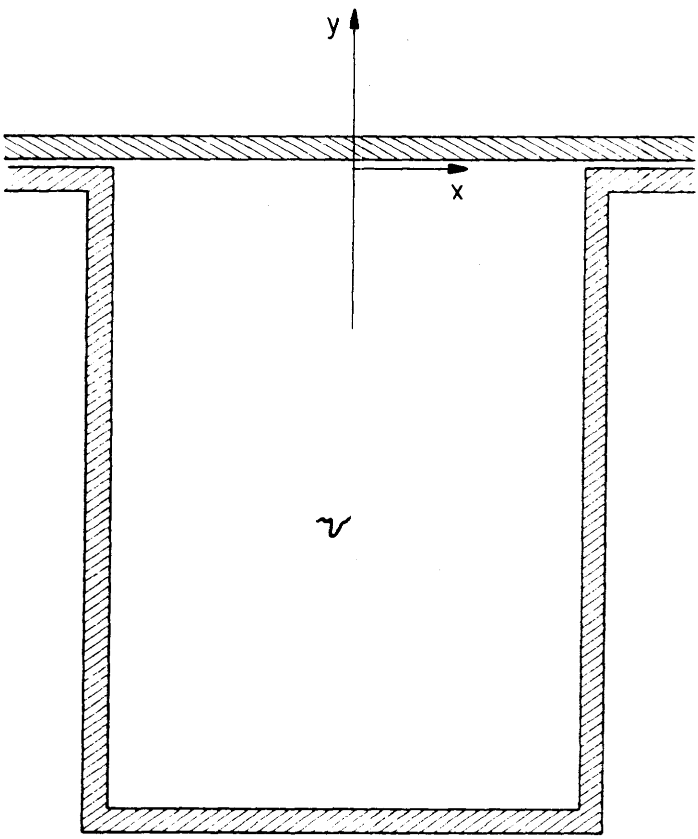

FIG. 11. *Fluid fills the region $\mathcal{V}$ and is set into motion by the steady motion of a covering plate or belt.*

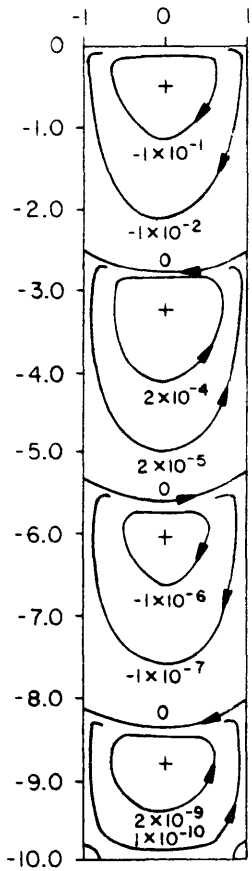

FIG. 12. *Level lines of the streamlines* (6.3) *of the flow in a cavity with a depth/width ratio of five. This figure corresponds to Fig.* 2.(c) *of Pan and Acrivos* (1967).

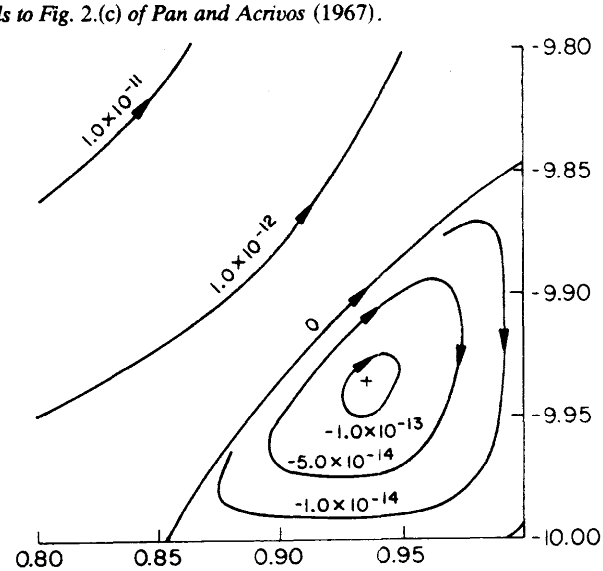

FIG. 13. *Enlarged view of the corner at* $(t, y) = (1, -H)$ *showing the first corner eddy*

TABLE 1

Convergence of the Papkovitch-Fadle series

$$\begin{pmatrix} \psi(t,0;N) \\ \psi(t,-H;N) \end{pmatrix} = \sum_{n=-N}^{N} \frac{\phi_1^{(n)}}{S_n^2} \begin{pmatrix} C_n + D_n e^{-S_n H} \\ C_n e^{-S_n H} + D_n \end{pmatrix} \sim \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \psi_{,y}(t,0;N) \\ \psi_{,y}(t,-H;N) \end{pmatrix} = \sum_{n=-N}^{N} \frac{\phi_1^{(n)}}{S_n} \begin{pmatrix} C_n - D_n e^{-S_n H} \\ C_n e^{-S_n H} - D_n \end{pmatrix} \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{for } N=20 \text{ and } H=10$$

t	$\psi(t,0;20)$	$\psi(t,-10;20)$	$\psi_{,y}(t,0;20)$	$\psi_{,y}(t,-10;20)$
0	1.704×10^{-4}	-1.100×10^{-16}	1.081	3.807×10^{-9}
0.05	-1.656×10^{-4}	1.058×10^{-16}	0.920	3.787×10^{-9}
0.10	1.512×10^{-4}	-9.289×10^{-17}	1.076	3.725×10^{-9}
0.15	-1.278×10^{-4}	7.221×10^{-17}	0.932	3.625×10^{-9}
0.20	9.607×10^{-5}	-4.435×10^{-17}	1.058	3.486×10^{-9}
0.25	-5.736×10^{-5}	1.101×10^{-17}	0.956	3.312×10^{-9}
0.30	1.335×10^{-5}	2.622×10^{-17}	1.028	3.106×10^{-9}
0.35	3.366×10^{-5}	-6.438×10^{-17}	0.991	2.872×10^{-9}
0.40	-8.068×10^{-5}	1.004×10^{-16}	0.988	2.614×10^{-9}
0.45	1.239×10^{-4}	1.297×10^{-16}	1.035	2.339×10^{-9}
0.50	-1.587×10^{-4}	1.473×10^{-16}	0.941	2.051×10^{-9}
0.55	1.795×10^{-4}	-1.466×10^{-16}	1.081	1.756×10^{-9}
0.60	-1.812×10^{-4}	1.212×10^{-16}	0.899	1.462×10^{-9}
0.65	1.545×10^{-4}	-6.481×10^{-17}	1.111	1.176×10^{-9}
0.70	-9.799×10^{-5}	-2.504×10^{-17}	0.893	9.046×10^{-9}
0.75	1.073×10^{-5}	1.422×10^{-16}	1.079	6.555×10^{-10}
0.80	9.631×10^{-5}	-2.595×10^{-16}	0.987	4.366×10^{-10}
0.85	-1.910×10^{-4}	3.083×10^{-16}	0.896	2.544×10^{-10}
0.90	2.082×10^{-4}	-1.580×10^{-16}	1.264	1.170×10^{-10}
0.95	-8.245×10^{-5}	-2.615×10^{-16}	0.673	3.059×10^{-11}
1.00	8.737×10^{-15}	2.547×10^{-23}	0.000	7.863×10^{-23}

Note added in proof. The factor $1/(N+1)$ in the definition of the Cesaro sums is incorrect and should be replaced by $1/N$. The graphs of $\bar{S}_N(t)$ (Figs. 2, 4, 5, 7, 9, 11) were drawn using the wrong definition. A correct graph can be obtained in each case by multiplying the given graph of $\bar{S}_N(t)$ by the factor $(N+1)/N$. The corrected graphs are then in even better agreement with the given data. Of course, theorems about the convergence of Cesaro sums of biorthogonal series have yet to be given.

REFERENCES

O. R. BURGGRAF (1966), *Analytical and numerical studies of the structure of steady separated flows*, J. Fluid Mech., 24, pp. 113–151.

G. H. HARDY (1902), *On the zeros of the integral function* $x - \sin x = \sum_{n=1}^{\infty} (-1)^{n-1} x^{2n+1}/(2n+1)!$, Messenger of Math., 31, pp. 161–165.

D. D. JOSEPH (1977), *The convergence of biorthogonal series for biharmonic and Stokes flow edge problems. Part I*, this Journal, 33, pp. 337–347.

D. D. JOSEPH AND L. STURGES (1975), *The free surface on a liquid filling a trench heated from its side*, J. Fluid Mech., 69, pp. 565–589.

F. LIU AND D. D. JOSEPH (1977), *Stokes flow in trenches wedge shaped*, Ibid., 80, pp. 443–464.

H. K. MOFFATT (1964), *Viscous and resistive eddies near a sharp corner*, Ibid., 18, pp. 1–18.

N. I. MUSKHELISHVELI (1953), *Some Basic Problems of the Mathematical Theory of Elasticity*, P. Noordhoff, Groningen, the Netherlands.

F. PAN AND A. ACRIVOS (1967), *Steady flows in rectangular cavities*, J. Fluid Mech., 28, pp. 643–655.

R. C. T. SMITH (1952), *The bending of a semi-infinite strip*, Austral. J. Sci. Res., 5, pp. 227–237.