G.M. de La Penha, L.A. Medeiros (eds.)
Contemporary Developments in Continuum Mechanics
and Partial Differential Equations

CONSTITUTIVE EQUATIONS AND FREE SURFACES

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Abstract

The general theory of perturbations of rigid body motions of simple fluids with applications to free surface problems is discussed. The general theory is utilized to explain phenomena exhibited in the movie "Novel Weissenberg Effects" by G.S. Beavers and D.D. Joseph.

Introduction.

The ideas behind the simultaneous perturbation of the domain and the constitutive equation may be explained without the extra, and largely incidental, mathematical complications which follow from analysis of the nasty equations which govern rheologically complicated materials. In a simpler setting, we shall first consider a model problem in which we show that the distortion of a free surface due to motion may be expressed in terms of unknown rheological constants. In the model problem we perturb the "state of rest". A rich theory of perturbations of the "rest state" of real viscoelastic fluids following along lines laid out in this lecture has already been given by Joseph and Beavers

(1977). Many of the simplifying features of the theory of perturbations of the rest state of a viscoelastic fluid are also present in the yet richer theory of perturbations of states of rigid motion of a viscoelastic fluid. In fact, states of rigid motions of a viscoelastic fluid are generalized rest states in the sense that the Cauchy strain $\underline{G}(s)$ and the extra stress $\mathfrak{J}$ both vanish on rigid motions. Moreover, in the theory given here, and by Joseph (1977), the same material functions suffice to describe the motions which perturb all states of steady rigid rotation including the rest state in which there is no rotation.

It is therefore possible to separate the problem of free surfaces on viscoelastic fluids which are close to steady rigid motions into two parts. In the first part (§1) we consider the model problem in which the ideas involved in the simultaneous perturbation of the domain and constitutive equation are exposed in a simple context. In the second part, (§2-7) we perturb states of steady rigid rotation of real viscoelastic fluids, but leave the domain fixed. When the domain is not fixed the canonical forms of the stress (§6) and the equations of motion (§7) are unchanged and the equations which hold on the free boundary may be easily derived by the methods used in the model problem (§7).

1. A model problem for domain perturbations of the rest state.

We suppose that there is an analytic function $\mathfrak{J}(\phi)$ which is not known except that in a neighborhood of $\phi = 0$ it has a Taylor series

$$\mathfrak{F}(\phi) = \frac{1}{2} a\phi^2 + \frac{1}{3!} b\phi^3 + \dots , \tag{1.1}$$

where $a = \mathfrak{F}_{\phi\phi}(0)$ and $b = \mathfrak{F}_{\phi\phi\phi}(0)$. We may think that $\mathfrak{F}(\phi)$ is representative of the nonlinear part of the stress in some fictitious material. Our idea is to find the Taylor coefficients for (1.1) by measuring the free surface induced by a dynamical process

$$F(\underline{x}) = \nabla^2\phi + \mathfrak{F}(\phi) = 0 \quad \text{in } \mathcal{V}_\delta , \tag{1.2}$$

subject to the condition that

$$G(\underline{x},\varepsilon) = \phi(\underline{x}) - g(\underline{x},\varepsilon) = 0 \quad \text{on } \partial\mathcal{V}_\delta , \tag{1.3}$$

where $\mathcal{V}_\delta$ is a bounded region of space depending on a parameter δ and $g(\underline{x},\varepsilon)$ are given data depending on a parameter ε.

It is instructive to carry out our analysis in easy stages. First we perturb ε, leaving δ fixed and assume that

$$\phi(\underline{x},\varepsilon) = \sum_0 \frac{\varepsilon^l}{l!} \phi_l(\underline{x}), \tag{1.4}$$

where, in $\mathcal{V}_\delta$,

$$\begin{aligned} &\nabla^2\phi_0 + \mathfrak{F}(\phi_0) = 0, \\ &\nabla^2\phi_1 + \mathfrak{F}_\phi(\phi_0)\phi_1 = 0, \\ &\nabla^2\phi_2 + \mathfrak{F}_\phi(\phi_0)\phi_2 + \mathfrak{F}_{\phi\phi}(\phi_0)\phi_1^2 = 0 \end{aligned} \tag{1.5}$$

and, on $\partial\mathcal{V}_\delta$;

$$G_n = \phi_n(\underline{x}) - g_n(\underline{x}) = 0, \tag{1.6}$$

where $g_n(\underline{x})$ is n^{th} derivative of $g(\underline{x},\varepsilon)$ at $\varepsilon = 0$. If we could solve (1.5) and (1.6) for ϕ_0 we could find ϕ_1, ϕ_2, etc. as the solution of linear boundary value problems. It would be hard to solve $(1.5)_1$, even if we knew $\mathfrak{F}(\phi_o)$, because it is nonlinear. But we cannot solve $(1.5)_1$ at all because we don't know $\mathfrak{F}(\phi_o)$ except as a power series in a small neighborhood of zero.

The "rest state" is now defined as the state for which $g_o(\underline{x})|_{\partial \mathcal{V}_\delta} = 0$. We couple this definition with the assumption that $(1.5)_1$ has no solutions $\phi_o \neq 0$ when $\phi_o(\underline{x}) \equiv 0$ $\phi_o(\underline{x})|_{\partial \mathcal{V}_\delta} = 0$. Then $\phi_o(\underline{x}) \equiv 0$ in $\mathcal{V}_\delta$ and, replacing $(1.5)_{2,3}$ we get

$$\nabla^2 \phi_1|_{\mathcal{V}_\delta} = 0, \quad \phi_1|_{\partial \mathcal{V}_\delta} = g_1;$$
$$\nabla^2 \phi_2 + a\phi_1^2|_{\mathcal{V}_\delta} = 0, \quad \phi_2|_{\partial \mathcal{V}_\delta} = g_2 \qquad (1.7)$$

etc. These problems are linear and easy to solve even when a is unknown (in fact, $\phi_2 = \phi_{21} + a\phi_{22}$, where ϕ_{21} and ϕ_{22} are independent of a). So we can use the solution which perturbs the rest state to find a and, with more work, we can get expressions which involve the other Taylor coefficients of (1.1) as well.

Now suppose that $\mathfrak{c}$ is fixed and δ varies. And suppose further that $\mathcal{V}_o$ is some convenient domain whose point of convenience is, say, that $\mathcal{V}_o$ has a high degree of symmetry. We are going to try to solve the dynamic problem in $\mathcal{V}_\delta$ as a series whose coefficients can be determined from boundary value problems posed on the symmetric domain $\mathcal{V}_o$.

First we map $\mathcal{V}_\delta$ into $\mathcal{V}_o$ with an invertible one to one mapping, which is analytic in δ and which takes points on $\partial \mathcal{V}_\delta$ into $\partial \mathcal{V}_o$:

$$\underline{x} = \underline{x}(\underline{x}_o, \delta) = \Sigma \frac{\delta^n}{n!} \underline{x}^{(n)}(\underline{x}_o) \quad \text{(analytic in } \delta\text{)},$$

$$\underline{x} = \underline{x}(\underline{x}_o,0) \quad \text{(identity)},$$
$$\underline{x}_o = \underline{x}^{-1}(\underline{x},\delta) \quad \text{(inverse)},$$
$$\partial \mathcal{V}_\delta \leftrightarrow \partial \mathcal{V}_o . \tag{1.8}$$

Let $H(\underline{x},\delta)$ be any function defined in the family of domains $\mathcal{V}_\delta$ and introduce the notation:

$$\tilde{H}(\delta) = H(\underline{x}(\underline{x}_o,\delta),\delta),$$
$$H^{[n]}(\underline{x}_o) = \frac{d^n}{d\delta^n}\tilde{H}(0)$$

and

$$H^{\langle n\rangle}(\underline{x}_o) = \frac{\partial^n H}{\partial\delta^n}(\underline{x},\delta)\Big|_{\substack{\delta=0\\ \underline{x}=\underline{x}_o}} .$$

Connection formulas (the chain rule) connect $H^{[n]}$ and $H^{\langle n\rangle}$:

$$H^{[0]}(\underline{x}_o) = H^{\langle 0\rangle}(\underline{x}_o) = H(\underline{x}_o,0),$$
$$H^{[1]}(\underline{x}_o) = H^{\langle 1\rangle}(\underline{x}_o) + \underline{x}^{[1]}\cdot\nabla H^{\langle 0\rangle},$$
$$H^{[2]}(\underline{x}_o) = H^{\langle 2\rangle}(\underline{x}_o) + 2\underline{x}^{[1]}\cdot\nabla H^{\langle 1\rangle} + \underline{x}^{[2]}\cdot\nabla H^{\langle 0\rangle} + (\underline{x}^{[1]}\cdot\nabla)^2 H^{\langle 0\rangle}, \tag{1.9}$$

etc. It follows from the equations that

$$\tilde{F}(\delta) = F(\underline{x}(\underline{x}_o,\delta),\delta) = 0,$$

when $\underline{x} \in \mathcal{V}_\delta$, $\underline{x}_o \in \mathcal{V}_o$, and that

$$\tilde{F}^{[n]}(0) = F^{[n]}(\underline{x}_o) = 0 \quad \text{in } \mathcal{V}_o . \tag{1.10}$$

We may easily establish by mathematical induction, using the the connection formulas, that

$$F^{\langle n\rangle}(\underline{x}_o) = 0 \quad \text{in } \mathcal{V}_o . \tag{1.11}$$

For example, $F^{\langle 0\rangle}(\underline{x}_o) = 0$ in $\mathcal{V}_o$ and $F^{[1]}(\underline{x}_o) = F^{\langle 1\rangle}(\underline{x}_o) + \underline{x}^{[1]}\cdot\nabla F^{\langle 0\rangle}(\underline{x}_o) = F^{\langle 1\rangle}(\underline{x}_o) = 0$. It is a bit more complicated at the boundary. Since $\tilde{G}(\delta) = 0$ on $\partial\mathcal{V}_\delta$, $G^{[m]}(\underline{x}_o) = 0$ on $\partial\mathcal{V}_o$ and tangential derivatives of $G^{[m]}$ on $\mathcal{V}_o$ must vanish but normal derivatives need not vanish. It follows that on $\partial\mathcal{V}_o$

$$G^{[m]}(\underline{x}_o) = \{(\frac{\partial}{\partial\varepsilon} + v_n \frac{\partial}{\partial n})^m G(\underline{x},\varepsilon)\}_{\substack{\varepsilon=0 \\ \underline{x}=\underline{x}_o}} = 0, \qquad (1.12)$$

where $\underline{n}$ is the outward normal to $\mathcal{V}_\delta$, $v_n = \underline{x}^{[1]}\cdot\underline{n}$ and $\underline{n}\cdot\nabla = \partial/\partial n$.

We may seek the solution of (1.2) and (1.3) in $\mathcal{V}_\delta$ as a series defined in $\mathcal{V}_o$

$$\phi(\underline{x}(\underline{x}_o,\delta),\delta) = \sum_0 \frac{\delta^n}{n!}\phi^{[n]}(\underline{x}_o), \qquad (1.13)$$

where, in $\mathcal{V}_o$,

$$\begin{aligned}
&\nabla^2\phi^{\langle 0\rangle} + \mathfrak{F}(\phi^{\langle 0\rangle}) = 0,\\
&\nabla^2\phi^{\langle 1\rangle} + \mathfrak{F}_\phi(\phi^{\langle 0\rangle})\phi^{\langle 1\rangle} = 0,\\
&\nabla^2\phi^{\langle 2\rangle} + \mathfrak{F}_\phi(\phi^{\langle 0\rangle})\phi^{\langle 2\rangle} + \mathfrak{F}_{\phi\phi}(\phi^{\langle 0\rangle})\phi^{\langle 1\rangle 2} = 0,
\end{aligned} \qquad (1.14)$$

etc., and on $\partial\mathcal{V}_o$

$$G^{[n]} = \phi^{[n]}(x_o) - g^{[n]}(x_o). \qquad (1.15)$$

The problems (1.14), (1.15) are like (1.5) and (1.6). We can't solve them because $\mathfrak{F}(\phi)$ is given only as a Taylor series (1.1) with unknown coefficients and an unknown circle of convergence.

The "rest state" for the domain perturbation, like the "rest state" for the perturbation of the boundary data may be defined by the condition $g^{[0]}(\underline{x}_o)|_{\partial\mathcal{V}_o} = 0$. This condition implies that $\phi^{[0]} = \phi^{\langle 0\rangle} = 0$ on $\partial\mathcal{V}_o$, hence, $\phi^{\langle 0\rangle} \equiv 0$ in $\mathcal{V}_o$ and

$$\nabla^2\phi^{\langle 1\rangle}\Big|_{\mathcal{V}_o} = 0, \quad \phi^{\langle 1\rangle}\Big|_{\partial\mathcal{V}_o} = g^{[1]}(\underline{x}_o), \tag{1.16}$$

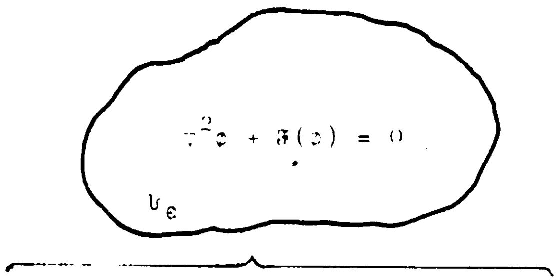
$$\nabla^2\phi^{\langle 2\rangle} + a\phi^{\langle 1\rangle^2}\Big|_{\mathcal{V}_o} = 0,$$

$$\phi^{[2]}\Big|_{\partial\mathcal{V}_o} = \phi^{\langle 2\rangle} + 2V_n\frac{\partial}{\partial n}\phi^{\langle 1\rangle} = g^{[2]}(\underline{x}_o), \tag{1.17}$$

etc. The linear problems (1.16), (1.17) and higher order problems are solvable and not too hard to actually solve, even when a is unknown.

In our rheological problems the boundary data (ϵ in our first example) perturb the boundary (δ in our second example). So we may put $\epsilon = \delta$ and construct a simple example of a domain perturbation of the rest state with a free surface. By a free surface we understand that there is a one (ϵ) parameter family of domains $\mathcal{V}_\epsilon$ which are unknown. Supposing now that our dynamical process (1.1) and (1.2) hold in $\mathcal{V}_\epsilon$ we might expect solutions in each and every $\mathcal{V}_\epsilon$ corresponding to some possibly small ϵ interval of the origin. But no, this will not be possible because in addition to (1.2) we pose an additional boundary condition, which is analogous to, but much simpler than, the condition that the jump in the normal component of stress is balanced by surface tension times mean curvature. Because we have this extra condition we can't solve (1.1) and (1.2) in every $\mathcal{V}_\epsilon$; the extra condition can be satisfied only when $\mathcal{V}_\epsilon$ is properly chosen.

As an example of the foregoing consider the two dimensional problem specified in polar coordinates (r,θ) in figure 1 where the boundary data $g(\theta,\epsilon)$ are given and correspond to a rest state $g(\theta,0) = g^{\langle 0\rangle}(\theta) = 0$. The dynamical process $\phi(r,\theta,\epsilon)$ and the function $f(\theta,\epsilon)$ which gives the boundary $r = 1 + f(\theta,\varepsilon)$ of $\mathcal{V}_\varepsilon$ are unknown. We remind the reader that our aim is to show how to find $\mathfrak{F}(\phi)$; that is, the Taylor coefficients in (1.1) by (fictitious) experimental measurements of the (made up) boundary $r = 1 + f(\theta,\varepsilon)$.



$$G_1(r,\theta,\varepsilon) = \phi(r,\theta,\varepsilon) - g(\theta,\varepsilon) = 0$$
$$G_2(r,\theta,\varepsilon) = \mathfrak{F}(\phi) - f(\theta,\varepsilon) = 0 \qquad (1.18)$$
on $r = 1 + f(\theta,\varepsilon)$.

Fig. 1. Model of a free surface problem

First we solve (1.18) when $\varepsilon = 0$ and we find that $\nabla^2\phi + \mathfrak{F}(\phi) = 0$ in $\mathcal{V}_0$ with $\phi(r,\theta,0) = 0$ on $r = 1 + f(\theta,0)$ has $\phi \equiv 0$ in $\mathcal{V}_0$. Then, since $\mathfrak{F}(0) = 0$, $f(\theta,0) = 0$, so that the reference configuration $\mathcal{V}_0$ is the unit circle $r_0 = 1$. We seek the solution of (1.18) in powers of ε.

$$\begin{pmatrix} \phi(r,\theta,\varepsilon) \\ f(\theta,\varepsilon) \end{pmatrix} = \sum_0 \frac{\varepsilon^n}{n!} \begin{pmatrix} \phi^{[n]}(r_0,\theta_0) \\ f^{[n]}(\theta_0) \end{pmatrix}$$

where $f^{[0]}(\theta_0) = \phi^{[0]}(r_0,\theta_0) = 0$ and $\mathcal{V}_\varepsilon$ and $\mathcal{V}_0$ are related by a shifting map

$$\theta = \theta_0$$
$$r = r_0(1 + f(\theta_0,\varepsilon)),$$

having all the properties required of (1.8). For the shifting map the deformation of $\mathcal{V}_o$ is along rays and

$$\underset{\sim}{x}^{[n]} = \underset{\sim}{e}_r r^{[n]} = \underset{\sim}{e}_r r_o f^{[n]}(\theta_o).$$

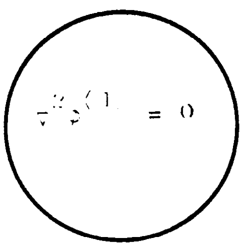
Note that for any function $f(\theta)$ of $\theta = \theta_o$ alone, we have $f^{\langle n\rangle}(\theta) \equiv f^{[n]}(\theta)$. Using the connection formulas (1.9) we find that

$$\phi^{[1]}(r_o,\theta_o) = \phi^{\langle 1\rangle}(r_o,\theta_o)$$

and

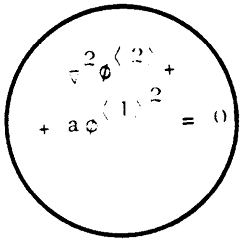
$$\phi^{[2]}(r_o,\theta_o) = \phi^{\langle 2\rangle}(r_o,\theta_o) + 2r_o f^{[1]}(\theta_o)\,\frac{\partial\phi^{\langle 1\rangle}}{\partial r_o}(r_o,\theta_o).$$

On the boundary of $\mathcal{V}_o, r_o = 1$ we have, from $(1.18)_3$ that $\mathfrak{F}_\phi(0)\phi^{\langle 1\rangle}(1,\theta_o) - f^{[1]}(\theta_o) = 0$. Since $\mathfrak{F}_\phi(0) = 0$, we find that $f^{[1]}(\theta_o) = 0$. The boundary value problems satisfied by $\phi^{\langle 1\rangle}(r_o,\theta_o)$ and $\phi^{\langle 2\rangle}(r_o,\theta_o)$ are given in figures 2 and 3.



$$\phi^{\langle 1\rangle}(1,\theta_o) = g^{[1]}(\theta_o)$$

Fig. 2. The problem satisfied by $\phi^{\langle 1\rangle}(r_o,\theta_o)$



$$\phi^{\langle 2\rangle}(1,\theta_o) = g^{[2]}(\theta_o)$$

$$\mathfrak{F}_{\phi\phi}(0)\phi^{[1]^2} = a\phi^{\langle 1\rangle^2} = f^{[2]}(\theta_o)$$

Fig. 3. The problem satisfied by $\phi^{\langle 2\rangle}(r_o,\theta_o)$

These problems are easy to solve. We find from figure 3 that

$$f^{[2]}(\theta_o) = a\phi^{\langle 1\rangle 2}(1,\theta_o),$$

where $\phi^{\langle 1\rangle}(r_o,\theta_o)$ is the solution of the problem shown in figure 2. It follows that the first approximation to the shape of $\mathcal{V}_\epsilon$ is given by

$$r = 1 + f(\theta,\epsilon) = 1 + a\phi^{\langle 1\rangle 2}(1,\theta)\epsilon^2 + O(\epsilon^3).$$

The next approximation depends on b as well as a. We may therefore deduce the values of derivatives of $\mathfrak{F}(\phi)$ at $\phi = 0$ by monitoring the changes in the shape of $\mathcal{V}_\epsilon$ when ϵ is near to zero.

2. Constitutive expressions which perturb the rest state of a simple fluid.

In our rheological studies the role of the nonlinear function $\underline{\mathfrak{F}}(\phi)$ is assumed by the nonlinear functional

$$\underline{\mathfrak{F}}[\underset{s=0}{\overset{\infty}{\underline{G}}}(s)] = \underline{T} + p\underline{1} \qquad (2.1)$$

giving the constitutively determined part of the stress $\underline{T}$ of an incompressible fluid. The argument functions in the domain of $\underline{\mathfrak{F}}$ are symmetric tensor valued functions of $s = t - \tau$ for fixed t and $\underline{x}$, called histories,

$$\underline{G}(s) = \underline{F}_t^T(\tau)\cdot\underline{F}_t(\tau) - \underline{1}, \quad \underline{G}(0) = \underline{0},$$

which are defined on the relative deformation tensor $\underline{F}_t(\tau) = \nabla_x\chi_t(\underline{x},\tau)$ where $\chi_t(\underline{x},\tau) = \xi(\underline{X},\tau)$ is the position vector of the particle $\underline{X}$ which is presently at $\underline{x}$. Eq. (2.1), which assumes that the stress depends only on the first spatial gradient of the deformation, is still too general to be used to solve the problems which lead to understanding how visco-elastic fluids respond to applied forces.

It is well known that the extra stress vanishes on the zero history $\underline{\mathfrak{F}}(0) = 0$. It is then natural to seek expansions of $\underline{\mathfrak{F}}(\underline{G}(s))$ in terms of functional expansions on the rest history

$$\underline{\mathfrak{F}}[\underline{G}(s)] = \underline{\mathfrak{F}}[0;\underline{G}(s)] + \underline{\mathfrak{F}}_1[0;\underline{G}(s),\underline{G}(s)] + \underline{\mathfrak{F}}_2[0;\underline{G}(s),\underline{G}(s),\underline{G}(s)] + \dots, \tag{2.2}$$

where

$$\underline{\mathfrak{F}}_n[0;\underline{G}_1(s),\underline{G}_2(s),\dots,\underline{G}_n(s)]$$

is an n linear form. We shall assume (Green and Rivlin (1957), Coleman-Noll (1961), Pipkin (1964) that these n-linear forms may be represented by iterated integrals of the form

$$\int_0^\infty \cdots \int_0^\infty K_{ijk\ell\dots mv}(s_1,s_2,\dots,s_n) \cdot G_{k\ell}(s_1)\dots G_{mv}(s_n)ds_1 ds_2 \dots ds_n.$$

It is very easy to find the isotropic forms of these integrals from invariance theory for a single tensor $\underline{G}(s)$ (see Exercise 94.7 in Joseph (1976)). If we introduce the notation $\mathfrak{F}^{(n)}(\underline{G}(s))$ for the partial sum of (2.2) after n terms it is easy to show that

$$\underline{\mathfrak{F}}^{(1)} = \int_0^\infty \zeta(s)\underline{G}(s)ds, \tag{2.3}$$

$$\underline{\mathfrak{F}}^{(2)} = \underline{\mathfrak{F}}^{(1)} + \int_0^\infty\int_0^\infty \{\beta(s_1,s_2)\underline{G}(s_1)\cdot\underline{G}(s_2) + \alpha(s_1,s_2)\,[\mathrm{tr}\,\underline{G}(s_1)]\underline{G}(s_2)\}\,ds_1 ds_2, \tag{2.4}$$

where $\beta(s_1,s_2) = \beta(s_2,s_1)$,

$$\mathfrak{F}^{(n)} = \mathfrak{F}^{(n-1)} + \int_0^\infty \cdots \int_0^\infty (n \text{ linear forms in } \underline{G}(s_i))ds_1 \ldots ds_n \quad (2.5)$$

So we may expect that the stress in any motion close to one in which $\underline{G}(s) = 0$ can be expressed in terms of the Fréchet stresses $\mathfrak{F}^{(n)}$.

But the stresses $\mathfrak{F}^{(n)}$ are not in the form which I call canonical for perturbation of the zero history. The canonical forms are the forms which the stress and equations of motion take when data giving rise to steady rotation are perturbed. We do not allow kinematics which are inconsistent with the equations of motion. For example, though it is well-known and easy to demonstrate that on all rigid body motions $\underline{G}(s) = 0$, only the steady rigid motions are compatible with the equations. Relative to a coordinate system translating with the translational velocity of the rigid body we may assume that body also rotates rigidly with velocity $\underline{\Omega} \wedge \underline{x}$. Then $\underline{G} = 0$, $\mathfrak{F}[0] = 0$ and the equations of motion

$$\rho\overset{\circ}{\underline{\Omega}} \wedge \underline{x} = -\nabla(p + \frac{1}{2}\rho|\underline{\Omega} \wedge \underline{x}|^2)$$

are solvable if and only if $\overset{\circ}{\underline{\Omega}} \wedge \underline{x}$ is a gradient; that is, if $\overset{\circ}{\underline{\Omega}} = 0$. So we are restricted to perturbations of rigid rotations with constant $\underline{\Omega}$.

To find the canonical forms, the forms which the stress and the equations of motion take when data giving rise to steady rigid rotation are perturbed, it will suffice to imagine that for $\underline{x} \in \partial\mathcal{V}(t)$, where $\mathcal{V}(t)$ is the region occupied by fluid, the prescribed boundary velocity

$$U(\underline{x},t,\epsilon) = \underline{\Omega} \wedge \underline{x} + \epsilon f(\underline{x},t), \quad \forall t \epsilon R,$$

is a steady rigid rotation plus an arbitrary part proportional to ϵ. Now we suppose that the solutions of all the governing equations depend on ϵ through the prescribed data and that they may be differentiated a certain number of times at $\epsilon = 0$. In the best case we would have analytic solutions and convergent power series in ϵ. In less good conditions we suppose that some low order partial sums are asymptotic to true solutions. In either event we must identify the boundary value problems which govern the derivatives of the solution at $\epsilon = 0$ and we call the stress and equations for these derivatives canonical; canonical in the sense that the derivatives are independent of ϵ. This natural method of doing perturbations requires that we consider only those forms of the stress which are compatible with the solutions of the equations, so after all is done we get a good theory with which we can actually compute solutions to problems. For example, we have already noted the only rigid body rotations compatible with the equations of motion are steady.

The canonical forms of the stress and the equations of motion are easiest to understand by actually deriving them. We shall find that at each stage of the perturbation we shall need to solve four linear partial differential equations for three components of velocity and a reaction pressure, as in the Navier-Stokes equations. The strain history comes in as an after-thought after the velocities are computed.

3. Kinematics for perturbations.

Starting generally, we first relate

$$\underline{\chi}_t(\underline{x},\tau,\epsilon) = \underline{\chi}_t^{\langle 0\rangle}(\underline{x},\tau) + \epsilon\underline{\chi}_t^{\langle 1\rangle}(\underline{x},\tau) + \epsilon^2\,\underline{\chi}_t^{\langle 2\rangle}(\underline{x},\tau) + \ldots \tag{3.1}$$

and

$$\underline{U}(\underline{\chi}_t(\underline{x},\tau,\epsilon),\tau,\epsilon) \equiv \underline{\tilde{U}}(\underline{x},\tau,\epsilon) = \underline{\tilde{U}}^{\langle 0\rangle}(\underline{x},\tau) + \epsilon\underline{\tilde{U}}^{\langle 1\rangle}(\underline{x},\tau) + \epsilon^2\,\underline{\tilde{U}}^{\langle 2\rangle}(\underline{x},\tau) + \ldots\,. \tag{3.2}$$

Since

$$\frac{\partial\underline{\chi}_t}{\partial t}(\underline{x},\tau,\epsilon) = \underline{\tilde{U}}(\underline{x},\tau,\epsilon) = \underline{U}(\underline{\chi}_t(\underline{x},\tau,\epsilon),\tau,\epsilon), \tag{3.3}$$

we have, assuming the particle label

$$\underline{x} = \underline{\chi}_t(\underline{x},t,\epsilon) \tag{3.4}$$

is independent of ϵ, that for $n = 0,1,2,\ldots,$

$$\frac{\partial\underline{\chi}_t^{\langle n\rangle}}{\partial\tau}(\underline{x},\tau) = \underline{\tilde{U}}^{\langle n\rangle}(\underline{x},\tau). \tag{3.5}$$

Moreover, using (3.1), (3.4) and (3.2), we find that

$$\underline{\chi}_t^{\langle 0\rangle}(\underline{x},t) = \underline{x}, \tag{3.6}$$

$$\underline{\chi}_t^{\langle n\rangle}(\underline{x},t) = 0 \tag{3.7}$$

and

$$\underline{\tilde{U}}^{\langle n\rangle}(\underline{x},t) = \underline{U}^{\langle n\rangle}(\underline{x},t).$$

The function $\underline{\tilde{U}}(\underline{x},\tau)$ is an auxiliary function used to facilitate our computation of particle paths.

To simplify notations, we define

$$\underline{\chi}^{\langle n\rangle}(\tau) \equiv \underline{\chi}_t^{\langle n\rangle}(\underline{x},\tau), \quad n=1,2,\ldots \tag{3.8}$$

and

$$\underline{\xi}(\tau) \equiv \underline{\chi}_t^{\langle 0\rangle}(\underline{x},\tau). \tag{3.9}$$

Then, using (3.3) and the chain rule, we find that

$$\underline{\xi}_{,\tau} = \tilde{\underline{U}}^{\langle 0\rangle}(\underline{x},\tau) = \underline{U}^{\langle 0\rangle}(\underline{\xi}(\tau),\tau),$$

$$\underline{\chi}_{,\tau}^{\langle 1\rangle} = \tilde{\underline{U}}^{\langle 1\rangle}(\underline{x},\tau) = \underline{U}^{\langle 1\rangle} + (\underline{\chi}^{\langle 1\rangle}\cdot\nabla_{\xi})\underline{U}^{\langle 0\rangle},$$

$$\underline{\chi}_{,\tau}^{\langle 2\rangle} = \tilde{\underline{U}}^{\langle 2\rangle}(\underline{x},\tau) = \underline{U}^{\langle 2\rangle} + (\underline{\chi}^{\langle 2\rangle}\cdot\nabla_{\xi})\underline{U}^{\langle 0\rangle} + (\underline{\chi}^{\langle 1\rangle}\cdot\nabla_{\xi})\underline{U}^{\langle 1\rangle}$$
$$+ \frac{1}{2}\chi_{\ell}^{\langle 1\rangle}\chi_{j}^{\langle 1\rangle}\partial^2 U^{\langle 0\rangle}/\partial\xi_{\ell}\partial\xi_{j},$$

$$\underline{\chi}_{,\tau}^{\langle n\rangle} = \tilde{\underline{U}}^{\langle n\rangle}(\underline{x},\tau) = \underline{U}^{\langle n\rangle} + (\underline{\chi}^{\langle n\rangle}\cdot\nabla_{\xi})\underline{U}^{\langle 0\rangle} + \text{other terms} \qquad (3.10)$$

where

$$\underline{U}^{\langle n\rangle} = \underline{U}^{\langle n\rangle}(\underline{\xi}(\tau),\tau).$$

The functions $\underline{\chi}^{\langle n\rangle}(\tau)$ may be computed by integrating (3.10) subject to the conditions (3.6) and (3.7).

We turn next to the computation of derivatives of the strain tensor. From the definition of the relative deformation gradient given in §2, we find, using (3.1) and (3.9) that

$$\underline{F}^{\langle n\rangle}(s) = \frac{1}{n!}\,\partial^n \underline{F}_t(s,\epsilon)/\partial\epsilon^n\big|_{\epsilon=0} = \nabla\underline{\chi}_t^{\langle n\rangle}(\underline{x},\tau) \qquad (3.11)$$

where

$$\underline{F}^{\langle 0\rangle}(0) = 1 \quad\text{and}\quad \underline{F}^{\langle n\rangle}(0) = \underline{0} \quad\text{for}\quad n > 0.$$

Then

$$\underline{G}(s,\epsilon) = \underline{G}^{\langle 0\rangle}(s) + \epsilon\underline{G}^{\langle 1\rangle}(s) + \epsilon^2\,\underline{G}^{\langle 2\rangle}(s) + \dots \qquad (3.12)$$

where

$$\underline{G}^{\langle 0\rangle}(s) = \underline{F}^{T\langle 0\rangle}(s)\cdot\underline{F}^{\langle 0\rangle}(s) - 1$$

and

$$\underline{G}^{\langle n\rangle}(s) = \sum_{\ell=0}^{n} \underline{F}^{T\langle \ell\rangle}(s)\cdot\underline{F}^{\langle n-\ell\rangle}(s). \qquad (3.13)$$

4. Functional derivatives of the stress and the equations governing the perturbation of special motions with arbitrary motions.

Suppose that $G(s,\epsilon)$ is the series given by (3.12). This series may be assumed to induce a functional expansion of the stress in powers of ϵ (see Joseph, 1976; p.197):

$$\underline{\mathfrak{F}}[\underline{G}(s,\epsilon)] = \underline{\mathfrak{F}}[\underline{G}^{\langle 0\rangle}] + \epsilon\underline{\mathfrak{F}}_1[\underline{G}^{\langle 0\rangle}|\underline{G}^{\langle 1\rangle}] +$$
$$\epsilon^2\{\underline{\mathfrak{F}}_2[\underline{G}^{\langle 0\rangle}|\underline{G}^{\langle 1\rangle},\underline{G}^{\langle 1\rangle}] + \underline{\mathfrak{F}}_1[\underline{G}^{\langle 0\rangle}|\underline{G}^{\langle 2\rangle}]\} +$$
$$\epsilon^3\{\underline{\mathfrak{F}}_3[\underline{G}^{\langle 0\rangle}|\underline{G}^{\langle 1\rangle},\underline{G}^{\langle 1\rangle},\underline{G}^{\langle 1\rangle}] + 2\underline{\mathfrak{F}}_2[\underline{G}^{\langle 0\rangle}|\underline{G}^{\langle 1\rangle},\underline{G}^{\langle 2\rangle}] +$$
$$\underline{\mathfrak{F}}_1[\underline{G}^{\langle 0\rangle}|\underline{G}^{\langle 3\rangle}]\} + O(\epsilon^4). \tag{4.1}$$

Apart from a factorial, $\mathfrak{F}_n$ is a functional derivative, typically a Fréchet derivative, evaluated on the history $\underline{G}^{\langle 0\rangle}(s)$ of the special solution. The linear arguments of these derivatives, those following the vertical bar, are to be determined sequentially by solving the perturbation equations of motion which have yet to be specified. The functional derivatives given in (4.1) are still too generally specified to be useful in the solution of problems. However, the first Fréchet derivative $\underline{\mathfrak{F}}_1[G^{\langle 0\rangle}|\cdot]$ may be assumed to be in integral form when $\underline{G}(s,\epsilon)$ lies in a $L^2_h(0,\infty)$ Hilbert space whose scalar product is defined by an integral with a weight $h(s)$, $h(s) \to 0$ as $s \to \infty$. Such a representation may be justified by appeal to the representation theorem of F. Riesz.

Identifying independent powers of ϵ in the expansion of the equations of motion, we may identify an ordered sequence of perturbation problems. The zeroth order problem is defined by (5.1), (5.2) and (5.3). At first order, we find that in $\mathcal{V}(t)$, $t > 0$

$$\rho\left[\frac{\partial \underline{U}^{\langle 1\rangle}}{\partial t} + (\underline{U}^{\langle 0\rangle}\cdot\nabla)\underline{U}^{\langle 1\rangle} + (\underline{U}^{\langle 1\rangle}\cdot\nabla)\underline{U}^{\langle 0\rangle}\right] = -\nabla p^{\langle 1\rangle} + \underline{\mathfrak{F}}_1[\underline{G}^{\langle 0\rangle}|\underline{G}^{\langle 1\rangle}] + \underline{f}^{\langle 1\rangle}(\underline{x},t), \tag{4.2}$$

$$\nabla\cdot\underline{U}^{\langle 1\rangle}(\underline{x},t) = 0 \tag{4.3}$$

and

$$\underline{\chi}^{\langle 1\rangle}(\tau) = \int_t^\tau [\underline{U}^{\langle 1\rangle} + (\underline{\chi}^{\langle 1\rangle}\cdot\nabla)\underline{U}^{\langle 0\rangle}]\,d\tau, \tag{4.4}$$

where, under the integral in (4.4), $\underline{U}^{\langle n\rangle} = \underline{U}^{\langle n\rangle}(\underline{\xi}(\tau'),\tau')$ for $n = 0$ and $n = 1$ and $\underline{\chi}^{\langle 1\rangle} = \underline{\chi}_t^{\langle 1\rangle}(\underline{x},\tau')$. On the boundary $\partial\mathcal{V}(t)$, $t > 0$,

$$\underline{U}^{\langle 1\rangle}(\underline{x},t) = \underline{q}^{\langle 1\rangle}(\underline{x},t). \tag{4.5}$$

The history of the velocity

$$\underline{U}^{\langle 1\rangle}(\underline{\xi}(\tau),\tau) \text{ is prescribed in } \mathcal{V}(\tau), \quad \tau \le 0. \tag{4.6}$$

Since $\underline{F}^{\langle 1\rangle}$ is the gradient of $\underline{\chi}^{\langle 1\rangle}$, (VI.2), with

$$\underline{G}^{\langle 1\rangle}(s) = \underline{F}^{T\langle 0\rangle}(s)\cdot\underline{F}^{\langle 1\rangle}(s) + \underline{F}^{T\langle 1\rangle}(s)\cdot\underline{F}^{\langle 0\rangle}(s), \tag{4.7}$$

(4.2), (4.3) and (4.4) may be viewed seven linear equations in the seven unknown functions $\underline{\chi}_t^{\langle 1\rangle}(\underline{x},\tau)$, $\underline{U}^{\langle 1\rangle}(\underline{x},t)$ and $p^{\langle 1\rangle}(\underline{x},t)$.

A similar linear problem for $\underline{\chi}^{\langle n\rangle}$, $\underline{U}^{\langle n\rangle}$ and $p^{\langle n\rangle}$, $(n \ge 2)$ arises at higher orders. If these problems are solvable, they are sequentially solvable and the motion and strain history may be generated as power series. These linear perturbation problems, with $\mathfrak{F}_1[\underline{G}^{\langle 0\rangle} \mid \cdot]$ represented by an integral, are not too general for mathematical studies of existence and uniqueness.

5. Kinematics of arbitrary motions perturbing steady rigid rotations of a simple fluid.

Now I am going to derive an algorithm for computing motions which perturb steady rigid rotations. I want solutions of the basic equations for which $\underline{G}^{\langle 0\rangle}(s) \equiv 0$. Rigid body motions have $\underline{G}^{\langle 0\rangle}(s) \equiv 0$ but, in general, such motions will not satisfy the equations because $\overset{\circ}{\underline{\Omega}} \wedge \underline{x}$ is not conservative (see (2.6)). I, therefore, set $\overset{\circ}{\underline{\Omega}} = 0$ and put

$$\underline{U}^{\langle 0\rangle}(\underline{\xi}(\tau),\tau) = \underline{\Omega} \wedge \underline{\xi}(\tau) \tag{5.1}$$

for $-\infty < \tau < t$ at all points in $\mathcal{V}(\tau)$. Then $\mathfrak{F}[\underline{G}^{\langle 0\rangle}(s)] \equiv 0$ and

$$p^{\langle 0\rangle}(\underline{x},t) + \frac{\rho}{2}\,|\underline{\Omega} \wedge \underline{x}|^2 = \text{const} \tag{5.2}$$

at each $\underline{x} \in \mathcal{V}(t)$ and at each and every instant $t > -\infty$. The path $\underline{\chi}_t^{\langle 0\rangle}(\underline{x},\tau) = \underline{\xi}(\tau)$ for $\tau \le t$ is obtained by integrating

$$\underline{\xi}_{,\tau} = \underline{\Omega} \wedge \underline{\xi}(\tau), \quad \underline{\xi}(t) = \underline{x}.$$

Without losing generality, we choose a fixed orthonormal basis $\underline{e}_1$, $\underline{e}_2$, $\underline{e}_3$ such that

$$\underline{\Omega} = \underline{e}_3\Omega \tag{5.3}$$

is a constant vector. Then the particle path is given by

$$\underline{\chi}_t^{\langle 0\rangle}(\underline{x},\tau) = \underline{\xi}(\tau) = \underline{Q}(\Omega s)\cdot\underline{x} \tag{5.4}$$

where $\underline{Q}(\Omega s)$ is the unique orthogonal tensor rotating the orthonormal basis $\hat{\underline{e}}_1(\Omega s)$, $\hat{\underline{e}}_2(\Omega s)$, $\underline{e}_3$ into $\underline{e}_1$, $\underline{e}_2$, $\underline{e}_3$; that is, $\underline{e}_i = \underline{Q}(\Omega s)\cdot\hat{\underline{e}}_i(\Omega s)$. Relative to the fixed basis,

$$\underline{Q}(\Omega s) = \underline{e}_i\ Q_{ij}(\Omega s)\underline{e}_j$$

where $Q_{ij}(\Omega s)$ $\underline{e}_1\cdot\underline{\Omega}(\Omega s)\cdot\underline{e}_j$; that is,

$$[Q_{ij}(\Omega s)] = \begin{pmatrix} \cos\Omega s & \sin\Omega s & 0 \\ -\sin\Omega s & \cos\Omega s & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{5.5}$$

It follows from (5.4) that

$$\underline{F}^{\langle 0\rangle} = \nabla\underline{\chi}_t^{\langle 0\rangle}(\underline{x},\tau) = \underline{Q}(\Omega s). \tag{5.6}$$

Hence,

$$\underline{G}^{\langle 0\rangle}(s) = \underline{Q}^T(\Omega s)\cdot\underline{Q}(\Omega s) - \underline{1} = \underline{0} \tag{5.7}$$

and $\underline{\mathfrak{J}}^{\langle 0\rangle} \equiv 0$.

The differential equations (3.10) for the perturbation coefficients $\underline{\chi}_{,\tau}^{\langle n\rangle} = \tilde{\underline{U}}^{\langle n\rangle}(\underline{x},\tau)$ giving the particle paths may now be expressed in terms of $\underline{\chi}^{\langle \ell\rangle} = \underline{\chi}_t^{\langle \ell\rangle}(\underline{x},\tau)$ and $\underline{U}^{\langle \ell\rangle} = \underline{U}^{\langle \ell\rangle}(\underline{\xi}(\tau),\tau)$. Using the identity $(\underline{\chi}^{\langle n\rangle}\cdot\nabla_\xi)\underline{U}^{\langle 0\rangle} = (\underline{\chi}^{\langle n\rangle}\cdot\nabla_\xi)(\underline{\Omega}\wedge\underline{\xi}) = \underline{\Omega}\wedge\underline{\chi}^{\langle n\rangle}$, we find that

$$\tilde{\underline{U}}^{\langle 0\rangle} - \underline{\Omega}\wedge\underline{\xi} = \underline{0},$$

$$\tilde{\underline{U}}^{\langle 1\rangle} - \underline{\Omega}\wedge\underline{\chi}^{\langle 1\rangle} = \underline{U}^{\langle 1\rangle}$$

$$\tilde{\underline{U}}^{\langle 2\rangle} - \underline{\Omega}\wedge\underline{\chi}^{\langle 2\rangle} = \underline{U}^{\langle 2\rangle} + (\underline{\chi}^{\langle 1\rangle}\cdot\nabla_\xi)\underline{U}^{\langle 1\rangle},$$

$$\tilde{\underline{U}}^{\langle 3\rangle} - \underline{\Omega}\wedge\underline{\chi}^{\langle 3\rangle} = \underline{U}^{\langle 3\rangle} + (\underline{\chi}^{\langle 2\rangle}\cdot\nabla_\xi)\underline{U}^{\langle 1\rangle} + (\underline{\chi}^{\langle 1\rangle}\cdot\nabla_\xi)\underline{U}^{\langle 2\rangle}$$

$$+ \frac{1}{2}\chi_\ell^{\langle 1\rangle}\ \chi_j^{\langle 1\rangle}\ \frac{\partial^2\underline{U}^{\langle 1\rangle}}{\partial\xi_\ell\,\partial\xi_j}$$

and

$$\underline{\tilde{U}}^{\langle n\rangle} - \underline{\Omega} \wedge \underline{\chi}^{\langle n\rangle} = \underline{\mathbb{U}}^{\langle n\rangle} - \text{other terms.} \qquad (5.8)$$

It follows that

$$\begin{aligned} \underline{\tilde{U}}(\underline{x},\tau,\varepsilon) - \underline{\Omega} \wedge \underline{\chi}_t(\underline{x},\tau,\varepsilon) &= \underline{\mathbf{W}}(\underline{\chi}_\tau(\underline{x},\tau,\varepsilon);\tau,\varepsilon) \\ &= \varepsilon \underline{U}^{\langle 1\rangle}(\underline{\xi}(\tau),\tau) + \varepsilon^2[\underline{U}^{\langle 2\rangle} + (\chi^{\langle 1\rangle}\cdot\nabla_{\xi})\underline{U}^{\langle 1\rangle}] + \dots \\ &+ \varepsilon^n[\underline{U}^{\langle n\rangle} + \text{other terms}] + \dots . \end{aligned} \qquad (5.9)$$

The most natural measure of deformation for rotating simple fluids is the time derivative of the Cauchy strain

$$\underline{\mathbf{J}}(s,\varepsilon) = -\frac{d}{ds}\,\underline{G}(s,\varepsilon) = \underline{F}^T_{,\tau}\cdot\underline{F} + \underline{F}^T\cdot\underline{F}_{,\tau} = \nabla\underline{\tilde{U}}^T\cdot\underline{F} + \underline{F}^T\cdot\nabla\underline{\tilde{U}}. \qquad (5.10)$$

We shall need the following expansion formula for

$$\underline{\mathbf{J}}(s,\varepsilon) = \nabla\underline{V}^T\cdot\underline{F} + \underline{F}^T\cdot\nabla\underline{V} = \varepsilon\,\underline{\mathbf{J}}^{\langle 1\rangle} + \varepsilon^2\,\underline{\mathbf{J}}^{\langle 2\rangle} + \dots \qquad (5.11)$$

where the $\underline{\mathbf{J}}^{\langle n\rangle}$ are defined in terms $\underline{Q}(\Omega s)$, $\underline{F}^{\langle \ell\rangle}$ and

$$\underline{L}^{\langle n\rangle} = \nabla[\underline{\tilde{U}}^{\langle n\rangle}(\underline{x},\tau) - \underline{\Omega} \wedge \underline{\chi}^{\langle n\rangle}], \quad n \geq 1. \qquad (5.12)$$

Equations (5.8) show that $\underline{L}^{\langle n\rangle}$ depends on $\underline{U}^{\langle \ell\rangle}(\underline{\xi}(\tau),\tau)$ for $\ell \leq n$ and on $\underline{\chi}^{\ell}_t(\underline{x},\tau)$ for $\ell < n$. In fact,

$$\underline{\mathbf{J}}^{\langle 1\rangle} = \underline{L}^{T\langle 1\rangle}\underline{Q}(\Omega s) + \underline{Q}^T(\Omega s)\cdot\underline{L}^{\langle 1\rangle} \qquad (5.13)$$

and for $n \geq 2$

$$\begin{aligned} \underline{\mathbf{J}}^{\langle n\rangle} &= \underline{L}^{T\langle n\rangle}\cdot\underline{Q}(\Omega s) + \underline{Q}^T(\Omega s)\cdot\underline{L}^{\langle n\rangle} \\ &+ \sum_{\ell=1}^{n-1}(\underline{L}^{T(n-\ell)}\cdot\underline{F}^{\langle \ell\rangle} + \underline{F}^{T\langle \ell\rangle}\cdot\underline{L}^{\langle n-\ell\rangle}). \end{aligned} \qquad (5.14)$$

To prove the expansion formula $(5.11)_2$, we must first show that (5.10) reduces to $(5.11)_1$. This reduction is a direct consequence of the identity

$$[\nabla(\underline{\Omega} \wedge \underline{\chi}_t)^T\cdot\underline{F} + \underline{F}^T\cdot\nabla(\underline{\Omega} \wedge \underline{\chi}_t)]_{ij} =$$

$$\frac{\partial}{\partial x_i}(\underline{\Omega} \wedge \underline{\chi}_t)_\ell \frac{\partial}{\partial x_j}(\underline{\chi}_t)_\ell + \frac{\partial}{\partial x_j}(\underline{\Omega} \wedge \underline{\chi}_t)_\ell \frac{\partial}{\partial x_i}(\underline{\chi}_t)_\ell =$$

$$\Omega_m \varepsilon_{\ell m n}[\frac{\partial}{\partial x_i}(\underline{\chi}_t)_n \frac{\partial}{\partial x_j}(\underline{\chi}_t)_\ell + \frac{\partial}{\partial x_j}(\underline{\chi}_t)_n \frac{\partial}{\partial x_i}(\underline{\chi}_t)_\ell] = 0.$$

To obtain (5.13) and (5.14), we expand $\underline{F}(s,\varepsilon) = \underline{Q}(\Omega s) + \sum_1 \varepsilon^\ell \underline{F}^{\langle \ell\rangle}$ and $\varepsilon\underline{V}$, using (5.9) and (5.8), and collect the coefficients of independent powers of ε in the induced expansion of (5.11).

Some further transformations of the tensors $\underline{\mathbb{J}}^{\langle n\rangle}(s)$ are used in the analysis. These transformations are motivated by the fact that the perturbation problems to be derived lead to the sequential determination of the velocity coefficients $\underline{U}^{\langle n\rangle}(\underline{\xi}(\tau),\tau)$ whose natural arguments are the components of the rotating vector $\underline{\chi}_t^{\langle 0\rangle}(\underline{x},\tau) = \underline{\xi}(\tau)$. Noting now that when $\tau = t$, $\xi = x$ and $\text{div}\ \underline{U}^{\langle n\rangle}(\underline{x},t) = 0$ in $\mathcal{V}(t)$, we find easily that

$$\text{div}_\xi\ \underline{U}^{\langle n\rangle}(\underline{\xi},\tau) = 0 \quad \text{in} \quad \mathcal{V}(\tau), \quad \tau \le t. \qquad (5.15)$$

Noting next that

$$\nabla(\cdot) = \nabla(\cdot)\cdot\underline{Q}(\Omega s), \quad \frac{\partial}{\partial x_i} = \frac{\partial}{\partial \xi_m}\frac{\partial \xi_m}{\partial x_i},$$

we find that

$$\underline{L}^{\langle \ell\rangle} = \nabla(\underline{\tilde{U}}^{\langle \ell\rangle} - \underline{\Omega} \wedge \underline{\chi}_t^{\langle \ell\rangle}) = \underline{\mathfrak{L}}^{\langle \ell\rangle}\cdot\underline{Q}(\Omega s) \qquad (5.16)$$

and

$$\underline{F}^{\langle \ell\rangle} = \nabla\underline{\chi}^{\langle \ell\rangle} = \underline{\mathfrak{F}}^{\langle \ell\rangle}\cdot\underline{Q}(\Omega s) \qquad (5.17)$$

where

$$\underline{\mathfrak{L}}^{\langle \ell\rangle} = \nabla_\xi[\underline{\tilde{U}} - \underline{\Omega} \wedge \underline{\chi}_t^{\langle \ell\rangle}] = \nabla_\xi[\underline{U}^{\langle \ell\rangle}(\underline{\xi},\tau) + \ldots]$$

and

$$\underline{f}^{\langle \ell \rangle} = \nabla_{\xi} \underline{\chi}^{\langle \ell \rangle} .$$

Using (5.8), we find that

$$\underline{\mathfrak{L}}^{\langle 1 \rangle} = \nabla_{\xi} \underline{U}^{\langle 1 \rangle}(\xi, \tau)$$

and

$$\begin{aligned} \underline{\mathfrak{L}}^{\langle 2 \rangle} &= \nabla_{\xi}[\underline{U}^{\langle 2 \rangle}(\underline{\xi}, \tau) + (\underline{\chi}^{\langle 1 \rangle} \cdot \nabla_{\xi})\underline{U}^{\langle 1 \rangle}(\underline{\xi}, \tau)] \\ &= \nabla_{\xi}\underline{U}^{\langle 2 \rangle} + (\underline{\chi}^{\langle 1 \rangle} \cdot \nabla_{\xi})\underline{\mathfrak{L}}^{\langle 1 \rangle} + \underline{\mathfrak{L}}^{\langle 1 \rangle} \cdot \underline{f}^{\langle 1 \rangle} . \end{aligned}$$

Finally, setting

$$\underline{A}^{\langle \ell \rangle} = \nabla_{\xi}\underline{U}^{\langle \ell \rangle} + \nabla_{\xi}\underline{U}^{\langle \ell \rangle^{T}}, \quad \ell \geq 1, \tag{5.18}$$

we find that

$$\underline{J}^{\langle 1 \rangle} = \underline{Q}^{T}(\Omega s) \cdot \underline{A}^{\langle 1 \rangle}(s) \cdot \underline{Q}(\Omega s), \tag{5.19}$$

$$\underline{J}^{\langle 2 \rangle} = \underline{Q}^{T}(\Omega s) \cdot (\underline{A}^{\langle 2 \rangle}(s) + \underline{B}(s)) \cdot \underline{Q}(\Omega s), \tag{5.20}$$

$$\underline{J}^{\langle \ell \rangle} = \underline{Q}^{T}(\Omega s) \cdot (\underline{A}^{\langle \ell \rangle}(s) + \ell.o.t.) \cdot \underline{Q}(\Omega s), \tag{5.21}$$

where

$$\underline{B}(s) = (\underline{\chi}^{\langle 1 \rangle} \cdot \nabla_{\xi})\underline{A}^{\langle 1 \rangle} + \underline{A}^{\langle 1 \rangle} \cdot \underline{f}^{\langle 1 \rangle} + \underline{f}^{T\langle 1 \rangle} \cdot \underline{A}^{\langle 1 \rangle} \tag{5.22}$$

and

$$\ell.o.t. = \text{lower order terms}.$$

6. Canonical forms for the stress.

My constitutive hypothesis is that the Fréchet derivatives of $\underline{\mathfrak{J}}[\underline{G}(s)]$ on the zero history can be represented by integrals. I also assume that kernels in these integrals vanish at a rate sufficient to justify integrating by parts; for example,

$$\underline{\mathfrak{F}}_1[0|\underline{G}(s)] = \int_0^\infty \frac{dG}{ds}\,\underline{G}(s)ds = \int_0^\infty \underline{G}(s)\underline{\mathbb{J}}(s)ds \equiv \tilde{\underline{\mathfrak{F}}}_1[0|\underline{\mathbb{J}}(s)] \tag{6.1}$$

and

$$\underline{\mathfrak{F}}_2[0|\underline{G}(s_1),\underline{G}(s_2)] =$$

$$\int_0^\infty\int_0^\infty [\frac{\partial^2\gamma}{\partial s_1\partial s_2}(s_1,s_2)\ \underline{G}(s_1)\cdot\underline{G}(s_2) +$$

$$\frac{\partial^2\hat{\alpha}}{\partial s_1\partial s_2}(s_1,s_2)\ [\mathrm{tr}\ \underline{G}(s_1)]\underline{G}(s_2)]ds_1ds_2 =$$

$$\int_0^\infty\int_0^\infty [\gamma(s_1,s_2)\underline{\mathbb{J}}(s_1)\cdot\underline{\mathbb{J}}(s_2) + \hat{\alpha}(s_1,s_2)[\mathrm{tr}\ \underline{\mathbb{J}}(s_1)]\underline{\mathbb{J}}(s_2)]ds_1ds_2 \equiv$$

$$\tilde{\underline{\mathfrak{F}}}_2\ [0\ |\ \underline{\mathbb{J}}(s_1),\underline{\mathbb{J}}(s_2)]. \tag{6.2}$$

Explicit expressions for $\tilde{\underline{\mathfrak{F}}}_3$ and $\tilde{\underline{\mathfrak{F}}}_4$ are given by Joseph & Beavers (1977). This integration by parts allows us to introduce $\underline{\mathbb{J}}(s) = -d\underline{G}(s)/ds$ as the fundamental measure of deformation and leads ultimately to a theory in which perturbation velocities are sequentially computed from four equations governing three components of velocity and the pressure, as in an incompressible, Navier-Stokes fluid. Assuming that the stress $\underline{\mathfrak{F}}[\underline{G}(s)]$ admits a Fréchet expansion in integrals with good kernels, we get

$$\begin{aligned}&\underline{\mathfrak{F}}[\underline{G}(s)] =\\ &\underline{\mathfrak{F}}_1[0|\underline{G}(s)] + \underline{\mathfrak{F}}_2[0|\underline{G}(s_1),\underline{G}(s_2)] + \underline{\mathfrak{F}}_3[0|\underline{G}(s_1),\underline{G}(s_2),\underline{G}(s_3)]+\dots\\ &\tilde{\underline{\mathfrak{F}}}_1[0|\underline{\mathbb{J}}(s)] + \tilde{\underline{\mathfrak{F}}}_2[0|\underline{\mathbb{J}}(s_1),\underline{\mathbb{J}}(s_2)] + \tilde{\underline{\mathfrak{F}}}_3[0|\underline{\mathbb{J}}(s_1),\underline{\mathbb{J}}(s_2),\underline{\mathbb{J}}(s_3)]+\dots=\\ &\tilde{\underline{\mathfrak{F}}}[\underline{\mathbb{J}}(s)].\end{aligned} \tag{6.3}$$

To obtain the canonical forms of the stress for the theory of rotating fluids, we identify independent coefficients in the series expansion of $\mathfrak{J}[J(s,\epsilon)]$ induced by the expansion (5.12) of $\underline{J}(s,\epsilon)$. This leads us to

$$\begin{aligned}\tilde{\underline{\mathfrak{J}}}[\underline{J}(s,\epsilon)] &= \epsilon\underline{\mathfrak{J}}^{\langle 1\rangle} + \epsilon^2\,\tilde{\underline{\mathfrak{J}}}^{\langle 2\rangle} + \epsilon^3\,\tilde{\underline{\mathfrak{J}}}^{\langle 3\rangle} + \ldots \\ &= \epsilon\,\tilde{\underline{\mathfrak{J}}}_1[0 \mid \underline{J}^{\langle 1\rangle}(s)] \\ &+ \epsilon^2\{\tilde{\underline{\mathfrak{J}}}_1[0|\underline{J}^{\langle 2\rangle}(s)] + \tilde{\underline{\mathfrak{J}}}_2[0|\underline{J}^{\langle 1\rangle}(s_1),\underline{J}^{\langle 1\rangle}(s_2)]\} \\ &+ \epsilon^3\{\tilde{\underline{\mathfrak{J}}}_1[0|\underline{J}^{\langle 3\rangle}(s)] + 2\tilde{\underline{\mathfrak{J}}}_2[0|\underline{J}^{\langle 1\rangle}(s_1),\underline{J}^{\langle 2\rangle}(s_2)] \\ &\quad + \tilde{\underline{\mathfrak{J}}}_3[0|\underline{J}^{\langle 1\rangle}(s_1), \underline{J}^{\langle 1\rangle}(s_2), \underline{J}^{\langle 1\rangle}(s_3)]\} \\ &\quad + O(\epsilon^4)\end{aligned} \tag{6.4}$$

where $\underline{J}^{\langle 1\rangle}$, $\underline{J}^{\langle 2\rangle}$ and $\underline{J}^{\langle \ell\rangle}$ are given by (5.19-21), $\tilde{\underline{\mathfrak{J}}}_1$ is given by (6.1) and

$$\tilde{\underline{\mathfrak{J}}}_2[0|\underline{J}^{\langle 1\rangle}(s_1),\underline{J}^{\langle 1\rangle}(s_2)] = \int_0^\infty\int_0^\infty \gamma(s_1,s_2)\underline{J}^{\langle 1\rangle}(s_1)\cdot\underline{J}^{\langle 1\rangle}(s_2)\,ds_1ds_2 \tag{6.5}$$

The term proportional to $\hat{\alpha}(s_1,s_2)$ vanishes in the second-order approximation because $\mathrm{tr}[\underline{J}^{\langle 1\rangle}] = \mathrm{tr}[\underline{Q}^T(\Omega s)\cdot\underline{A}^{\langle 1\rangle}\cdot\underline{Q}(\Omega s)] =$ $= \mathrm{tr}\,\underline{A}^{\langle 1\rangle} = \mathrm{div}_\xi\,\underline{U}^{\langle 1\rangle}(\underline{\xi},\tau) = 0$.

The canonical forms of the stress for perturbations of steady rigid rotation are given through order two by

$$\begin{aligned}\tilde{\underline{\mathfrak{J}}}[\underline{J}(s,\epsilon)] &= \epsilon\int_0^\infty \underline{G}(s)\;\underline{Q}^T(\Omega s)\cdot\underline{A}^{\langle 1\rangle}(s)\cdot\underline{Q}^T(\Omega s)ds\; + \\ &\epsilon^2\{\int_0^\infty \underline{G}(s)\;\underline{Q}^T(\Omega s)\cdot[\underline{A}^{\langle 2\rangle}(s) + \underline{B}(s)]\cdot\underline{Q}(\Omega s)ds\; + \\ &\int_0^\infty\int_0^\infty \gamma(s_1,s_2)\underline{Q}^T(\Omega s_1)\cdot\underline{A}^{\langle 1\rangle}(s_1)\cdot\underline{Q}(\Omega s_1)\cdot\underline{Q}^T(\Omega s_2)\cdot\underline{A}^{\langle 1\rangle}(s_2)\cdot\underline{Q}(\Omega s_2)\,ds_1ds_2 + O(\epsilon^3)\end{aligned} \tag{6.6}$$

where $\underline{A}^{\langle \ell\rangle}(s)$ is defined by (5.18) and $\underline{B}(s)$ by (5.22). The higher order stresses are not hard to derive. They are in the form

$$\tilde{\mathfrak{Z}}^{\langle n\rangle} = \int_0^\infty \underline{G}(s)\underline{Q}^T(\Omega s)\cdot\underline{A}^{\langle n\rangle}\cdot\underline{Q}(\Omega s)ds$$
$$+ \text{ lower-order terms.} \tag{6.7}$$

7. Canonical forms of the equations of motion.

After expanding in powers of ϵ, we find that

$$\rho\left[\frac{\partial \underline{U}^{\langle n\rangle}}{\partial t}(x,t) + \sum_{\ell=0}^{n} (\underline{U}^{\langle n-\ell\rangle}(\underline{x},t)U^{\ell}(\underline{x},t)\right] = -\nabla p^{\langle n\rangle}(\underline{x},t)$$
$$+ (\text{div } \tilde{\mathfrak{Z}}^{\langle n\rangle})(\underline{x},t). \tag{7.1}$$

When $n = 0$, $\underline{U}^{\langle 0\rangle}(\underline{x};t) = \underline{\Omega}\wedge\underline{x}$, $\mathfrak{Z}^{\langle 0\rangle} = \partial\underline{U}^{\langle 0\rangle}/\partial t = 0$, and $p^{\langle 0\rangle} + 1/2\ \rho|\underline{\Omega}\wedge\underline{x}|^2$ is constant. Now we shall demonstrate that

$$\text{div } \tilde{\mathfrak{Z}}^{\langle n\rangle} = \int_0^\infty \underline{G}(s)\underline{Q}^T(\Omega s)\cdot\nabla_\xi^2\underline{U}^{\langle n\rangle}(\underline{\xi}(t-s),t-s)ds$$
$$+ \text{ lower-order terms.} \tag{7.2}$$

To establish (7.2), we first show that for any tensor $\underline{M}(\underline{x},\tau) = \underline{\mathfrak{m}}(\underline{\xi},\tau)$, we have

$$\text{div}[\underline{Q}^T(\Omega s)\cdot\underline{M}\cdot\underline{Q}(\Omega s)] = \underline{Q}^T(\Omega s)\cdot\text{div}_\xi\ \underline{\mathfrak{m}}. \tag{7.3}$$

Taking components in the fixed cartesian basis, we find that

$$(\text{div } \underline{Q}^T\cdot\underline{M}\cdot\underline{Q})_i = \frac{\partial}{\partial x_j}(\underline{Q}^T\cdot\underline{M}\cdot\underline{Q})_{ij} = Q^T_{i\ell}\left(\frac{\partial M_{\ell n}}{\partial x_j}\right)Q_{nj}$$
$$= Q^T_{i\ell}\frac{\partial \mathfrak{m}_{\ell n}}{\partial \xi_k}\frac{\partial \xi_k}{\partial x_j}Q_{nj} = Q^T_{i\ell}\frac{\partial \mathfrak{m}_{\ell n}}{\partial \xi_n} = (\underline{Q}^T\cdot\text{div}_\xi\ \underline{\mathfrak{m}})_i$$

where we have used $Q_{kj} = \partial\xi_k/\partial x_j$. Since $\partial\underline{U}_i^{\langle n\rangle}(\underline{\xi},\tau)/\partial\xi_i = \underline{0}$, we may also verify that

$$(\mathrm{div}_{\xi}\ \underline{A}^{\langle n\rangle})_i = \partial A^{\langle n\rangle}_{ij}/\partial\xi_j = \nabla^2_{\xi}\ U^{\langle n\rangle}_i(\underline{\xi},\tau). \qquad (7.4)$$

Combining (7.3) and (7.4) with (6.7), we prove (7.2).

Equations (7.1) may be expressed in terms of the components $U^{\langle n\rangle}_i$ of $\underline{U}^{\langle n\rangle} = \underline{e}_i\ U^{\langle n\rangle}_i$ relative to the fixed cartesian basis $\underline{e}_1, \underline{e}_2, \underline{e}_3$. To express (7.2) in terms of $U^{\langle n\rangle}_i$, we note that

$$\underline{Q}^T(\Omega s)\cdot\underline{U}^{\langle n\rangle} = \underline{Q}^T(\Omega s)\cdot\underline{e}_i\ U^{\langle n\rangle}_i = \hat{\underline{e}}_i(s)U^{\langle n\rangle}_i. \qquad (7.5)$$

Equations (7.1) and $\mathrm{div}\ \underline{U}^{\langle n\rangle}(\underline{x},t) = 0$ hold identically in $\mathcal{V}(t)$ for $t > 0$ and $\underline{U}^{\langle n\rangle}(\underline{x},t)$ is prescribed in $\mathcal{V}(t)$ when $t \le 0$. Boundary conditions, say $\underline{U}^{\langle n\rangle}(\underline{x},t) = g^{\langle n\rangle}(\underline{x},t)$ for $\underline{x} \in \partial\mathcal{V}(t)$ and $t \ge 0$, are also prescribed. Noting that $(\underline{U}^{\langle n\rangle}\cdot\nabla)\underline{U}^{\langle 0\rangle} = \underline{\Omega} \wedge \underline{U}^{\langle n\rangle}$, we may write the perturbation problems in sequence. When $n = 1$,

$$\rho[\frac{\partial\underline{U}^{\langle 1\rangle}}{\partial t} + (\underline{\Omega} \wedge \underline{x})\cdot\nabla\underline{U}^{\langle 1\rangle} + \underline{\Omega} \wedge \underline{U}^{\langle 1\rangle}] + \nabla p^{\langle 1\rangle}$$
$$- \int_0^\infty \underline{G}(s)\underline{Q}^T(\Omega s)\cdot\nabla^2_{\xi}\ \underline{U}^{\langle 1\rangle}(\underline{\xi}(t-s),t-s)ds=0. \qquad (7.6)$$

The initial-history problem associated with (7.6) determine $\underline{U}^{\langle 1\rangle}(x,t)$ and $p^{\langle 1\rangle}(\underline{x},t)$. The solution is independent of the particle path $\underline{\chi}^{\langle 1\rangle}(\underline{x},\tau)$ at first order. The particle path may be computed as an afterthought once $\underline{U}^{\langle 1\rangle}(\underline{x},t)$ is known:

$$\underline{\chi},^{\langle 1\rangle}_{\tau} = \underline{\Omega} \wedge \underline{\chi}^{\langle 1\rangle} + \underline{U}^{\langle 1\rangle}(\underline{\xi}(\tau),\tau), \quad \underline{\chi}^{\langle 1\rangle}(\underline{x},\tau) = 0. \qquad (7.7)$$

When $n = 2$,

$$
\begin{aligned}
&\left[\frac{\partial \underline{U}^{\langle 2\rangle}}{\partial t} + (\underline{\Omega} \wedge \underline{x})\cdot\nabla\underline{U}^{\langle 2\rangle} + \underline{\Omega} \wedge \underline{U}^{\langle 2\rangle}\right] + \nabla p^{\langle 2\rangle} \\
&- \int_0^\infty \underline{G}(s)\underline{Q}^T(\Omega s)\cdot\nabla_{\xi}^2\, \underline{U}^{\langle 2\rangle}(\underline{\xi}(t-s),t-s)ds = -\rho\underline{U}^{\langle 1\rangle}\,\nabla\underline{U}^{\langle 1\rangle} \\
&\qquad + \int_0^\infty G(s)\,\underline{Q}^T(\Omega s)\cdot\mathrm{div}_{\xi}\,\underline{B}(s)ds \\
&+ \mathrm{div}\int_0^\infty\int_0^\infty \gamma(s_1,s_2)\underline{\mathbb{J}}^{\langle 1\rangle}(s_1)\cdot\underline{\mathbb{J}}^{\langle 1\rangle}(s_2)ds_1ds_2 \qquad (7.8)
\end{aligned}
$$

where $\mathbb{J}^{\langle 1\rangle}$ is given by (6.19) and $\underline{B}(s)$ by (6.22). The terms on the right of (7.8) are known when $\underline{U}^{\langle 1\rangle}$ and $\underline{\chi}^{\langle 1\rangle}$ are known. Hence, we may solve the initial-history problem associated with (7.8) for $\underline{U}^{\langle 2\rangle}(\underline{x},t)$ and $p^{\langle 2\rangle}(\underline{x},t)$. Then we can compute the path at second order:

$$
\begin{aligned}
\underline{\chi}^{\langle 2\rangle}_{,\tau} &= \underline{\Omega} \wedge \underline{\chi}^{\langle 2\rangle} + \underline{U}^{\langle 2\rangle}(\underline{\xi}(\tau),\tau) \\
&+ (\underline{\chi}^{\langle 1\rangle}\cdot\nabla_{\xi})\underline{U}^{\langle 1\rangle}(\underline{\xi}(\tau),\tau),\ \underline{\chi}^{\langle 2\rangle}(\underline{x},t) = 0. \qquad (7.9)
\end{aligned}
$$

It follows that at each order we may compute three velocity components $\underline{U}^{\langle n\rangle}(\underline{x},t)$ and a pressure $p^{\langle n\rangle}(\underline{x},t)$ from an inhomogeneous, linear, initial-history problem associated with (7.1) and $\mathrm{div}\ \underline{U}^{\langle n\rangle} = 0$. The particle path $\underline{\chi}^{\langle n\rangle}$ appears as an auxiliary quantity which may be computed when $\underline{U}^{\langle \ell\rangle}$, $\ell < n$ is known. In other words, at each stage of the sequence, we solve four equations in four unknowns.

The mathematical problem defined by this perturbation sequence may be stated as follows: Given $f(\underline{x},t)$ in $\mathcal{V}(t)$ for $t > 0$, $h(\underline{x},t)$ in $\mathcal{V}(t)$ for $t \le 0$ and $q(\underline{x},t)$ on $\partial\mathcal{V}(t)$ for $t > 0$, find $\underline{a}(\underline{x},t)$ and $\phi(\underline{x},t)$ such that $\mathrm{div}\ \underline{a}(\underline{x},t) = 0$ and $\underline{a}(\underline{x},t) = \underline{h}(\underline{x},t)$ in $\mathcal{V}(t)$ for $t \le 0$,

$\underline{a}(\underline{x},t) = \underline{q}(\underline{x},t)$ for $x \in \partial\mathcal{V}(t)$, $t > 0$ and

$$\rho\{\frac{\partial \underline{a}}{\partial t} + (\underline{\Omega} \wedge \underline{x})\cdot\nabla\underline{a} + \underline{\Omega} \wedge \underline{a}\} + \nabla\phi$$
$$- \int_0^\infty \underline{G}(s)Q^T(\Omega s)\cdot\nabla_{\xi}^2\, a(\underline{\xi}(t-s),t-s)ds = \underline{f}(\underline{x},t).$$

It is known (Slemrod, 1976) that under very mild conditions on the shear-relaxation modulus $G(s)$, the problem with $\Omega = 0$ has a unique solution. The stability result proved by Joseph (1977) suggests that if a solution of this problem exists when $\Omega \neq 0$, then it is unique.

Finally, I note that in limit $\Omega \to 0$ the theory of rotating simple fluids collapses into my previous theory of perturbations of the state of rest (Joseph, 1976).

The canonical forms of the stress and the equations of motion take a particularly simple form in a rotationally symmetric coordinate system defined by the circular paths of particles at zeroth order. These equations are derived and their properties are discussed and some rheological problems are solved by Joseph (1977).

In the derivation of the canonical forms of the equations of motion we assumed that the particle labels $\underline{x} = \chi_t(\underline{x},t,\varepsilon)$ used in the backward integration of path lines $\underline{\chi}_t(\underline{x},\tau,\varepsilon)$ are independent of ε. If the boundary $\partial\mathcal{V}_\varepsilon(t)$ of the domain $\mathcal{V}_\varepsilon(t)$ depends on ε, as in a free surface problem, we proceed as in the model problem except that the mapping function will depend parametrically on the time; that is, we map $\mathcal{V}_\varepsilon(t)$ into $\mathcal{V}_o$ where $\mathcal{V}_o$ is independent of time with an invertible mapping $\underline{x} = \underline{x}(\underline{x}_o,t,\varepsilon)$ possessing the properties that $\underline{x}_o = \underline{x}(\underline{x}_o, t, 0)$ is independent of time, and if $\underline{x}_o \in \partial \mathcal{V}_o$, then $\underline{x} \in \partial \mathcal{V}_\epsilon(t)$. After carrying out analysis like that given in §2, we can derive exactly the same canonical equations with $\underline{x}_o$ replacing $\underline{x}$. All equations are posed on the domain $\mathcal{V}_o$. The free surface equations are set on $\mathcal{V}_o$ and, since the real free surface depends on time, these equations can contain derivatives of the boundary values of the mapping function.

The main applications of the theory just described to free surface problems have so far been confined to perturbations of the rest state (see Joseph & Beavers, 1977 for a review of these applications). The most successful application so far has been to rod climbing (the Weissenberg effect). On the basis of the second order theory alone we have been able to explain and even to predict many of the novel effects which appear in the movie by G.S. Beavers and myself. These effects are peculiar to non-Newtonian fluids like STP. They include climbing on rotating rods, a Hopf bifurcation (the breathing instability) of the steady axi-symmetric climb, the critical radius, the big effect of temperature and adhesion, a normal stress amplifier (how to remove gravity on earth), buckling of fuid towers, the mean climb on an oscillating rod and the symmetry breaking bifurc-ation of axisymmetric time-periodic flow (the flower instabi-lity).

This work was supported by the U.S. Army Research Office and under NSF Grant 19047.

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