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STOKES FLOW IN A TRENCH BETWEEN CONCENTRIC CYLINDERS*

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Abstract. In this paper we develop a separation of variables theory for solving problems of Stokes flow in annular trenches bounded by horizontal parallel planes and concentric vertical cylinders. The theory leads to a new set of Stokes flow eigenfunctions, adjoint eigenfunctions, biorthogonality conditions and an algorithm for the computation of the coefficients in an eigenfunction expansion of edge data prescribed on the horizontal boundaries. To illustrate the algorithm we compute the motion and the shape of the free surface on a liquid in the space between two cylinders which are maintained at different temperatures. The solution is constructed as a domain perturbation of the rest state in powers of the temperature difference. At the lowest significant order the problem is reduced to a Stokes flow problem of the desired type with edge data prescribed on the horizontal boundaries.

1. Introduction. The aim of this paper is to contribute to a "separation of variables" theory for Stokes flows in cavities of simple configuration. Generality in a "separation of variables" theory is associated with the applicability of the techniques to many problems in many domains of simple shape. We claim this kind of generality for the theory given here. The techniques developed here owe much to the excellent ideas which R. C. T. Smith [10] introduced in his study of stresses in a semi-infinite strip clamped at its side and loaded at its top edge. Smith's ideas were used by Joseph and Fosdick [6] to study a narrow gap approximation for secondary motions generated in the problem of the free surface on a liquid between cylinders rotating at different speeds. A more complete analysis, including numerical analysis, of the problem of Stokes flow in rectangular trenches was given by Joseph and Sturges [7] in their study of the free surface on a liquid filling a rectangular trench heated from its side. In that paper it is shown that Smith's biorthogonal series are formally analogous to complex Fourier series and though the biorthogonal eigenfunctions are much more complicated than circular functions, the "Fourier coefficients" may be computed by simple algorithms. Joseph and Sturges [7] also showed how the eigenfunction expansions should be used to compute solutions when the rectangular strip is not semi-infinite but, instead, has a solid bottom.

Smith [10] also established conditions on the edge data sufficient to guarantee the convergence of the biorthogonal series. But Smith's conditions are too restrictive for applications. Joseph [4] and Joseph and Sturges [5] showed that the biorthogonal series converge for all types of edge data which might be expected in applications.

The same type of biorthogonal expansions were used by Joseph [3] in a study of the free surface on the round edge of a flowing liquid filling a torsion flow viscometer. This is the first case where this type of eigenfunction expansion arises for a Stokes flow problem which is not biharmonic. Similar eigenfunction expansions are required for the axisymmetric problems of Stokes flow in a wedge shaped trench studied by Liu and Joseph [8] and for the problem of axisymmetric flow in a cone studied by Liu and Joseph [14]. The study of the free surface on a viscoelastic

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fluid between oscillating planes (Sturges and Joseph [11]) also falls within the domain of application of the biorthogonal series. This problem may be reduced to the study of $\nabla^4\psi+\lambda^2\nabla^2\psi=0$ (λ^2 is complex) where ψ and the normal derivative of ψ vanish on the side-walls.

The list of problems given in the last paragraph is a small sample of the problems which can be solved by biorthononal eigenfunction expansions. The eigenfunctions required in these different problems depend on the given data and on the domain of flow; though the data and domains of flow differ from problem to problem, the expansions for different problems share common properties which appear to be intrinsic to Stokes flow in cavities.

In this paper we shall develop the theory of biorthogonal eigenfunction expansions for problems which are posed between concentric cylinders. The details of the derivation of the eigenfunctions, asymptotic expansions, limiting cases and computational algorithms are given in the appendices. In the text we show, by example, how to apply the theory. Our sample problem is to describe the motion and the shape of the free surface on the top of a liquid which fills the space between two concentric cylinders with prescribed constant but unequal, temperatures. In the limit, as the ratio of the gap width to the radius of the inner cylinder tends to zero, our problem reduces to the problem of the free surface on a liquid in a rectangular trench studied by Joseph and Sturges [7]. We use the methods employed by Joseph and Sturges to construct the solution of our problem as a domain perturbation of the rest state in powers of the temperature difference. At the lowest significant order the problem is reduced to a Stokes flow problem of the desired type with edge data prescribed on the horizontal boundaries.

2. Mathematical formulation. We consider the steady motion of a Newtonian liquid between concentric cylinders of radius $\tilde{b}$ and $\tilde{a}$, $\tilde{b}>\tilde{a}$. The liquid fills the annulus up to a mean height $\tilde{D}$; the total volume of the liquid is $\pi(\tilde{b}^2-\tilde{a}^2)\tilde{D}$. Cylindrical coordinates (r,θ,z) are located on the axis of the cylinders at a height $\tilde{D}$ from the flat bottom (see Fig. 1). The pressure of the atmosphere at the top of the liquid is p_a and the reduced pressure in the fluid is designated as

$$\Phi=p(r,z)-p_a+\rho g z,$$

where $p(r,z)$ is the pressure in the fluid, ρ is the density and g is the acceleration of gravity. Since the fluid is incompressible, the volume $\pi(\tilde{b}^2-\tilde{a}^2)\tilde{D}$ occupied by the fluid is an invariant and the free surface

$$z=h(r)$$

must have a zero mean value

$$0=\int_{\tilde{a}}^{\tilde{b}} 2\pi r h(r)\,dr.$$

The temperature on the wall at $r=\tilde{a}$ and $r=\tilde{b}$ is T_0 and $T_0+\varepsilon$, respectively. We will assume that the bottom of the cylinder is insulated and since the thermal conductivity of liquids is generally so much greater than the thermal conductivity of air, the free surface of the liquid-air interface is also assumed to be insulated. We designate the temperature as $T(r,z)=\Theta(r,z)+T_0$, the solenoidal velocity by $\mathbf{u}=\mathbf{e}_r u+\mathbf{e}_z w$, the stress deviator by $\mathbf{S}=2\mu\mathbf{D}[\mathbf{u}]$ where $\mathbf{D}$ is the rate-of-strain

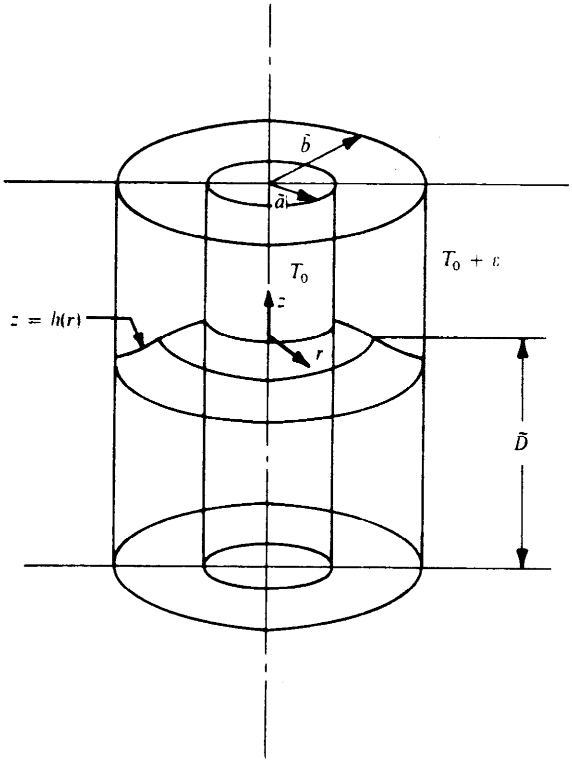

FIG. 1. *A fixed volume $\pi(\tilde{b}^2 - \tilde{a}^2)\tilde{D}$ of liquid fills a cylindrical annulus heated from its side up to a mean height $\tilde{D}$ from a flat bottom. The origin of coordinates (r, θ, z) is set in the plane of the mean height. The rigid bottom of the cylinder and the air at the free surface are assumed to be insulated relative to the liquid. Dimensionless coordinates (t, θ, y), values a, b, D and functions $\mathcal{P}(t, y)$, $\Psi(t, y)$, are defined by* (3.6).

tensor and μ is the viscosity. The variables $\Theta(r, z)$, $\mathbf{u}(r, z)$ and $\Phi(r, z)$ are defined in

$$\mathcal{V}_\varepsilon = \mathcal{V}_\varepsilon[r, z|, \quad \tilde{a} \leqq r \leqq \tilde{b}, \quad -\tilde{D} \leqq z \leqq h(r; \varepsilon)],$$

where the dependence of the free surface $h(r; \varepsilon)$ on the temperature difference ε is to be determined. On the free surface, the normal component of the velocity vanishes; $\mathbf{u} \cdot \mathbf{n} = 0$, the normal component of the temperature gradient vanishes; $\mathbf{n} \cdot \nabla T = 0$, the shear stress vanishes; $S_{nt} = 0$ and the normal component of the stress jump is balanced by surface tension

$$S_{nn} - p + p_a = S_{nn} - \Phi + \rho g h = \frac{\sigma}{r}\left(\frac{rh'}{\sqrt{1+h'^2}}\right)'.$$

We shall assume that the fluid sticks to a sharp edge or makes a flat contact with the vertical walls: $h'(a) = h'(b) = 0$ or $h(a) = h(b) = 0$. The Oberbeck–Boussinesq equations are assumed to govern the motion in $\mathcal{V}_\varepsilon$,

$$\rho \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla\Phi + \mathbf{e}_z \rho g \alpha \Theta + \mu \nabla^2 \mathbf{u}, \qquad \mathbf{u} \cdot \nabla\Theta = \kappa \nabla^2 \Theta, \tag{2.1}$$

where α is the coefficient of thermal expansivity and κ is the coefficient of thermal diffusivity.

Now we shall summarize the governing equations. As a notational convenience we shall define two sets of functions, A_ε and B.

$$A_\varepsilon = [\mathbf{u}(r,z), \Theta(r,z) | \operatorname{div} \mathbf{u} = 0 \text{ in } \mathcal{V}_\varepsilon, \mathbf{u}(\tilde{a}, z) = \mathbf{u}(\tilde{b}, z) = 0, \\ \mathbf{u}(r, -\tilde{D}) = 0, \Theta(\tilde{a}, z) = 0, \frac{\partial \Theta}{\partial z}(r, -\tilde{D}) = 0],$$

$$B = [h(r) | 0 = \int_{\tilde{a}}^{\tilde{b}} rh(r)\,dr,\ h(\tilde{a}) = h(\tilde{b}) = 0 \text{ or } h'(\tilde{a}) = h'(\tilde{b}) = 0].$$

We shall seek solutions

$$(2.2) \qquad [\mathbf{u}(r, z; \varepsilon), \Theta(r, z; \varepsilon)] \in A_\varepsilon \quad \text{and} \quad \Phi(r, z)$$

of equations (2.1) for which

$$(2.3) \qquad \Theta(\tilde{b}, z; \varepsilon) = \varepsilon$$

and, on $z = h(r; \varepsilon)$,

$$(2.4) \qquad \begin{aligned} \frac{\partial \Theta}{\partial z} - h' \frac{\partial \Theta}{\partial r} &= 0, \\ w - h'u &= 0, \end{aligned}$$

$$(2.5) \qquad \begin{aligned} (S_{zz} - S_{rr})h' + (1 - h'^2)S_{rz} &= 0, \\ S_{zz} - h'S_{zr} - \Phi &= \frac{\sigma}{r}\left(\frac{rh'}{\sqrt{1+h'^2}}\right)' - \rho g h, \end{aligned}$$

where $h \in B$.

3. Perturbation equations. We are now going to set down the equations governing the first order perturbation problem. The derivation of these equations and of the ones governing higher order approximations may be obtained by the methods outlined in the paper by Joseph and Sturges [7]. The solution at each order is defined on the flat domain, $\mathcal{V}_0 = \mathcal{V}_0(r_0, z_0 | \tilde{a} \leqq r_0 \leqq \tilde{b}, -\tilde{D} \leqq z_0 \leqq 0)$. To obtain the solution in deformed domain we replace r_0 with r and z_0 with $(z-h)\tilde{D}/(\tilde{D}+h)$ where, to first order, $h(r;\varepsilon) = h^{(1)}(r_0)\varepsilon$ (see Joseph and Sturges [7]). However, for simplicity, we drop subscript 0's where it is apparent that the problem is considered on the reference domain. Now, the first order approximation may also be obtained by linearization leading to Stokes equations

$$[\mathbf{u}, \Theta, \Phi, h] \to \varepsilon[\mathbf{u}^{(1)}, \Theta^{(1)}, \Phi^{(1)}, h^{(1)}],$$

where

$$[u^{\langle 1\rangle}(r,z), \Theta^{\langle 1\rangle}(r,z)] \in A_0, \qquad h^{\langle 1\rangle} \in B,$$

$$0 = -\nabla\Phi^{\langle 1\rangle} + \mathbf{e}_z \rho\alpha g \Theta^{\langle 1\rangle} + \mu \nabla^2 \mathbf{u}^{\langle 1\rangle} \quad \text{in } \mathscr{V}_0,$$

$$0 = \kappa \nabla^2 \Theta^{\langle 1\rangle} \quad \text{in } \mathscr{V}_0,$$

$$\Theta^{\langle 1\rangle}(\tilde{b}, z) = 1, \qquad -\tilde{D} \leqq z \leqq 0,$$

$$w^{\langle 1\rangle} = \mu\left[\frac{\partial u^{\langle 1\rangle}}{\partial z} + \frac{\partial w^{\langle 1\rangle}}{\partial r}\right] = 0 \quad \text{on } z = 0$$

and

$$2\mu \frac{\partial w^{\langle 1\rangle}}{\partial z} - \Phi^{\langle 1\rangle} = \sigma\left(h^{\langle 1\rangle\prime\prime} + \frac{1}{r} h^{\langle 1\rangle\prime}\right) - \rho g h^{\langle 1\rangle} \quad \text{on } z = 0.$$

We find that

$$\Theta^{\langle 1\rangle}(r) = \frac{\ln(\tilde{a}/r)}{\ln(\tilde{a}/\tilde{b})} \tag{3.1}$$

and

$$0 = -\nabla\Phi^{\langle 1\rangle} + \mathbf{e}_z \rho\alpha g \frac{\ln(\tilde{a}/r)}{\ln(\tilde{a}/\tilde{b})} + \mu \nabla^2 \mathbf{u}^{\langle 1\rangle}. \tag{3.2}$$

We introduce a stream function $\psi(r, z)$,

$$r u^{\langle 1\rangle} = \frac{\partial \psi}{\partial z}, \qquad r w^{\langle 1\rangle} = -\frac{\partial \psi}{\partial r} \tag{3.3}$$

and find that

$$\mathscr{L}^2 \psi = \frac{-\rho\alpha g}{\mu \ln(\tilde{a}/\tilde{b})} \quad \text{in } \mathscr{V}_0, \tag{3.4}$$

where

$$\mathscr{L}(\cdot) = r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial(\cdot)}{\partial r}\right) + \frac{\partial^2(\cdot)}{\partial z^2}.$$

The boundary conditions to be satisfied by ψ are such that

$$\begin{gathered} \psi(\tilde{a}, z), \quad \psi(\tilde{b}, z), \quad \frac{\partial \psi}{\partial r}(\tilde{a}, z), \quad \frac{\partial \psi}{\partial r}(\tilde{b}, z), \\ \psi(r, -\tilde{D}), \quad \frac{\partial \psi}{\partial z}(r, -\tilde{D}), \quad \psi(r, 0), \quad \frac{\partial^2 \psi}{\partial z^2}(r, 0) = 0. \end{gathered} \tag{3.5}$$

Equations (3.4) and (3.5) suffice to determine the function ψ uniquely. Given this function, $\mathbf{u}^{\langle 1\rangle}$ may be computed from (3.3) and $\Phi^{\langle 1\rangle}$ from (3.2). The first order correction for the height rise is given by

$$\sigma\left(h^{\langle 1\rangle\prime\prime} + \frac{1}{r} h^{\langle 1\rangle\prime}\right) - \rho g h^{\langle 1\rangle} = -\Phi^{\langle 1\rangle} - \frac{2\mu}{r} \frac{\partial^2 \psi}{\partial r \partial z}.$$

To reformulate the problem (3.4) and (3.5) as an edge problem, we introduce the following change of variables:

$$\eta = \tilde{a}/\tilde{b},$$

$$(t, y) = \frac{2}{\tilde{b}(1-\eta)}(r, z),$$

(3.6a) $$a \equiv \frac{2\eta}{1-\eta} \leqq t \leqq \frac{2}{1-\eta} \equiv b, \qquad \mathscr{V}_0(t, y),$$

$$-D \equiv \frac{-2\tilde{D}}{\tilde{b}(1-\eta)} \leqq y \leqq 0,$$

(3.6b) $$\mathscr{P} = -\frac{16}{\rho\alpha g\tilde{b}} \frac{\ln \eta}{1-\eta} \Phi^{\langle 1 \rangle},$$

(3.6c) $$\Psi = -\frac{256\,\mu}{\rho\alpha g\tilde{b}^4} \frac{\ln \eta}{(1-\eta)^4} \psi - \left\{ A_0 + \left[B_0 + C_0 \ln\left(\frac{1-\eta}{2\eta} t\right) \right] t^2 + D_0 t^4 + t^4 \ln\left(\frac{1-\eta}{2\eta} t\right) \right\},$$

where

$$A_0 = \frac{4}{(1-\eta)^4} \frac{\eta^2}{1-\eta^2} \frac{-(1-\eta^2)^2 + 4\eta^2(\ln \eta)^2}{1-\eta^2+(1+\eta^2)\ln \eta},$$

$$B_0 = \frac{1}{(1-\eta)^2} \frac{1}{1-\eta^2} \frac{(1-\eta^2)^2(1+\eta^2) - 2(1-\eta^2)^2\eta^2 \ln \eta - 8\eta^2(\ln \eta)^2}{1-\eta^2+(1+\eta^2)\ln \eta},$$

$$C_0 = \frac{2}{(1-\eta)^2} \frac{-(1-\eta^2)(1+\eta^2) - 4\eta^2 \ln \eta}{1-\eta^2+(1+\eta^2)\ln \eta},$$

$$D_0 = \frac{1}{4} \frac{1}{1-\eta^2} \frac{-(1-\eta^2)^2 + 2(1-\eta^2)^2 \ln \eta + 4(\ln \eta)^2}{1-\eta^2+(1+\eta^2)\ln \eta}.$$

The dimensionless height correction $H(t)$ is related to $h^{\langle 1 \rangle}$ by

(3.6d) $$H(t) = \frac{-64\sigma \ln \eta}{\rho g\alpha\tilde{b}^3(1-\eta)^3} h^{\langle 1 \rangle}(r).$$

In the new variables

(3.7a) $$L^2\Psi = 0 \quad \text{in } \mathscr{V}_0(t, y),$$

where

$$L(\cdot) \equiv \frac{\partial^2(\cdot)}{\partial t^2} - \frac{1}{t}\frac{\partial(\cdot)}{\partial t} + \frac{\partial^2(\cdot)}{\partial y^2}.$$

Ψ satisfies the boundary conditions

(3.7b) $$\left[\Psi(a, y), \Psi(b, y), \frac{\partial \Psi}{\partial t}(a, y), \frac{\partial \Psi}{\partial t}(b, y) \right] = 0$$

and edge conditions

$$\frac{\partial^2\Psi(t,0)}{\partial y^2}=\frac{\partial\Psi}{\partial y}(t,-D)=0, \tag{3.7c}$$

$$\Psi(t,0)=\Psi(t,-D)=-\left\{A_0+\left(B_0+C_0\ln\left(\frac{1-\eta}{2\eta}t\right)\right)t^2+D_0t^4 + t^4\ln\left(\frac{1-\eta}{2\eta}t\right)\right\}. \tag{3.7d}$$

Given Ψ, the dimensionless pressure $\mathcal{P}$ may be found from the equations

$$\frac{\partial\mathcal{P}}{\partial t}=\frac{1}{2}\nabla^2\left(\frac{1}{t}\frac{\partial\Psi}{\partial y}\right)-\frac{1}{2t^3}\frac{\partial\Psi}{\partial y}, \tag{3.8a}$$

$$\frac{\partial\mathcal{P}}{\partial y}=-\frac{1}{2}\nabla^2\left(\frac{1}{t}\frac{\partial\Psi}{\partial t}\right)-(8D_0+10). \tag{3.8b}$$

The dimensionless height rise coefficient satisfies the equation

$$H''+\frac{1}{t}H'-m^2H=-\left(\mathcal{P}+\frac{1}{t}\frac{\partial^2\Psi}{\partial t\partial y}\right), \tag{3.9}$$

where

$$m^2=\frac{\rho g\tilde{b}^2(1-\eta)^2}{4\sigma}.$$

Equations (3.7) define an edge problem for Stokes flow between cylinders. The free surface problem (3.9) may be solved by integration after Ψ is determined provided that suitable boundary conditions are prescribed for H. We shall consider contact line conditions

$$H(a)=H(b)=0 \tag{3.10a}$$

or contact angle conditions, with a prescribed flat contact,

$$H'(a)=H'(b)=0. \tag{3.10b}$$

The mean value of the height rise must vanish:

$$\int_a^b tH(t)\,dt=0. \tag{3.11}$$

4. Solution of the edge problem. The solution of problem (3.7) is given by

$$\Psi(t,y)=\lim_{N\to\infty}\sum_{-N}^{N}[C_n e^{p_n y}+D_n e^{-p_n y}]\phi_1^{(n)}(t)/p_n^2, \tag{4.1}$$

where $\phi_1^{(n)}(t)$ are strip eigenfunctions for Stokes flow between concentric cylinders: $L^2e^{\pm p_n y}\phi_1^{(n)}=0$ and $\phi_1^{(n)}$ satisfies the boundary conditions (3.7b). The strip eigenfunctions are given by (see A.9)

$$\phi_1^{(n)}(t)=A_1^{(n)}tJ_1(p_nt)+A_2^{(n)}tY_1(p_nt)+A_3^{(n)}t^2J_0(p_nt) + A_4^{(n)}t^2Y_0(p_nt), \tag{4.2}$$

where the coefficients $A_i^{(n)}$ are defined by (A.13) and the eigenvalues p_n appear as the first quadrant roots of (A.12). The eigenvalues may be arranged in a sequence corresponding to increasing size of the real part of p_n, $n = 1, 2, 3, \cdots$. The roots of (A.12) are symmetrically located in the complex p plane; we choose p_n, $n > 0$, as first quadrant roots and define

$$p_{-n} = \bar{p}_n, \tag{4.3}$$

where the overbar designates complex conjugate. Then

$$\phi_1^{(-n)}(t) = \bar{\phi}_1^{(n)}(t) \qquad (n = \pm 1, \pm 2, \cdots)$$

is a strip eigenfunction belonging to the eigenvalue p_{-n}. Since the given edge data (3.7d) are real

$$C_{-n} = \bar{C}_n, \qquad D_{-n} = \bar{D}_n \qquad (n = \pm 1, \pm 2, \cdots). \tag{4.4}$$

It is necessary to specify how to compute the Fourier coefficients (4.4) and it is easiest to compute the coefficients when the cylinders are semi-infinite.

4.1. Solution of the edge problem between semi-infinite concentric cylinders. When $D \to \infty$, we replace the boundary conditions on $y = -D$ with the requirement that $\Psi \to 0$ as $y \to -\infty$. Therefore, $D_n = 0$ for all n. Substitution of (4.1) with $D_n = 0$ into (3.7c,d) leads us to the relation

$$\begin{bmatrix} \dfrac{\partial^2 \Psi(t,0)}{\partial y^2} \\ \Psi(t,0) \end{bmatrix} = -\begin{bmatrix} 0 \\ A_0 + \left(B_0 + C_0 \ln\left(\dfrac{1-\eta}{2\eta}t\right)\right)t^2 + \left(D_0 + \ln\left(\dfrac{1-\eta}{2\eta}t\right)\right)t^4 \end{bmatrix}$$
$$= \sum_{-\infty}^{\infty} C_n \begin{bmatrix} \phi_1^{(n)}(t) \\ \phi_1^{(n)}(t)/p_n^2 \end{bmatrix}.$$

This edge condition may be put into standard form:

$$\begin{bmatrix} \dfrac{\partial^2 \Psi}{\partial y^2}(t,0) \\ t\dfrac{\partial}{\partial t}\dfrac{1}{t}\dfrac{\partial \Psi}{\partial t}(t,0) \end{bmatrix} = -\begin{bmatrix} 0 \\ 2C_0 + (8D_0+6)t^2 + 8t^2 \ln\left(\dfrac{1-\eta}{2\eta}t\right) \end{bmatrix} = \sum_{-\infty}^{\infty} C_n \begin{bmatrix} \phi_1^{(n)}(t) \\ \phi_2^{(n)}(t) \end{bmatrix}, \tag{4.5}$$

where

$$\phi_2^{(n)} = \frac{1}{p_n^2} t \frac{d}{dt}\left[\frac{1}{t}\frac{d\phi_1^{(n)}}{dt}\right].$$

To determine the constants C_n, we introduce the vectors:

$$\mathbf{f} = -\begin{bmatrix} 0 \\ 2C_0 + (8D_0+6)t^2 + 8t^2 \ln\left(\frac{1-\eta}{2\eta}t\right) \end{bmatrix},$$

$$\boldsymbol{\phi}^{(n)} = \begin{bmatrix} \phi_1^{(n)}(t) \\ \phi_2^{(n)}(t) \end{bmatrix},$$

$$\boldsymbol{\psi}^{(n)} = [\psi_1^{(n)}(t), \psi_2^{(n)}(t)],$$

where $\boldsymbol{\phi}^{(n)}$ and $\boldsymbol{\psi}^{(n)}$ are defined through the eigenvalue problem

$$t\frac{d}{dt}\left(\frac{1}{t}\frac{d}{dt}\boldsymbol{\phi}^{(n)}\right) + p_n^2 \mathbf{A} \cdot \boldsymbol{\phi}^{(n)} = 0, \tag{4.6a}$$

$$t\frac{d}{dt}\left(\frac{1}{t}\frac{d}{dt}\boldsymbol{\psi}^{(n)}\right) + p_n^2 \boldsymbol{\psi}^{(n)} \cdot \mathbf{A} = 0, \tag{4.6b}$$

$$\left[\phi_1^{(n)}, \psi_2^{(n)}, \frac{d\phi_1^{(n)}}{dt}, \frac{d\psi_2^{(n)}}{dt}\right] = 0 \quad \text{at } t = a, b, \tag{4.6c}$$

and

$$\mathbf{A} = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}.$$

$\boldsymbol{\psi}^{(n)}(t)$ is the adjoint eigenvector, relative to $\mathbf{A}$, belonging to p_n. The eigenvectors $\boldsymbol{\phi}^{(n)}$ and $\boldsymbol{\psi}^{(n)}$ are given in explicit forms by (C.8). The eigenvalues p_n are determined by the methods discussed in Appendix B.

To compute C_n in (4.5) we use the biorthogonality property (C.7) which may be written as

$$\int_a^b \frac{1}{t}\boldsymbol{\psi}^{(m)} \cdot \mathbf{A} \cdot \boldsymbol{\phi}^{(n)}\, dt = K_n \delta_{nm}, \tag{4.7}$$

where δ_{nm} is Kronecker's delta and K_n is given by (C.9). We find that

$$C_n = I_n / K_n, \tag{4.8}$$

where

$$I_n = \int_a^b t^{-1}[\psi_1^{(n)}, \psi_2^{(n)}] \cdot \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ -2C_0 - (8D_0+6)t^2 - 8t^2 \ln\left(\frac{1-\eta}{2\eta}t\right) \end{bmatrix} dt \tag{4.9}$$

$$= \frac{32}{p_n^4}[A_3^{(n)} J_0(p_n b) + A_4^{(n)} Y_0(p_n b) - A_3^{(n)} J_0(p_n a) - A_4^{(n)} Y_0(p_n a)].$$

In evaluating the integral we used the formulas derived in Appendix D. Finally,

$$C_n = \frac{-16\{[A_3^{(n)}J_0(p_nb)+A_4^{(n)}Y_0(p_nb)] - [A_3^{(n)}J_0(p_na)+A_4^{(n)}Y_0(p_na)]\}}{p_n^2\{b^2[A_3^{(n)}J_1(p_nb)+A_4^{(n)}Y_1(p_nb)]^2 - a^2[A_3^{(n)}J_1(p_na)+A_4^{(n)}Y_1(p_na)]^2\}}, \tag{4.10}$$

where $A_3^{(n)}$ and $A_4^{(n)}$ are given by (A.13c,d).

The numerical convergence of the partial sum

$$-\begin{bmatrix} 0 \\ 2C_0+(8D_0+6)t^2+8t^2\ln\left(\frac{1-\eta}{2\eta}t\right)\end{bmatrix} = \sum_{-N}^{N} C_n\begin{bmatrix}\phi_1^{(n)}(t)\\ \phi_2^{(n)}(t)\end{bmatrix} \tag{4.11}$$

is very rapid (Tables 1, 2, 3, 4) and for practical purposes the series has converged after three terms. Mathematical convergence of the series (4.11) follows easily from the asymptotic representations given in Appendix B, followed by estimates of the type used in the paper of Joseph [4].[1]

TABLE 1
Convergence of the partial sums (4.11) *for* $\eta = 0.8$

$$\begin{bmatrix} f \\ g\end{bmatrix} = \begin{bmatrix} 0 \\ -(2C_0+(8D_0+6)t^2+8t^2\ln\left(\frac{1-\eta}{2\eta}t\right)\end{bmatrix} \sim \sum_{-N}^{N} C_n\begin{bmatrix}\phi_1^{(n)}(t)\\ \phi_2^{(n)}(t)\end{bmatrix}$$

t	$g(t)$	$N=1$	$N=3$	$N=5$	$N=7$	$N=9$
10.0	−5.558	−5.255	−5.478	−5.526	−5.542	−5.549
9.8	−2.551	−2.666	−2.563	−2.529	−2.531	−2.542
9.6	−0.238	−0.287	−0.201	−0.247	−0.255	−0.238
9.4	1.394	1.438	1.377	1.387	1.407	1.390
9.2	2.358	2.418	2.345	2.370	2.348	2.363
9.0	2.668	2.709	2.682	2.656	2.677	2.663
8.8	2.339	2.364	2.328	2.352	2.330	2.344
8.6	1.384	1.375	1.373	1.375	1.397	1.381
8.4	−0.181	−0.269	−0.146	−0.187	−0.199	−0.182
8.2	−2.342	−2.438	−2.361	−2.322	−2.321	−2.332
8.0	−5.083	−4.699	−4.993	−5.048	−5.066	−5.073

[1] The formal part of the solution developed carries over to edge problems in which the given data

$$\mathbf{f} = \begin{bmatrix} f(t) \\ g(t)\end{bmatrix} = \begin{bmatrix} \partial^2\Psi/\partial y^2 \\ t\partial\left(\frac{1}{t}\partial\Psi/\partial t\right)\Big/\partial t\end{bmatrix}_{y=0}$$

is compatible with side wall boundary conditions $\Psi = \partial\Psi/\partial t = 0$ at $t = a,b$. This condition may be written as

$$\int_a^b \frac{1}{t} g(t)\,dt = \int_a^b tg(t)\,dt = 0.$$

Mathematical convergence with such arbitrary data has not yet been established (see Smith [10, p. 231]). The data which arise in the free surface problems considered here and in the similar problems (see Joseph [4]) are "good" data for which convergence can be established.

TABLE 1 (cont.)

t	$f(t)$	$N=1$	$N=3$	$N=5$	$N=7$	$N=9$
10.0	0	0	0	0	0	0
9.8	0	−0.100	−0.076	−0.045	−0.019	−0.001
9.6	0	−0.169	−0.029	0.027	0.019	−0.004
9.4	0	−0.096	0.051	0	−0.015	0.006
9.2	0	0.035	0.009	−0.016	0.013	−0.007
9.0	0	0.096	−0.039	0.020	−0.011	0.007
8.8	0	0.034	0.013	−0.018	0.013	−0.007
8.6	0	−0.091	0.048	0.004	−0.016	0.005
8.4	0	−0.158	−0.035	0.026	0.020	−0.003
8.2	0	−0.091	−0.077	−0.047	−0.020	−0.002
8.0	0	0	0	0	0	0

TABLE 2

Convergence of the partial sums (4.11) *for* $\eta = 0.5$

$$\begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} 0 \\ -\left(2C_0+(8D_0+6)t^2+8t^2\ln\left(\frac{1-\eta}{2\eta}t\right)\right) \end{bmatrix} \sim \sum_{-N}^{N} C_n \begin{bmatrix} \phi_1^{(n)}(t) \\ \phi_2^{(n)}(t) \end{bmatrix}$$

t	$g(t)$	$N=1$	$N=3$	$N=5$	$N=7$	$N=9$
4.0	−5.938	−5.737	−5.499	−5.691	−5.705	−5.711
3.8	−2.708	−2.848	−2.831	−2.709	−2.713	−2.725
3.6	−0.266	−0.273	−0.379	−0.444	−0.486	−0.431
3.4	1.421	1.527	1.428	1.378	1.397	1.379
3.2	2.391	2.493	2.451	2.498	2.475	2.491
3.0	2.682	2.728	2.710	2.685	2.705	2.691
2.8	2.336	2.328	2.368	2.374	2.351	2.366
2.6	1.396	1.333	1.420	1.473	1.497	1.482
2.4	−0.089	−0.216	−0.130	−0.157	−0.172	−0.156
2.2	−2.069	−2.149	−2.162	−2.069	−2.066	−2.076
2.0	−4.488	−4.041	−4.208	−4.344	−4.364	−4.371

t	$f(t)$	$N=1$	$N=3$	$N=5$	$N=7$	$N=9$
4.0	0	0	0	0	0	0
3.8	0	−0.109	−0.093	−0.027	0	0.016
3.6	0	−0.183	−0.177	0.004	−0.009	−0.031
3.4	0	−0.104	−0.119	−0.150	−0.162	−0.140
3.2	0	0.033	0.024	−0.101	−0.073	−0.094
3.0	0	0.095	0.087	0.147	0.115	0.134
2.8	0	0.034	0.042	0.103	0.136	0.116
2.6	0	−0.086	−0.072	−0.135	−0.159	−0.137
2.4	0	−0.145	−0.150	−0.189	−0.192	−0.215
2.2	0	−0.081	−0.094	−0.044	−0.017	0.002
2.0	0	0	0	0	0	0

TABLE 3
Convergence of the partial sums (4.11) *for* $\eta = 0.25$

$$\begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} 0 \\ -\left(2C_0 + (8D_0+6)t^2 + 8t^2 \ln\left(\frac{1-\eta}{2\eta}t\right)\right) \end{bmatrix} \sim \sum_{-N}^{N} C_n \begin{bmatrix} \phi_1^{(n)}(t) \\ \phi_2^{(n)}(t) \end{bmatrix}$$

t	$g(t)$	$N=1$	$N=3$	$N=5$	$N=7$	$N=9$
2.667	−6.302	−6.270	−6.261	−6.392	−6.426	−6.438
2.467	−2.842	−3.027	−2.836	−2.760	−2.741	−2.746
2.267	−0.267	−0.207	−0.219	−0.193	−0.230	−0.227
2.067	1.479	1.689	1.449	1.522	1.552	1.555
1.867	2.455	2.632	2.430	2.424	2.417	2.411
1.667	2.726	2.783	2.739	2.736	2.732	2.736
1.467	2.365	2.305	2.361	2.381	2.375	2.372
1.267	1.454	1.307	1.457	1.401	1.438	1.444
1.067	0.086	−0.101	0.117	0.101	0.067	0.077
0.867	−1.627	−1.691	−1.663	−1.594	−1.580	−1.588
0.667	−3.552	−3.058	−3.441	−3.536	−3.563	−3.574

t	$f(t)$	$N=1$	$N=3$	$N=5$	$N=7$	$N=9$
2.667	0	0	0	0	0	0
2.467	0	−0.127	−0.077	−0.052	−0.023	0
2.267	0	−0.215	−0.014	−0.166	−0.137	−0.155
2.067	0	−0.131	0.062	0.045	−0.008	−0.002
1.867	0	0.020	0.003	0.092	0.117	0.120
1.667	0	0.094	0.041	−0.027	−0.019	−0.024
1.467	0	0.044	0.022	−0.074	−0.051	−0.042
1.267	0	−0.064	0.042	0.056	0.002	−0.014
1.067	0	−0.117	−0.050	0.052	0.072	0.047
0.867	0	−0.064	−0.075	−0.052	−0.023	0
0.667	0	0	0	0	0	0

TABLE 4
Convergence of the partial sums (4.11) *for* $\eta = 0.08$

$$\begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} 0 \\ -\left(2C_0 + (8D_0+6)t^2 + 8t^2 \ln\left(\frac{1-\eta}{2\eta}t\right)\right) \end{bmatrix} \sim \sum_{-N}^{N} C_n \begin{bmatrix} \phi_1^{(n)}(t) \\ \phi_2^{(n)}(t) \end{bmatrix}$$

t	$g(t)$	$N=1$	$N=3$	$N=5$	$N=7$	$N=9$
2.174	−6.629	−6.862	−6.643	−6.430	−6.472	−6.488
1.974	−2.958	−3.220	−2.929	−3.052	−3.024	−3.026
1.774	−0.249	−0.082	−0.173	−0.233	−0.275	−0.279
1.574	1.567	1.956	1.517	1.549	1.485	1.497
1.374	2.567	2.873	2.511	2.567	2.561	2.549
1.174	2.838	2.907	2.847	2.850	2.856	2.861
0.974	2.481	2.322	2.502	2.474	2.462	2.463
0.774	1.616	1.323	1.640	1.592	1.629	1.626
0.574	0.391	0.107	0.407	0.423	0.381	0.389
0.374	−0.998	−1.060	−1.052	−1.004	−0.982	−0.988
0.174	−2.268	−1.820	−2.153	−2.210	−2.235	−2.246

TABLE 4 (cont.)

t	$f(t)$	$N=1$	$N=3$	$N=5$	$N=7$	$N=9$
2.174	0	0	0	0	0	0
1.974	0	−0.161	−0.088	−0.080	−0.054	−0.031
1.774	0	−0.288	−0.010	−0.008	0.031	0.018
1.574	0	−0.203	0.084	0.032	0.126	0.123
1.374	0	−0.015	0.014	0.005	−0.012	−0.002
1.174	0	0.109	−0.044	−0.024	−0.038	−0.047
0.974	0	0.100	0.013	−0.006	0.035	0.024
0.774	0	0.011	0.029	0.048	−0.011	0.007
0.574	0	−0.050	−0.055	−0.013	0.015	−0.011
0.374	0	−0.032	−0.063	−0.066	−0.042	−0.019
0.174	0	0	0	0	0	0

In Fig. 2 we have plotted the level lines of (4.1) which gives the stream function $\Psi(t, y)$ for the edge problem. We have also shown the true streamlines ψ of the flow.

$$-\frac{256\mu \ln \eta}{\rho\alpha g b^4(1-\eta)^4}\psi = A_0 + \left(B_0 + C_0 \ln\left(\frac{1-\eta}{2\eta}t\right)\right)t^2 + D_0 t^4 + t^4 \ln\left(\frac{1-\eta}{2\eta}t\right) + \sum_{-\infty}^{\infty} \frac{C_n}{p_n^2} e^{p_n y}\phi_1^{(n)}(t), \tag{4.12}$$

where C_n is defined by (4.10). The streamlines are given in the reference domain; they cannot be plotted in the physical domain without first finding the first order height correction (see § 5 of Joseph and Sturges [7]). The streamlines shown in Fig. 2 show that the edge eddies exist, to turn the stream around at the free edge.

The pressure corresponding to (4.1) with $D_n = 0$ can be obtained from (3.8) as

$$\mathcal{P} = -(8D_0 + 10)y + \sum_{-\infty}^{\infty} \frac{C_n e^{p_n y}}{p_n}\{A_3^{(n)} J_0(p_n t) + A_4^{(n)} Y_0(p_n t)\} + C, \tag{4.13}$$

where C is a constant to be determined from (3.11).

4.2. Solution of the edge problem between cylinders of finite depth. We are now considering problem (3.7) with $D < \infty$. There is an edge at both the bottom and the top. There are now two sets of coefficients (C_n and D_n) to be determined from the edge conditions (3.7c) and (3.7d). At the top ($y = 0$), we find, using (4.1), that

$$\begin{bmatrix} \frac{\partial^2 \Psi}{\partial y^2}(t, 0) \\ t\frac{\partial}{\partial t}\frac{1}{t}\frac{\partial \Psi}{\partial t}(t, 0) \end{bmatrix} = \begin{bmatrix} 0 \\ -\left\{2C_0 + (8D_0 + 6)t^2 + 8t^2 \ln\left(\frac{1-\eta}{2\eta}t\right)\right\} \end{bmatrix} = \sum_{-\infty}^{\infty} (C_n + D_n)\begin{bmatrix} \phi_1^{(n)}(t) \\ \phi_2^{(n)}(t) \end{bmatrix}. \tag{4.14}$$

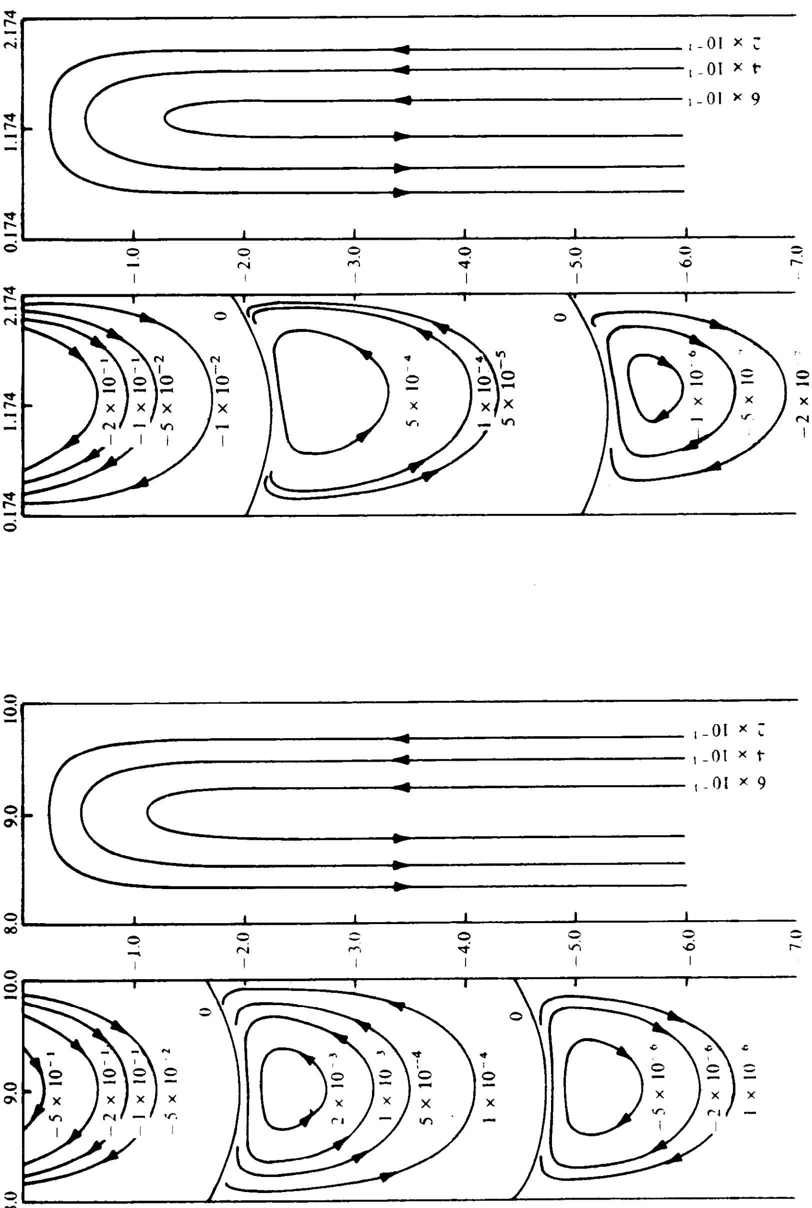

FIG. 2 (a). $\eta = 0.8$

FIG. 2 (b). $\eta = 0.08$

FIG. 2. *Levels lines of the edge eddies* (4.1) *and streamlines* (4.12) *of the flow in the reference domain for the infinitely deep trench*

At the bottom, $y=-D$, we have

$$\begin{bmatrix} \frac{\partial \Psi}{\partial y}(t,-D) \\ \Psi(t,-D) \end{bmatrix} = \begin{bmatrix} 0 \\ -\left\{A_0+\left(B_0+C_0 \ln\left(\frac{1-\eta}{2\eta}t\right)\right)t^2+D_0t^4+t^4\ln\left(\frac{1-\eta}{2\eta}t\right)\right\} \end{bmatrix}$$

$$= \sum_{-\infty}^{\infty} \begin{bmatrix} (C_n e^{-p_nD} - D_n e^{p_nD})\phi_1^{(n)}(t)/p_n \\ (C_n e^{-p_nD} + D_n e^{p_nD})\phi_1^{(n)}(t)/p_n^2 \end{bmatrix},$$

which, after differentiating the bottom row twice, can be written as

$$\begin{bmatrix} \frac{\partial \Psi}{\partial y}(t,-D) \\ t\frac{\partial}{\partial t}\frac{1}{t}\frac{\partial \Psi}{\partial t}(t,-D) \end{bmatrix} = \begin{bmatrix} 0 \\ -\left\{2C_0+(8D_0+6)t^2+8t^2\ln\left(\frac{1-\eta}{2\eta}t\right)\right\} \end{bmatrix}$$

$$= \sum_{-\infty}^{\infty} \begin{bmatrix} (C_n e^{-p_nD} - D_n e^{p_nD})\phi_1^{(n)}(t)/p_n \\ (C_n e^{-p_nD} + D_n e^{p_nD})\phi_2^{(n)}(t) \end{bmatrix}. \tag{4.15}$$

The biorthogonality computation which led to (4.10) now yields

$$C_n+D_n=\frac{I_n}{K_n}=-\frac{16}{p_n^2}\cdot\frac{\{[A_3^{(n)}J_0(p_nb)+A_4^{(n)}Y_0(p_nb)]-[A_3^{(n)}J_0(p_na)+A_4^{(n)}Y_0(p_na)]\}}{\{b^2[A_3^{(n)}J_1(p_nb)+A_4^{(n)}Y_1(p_nb)]^2-a^2[A_3^{(n)}J_1(p_na)+A_4^{(n)}Y_1(p_na)]^2\}}. \tag{4.16}$$

The same computation, applied to (4.15) where the boundary conditions are of a different type, defines C_n and D_n implicitly through an infinite set of algebraic equations linear in C_n and D_n:

$$\int_a^b \frac{1}{t}[\psi_1^{(n)}(t),\psi_2^{(n)}(t)]\begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ -\left\{2C_0+(8D_0+6)t^2+8t^2\ln\left(\frac{1-\eta}{2\eta}t\right)\right\} \end{bmatrix} dt$$

$$=\int_a^b \frac{1}{t}[\psi_1^{(n)}(t),\psi_2^{(n)}(t)]\begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}$$

$$\cdot \sum_{-\infty}^{\infty} \begin{bmatrix} (C_n e^{-p_nD} - D_n e^{p_nD})\phi_1^{(n)}(t)/p_n \\ (C_n e^{-p_nD} + D_n e^{p_nD})\phi_2^{(n)}(t) \end{bmatrix} dt,$$

that is,

$$I_n = \sum_{-\infty}^{\infty} C_l e^{-p_l D} \int_a^b \frac{1}{t}\left\{[\psi_1^{(n)}(t), \psi_2^{(n)}(t)]\begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} \phi_1^{(l)}(t) \\ \phi_2^{(l)}(t) \end{bmatrix} + \frac{1-p_l}{p_l}\psi_2^{(n)}(t)\phi_1^{(l)}(t)\right\} dt$$

$$+ \sum_{-\infty}^{\infty} D_l e^{p_l D} \int_a^b \frac{1}{t}\left\{[\psi_1^{(n)}(t), \psi_2^{(n)}(t)]\begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} \phi_1^{(l)}(t) \\ \phi_2^{(l)}(t) \end{bmatrix} - \frac{1+p_l}{p_l}\psi_2^{(n)}(t)\phi_1^{(l)}(t)\right\} dt$$

$$(4.17) \qquad = (C_n e^{-p_n D} + D_n e^{p_n D})K_n + \sum_{-\infty}^{\infty}\left(C_l e^{-p_l D} - \frac{1+p_l}{1-p_l} D_l e^{p_l D}\right) \cdot \frac{1-p_l}{p_l}\int_a^b \frac{1}{t}\psi_2^{(n)}(t)\phi_1^{(l)}(t)\, dt$$

$$= \sum_{-\infty}^{\infty} A_{ln}\left(C_l e^{-p_l D} - \frac{1+p_l}{1-p_l} D_l e^{p_l D}\right) + (C_n e^{-p_n D} + D_n e^{p_n D})K_n,$$

where

$$A_{ln} = \frac{1-p_l}{p_l}\int_a^b \frac{1}{t}\phi_1^{(l)}(t)\psi_2^{(n)}(t)\, dt.$$

We can combine (4.16) and (4.17) to get

$$(4.18) \qquad K_n D_n(e^{-p_n D} - e^{p_n D}) + \sum_{-\infty}^{\infty} A_{ln} D_l\left(e^{-p_l D} + \frac{1+p_l}{1-p_l} e^{p_l D}\right) = I_n(e^{-p_n D} - 1) + \sum_{-\infty}^{\infty} A_{ln}\frac{I_l}{K_l} e^{-p_l D}$$

for $n = \pm 1, \pm 2, \cdots$.

We solved (4.18) by truncation and checked the covergence of the solution of the truncated equations numerically. In all cases considered by us convergence is very rapid (Tables 5 and 6). In deep cylinders the factor $e^{p_l D}$ in the first term of (4.18) becomes large; it is convenient for computations in the deep trench to be worked with the coefficients $\hat{D}_l = e^{p_l D} D_l$ which satisfy the relation

$$(4.19) \qquad K_n \hat{D}_n(e^{-2p_n D} - 1) + \sum_{-\infty}^{\infty} A_{ln}\hat{D}_l\left(e^{-2p_l D} + \frac{1+p_l}{1-p_l}\right) = I_n(e^{-p_n D} - 1) + \sum_{-\infty}^{\infty} A_{ln}\frac{I_l}{K_l} e^{-p_l D}.$$

The $\hat{D}_n$ are computed from (4.19); then the C_n are computed from (4.16).

We computed coefficients and plotted level lines of the edge stream function $\Psi(t, y)$ satisfying (3.7) and the true stream function $\psi(t, y)$ given by

$$-\frac{256\mu \ln \eta}{\rho\alpha g\tilde{b}^4(1-\eta)^4}\psi = \left[A_0+\left(B_0+C_0 \ln\left(\frac{1-\eta}{2\eta}t\right)\right)t^2+D_0t^4 + t^4 \ln\left(\frac{1-\eta}{2\eta}t\right)\right]+\Psi(t, y). \tag{4.20}$$

The effect of edge eddies at the top and bottom of the fluid extends into the interior for a distance about three gap widths. The effect of these eddies is most easily ascertained by the inspection of Figs. 3, 4 and 5.

The pressure corresponding to (4.1) can be obtained from (3.8) as

$$\begin{aligned}\mathcal{P} = C-(8D_0+10)y + \sum_{-N}^{N} p_n^{-1}(C_n e^{p_n y} - D_n e^{-p_n y})\\ \cdot [A_3^{(n)}J_0(p_n t)+A_4^{(n)}Y_0(p_n t)],\end{aligned} \tag{4.21}$$

where C is a constant to be determined from the zero mean condition (3.11) for $H(t)$.

5. The shape of the free surface. We turn next to the calculation of the fist approximation giving the change of the shape of the free surface

$$h(r)=\varepsilon h^{(1)}(r) = -\frac{\rho g\alpha\tilde{b}^3(1-\eta)^3}{64\sigma \ln \eta}\varepsilon H(t).$$

The unknown function $H(t)$ can be computed from (3.9), (3.10) and (3.11) when the pressure $\mathcal{P}$ and stream function Ψ are known. We find that

$$\begin{aligned}H''+\frac{1}{t}H'-m^2H = -\sum_{-N}^{N}(C_n-D_n)\{[A_1^{(n)}J_0(p_n t)+A_2^{(n)}Y_0(p_n t)]\\ -t[A_3^{(n)}J_1(p_n t)+A_4^{(n)}Y_1(p_n t)]\\ +\frac{3}{p_n}[A_3^{(n)}J_0(p_n t)+A_4^{(n)}Y_0(p_n t)]\}-C.\end{aligned} \tag{5.1}$$

The general solution of (5.1) is

$$\begin{aligned}H = B_1I_0(mt)+B_2K_0(mt)+C/m^2-\sum_{-N}^{N}(C_n-D_n)\frac{1}{m^2+p_n^2}\\ \cdot\Big\{-[A_1^{(n)}J_0(p_n t)+A_2^{(n)}Y_0(p_n t)]\\ +t[A_3^{(n)}J_1(p_n t)+A_4^{(n)}Y_1(p_n t)]\\ +\left[\frac{2p_n}{m^2+p_n^2}-\frac{3}{p_n}\right][A_3^{(n)}J_0(p_n t)+A_4^{(n)}Y_0(p_n t)]\Big\}.\end{aligned} \tag{5.2}$$

TABLE 5

Convergence of the partial sums (3.7d) for $\eta = 0.8$ *in an annular trench of finite depth* $D = 2$ (*Gap/Depth* = 1)

$$\begin{bmatrix} \Psi(t,0;N) \\ \Psi(t,-D;N) \end{bmatrix} = \sum_{\substack{-N \\ N\neq 0}}^{N} \frac{\phi_1^{(n)}(t)}{p_n^2} \begin{bmatrix} C_n + D_n \\ C_n e^{-p_n D} + D_n e^{p_n D} \end{bmatrix}$$

$$\sim -\left\{ A_0 + \left(B_0 + C_0 \ln\left(\frac{1-\eta}{2\eta} t\right)\right) t^2 + \left(D_0 + \ln\left(\frac{1-\eta}{2\eta} t\right)\right) t^4 \right\} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

t	$-\left\{A_0 + \left(B_0 + C_0 \ln\left(\frac{1-\eta}{2\eta} t\right)\right) t^2 + \left(D_0 + \ln\left(\frac{1-\eta}{2\eta} t\right)\right) t^4\right\}$	$\Psi(t,0;1)$	$\Psi(t,0;5)$
8.0	0	0	0
8.2	−0.0831	−0.0801	−0.0833
8.4	−0.2653	−0.2614	−0.2653
8.6	−0.4614	−0.4599	−0.4613
8.8	−0.6081	−0.6099	−0.6082
9.0	−0.6658	−0.6698	−0.6658
9.2	−0.6191	−0.6236	−0.6192
9.4	−0.4781	−0.4809	−0.4780
9.6	−0.2799	−0.2795	−0.2799
9.8	−0.0893	−0.0875	−0.0895
10.0	0	0	0

t	$-\left\{A_0 + \left(B_0 + C_0 \ln\left(\frac{1-\eta}{2\eta} t\right)\right) t^2 + \left(D_0 + \ln\left(\frac{1-\eta}{2\eta} t\right)\right) t^4\right\}$	$\Psi(t,-D;1)$	$\Psi(t,-D;5)$
8.0	0	0	0
8.2	−0.0831	−0.0917	−0.0833
8.4	−0.2653	−0.2791	−0.2653
8.6	−0.4614	−0.4641	−0.4614
8.8	−0.6081	−0.5928	−0.6080
9.0	−0.6658	−0.6426	−0.6660
9.2	−0.6191	−0.6062	−0.6189
9.4	−0.4781	−0.4853	−0.4781
9.6	−0.2799	−0.2985	−0.2799
9.8	−0.0893	−0.1003	−0.0894
10.0	0	0	0

TABLE 6

Convergence of the partial sums (3.7d) for $\eta = 0.08$ *in an annular trench of finite depth* $D = 2$ (*Gap/Depth* = 1)

$$\begin{bmatrix} \Psi(t, 0; N) \\ \Psi(t, -D; N) \end{bmatrix} = \sum_{\substack{-N \\ N \neq 0}}^{N} \frac{\phi_1^{(n)}(t)}{p_n^2} \begin{bmatrix} C_n + D_n \\ C_n e^{-p_n D} + D_n e^{p_n D} \end{bmatrix}$$

$$\sim -\left\{A_0 + \left(B_0 + C_0 \ln\left(\frac{1-\eta}{2\eta}t\right)\right)t^2 + \left(D_0 + \ln\left(\frac{1-\eta}{2\eta}t\right)\right)t^4\right\}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

t	$-\left\{A_0 + \left(B_0 + C_0 \ln\left(\frac{1-\eta}{2\eta}t\right)\right)t^2 + \left(D_0 + \ln\left(\frac{1-\eta}{2\eta}t\right)\right)t^4\right\}$	$\Psi(t, 0; 1)$	$\Psi(t, 0; 5)$
0.174	0	0	0
0.374	−0.0497	−0.0427	−0.0498
0.574	−0.1887	−0.1705	−0.1891
0.774	−0.3703	−0.3486	−0.3701
0.974	−0.5352	−0.5218	−0.5350
1.174	−0.6308	−0.6343	−0.6309
1.374	−0.6234	−0.6440	−0.6234
1.574	−0.5069	−0.5349	−0.5068
1.774	−0.3103	−0.3311	−0.3106
1.974	−0.1029	−0.1094	−0.1024
2.174	0	0	0

t	$-\left\{A_0 + \left(B_0 + C_0 \ln\left(\frac{1-\eta}{2\eta}t\right)\right)t^2 + \left(D_0 + \ln\left(\frac{1-\eta}{2\eta}t\right)\right)t^4\right\}$	$\Psi(t, -D; 1)$	$\Psi(t, -D; 5)$
0.174	0	0	0
0.374	−0.0497	−0.0462	−0.0499
0.574	−0.1887	−0.1740	−0.1880
0.774	−0.3703	−0.3409	−0.3704
0.974	−0.5352	−0.4992	−0.5361
1.174	−0.6308	−0.6084	−0.6307
1.374	−0.6234	−0.6344	−0.6232
1.574	−0.5069	−0.5521	−0.5067
1.774	−0.3103	−0.3632	−0.3093
1.974	−0.1029	−0.1286	−0.1041
2.174	0	0	0

The last step in the evaluation of the height correction at first order is the application of the side-wall conditions (3.10) and (3.11). When the trench is filled to the top with liquid and the liquid adheres to sharp edge (3.10a), we find that

$$B_1=\frac{-1}{m^2\cdot\operatorname{Det}(B_1,B_2,C)}\sum_{-N}^{N}(C_n-D_n)\frac{(m^2-p_n^2)}{p_n^2(m^2+p_n^2)^2}$$
$$\cdot\Big\{-[b(A_3^{(n)}J_1(p_nb)+A_4^{(n)}Y_1(p_nb))-a(A_3^{(n)}J_1(p_na)$$
$$+A_4^{(n)}Y_1(p_na))][K_0(mb)-K_0(ma)]-\frac{p_n}{m}[(A_3^{(n)}J_0(p_nb)$$
$$+A_4^{(n)}Y_0(p_nb))-(A_3^{(n)}J_0(p_na)+A_4^{(n)}Y_0(p_na))][bK_1(mb)-aK_1(ma)]$$
$$+\frac{b^2-a^2}{2}p_n[(A_3^{(n)}J_0(p_na)+A_4^{(n)}Y_0(p_na))K_0(mb)$$
$$-(A_3^{(n)}J_0(p_nb)+A_4^{(n)}Y_0(p_nb))K_0(ma)]\Big\},$$

(5.3)

$$B_2=\frac{-1}{m^2\cdot\operatorname{Det}(B_1,B_2,C)}\sum_{-N}^{N}(C_n-D_n)\frac{(m^2-p_n^2)}{p_n^2(m^2+p_n^2)^2}$$
$$\cdot\Big\{[b(A_3^{(n)}J_1(p_nb)+A_4^{(n)}Y_1(p_nb))-a(A_3^{(n)}J_1(p_na)$$
$$+A_4^{(n)}Y_1(p_na))][I_0(mb)-I_0(ma)]-\frac{p_n}{m}[A_3^{(n)}J_0(p_nb)$$
$$+A_4^{(n)}Y_0(p_nb)-(A_3^{(n)}J_0(p_na)+A_4^{(n)}Y_0(p_na))][bI_1(mb)-aI_1(ma)]$$
$$-\frac{b^2-a^2}{2}p_n[(A_3^{(n)}J_0(p_na)+A_4^{(n)}Y_0(p_na))I_0(mb)$$
$$-(A_3^{(n)}J_0(p_nb)+A_4^{(n)}Y_0(p_nb))I_0(ma)]\Big\},$$

$$C=\frac{-1}{\operatorname{Det}(B_1,B_2,C)}\sum_{-N}^{N}(C_n-D_n)\frac{(m^2-p_n^2)}{p_n^2(m^2+p_n^2)^2}$$
$$\cdot\Big\{-\frac{p_n}{m^2}[(A_3^{(n)}J_0(p_nb)+A_4^{(n)}Y_0(p_nb))+(A_3^{(n)}J_0(p_na)$$
$$+A_4^{(n)}Y_0(p_na))]+\frac{p_n}{m}[I_1(mb)K_0(ma)+K_1(mb)I_0(ma)]$$
$$\cdot b[A_3^{(n)}J_0(p_nb)+A_4^{(n)}Y_0(p_nb)]+\frac{p_n}{m}[I_1(ma)K_0(mb)$$
$$+K_1(ma)I_0(mb)]a[A_3^{(n)}J_0(p_na)+A_4^{(n)}Y_0(p_na)]$$
$$+[b(A_3^{(n)}J_1(p_nb)+A_4^{(n)}Y_1(p_nb))-a(A_3^{(n)}J_1(p_na)$$
$$+A_4^{(n)}Y_1(p_na))][I_0(ma)K_0(mb)-I_0(mb)K_0(ma)]\Big\},$$

where

$$\operatorname{Det}(B_1, B_2, C) = \frac{1}{m^2}\Big\{-\frac{2}{m^2} + \frac{b^2-a^2}{2}[I_0(ma)K_0(mb) - K_0(ma)I_0(mb)] + \frac{b}{m}[I_1(mb)K_0(ma) + K_1(mb)I_0(ma)] + \frac{a}{m}[I_1(ma)K_0(mb) + K_1(ma)I_0(mb)]\Big\}$$

and

$$\begin{aligned}
H(t) = {} & \frac{-1}{m^2\cdot \operatorname{Det}(B_1,B_2,C)}\sum_{-N}^{N}\frac{(C_n-D_n)(m^2-p_n^2)}{p_n^2(m^2+p_n^2)^2} \\
& \cdot\{b[A_3^{(n)}J_1(p_nb)+A_4^{(n)}Y_1(p_nb)] - a[A_3^{(n)}J_1(p_na)+A_4^{(n)}Y_1(p_na)]\} \\
& \cdot\{[-K_0(mb)+K_0(ma)]I_0(mt) + [I_0(mb)-I_0(ma)]K_0(mt) \\
& \quad + [I_0(ma)K_0(mb) - K_0(ma)I_0(mb)]\} \\
& + \frac{-1}{m^2\cdot \operatorname{Det}(B_1,B_2,C)}\sum_{-N}^{N}\frac{(C_n-D_n)(m^2-p_n^2)}{p_n(m^2+p_n^2)^2} \\
& \qquad \cdot[A_3^{(n)}J_0(p_nb)+A_4^{(n)}Y_0(p_nb)] \\
& \cdot\Big\{-\frac{b}{m}[K_1(mb)I_0(mt)+I_1(mb)K_0(mt)] + \frac{a}{m}[K_1(ma)I_0(mt) \\
& \quad + I_1(ma)K_0(mt)] + \frac{b}{m}[I_1(mb)K_0(ma)+K_1(mb)I_0(ma)] \\
& \quad - \frac{b^2-a^2}{2}[K_0(ma)I_0(mt) - I_0(ma)K_0(mt)] - \frac{1}{m^2}\Big\} \\
& + \frac{-1}{m^2\cdot \operatorname{Det}(B_1,B_2,C)}\sum_{-N}^{N}\frac{(C_n-D_n)(m^2-p_n^2)}{p_n(m^2+p_n^2)^2} \\
& \qquad \cdot[A_3^{(n)}J_0(p_na)+A_4^{(n)}Y_0(p_na)] \\
& \cdot\Big\{\frac{b}{m}[K_1(mb)I_0(mt)+I_1(mb)K_0(mt)] - \frac{a}{m}[K_1(ma)I_0(mt) \\
& \quad + I_1(ma)K_0(mt)] + \frac{a}{m}[I_1(ma)K_0(mb)+K_1(ma)I_0(mb)] \\
& \quad + \frac{b^2-a^2}{2}[K_0(mb)I_0(mt) - I_0(mb)K_0(mt)] - \frac{1}{m^2}\Big\} \\
& + \sum_{-N}^{N}\frac{C_n-D_n}{m^2+p_n^2}\Big\{[A_1^{(n)}J_0(p_nt)+A_2^{(n)}Y_0(p_nt)] \\
& \qquad - t[A_3^{(n)}J_1(p_nt)+A_4^{(n)}Y_1(p_nt)] \\
& \qquad + \frac{3m^2+p_n^2}{p_n(m^2+p_n^2)}[A_3^{(n)}J_0(p_nt)+A_4^{(n)}Y_0(p_nt)]\Big\}.
\end{aligned} \tag{5.4}$$

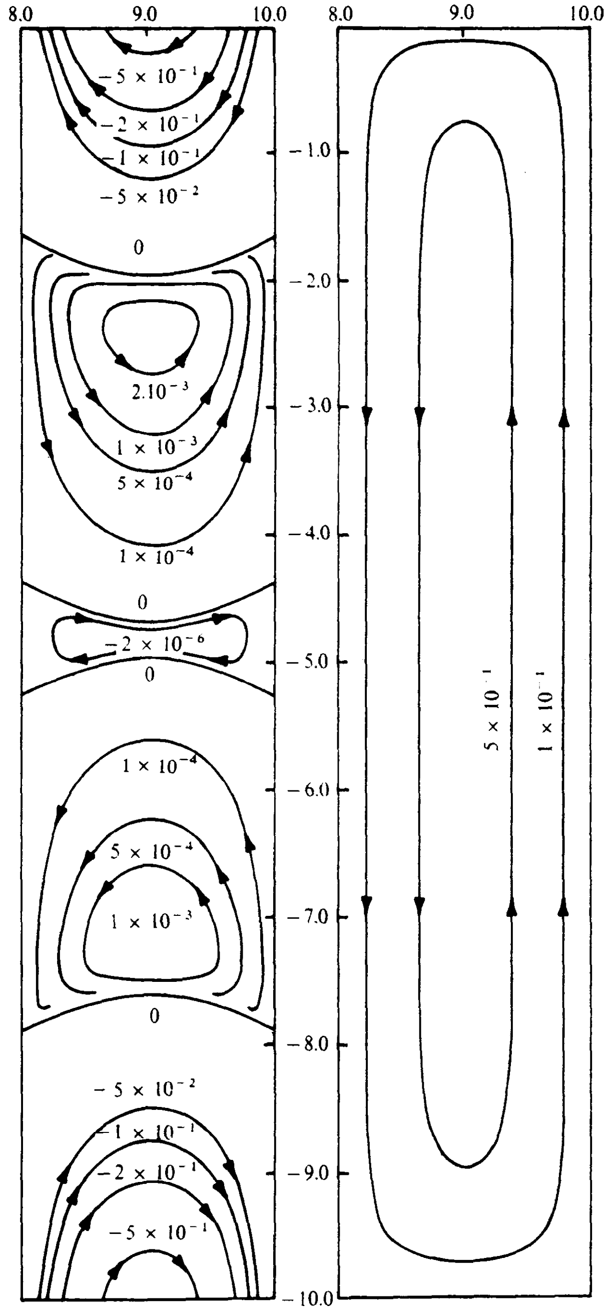

FIG. 3 (a). $\eta = 0.8$

FIG. 3. *Level lines of the edge eddies* (4.1) *and streamlines* (4.20) *in the reference domain for the cylindrical annulus of finite depth with a depth/width ratio of five*

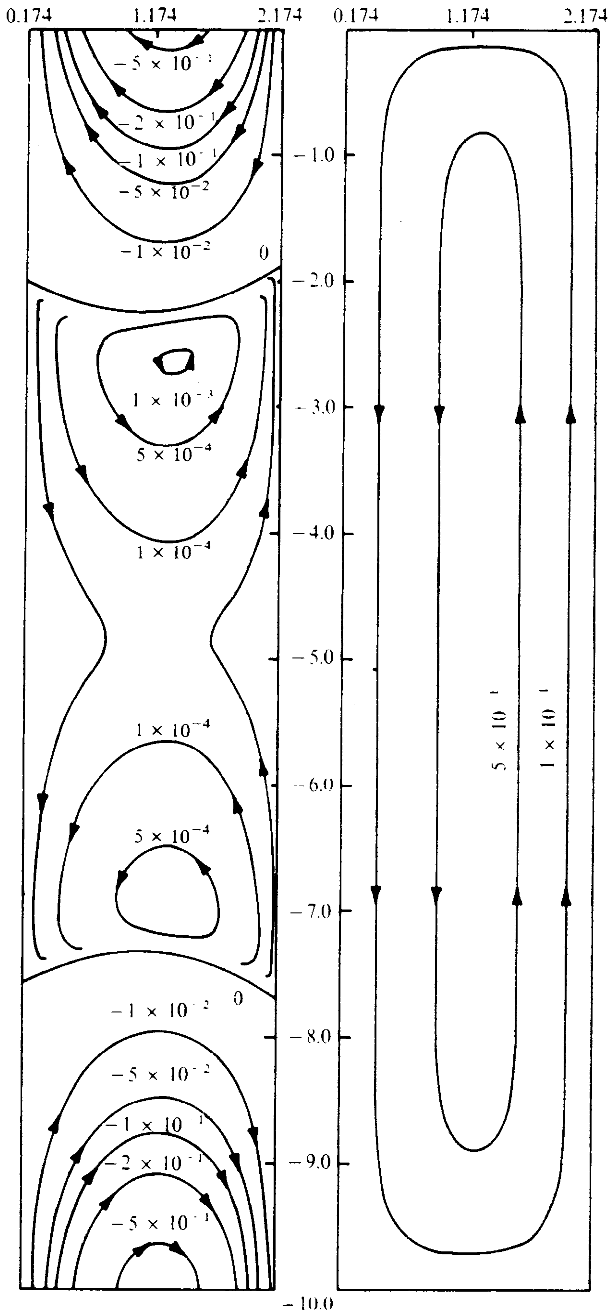

FIG. 3 (b). $\eta = 0.08$

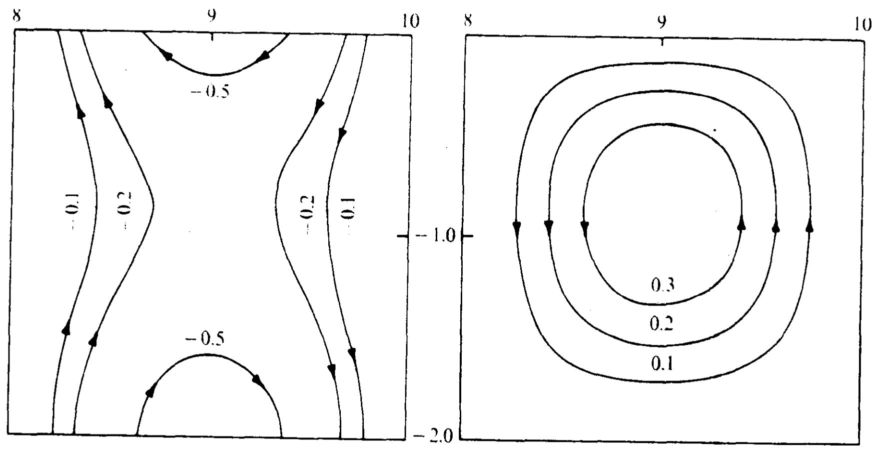

FIG. 4 (a). $\eta = 0.8$

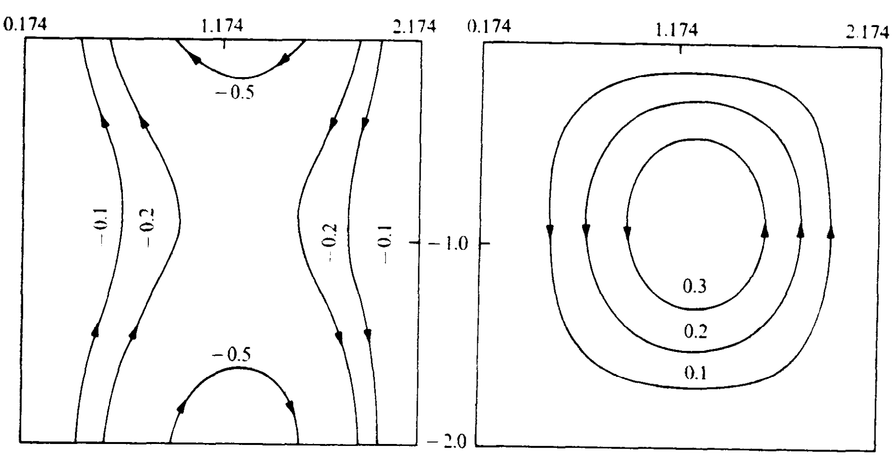

FIG. 4 (b). $\eta = 0.08$

FIG. 4. *Level lines of the edge eddies* (4.1) *and streamlines* (4.20) *in the reference domain for the cylindrical annulus of finite depth with a depth/width ratio of* 1

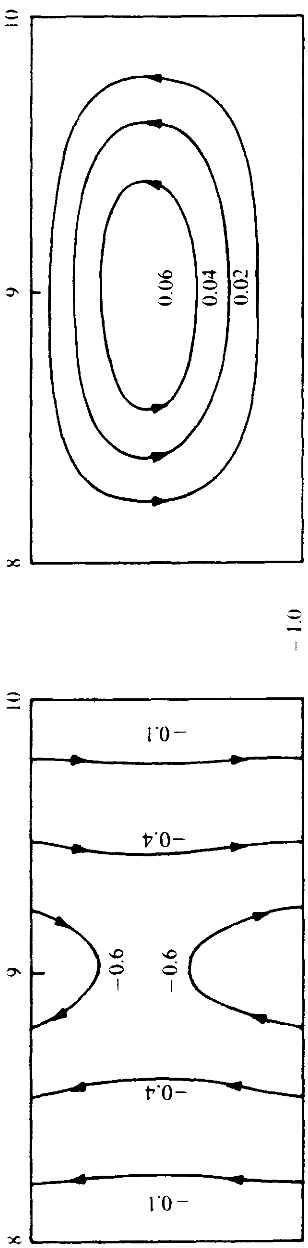

FIG. 5 (a). $\eta = 0.8$

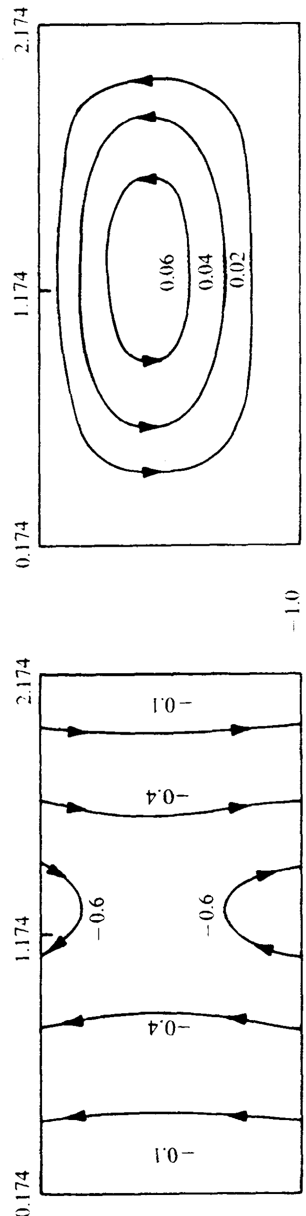

FIG. 5 (b). $\eta = 0.08$

FIG. 5. *Level lines of the edge eddies* (4.1) *and streamlines* (4.20) *in the reference domain for the cylindrical annulus of finite depth with a depth/width ratio of* 1/2

When the trench is partially filled with liquid and an angle of flat contact is specified, we find, using (5.2), (3.10b) and (3.11), that

$$B_1 = \frac{m}{\operatorname{Det}(B_1, B_2)} \sum_{-N}^{N} (C_n - D_n) \frac{(3m^2 + p_n^2)}{(m^2 + p_n^2)^2} \cdot \{-[A_3^{(n)} J_1(p_n a) + A_4^{(n)} Y_1(p_n a)] K_1(mb) + [A_3^{(n)} J_1(p_n b) + A_4^{(n)} Y_1(p_n b)] K_1(ma)\},$$

(5.5)
$$B_2 = \frac{m}{\operatorname{Det}(B_1, B_2)} \sum_{-N}^{N} (C_n - D_n) \frac{(3m^2 + p_n^2)}{(m^2 + p_n^2)^2} \cdot \{-[A_3^{(n)} J_1(p_n a) + A_4^{(n)} Y_1(p_n a)] I_1(mb) + [A_3^{(n)} J_1(p_n b) + A_4^{(n)} Y_1(p_n b)] I_1(ma)\},$$

$$C = \frac{-2}{b^2 - a^2} \sum_{-N}^{N} (C_n - D_n) \frac{1}{p_n^2} \{b[A_3^{(n)} J_1(p_n b) + A_4^{(n)} Y_1(p_n b)] - a[A_3^{(n)} J_1(p_n a) + A_4^{(n)} Y_1(p_n a)]\},$$

where

$$\operatorname{Det}(B_1, B_2) = -m^2 [I_1(ma) K_1(mb) - K_1(ma) I_1(mb)]$$

and

(5.6)
$$\begin{aligned} H(t) = {} & \frac{-m}{\operatorname{Det}(B_1, B_2)} \sum_{-N}^{N} (C_n - D_n) \frac{(3m^2 + p_n^2)}{(m^2 + p_n^2)^2} [A_3^{(n)} J_1(p_n a) \\ & + A_4^{(n)} Y_1(p_n a)][K_1(mb) I_0(mt) + I_1(mb) K_0(mt)] \\ & + \frac{m}{\operatorname{Det}(B_1, B_2)} \sum_{-N}^{N} (C_n - D_n) \frac{(3m^2 + p_n^2)}{(m^2 + p_n^2)^2} [A_3^{(n)} J_1(p_n b) \\ & + A_4^{(n)} Y_1(p_n b)][K_1(ma) I_0(mt) + I_1(ma) K_0(mt)] \\ & + \frac{1}{m^2} \frac{2}{b^2 - a^2} \sum_{-N}^{N} (C_n - D_n) \frac{1}{p_n^2} \{-b[A_3^{(n)} J_1(p_n b) \\ & + A_4^{(n)} Y_1(p_n b)] + a[A_3^{(n)} J_1(p_n a) + A_4^{(n)} Y_1(p_n a)]\} \\ & + \sum_{-N}^{N} \frac{C_n - D_n}{m^2 + p_n^2} \Big\{ [A_1^{(n)} J_0(p_n t) + A_2^{(n)} Y_0(p_n t)] \\ & - t[A_3^{(n)} J_1(p_n t) + A_4^{(n)} Y_1(p_n t)] \\ & + \frac{3m^2 + p_n^2}{p_n(m^2 + p_n^2)} [A_3^{(n)} J_0(p_n t) + A_4^{(n)} Y_0(p_n t)] \Big\}. \end{aligned}$$

In Fig. 6 we have plotted the correction coefficient H for the free surface in an infinitely deep annulus (see Tables 7 and 8 for computed values). This coefficient is the dimensionless form of $h^{(1)}(r)$; and the free surface is given by $h(r; \varepsilon) = h^{(1)}\varepsilon + O(\varepsilon^2)$. To obtain the streamlines in the deformed domain $\mathcal{V}_\varepsilon$ we merely invert the mapping $\mathcal{V}_\varepsilon \to \mathcal{V}_0$ using a linear shifting transformation in semi-infinite domains and a linear scaling transformation when the length of the cylinders is finite. The inversions have been discussed in detail by Joseph and Sturges [7] and will not be repeated here.

TABLE 7

The correction coefficients for the free surface on a liquid in an infinitely deep annular trench with $\eta = 0.8$. *The first columns are for the prescribed contact line problem* (5.4). *The second columns are for the prescribed contact angle problem* (5.6) *with a flat contact.*

	$4m^2 = 0.1$		$4m^2 = 10$	
8.000	−.0000	−.1561	−.0000	−.0768
8.100	−.0147	−.1550	−.0122	−.0766
8.200	−.0262	−.1508	−.0215	−.0752
8.300	−.0340	−.1431	−.0276	−.0721
8.400	−.0380	−.1317	−.0305	−.0669
8.500	−.0382	−.1166	−.0304	−0.597
8.600	−.0350	−.0983	−.0278	−.0506
8.700	−.0291	−.0774	−.0229	−.0399
8.800	−.0209	−.0544	−.0165	−.0281
8.900	−.0114	−.0300	−.0089	−.0154
9.000	−.0011	−.0050	−.0009	−.0024
9.100	.0090	.0197	.0071	.0104
9.200	.0184	.0436	.0144	.0228
9.300	.0262	.0658	.0206	.0342
9.400	.0318	.0858	.0251	.0443
9.500	.0346	.1030	.0276	.0528
9.600	.0343	.1169	.0275	.0594
9.700	.0306	.1274	.0248	.0641
9.800	.0235	.1343	.0191	.0668
9.900	.0130	.1380	.0107	.0680
10.000	−.0000	.1390	−.0000	.0582

TABLE 8

The correction coefficients for the free surface on a liquid in an infinitely deep annular trench with $\eta = 0.08$. *The first columns are for the prescribed contact line problem* (5.4). *The second columns are for the prescribed contact angle problem* (5.6) *with a flat contact.*

	$4m^2 = 0.1$		$4m^2 = 10$	
.174	−.0000	−1.4682	−.0000	−.8487
.274	−.2572	−1.4721	−.2083	−.8437
.374	−.4002	−1.4251	−.3212	−.8205
.474	−.4690	−1.3449	−.3733	−.7762
.574	−.4842	−1.2354	−.3824	−.7130
.674	−.4599	−1.1019	−.3608	−.6343
.774	−.4069	−.9502	−.3174	−.5444
.874	−.3342	−.7862	−.2696	−.4472
.974	−.2494	−.6153	−.1933	−.3463
1.074	−.1594	−.4428	−.1235	−.2449
1.174	−.0700	−.2734	−.0545	−.1460
1.274	.0135	−.1116	.0099	−.0526
1.374	.0862	.0383	.0661	.0331
1.474	.1443	.1731	.1112	.1089
1.574	.1844	.2898	.1428	.1730
1.674	.2055	.3878	.1603	.2258
1.774	.2048	.4645	.1609	.2653
1.874	.1825	.5203	.1445	.2923
1.974	.1395	.5565	.1115	.3081
2.074	.0770	.5744	.0621	.3139
2.174	.0000	.5790	.0000	.3147

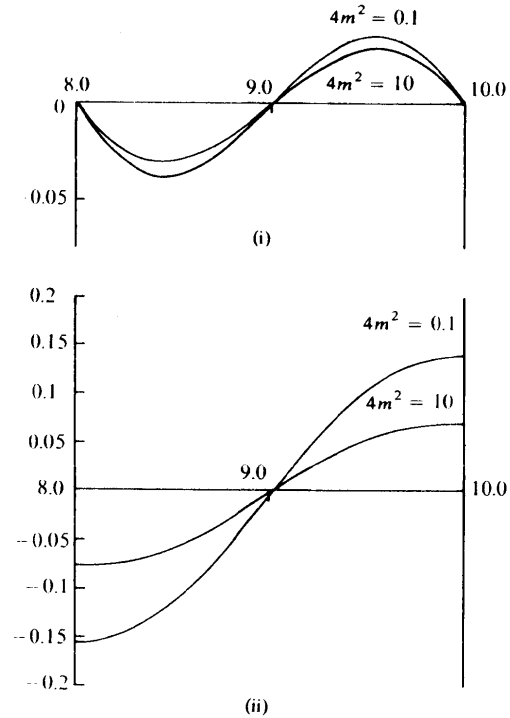

FIG. 6 (a). $\eta = 0.8$

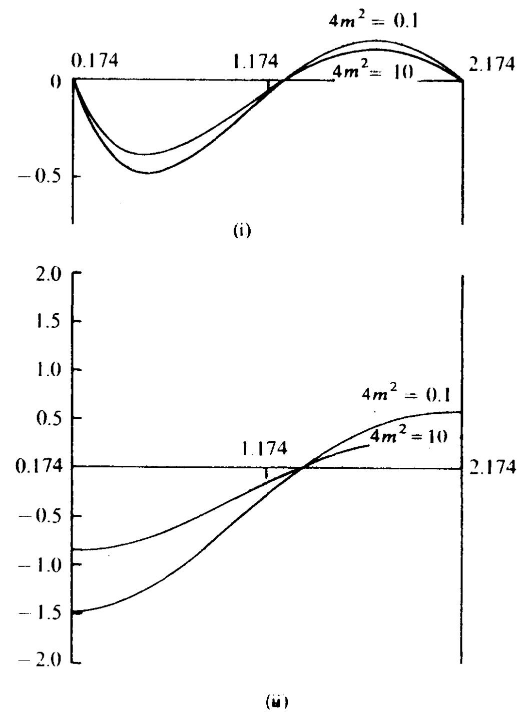

FIG. 6 (b). $\eta = 0.08$

FIG. 6. *Graphs of the free surface correction $H(t)$ for the infinitely deep annular trenches.*
(i) *The free surface adheres to the upper edge of the cylinder.*
(ii) *The free surface makes a horizontal contact with the cylinder walls.*

Appendix A: Eigenfunctions for axisymmetric Stokes flow between two concentric cylinders. Separable solutions of $\Psi(t, y)=T(t)Y(y)$ of

$$\text{(A.1)} \qquad \left[\frac{\partial^2}{\partial t^2}-\frac{1}{t}\frac{\partial}{\partial t}+\frac{\partial^2}{\partial y^2}\right]^2\Psi=0, \qquad -\infty<y<\infty, \quad 0<a\leqq t\leqq b$$

and governed by the reduced equation

$$\text{(A.2)} \qquad \frac{1}{T}\left(T^{\text{iv}}-\frac{2}{t}T'''+\frac{3}{t^2}T''-\frac{3}{t^3}T'\right)+\frac{2}{T}\left(T''-\frac{T'}{t}\right)\frac{Y''}{Y}+\frac{Y^{\text{iv}}}{Y}=0.$$

Equation (A.2) can hold if and only if

$$\text{(A.3)} \qquad \frac{T''}{T}-\frac{1}{t}\frac{T'}{T}=\alpha f(t)+\beta$$

and

$$\text{(A.4)} \qquad \frac{Y''}{Y}=\gamma g(y)+\delta,$$

where α, β, γ and δ are complex constants with $\alpha\gamma=0$ and f and g are arbitrary functions of a single variable. In the analysis we will introduce complex constants p, c_i and $A_i (i=1, 2, \cdots)$; the c_i are integration constants and p lies in the first quadrant but is otherwise unrestricted. The following list exhausts the possibilities associated with (A.2, 3, 4):

(I) $\alpha=\beta=0$. Equation (A.3) shows that

$$T(t)=\text{linear combination of } (c_1, t^2)$$

and (A.3) and (A.2) show that

$$Y(y)=\text{linear combination of } (c_2, y, y^2, y^3).$$

(II) $\alpha=0$, $\operatorname{Im}\beta<0$, $\beta=-p^2$. Integrating (A.3) we find that

$$T(t)=\text{linear combination of } (tJ_1(pt), tY_1(pt)),$$

where J_1 and Y_1 are Bessel functions of the first and second kind. Equations (A.3) and (A.2) imply that

$$Y(y)=\text{linear combination of } (e^{-py}, ye^{-py}, e^{py}, ye^{py}).$$

(III) $\alpha=0$, $\operatorname{Im}\beta>0$, $\beta=p^2$. Integrating (A.3) we find that

$$T(t)=\text{linear combination of } (tI_1(pt), tK_1(pt)),$$

where I_1 and K_1 are the modified Bessel functions. Equations (A.3) and (A.2) imply that

$$Y(y)=\text{linear combination of } (\cos py, y\cos py, \sin py, y\sin py).$$

(IV) $\gamma=\delta=0$.

$$Y(y)=\text{linear combination of } (c_3, y).$$

$$T(t)=\text{linear combination of } (c_4, t^2, t^2\ln t, t^4).$$

(V) $\gamma=0$, $\operatorname{Im}\delta<0$, $\delta=-p^2$. Integrating (A.4) we find that

$$Y(y)=\text{linear combination of } (\cos py, \sin py).$$

Combining $Y''=-p^2Y$ with (A.2) we find that

$$\left(\frac{d^2}{dt^2}-\frac{1}{t}\frac{d}{dt}-p^2\right)^2 T=0 \tag{A.5}$$

and consequently

$$T''-\frac{1}{t}T'-p^2T=c_5tI_1(pt)+c_6tK_1(pt). \tag{A.6}$$

Equation (A.6) may be solved by the method of variation of constants

$$T(t)=\text{linear combination }(tI_1(pt),\, tK_1(pt),\, t^2I_0(pt),\, t^2K_0(pt)).$$

(VI) $\gamma=0$, $\operatorname{Im}\delta>0$, $\delta=p^2$. Integrating (A.4) we find that

$$Y(y)=\text{linear combination of }(e^{py}, e^{-py}).$$

Combining $Y''=p^2Y$ with (A.2) we find that

$$\hat{L}^2T=0,\qquad \hat{L}(\cdot)\equiv\frac{d^2(\cdot)}{dt^2}-\frac{1}{t}\frac{d(\cdot)}{dt}+p^2(\cdot), \tag{A.7}$$

$$\hat{L}T=c_7tJ_1(pt)+c_8tY_1(pt). \tag{A.8}$$

The solution of (A.8) is

$$\begin{aligned} T(t)&=A_1tJ_1(pt)+A_2tY_1(pt)+A_3t^2J_0(pt)+A_4t^2Y_0(pt)\\ &\equiv\phi_1(t;p). \end{aligned} \tag{A.9}$$

The linear combinations (A.9) may be formed into a countably infinite set of eigenfunctions $\phi_1^{(n)}(t)\equiv\phi_1(t,p_n)$ associated with eigenvalues p_n. The eigenfunctions $\phi_1^{(n)}(t)$ are the combinations (A.9) which satisfy the homogeneous side-wall conditions

$$T(a)=T(b)=T'(a)=T'(b)=0, \tag{A.10}$$

where

$$\begin{aligned} T'(t)&=A_1ptJ_0(pt)+A_2ptY_0(pt)+A_3[2tJ_0(pt)-pt^2J_1(pt)]\\ &\quad+A_4[2tY_0(pt)-pt^2Y_1(pt)]. \end{aligned}$$

The matrix of the homogeneous linear equations (A.10) is designated as

$$[\underline{F}]=\begin{bmatrix} aJ_1(pa) & aY_1(pa) & a^2J_0(pa) & a^2Y_0(pa)\\ bJ_1(pb) & bY_1(pb) & b^2J_0(pb) & b^2Y_0(pb)\\ paJ_0(pa) & paY_0(pa) & 2aJ_0(pa)-pa^2J_1(pa) & 2aY_0(pa)-pa^2Y_1(pa)\\ pbJ_0(pb) & pbY_0(pb) & 2bJ_0(pb)-pb^2J_1(pb) & 2bY_0(pb)-pb^2Y_1(pb) \end{bmatrix} \tag{A.11}$$

The equations

$$\sum_{j=1}^{4} F_{ij}A_j=0 \qquad (i=1,2,3,4)$$

are solvable if and only if

$$\det \underline{F}=0. \tag{A.12}$$

Given (A.12), the A_i may be determined to within an arbitrary multiplicative constant which we choose so as to facilitate the passage to the narrow gap limit $\eta \to 1$. Then

(A.13a) $$A_1 = \frac{1}{ab}\begin{vmatrix} F_{12} & F_{13} & F_{14} \\ F_{32} & F_{33} & F_{34} \\ F_{42} & F_{43} & F_{44} \end{vmatrix},$$

(A.13b) $$A_2 = \frac{1}{ab}\begin{vmatrix} -F_{11} & F_{13} & F_{14} \\ -F_{31} & F_{33} & F_{34} \\ -F_{41} & F_{43} & F_{44} \end{vmatrix},$$

(A.13c) $$A_3 = \frac{1}{ab}\begin{vmatrix} F_{12} & -F_{11} & F_{14} \\ F_{32} & -F_{31} & F_{34} \\ F_{42} & -F_{41} & F_{44} \end{vmatrix},$$

(A.13d) $$A_4 = \frac{1}{ab}\begin{vmatrix} F_{12} & F_{13} & -F_{11} \\ F_{32} & F_{33} & -F_{31} \\ F_{42} & F_{43} & -F_{41} \end{vmatrix}.$$

There are countably infinite number of eigenvalues p_n of (A.12) which are symmetrically located in the complex p plane. We make use of the eigenvalues with positive real parts. The eigenvalues p_n $(n = 1, 2, \cdots)$ are the first quadrant roots of (A.12) ordered according to the size of their real parts. The asymptotic distribution of eigenvalues for large values of n is given by (B.8) and (B.9). The eigenvalues with a negative index are defined by

$$p_{-n} = \bar{p}_n.$$

Then, using (A.9) and (A.13), we have

$$\phi_1^{(-n)}(t) = \bar{\phi}_1^{(n)}(t) = \phi(t; p_{-n}).$$

Appendix B: Computation and asymptotic properties of the eigenvalues. It is apparent from Table B.1 that most of the eigenvalues p_n, $n=1, 2, 3, \cdots$, have larger real parts. It is therefore useful to consider the simpler mathematical expressions which arise when the Bessel functions are represented by Hankel's asymptotic expansions for large arguments (see Abramowitz and Stegun [1, p. 364])

(B.1)
$$J_\nu(z) = \sqrt{\frac{2}{\pi z}}\left\{P(\nu, z)\cos\left(z - \left(\frac{1}{2}\nu + \frac{1}{4}\right)\pi\right) - Q(\nu, z)\sin\left(z - \left(\frac{1}{2}\nu + \frac{1}{4}\right)\pi\right)\right\},$$

$$Y_\nu(z) = \sqrt{\frac{2}{\pi z}}\left\{P(\nu, z)\sin\left(z - \left(\frac{1}{2}\nu + \frac{1}{4}\right)\pi\right) + Q(\nu, z)\cos\left(z - \left(\frac{1}{2}\nu + \frac{1}{4}\right)\pi\right)\right\},$$

where

$$P(\nu, z) \sim \sum_{k=0}^{\infty} \frac{(-)^k(\nu, 2k)}{(2z)^{2k}} = 1 - \frac{(\lambda-1)(\lambda-9)}{2!(8z)^2} + \frac{(\lambda-1)(\lambda-9)(\lambda-25)(\lambda-49)}{4!(8z)^4} - \cdots,$$

$$Q(\nu, z) \sim \sum_{k=0}^{\infty} \frac{(-)^k(\nu, 2k+1)}{(2z)^{2k+1}} = \frac{\lambda-1}{8z} - \frac{(\lambda-1)(\lambda-9)(\lambda-25)}{3!(8z)^3} + \cdots$$

and with $4\nu^2$ denoted by λ. We are interested in values $\nu=0$ and $\nu=1$. The representations (B.1) are substituted into (A.9), (A.12) and (A.13). To achieve a nontrivial result it is necessary, because of cancellations, to retain terms of $O(z^{-2})$. After a very long and demanding computation, the details of which are given in the dissertation of J. Yoo [13] we find that

$$\text{(B.2a)} \qquad \frac{1}{ab}\det \underset{\sim}{F} = \frac{4}{\pi^2 p_n^2}[4p_n^2 - \sin^2 2p_n] + O\left[\frac{1}{|ap_n|}\right] = 0$$

and

$$\begin{aligned} \phi_1^{(n)}(t) &= t[A_1^{(n)}J_1(p_n t) + A_2^{(n)}Y_1(p_n t)] + t^2[A_3^{(n)}J_0(p_n t) + A_4^{(n)}Y_0(p_n t)] \\ \text{(B.2b)} \qquad &= \frac{4}{\pi^2 p_n^2}\sqrt{\frac{t}{b}}\{2p_n^2(t-a)\cos p_n(b-t) - \sin 2p_n \sin p_n(t-a) \\ &\qquad + 2p_n \sin p_n(b-t) - p_n(b-t)\sin 2p_n \cos p_n(t-a)\} \\ &\quad + O\left[\frac{1}{|p_n a|}\right]. \end{aligned}$$

The limiting forms of (B.2a) and (B.2b) as

$$\text{(B.3)} \qquad \frac{1}{|p_n|a} = \frac{1-\eta}{2\eta|p_n|} \to 0$$

are given by

$$\text{(B.4a)} \qquad (2p_n + \sin 2p_n)(2p_n - \sin 2p_n) = 0$$

and

$$\begin{aligned} \phi_1^{(n)} = \frac{4}{\pi^2 p_n^2}\left[1 + \frac{(z-1)(1-\eta)}{2}\right]^{1/2} &\{(2p_n - \sin 2p_n)[p_n \cos p_n \cos p_n z \\ \text{(B.4b)} \qquad & + \sin p_n \cos p_n z + p_n z \sin p_n \sin p_n z] \\ & + (2p_n + \sin 2p_n)[p_n \sin p_n \sin p_n z - \cos p_n \sin p_n z \\ & + p_n z \cos p_n \cos p_n z]\}, \end{aligned}$$

where we have put

$$z = t - \frac{a+b}{2} = t - \frac{1+\eta}{1-\eta}, \qquad -1 \leqq z \leqq 1. \tag{B.5}$$

If (B.4a) is satisfied by

$$2p_n + \sin 2p_n = 0, \tag{B.6a}$$

then (B.4b) reduces to the *even* eigenfunctions

$$\phi_1^{(n)} = \frac{4}{\pi^2 p_n^2}\left[1 + \frac{(z-1)(1-\eta)}{2}\right]^{1/2} (4 \sin^2 p_n)(p_n \sin p_n \cos p_n z - p_n z \cos p_n \sin p_n z). \tag{B.6b}$$

If (B.4a) is satisfied by

$$2p_n - \sin 2p_n = 0, \tag{B.7a}$$

then (B.4b) reduces to the *odd* eigenfunctions

$$\phi_1^{(n)} = \frac{4}{\pi^2 p_n^2}\left[1 + \frac{(z-1)(1-\eta)}{2}\right]^{1/2} (-4 \cos^2 p_n)(p_n \cos p_n \sin p_n z - p_n z \sin p_n \cos p_n z). \tag{B.7b}$$

The eigenvalues (B.6a) and (B.7a) are the eigenvalues for the even and odd Fadle–Papkovich eigenfunctions, respectively. These eigenfunctions are given by (B.6b) and (B.7b) with $\eta = 1$ (see Joseph and Sturges [7]).

It is, of course, expected that in the limit of narrow gaps, $\eta \to 1$, the strip eigenfunctions and eigenvalues for the annulus should reduce to strip eigenfunctions and eigenvalues for the plane channel. The limit result specified under (B.3) goes further than this. It asserts that for any fixed value of $\eta > 0$ the eigenvalues reduce to the Fadle–Papkovich eigenvalues and the eigenfunctions reduce, almost, to the Fadle–Papkovich eigenfunctions for every sufficiently large eigenvalue $|p_n|$. It follows that the large eigenvalues should be relatively independent of variations in the radius ratio. This property is clearly evident in Table B.1 of eigenvalues. A second consequence of the limit result is that the asymptotic expansions of the Fadle–Papkovich eigenvalues (Hillman and Salzer [2]; Robbins and Smith [9]) hold relative to the wide gap problem whenever $|p_n|$ is sufficiently large. When n is large, the first quadrant roots of $2p_n - \sin 2p_n = 0$ are given asymptotically by

$$2p_n = (2n + \tfrac{1}{2})\pi + i \log (4n+1)\pi \tag{B.8}$$

and the first quadrant roots of $2p_n + \sin 2p_n = 0$ by

$$2p_n = (2n - \tfrac{1}{2})\pi + i \log (4n-1)\pi. \tag{B.9}$$

In Table B.1 we have listed the first 20 eigenvalues of Fadle–Papkovich eigenfunctions ($\eta = 1$). The values given in Table B.1 for $\eta = 1$ serve as initial guesses for an iterative scheme for the roots of (A.12) which give the eigenvalues when $0 < \eta < 1$. The iterative scheme uses Muller's method with deflation. The details of iteration along with numerical programs is given in Yoo's thesis [13]. Representative results are displayed in Table B.1.

TABLE B.1

Real and imaginary parts of the first 20 eigenvalues p_n for various radius ratios

$\eta = 1$		$\eta = 0.8$		$\eta = 0.5$		$\eta = 0.25$		$\eta = 0.08$	
Re p_n	Im p_n	Re p_n	Im p_n	Re p_n	Im p_n	Re p_n	Im p_n	Re p_n	Re p_n
2.10620	1.12536	2.10791	1.12444	2.12186	1.11652	2.15891	1.09110	2.21749	1.02813
3.74884	1.38434	3.74993	1.38393	3.75923	1.38006	3.78746	1.36368	3.84328	1.30218
5.35627	1.55157	5.35707	1.55134	5.36406	1.54903	5.38706	1.53806	5.44128	1.48506
6.95000	1.67610	6.95062	1.67595	6.95620	1.67441	6.97550	1.66666	7.02760	1.62206
8.53668	1.77554	8.53721	1.77543	8.54185	1.77433	8.55841	1.76859	8.60795	1.73116
10.11926	1.85838	10.11971	1.85830	10.12368	1.85747	10.13813	1.85306	10.18493	1.82150
11.69918	1.92940	11.69957	1.92934	11.70304	1.92869	11.71583	1.92520	11.75991	1.89839
13.27727	1.99157	13.27762	1.99152	13.28070	1.99100	13.29216	1.98817	13.33363	1.96521
14.85406	2.04685	14.85437	2.04681	14.85714	2.04638	14.86752	2.04404	14.90653	2.02422
16.42987	2.09663	16.43015	2.09659	16.43266	2.09623	16.44214	2.09426	16.47888	2.07702
18.00493	2.14189	18.00519	2.14186	18.00749	2.14155	18.01620	2.13988	18.05086	2.12477
19.57941	2.18340	19.57964	2.18337	19.58176	2.18311	19.58982	2.18166	19.62258	2.16833
21.15341	2.22172	21.15363	2.22170	21.15560	2.22147	21.16309	2.22021	21.19411	2.20837
22.72704	2.25732	22.72724	2.25730	22.72907	2.25710	22.73608	2.25599	22.76550	2.24542
24.30034	2.29055	24.30053	2.29053	24.30225	2.29035	24.30882	2.28937	24.33678	2.27988
25.87338	2.32171	25.87356	2.32170	25.87518	2.32154	25.88137	2.32066	25.90799	2.31210
27.44620	2.35105	27.44637	2.35103	27.44789	2.35089	27.45374	2.35010	27.47914	2.34234
29.01883	2.37876	29.01899	2.37874	29.02043	2.37861	29.02597	2.37790	29.05025	2.37084
30.59130	2.40501	30.59145	2.40500	30.59282	2.40488	30.59808	2.40423	30.62131	2.39778
32.16362	2.42996	32.16376	2.42995	32.16506	2.42984	32.17008	2.42925	32.19235	2.42333

Appendix C: Biorthogonality. Following a method introduced by Smith [10] in a different but related problem we note that

$$\left(\frac{\partial^2}{\partial t^2}-\frac{1}{t}\frac{\partial}{\partial t}\right)\frac{\partial^2\Psi}{\partial y^2}=\frac{\partial^2}{\partial y^2}\left(\frac{\partial^2\Psi}{\partial t^2}-\frac{1}{t}\frac{\partial\Psi}{\partial t}\right) \tag{C.1}$$

and define

$$\phi_2(t;p)=\frac{t}{p^2}\frac{d}{dt}\left[\frac{1}{t}\frac{d\phi_1(t;p)}{dt}\right]=\frac{1}{p^2}\left[\phi_1''-\frac{1}{t}\phi_1'\right]. \tag{C.2}$$

Since Ψ is a series of terms of the form

$$e^{\pm py}\phi_1(t;p),$$

we find, from (A.1) and (C.2) that

$$\phi_2''-\frac{1}{t}\phi_2'+p^2[2\phi_2+\phi_1]=0. \tag{C.3}$$

Equations (C.2) and (C.3) may be written as

$$t\frac{d}{dt}\left(\frac{1}{t}\frac{d}{dt}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}\right)+p^2\begin{pmatrix}0 & -1\\ 1 & 2\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}=0 \tag{C.4a}$$

with boundary conditions

$$\phi_1=\frac{d\phi_1}{dt}=0 \quad \text{at } t=a, b. \tag{C.4b}$$

To define the eigenvalue problem which is adjoint to (C.4) we define a generalized Wronskian

$$\begin{aligned}W&=[\psi_1,\psi_2]\begin{pmatrix}\phi_1'\\ \phi_2'\end{pmatrix}-[\psi_1',\psi_2']\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}\\ &=\psi_1\phi_1'+\psi_2\phi_2'-\psi_1'\phi_1-\psi_2'\phi_2.\end{aligned} \tag{C.5}$$

The adjoint eigenfunction $[\psi_1,\psi_2]$ is the function which for the given eigenvalue p and eigenfunction $\begin{bmatrix}\phi_1(t;p)\\ \phi_2(t;p)\end{bmatrix}$ makes $W\equiv 0$. We note that $W(a)=W(b)=0$ when

$$\psi_2(a)=\psi_2(b)=\psi_2'(a)=\psi_2'(b)=0. \tag{C.6a}$$

Moreover, $W/t=$ constant whenever

$$t\frac{d}{dt}\left(\frac{1}{t}\frac{d}{dt}[\psi_1,\psi_2]\right)+p^2[\psi_1,\psi_2]\begin{bmatrix}0 & -1\\ 1 & 2\end{bmatrix}=0. \tag{C.6b}$$

The eigenvalue problem (C.6) is adjoint to (C.4). *If* $p=p_n$ is an eigenvalue of (C.4) with an eigenvector

$$\begin{bmatrix}\phi_1^{(n)}\\ \phi_2^{(n)}\end{bmatrix},$$

then $[\psi_1^{(n)},\psi_2^{(n)}]$ is an eigenvector of (C.6) with the same eigenvalue. Suppose that

p_n and p_m, $m \neq n$ are different eigenvalues. Then

$$(p_m^2 - p_n^2)\int_a^b \frac{1}{t}[\psi_1^{(m)}, \psi_2^{(m)}]\begin{bmatrix}0 & -1\\ 1 & 2\end{bmatrix}\begin{bmatrix}\phi_1^{(n)}\\ \phi_2^{(n)}\end{bmatrix} dt = 0.$$

When $p_n = p_m$ we define

$$K_n = \int_a^b \frac{1}{t}[\psi_1^{(n)}, \psi_2^{(n)}]\begin{bmatrix}0 & -1\\ 1 & 2\end{bmatrix}\begin{bmatrix}\phi_1^{(n)}\\ \phi_2^{(n)}\end{bmatrix} dt.$$

The following biorthogonality conditions holds

$$\int_a^b \frac{1}{t}[\psi_1^{(m)}, \psi_2^{(m)}]\begin{bmatrix}0 & -1\\ 1 & 2\end{bmatrix}\begin{bmatrix}\phi_1^{(n)}\\ \phi_2^{(n)}\end{bmatrix} dt = K_n\,\delta_{mn}. \tag{C.7}$$

Using (A.9), with $p = p_n$, $\phi_1(t; p_n) = \phi_1^{(n)}(t)$, and (C.6) we find that

$$\begin{aligned}
\phi_1^{(n)} &= A_1^{(n)} tJ_1(p_n t) + A_2^{(n)} tY_1(p_n t) + A_3^{(n)} t^2 J_0(p_n t)\\
&\quad + A_4^{(n)} t^2 Y_0(p_n t),\\
\psi_2^{(n)} &= \phi_1^{(n)},\\
\phi_2^{(n)} &= -\left(A_1^{(n)} + \frac{2}{p_n}A_3^{(n)}\right) tJ_1(p_n t) - \left(A_2^{(n)} + \frac{2}{p_n}A_4^{(n)}\right) tY_1(p_n t)\\
&\quad - A_3^{(n)} t^2 J_0(p_n t) - A_4^{(n)} t^2 Y_0(p_n t),\\
\psi_1^{(n)} &= \left(A_1^{(n)} - \frac{2}{p_n}A_3^{(n)}\right) tJ_1(p_n t) + \left(A_2^{(n)} - \frac{2}{p_n}A_4^{(n)}\right) tY_1(p_n t)\\
&\quad + A_3^{(n)} t^2 J_0(p_n t) + A_4^{(n)} t^2 Y_0(p_n t).
\end{aligned} \tag{C.8}$$

Using (C.8), (C.4b) and the integration formulas in Appendix D we compute

$$\begin{aligned}
K_n &= -\frac{2b^2}{p_n^2}[A_3^{(n)} J_1(p_n b) + A_4^{(n)} Y_1(p_n b)]^2\\
&\quad + \frac{2a^2}{p_n^2}[A_3^{(n)} J_1(p_n a) + A_4^{(n)} Y_1(p_n a)]^2.
\end{aligned} \tag{C.9}$$

Appendix D: Indefinite integrals of products of Bessel functions. Let $\mathscr{C}_\mu(kz)$ and $\bar{\mathscr{C}}_\nu(lz)$ denote any two cylinder functions of orders μ and ν, respectively. When $\mu \geqq 1$, we have

$$\begin{aligned}
\int^t z\mathscr{C}_\mu(kz)\bar{\mathscr{C}}_\mu(lz)\,dz &= \frac{t}{k^2 - l^2}\{-k\mathscr{C}_{\mu-1}(kt)\bar{\mathscr{C}}_\mu(lt)\\
&\quad + l\mathscr{C}_\mu(kt)\bar{\mathscr{C}}_{\mu-1}(lt)\},
\end{aligned} \tag{D.1}$$

$$\begin{aligned}
\int^t z^2\mathscr{C}_{\mu-1}(kz)\bar{\mathscr{C}}_\mu(lz)\,dz &= \frac{t^2}{k^2 - l^2}\{k\mathscr{C}_\mu(kt)\bar{\mathscr{C}}_\mu(lt)\\
&\quad + l\mathscr{C}_{\mu-1}(kt)\bar{\mathscr{C}}_{\mu-1}(lt)\}\\
&\quad - \frac{2lt}{(k^2 - l^2)^2}\{k\mathscr{C}_\mu(kt)\bar{\mathscr{C}}_{\mu-1}(lt) - l\mathscr{C}_{\mu-1}(kt)\bar{\mathscr{C}}_\mu(lt)\}\\
&\quad - \frac{2(\mu -)t}{k^2 - l^2}\mathscr{C}_{\mu-1}(kt)\bar{\mathscr{C}}_\mu(lt).
\end{aligned} \tag{D.2}$$

When $\mu \geqq 0$,

$$\begin{aligned}(\text{D.3})\quad &\int^t z^3\mathscr{C}_\mu(kz)\bar{\mathscr{C}}_\mu(lz)\,dz\\
&=\frac{t^3}{k^2-l^2}\{k\mathscr{C}_{\mu+1}(kt)\bar{\mathscr{C}}_\mu(lt)-l\mathscr{C}_\mu(kt)\bar{\mathscr{C}}_{\mu+1}(lt)\}\\
&\quad+\frac{2t^2}{(k^2-l^2)^2}\{2kl\mathscr{C}_{\mu+1}(kt)\bar{\mathscr{C}}_{\mu+1}(lt)+(k^2+l^2)\mathscr{C}_\mu(kt)\bar{\mathscr{C}}_\mu(lt)\}\\
&\quad-\frac{4(k^2+l^2)t}{(k^2-l^2)^3}\{k\mathscr{C}_{\mu+1}(kt)\bar{\mathscr{C}}_\mu(lt)-l\mathscr{C}_\mu(kt)\bar{\mathscr{C}}_{\mu+1}(lt)\}\\
&\quad-\frac{4\mu t}{(k^2-l^2)^2}\{k\mathscr{C}_{\mu+1}(kt)\bar{\mathscr{C}}_\mu(lt)+l\mathscr{C}_\mu(kt)\bar{\mathscr{C}}_{\mu+1}(lt)\}.\end{aligned}$$

When $\mu \geqq 1$,

$$\begin{aligned}(\text{D.4})\quad \int^t z\mathscr{C}_\mu(kz)\bar{\mathscr{C}}_\mu(kz)\,dz &=\frac{t^2}{2}\{\mathscr{C}_\mu(kt)\bar{\mathscr{C}}_\mu(kt)+\mathscr{C}_{\mu-1}(kt)\bar{\mathscr{C}}_{\mu-1}(kt)\}\\
&\quad-\frac{\mu t}{2k}\{\mathscr{C}_{\mu-1}(kt)\bar{\mathscr{C}}_\mu(kt)+\mathscr{C}_\mu(kt)\bar{\mathscr{C}}_{\mu-1}(kt)\},\end{aligned}$$

$$\begin{aligned}(\text{D.5})\quad &\int^t z^2\mathscr{C}_{\mu-1}(kz)\bar{\mathscr{C}}_\mu(kz)\,dz\\
&=\frac{t^3}{4}\{\mathscr{C}_{\mu-1}(kt)\bar{\mathscr{C}}_\mu(kt)-\mathscr{C}_\mu(kt)\bar{\mathscr{C}}_{\mu-1}(kt)\}\\
&\quad+\frac{t^2}{2k}\{\mu\mathscr{C}_\mu(kt)\bar{\mathscr{C}}_\mu(kt)+(\mu-1)\mathscr{C}_{\mu-1}(kt)\bar{\mathscr{C}}_{\mu-1}(kt)\}\\
&\quad-\frac{\mu(\mu-1)t}{4k^2}\{3\mathscr{C}_{\mu-1}(kt)\ \bar{\mathscr{C}}_\mu(kt)+\mathscr{C}_\mu(kt)\bar{\mathscr{C}}_{\mu-1}(kt)\}.\end{aligned}$$

When $\mu \geqq 0$,

$$\begin{aligned}(\text{D.6})\quad &\int^t z^3\mathscr{C}_\mu(kz)\bar{\mathscr{C}}_\mu(kz)\,dz\\
&=\frac{t^4}{6}\{\mathscr{C}_{\mu+1}(kt)\bar{\mathscr{C}}_{\mu+1}(kt)+\mathscr{C}_\mu(kt)\bar{\mathscr{C}}_\mu(kt)\}\\
&\quad-\frac{(\mu-1)t^3}{6k}\{\mathscr{C}_{\mu+1}(kt)\bar{\mathscr{C}}_\mu(kt)+\mathscr{C}_\mu(kt)\bar{\mathscr{C}}_{\mu+1}(kt)\}\\
&\quad+\frac{(\mu-1)t^2}{3k^2}\{(\mu+1)\mathscr{C}_{\mu+1}(kt)\bar{\mathscr{C}}_{\mu+1}(kt)+\mu\mathscr{C}_\mu(kt)\bar{\mathscr{C}}_\mu(kt)\}\\
&\quad-\frac{\mu(\mu^2-1)t}{3k^3}\{\mathscr{C}_{\mu+1}(kt)\bar{\mathscr{C}}_\mu(kt)+\mathscr{C}_\mu(kt)\bar{\mathscr{C}}_{\mu+1}(kt)\}.\end{aligned}$$

Equations (D.1) and (D.4) are well known (see Watson [12, p. 134]). The other relations are also probably well known; in any event, they may be derived as follows: To obtain (D.2) we first note that

$$z^{\mu}\mathscr{C}_{\mu-1}(kz)=\frac{1}{k}\frac{d}{dz}(z^{\mu}\mathscr{C}_{\mu}(kz)).$$

Then integrating by parts and using (D.1) we find that

$$\begin{aligned}\int^{t} z^{2}\mathscr{C}_{\mu-1}(kz)\bar{\mathscr{C}}_{\mu}(lz)\,dz &= \int^{t}[z^{\mu}\mathscr{C}_{\mu-1}(kz)][z^{2}z^{-\mu}\bar{\mathscr{C}}_{\mu}(lz)]\,dz\\ &=\frac{t^{2}}{k}\mathscr{C}_{\mu}(kt)\bar{\mathscr{C}}_{\mu}(lt)-\frac{2t}{k(k^{2}-l^{2})}\{k\mathscr{C}_{\mu+1}(kt)\bar{\mathscr{C}}_{\mu}(lt)\\ &\quad -l\mathscr{C}_{\mu}(kt)\bar{\mathscr{C}}_{\mu+1}(lt)\}+\frac{l}{k}\int^{t} z^{2}\mathscr{C}_{\mu}(kz)\bar{\mathscr{C}}_{\mu+1}(lz)\,dz.\end{aligned}\tag{D.7}$$

Again integrating by parts, but using different parts, we find that

$$\begin{aligned}\int^{t} z^{2}\mathscr{C}_{\mu-1}(kz)\bar{\mathscr{C}}_{\mu}(lz)\,dz &= \int^{t}[z^{-(\mu-1)}\mathscr{C}_{\mu-1}(kz)][z^{\mu+1}\bar{\mathscr{C}}_{\mu}(lz)]\,dz\\ &=\frac{t^{2}}{l}\mathscr{C}_{\mu-1}(kt)\bar{\mathscr{C}}_{\mu+1}(lt)+\frac{k}{l}\int^{t} z^{2}\mathscr{C}_{\mu}(kz)\bar{\mathscr{C}}_{\mu+1}(lz)\,dz.\end{aligned}\tag{D.8}$$

The difference k/l (D.7) $-l/k$ (D.8) followed by application of recursion relations for cylinder functions leads to (D.2). Equation (D.5) follows from an application of l'Hôpital's rule to (D.2) in the limit $l\to k$. Similar computations lead to (D.3) and (D.6).

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