*Offprint from "Archive for Rational Mechanics and Analysis",*
*Volume 66, Number 4, 1977, P. 311—344*


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## Contents



## I. Introduction

In this paper I derive iterative procedures for the sequential computation of velocity fields and strain histories of motions of incompressible simple fluids driven by arbitrary, time-dependent prescribed data. The arbitrary data is a small perturbation of special data giving rise to special exact solutions of the equations of motion. My goal, which is achieved when the special data gives rise to steady rigid rotation $\boldsymbol{\Omega} \wedge \boldsymbol{x}$, $\boldsymbol{\Omega}=$ constant, is to reduce the perturbation equations to canonical form; that is, to show that the three equations of momentum and the equation of incompressibility are four equations in three unknown components of the perturbation velocity $\boldsymbol{U}^{\langle n\rangle}(\boldsymbol{x},t)$ and reaction pressure $p^{\langle n\rangle}(\boldsymbol{x},t)$. Here $\boldsymbol{U}^{\langle n\rangle}= \frac{1}{n!}\partial^n \boldsymbol{U}(x,t,\varepsilon)/\partial\varepsilon^n$ is evaluated at $\varepsilon=0$ where $\varepsilon$ is a small parameter in the prescribed data such that when $\varepsilon=0$, $\boldsymbol{U}(\boldsymbol{x},t,0)=\boldsymbol{U}^{\langle 0\rangle}=\boldsymbol{\Omega}\wedge\boldsymbol{x}$. For example, the motion may be driven by boundary data in the form $\boldsymbol{U}(\boldsymbol{x},t,\varepsilon)=\boldsymbol{\Omega}\wedge\boldsymbol{x}+\varepsilon\boldsymbol{q}(\boldsymbol{x},t)$ for $\boldsymbol{x}\in\partial\mathscr{V}(t)$ where $\mathscr{V}(t)$ is the region occupied by fluid. In the canonical theory, as in the Navier Stokes theory, we can find $U^{\langle n\rangle}$ and $p^{\langle n\rangle}$ by solving four equations in four unknowns, The particle paths $\chi^{\langle n\rangle}$ which determine the strain history are for the forcing terms in the equations which determine $U^{\langle n+1\rangle}$ and $p^{\langle n+1\rangle}$. But $U^{\langle n\rangle}$ is independent of $\chi^{\langle n\rangle}$ and $\chi^{\langle n\rangle}$ is given by quadrature involving $U^{\langle n\rangle}$ and other known terms of lower order. The history of the *strain* (given in terms $\chi^{\langle n\rangle}$) is passive; that is, it enters into the equations of motion through forcing terms computed on solutions of lower order. But the history of the *velocity* is active; that is, it enters into the dominant term from the stress in the linear operator which needs inverting at each stage of the perturbation.

The velocity derivatives which enter into the hereditary integrals giving the stress are in the form $U^{\langle n\rangle}(\chi^{\langle 0\rangle}(x,\tau),\tau)$ where $-\infty<\tau\leqq t$, $x$ is the particle which coincides with the field point $x$ in the present configuration and $\chi^{\langle 0\rangle}$ is determined by integration of $d\chi^{\langle 0\rangle}/d\tau=\Omega\wedge\chi^{\langle 0\rangle}$ subject to the condition that $\chi^{\langle 0\rangle}(x,t)=x$. The history of the velocity derivative $U^{\langle n\rangle}$ belonging to $x$ is therefore computed for $\chi^{\langle 0\rangle}(x,\tau)$ on circles of fixed radius and elevation around the fixed axis $e_3=\Omega/|\Omega|$ of rotation. For such motions the only coordinate which changes with $s=t-\tau$ is the angular displacement $\theta^{\langle 0\rangle}=\theta-\Omega s$. The perturbation equations are therefore naturally framed in coordinates having axial symmetry; for example, in cylindrical coordinates $(r,\theta,z)$. In these coordinates, I find that

$$U^{\langle 0\rangle}=e_\theta r\Omega,\qquad p^{\langle 0\rangle}+\rho\Omega^2 r=\text{const},$$

and I introduce the vectors

$$\begin{aligned}\tilde{U}^{\langle n\rangle}(s)&=e_r(\theta)\,U_r^{\langle n\rangle}(r,\theta-\Omega s,z,t-s)\\&\quad+e_\theta(\theta)\,U_\theta^{\langle n\rangle}(r,\theta-\Omega s,z,t-s)\\&\quad+e_3\,U_z^{\langle n\rangle}(r,\theta-\Omega s,z,t-s)\\&\equiv e_l(\theta)\,U_l^{\langle n\rangle}(r,\theta-\Omega s,z,t-s),\end{aligned}$$

and

$$U^{\langle n\rangle}=\tilde{U}^{\langle n\rangle}(0),$$

for $n>1$. The four equations and four unknowns to which I have already alluded are

$$\text{div}\,U^{\langle n\rangle}=0 \tag{I.1}$$

and

$$\rho(\partial_t+\Omega\partial_\theta+\Omega\wedge)U^{\langle n\rangle}+\nabla p^{\langle n\rangle}-\int_0^\infty G(s)\nabla^2\tilde{U}^{\langle n\rangle}(s)\,ds=f^{\langle n\rangle}. \tag{I.2}$$

Here $G(s)$ is the shear relaxation modulus, $f^{\langle n\rangle}$ depends on lower-order fields $l<n$; for example, $f^{\langle 1\rangle}=0$,

$$f^{\langle 2\rangle}=\text{div}\left\{\int_0^\infty G(s)A_2(s)\,ds+\int_0^\infty\int_0^\infty \gamma(s_1,s_2)A_1(s_1)\cdot A_1(s_2)\,ds_1\,ds_2\right\}$$

where

$$\begin{aligned}A_1(s)&=\nabla\tilde{U}^{\langle 1\rangle}(s)+\nabla^T\tilde{U}^{\langle 1\rangle}(s),\\A_2(s)&=\chi^{\langle 1\rangle}\cdot\nabla A_1(s)+A_1(s)\cdot\nabla\chi^{\langle 1\rangle}+(A_1(s)\cdot\nabla\chi^{\langle 1\rangle})^T\end{aligned}$$

and

$$\text{(I.3)} \qquad \frac{d\boldsymbol{\chi}^{\langle 1\rangle}(\boldsymbol{x},\tau)}{d\tau}=\boldsymbol{e}_l(\theta-\Omega s)\,U_l^{\langle 1\rangle}(r,\theta-\Omega s,z,t-s), \qquad \boldsymbol{\chi}^{\langle 1\rangle}(\boldsymbol{x},t)=0.$$

$\boldsymbol{U}^{\langle 1\rangle}$ and $p^{\langle 1\rangle}$ are determined by inverting (I.1) and (I.2) for prescribed boundary values and initial-history. Then $\boldsymbol{\chi}^{\langle 1\rangle}$ follows from (I.3) and we may compute $\boldsymbol{f}^{\langle 2\rangle}$. $\boldsymbol{U}^{\langle 2\rangle}$ and $p^{\langle 2\rangle}$ are then determined by inverting (I.1) and (I.2) (when $n=2$). Then we get $\boldsymbol{\chi}^{\langle 2\rangle}$ from an equation like (I.3) and compute $\boldsymbol{f}^{\langle 3\rangle}$, and so on.

The perturbation equations have many interesting properties:

(1) The principal (linear) parts of the equations (I.1) and the left side of (I.2) do not depend on any constitutive assumption less general than that the stress be Fréchet differentiable on the zero history in Hilbert space topologies defined by integrals. An integral representation for the stress is then implied by the Reisz theorem.

(2) The theory of perturbations of the rest state (JOSEPH, 1976) is a special case of the present theory with $\Omega=0$. When $\Omega=0$ and $G(s)$ is physically reasonable the rest state is linearly stable and solutions of the perturbation problems exist and are unique (SLEMROD, 1976).

(3) Rigid rotation is linearly stable if $G(s)$ is physically reasonable; see §XII. Hence, we expect uniqueness (and existence) for $\Omega\neq 0$.

(4) If the velocity $\boldsymbol{U}^{\langle n\rangle}(r,\theta,z)$ at a fixed point $(r,\theta,z)$ is steady, the history of the particle which is presently at that fixed point is unsteady because it was at a different place $(r,\theta-\Omega s,z)$ earlier on, and its velocity $\tilde{\boldsymbol{U}}^{\langle n\rangle}(r,\theta-\Omega s,z)$ at that place was different. The same remark goes for steady flow in a rotating basis $(r,\theta(t),z)$ but, of course, the inertial terms in (I.2) take a different form in the rotating basis.

(5) If the flow is axisymmetric in a fixed or rotating basis, the perturbation fields are independent of the angular variable $\theta$ and $\theta-\Omega s$. The terms in the equations of motion are then identical to those which arise in the theory of perturbations of the rest state. If, in addition, the flow is steady in some fixed or rotating basis, the perturbation equations for particle paths may be integrated in terms of the steady, but unknown, velocity, the time variables $t$ and $\tau$ appear only in the combination $s=t-\tau$, and the stresses may be expressed in terms of the Rivlin-Ericksen tensors and constants familiar in the theory of slow steady motions. The constants are defined by various moments of the kernel functions in the integrals giving the stress (see X.22, 23, 24).

(6) Various simplifications of the equations result from decomposing solutions of (I.2) and (I.3) as a Fourier series in $\theta$. For time-periodic data, the equations may be further simplified by decomposing solutions into a Fourier series in time (see §11). These decompositions lead to an infinite number of different "complex viscosities" of the form

$$\eta^*(\omega m+\Omega l)=\int_0^\infty G(s)\,e^{-i(\omega m+\Omega l)s}\,ds$$

where $l$ is the index of azimuthal periodicity and $m\omega$ the $m^{\text{th}}$ harmonic of the fundamental frequency. The "complex viscosity" is not a material function because $\Omega$, $\omega$, $m$ and $l$ depend on the prescribed data. Some results of WALTERS (1970) and

ABBOTT & WALTERS (1970) for $m=0$ and $l=1$ are generalized by the present analysis.
(7) I call (I.2) with $n=1(f^{\langle 1\rangle}=0)$ the linearized equation for rotating simple fluids. In § XIV I show that the well known linearized theory of rotating Newtonian fluids for large $\Omega$ and small $\varepsilon$ applies to (I.1) and (I.2) when the boundary data is steady. In XIV and XV, I consider the orthogonal rheometer and show that when $\Omega$ is large the stress in the interior of the fluid may be neglected and inertia dominates. At the boundary there is a layer in which the stress is important. The thickness of this layer is much smaller in a viscoelastic fluid than in a Newtonian fluid; the ratio of thicknesses tends to zero with $1/\sqrt{\Omega}$ as $\Omega\to\infty$.

## II. Notations

The list of notations given below defines primitive quantities which are used in NOLL's theory of incompressible simple fluids. I am assuming that readers already understand the elementary concepts implied by these definitions.

| | |
|---|---|
| $\tau, t, s$ | Times; $t$ is the present time, $\tau \leqq t$ is an earlier time and $s=t-\tau$. |
| $\boldsymbol{\chi}_t(\boldsymbol{x},\tau)$ | Position of the particle presently at $\boldsymbol{x}=\boldsymbol{\chi}_t(\boldsymbol{x},t)$ at time $\tau<t$. |
| $\nabla=\partial/\partial\boldsymbol{x}$, $\operatorname{div}=\nabla\cdot$ | Gradient, divergence with respect to $\boldsymbol{x}$. |
| $\nabla_\xi=\partial/\partial\boldsymbol{\xi}$, $\operatorname{div}_\xi=\nabla_\xi$ | Gradient, divergence with respect to $\boldsymbol{\xi}$. |
| $\boldsymbol{F}(s)=\nabla\boldsymbol{\chi}_t(\boldsymbol{x},t-s)$ | Relative deformation gradient. |
| $\boldsymbol{G}(s)=\boldsymbol{F}^T(s)\cdot\boldsymbol{F}(s)-1$ | Cauchy-strain tensor minus $\mathbf{1}$ at time $\tau$. |
| $\mathbb{J}(s)=-\dot{\boldsymbol{G}}(s)$ | Derivative with respect to $s$. |
| $\mathscr{F}\left[\overset{\infty}{\underset{s=0}{\boldsymbol{G}}}(s)\right]\equiv\mathscr{F}[\boldsymbol{G}(s)]$ | Extra stress, $\mathscr{F}[0]\equiv 0$. |
| $\boldsymbol{T}=-p\mathbf{1}+\mathscr{F}[\boldsymbol{G}(s)]$ | The stress $-p\mathbf{1}$ is any convenient isotropic part of $\boldsymbol{T}$ consistent with $\mathscr{F}[0]=0$. $p$ is constitutively indeterminate and is to be determined from the solution of the initial boundary-value problem. |
| $\mathscr{F}_n[\boldsymbol{G}_0(s)\lvert\boldsymbol{a}_1,\boldsymbol{a}_2,\ldots,\boldsymbol{a}_n]=\boldsymbol{n}!\,\mathscr{F}_{(n)}$ | $n!$ times the $n^{\text{th}}$ functional derivative of $\mathscr{F}[\boldsymbol{G}(s)]$ evaluated on $\boldsymbol{G}_0$. $\mathscr{F}_n$ is multilinear and symmetric in the $n$ tensors $\boldsymbol{a}_i(s)$. |

The quantities listed below are defined and explained in the text. They are listed here for the convenience of the reader.

| | |
|---|---|
| $\tilde{\mathscr{F}}_n[0\lvert\mathbb{J}(s_1)\ldots,\mathbb{J}(s_n)]$ | Functional derivative defined by (VIII.1, 2). |
| $\boldsymbol{A}^{\langle n\rangle}$, $\mathbb{J}^{\langle n\rangle}$, $\boldsymbol{B}=\boldsymbol{A}_2(s)$ | Symmetric tensors defined by (VII.18–22). |
| $\boldsymbol{Q}(\Omega s)$ | Orthogonal tensor whose matrix relative to the fixed orthonormal basis $(\boldsymbol{e}_1,\boldsymbol{e}_2,\boldsymbol{e}_3)$ is |

$$[\boldsymbol{Q}(\Omega s)]=\begin{pmatrix}\cos\Omega s & \sin\Omega s & 0\\ -\sin\Omega s & \cos\Omega s & 0\\ 0 & 0 & 1\end{pmatrix}.$$

$(\boldsymbol{e}_1(\theta), \boldsymbol{e}_2(\theta), \boldsymbol{e}_3) = (\boldsymbol{e}_r(\theta), \boldsymbol{e}_\theta(\theta), \boldsymbol{e}_3)$ Orthonormal base vectors in cylindrical coordinates $(r, \theta, z)$. $\boldsymbol{e}_3 \equiv \boldsymbol{e}_z$ is independent of $\theta$ and $(\boldsymbol{e}_1, \boldsymbol{e}_2) = (\boldsymbol{e}_1(0), \boldsymbol{e}_2(0))$.

$A_{ij}\langle\theta\rangle = \boldsymbol{e}_i(\theta) \cdot \boldsymbol{A} \cdot \boldsymbol{e}_j(\theta)$ Physical component of $\boldsymbol{A}$ in the basis $(\boldsymbol{e}_r(\theta), \boldsymbol{e}_\theta(\theta), \boldsymbol{e}_3)$; for example, $A_{12}\langle\theta\rangle = A_{r\theta} = \boldsymbol{e}_r(\theta) \cdot \boldsymbol{A} \cdot \boldsymbol{e}_\theta(\theta)$.

$\hat{A}_{ij} = A_{ij}\langle\theta - \Omega s\rangle = \boldsymbol{e}_i(\theta - \Omega s) \cdot \boldsymbol{A} \cdot \boldsymbol{e}_j(\theta - \Omega s)$.

The following quantities are material functions, moments of material functions and material parameters. They are defined at the equation listed with the symbol. $G(s)$, see (VIII.1); $\hat{\alpha}(s_1, s_2)$ and $\gamma(s_1, s_2)$, see (VIII.2); $\mu$, see (X.22); $\nu = \mu/\rho$, $\alpha_1$ and $\alpha_2$, see (X.23, 24); $\eta^*(\Omega)$, see (XI.8).

## III. The Initial-History Problem

In this section I propose one formulation of an initial-history problem for an incompressible simple fluid. This formulation is tentative since conventional methods of analysis are unavailing in problems in which the extra stress $\mathscr{F}[\boldsymbol{G}(s)]$ is so generally specified.

I suppose that at each instant $t > 0$, the fluid is driven by a prescribed body force $\boldsymbol{f}(\boldsymbol{x}, t)$ in the region $\mathscr{V}(t)$ occupied by the fluid and by the motion of the boundary $\partial\mathscr{V}(t)$. Then

$$\left[\frac{\partial \boldsymbol{U}}{\partial t} + (\boldsymbol{U} \cdot \nabla)\boldsymbol{U}\right] = -\nabla p + \nabla \cdot \mathscr{F}[\boldsymbol{G}(s)] + \rho \boldsymbol{f}(\boldsymbol{x}, t) \tag{III.1}$$

and

$$\nabla \cdot \boldsymbol{U}(\boldsymbol{x}, t) = 0 \tag{III.2}$$

hold in $\mathscr{V}(t)$, $t > 0$ and, on the boundary,

$$\boldsymbol{U}(\boldsymbol{x}, t) = \boldsymbol{q}(\boldsymbol{x}, t), \qquad \boldsymbol{x} \in \partial\mathscr{V}(t). \tag{III.3}$$

When $\tau \leqq 0$, the history of the motion of the fluid in $\mathscr{V}(\tau)$ and on the boundary are prescribed arbitrarily.

$$\boldsymbol{U}(\boldsymbol{\chi}_t(\boldsymbol{x}, \tau), \tau) \quad \text{is prescribed in } \mathscr{V}(\tau), \ \tau \leqq 0. \tag{III.4}$$

The relative position at time $\tau < t$ of the particle at $\boldsymbol{x}$ enters into (III.1) through the argument $\boldsymbol{G}(s)$ of the extra stress $\mathscr{F}$. It is then possible to regard (III.1), (III.2) and the path equations $\boldsymbol{U} = \partial\boldsymbol{\chi}_t/\partial\tau$ as seven equations for seven unknown scalars $\boldsymbol{U}$, $p$ and $\boldsymbol{\chi}_t$. I call this system of seven equations, subject to conditions (III.3) and (III.4), the initial-history problem.

Though there are no existence or uniqueness theorems known for the initial-history problem under general circumstances, it is possible to give some special solutions which do not require more assumptions about the relation between $\mathscr{F}$ and $\boldsymbol{G}(s)$. Some progress toward an explicit and tractable theory for the initial-history problem can be made by perturbing the special solutions.

## IV. Special Solutions

We should imagine that the data depends on a parameter $\varepsilon$ and that when $\varepsilon = 0$,

$$\text{(IV.1)} \quad \rho\left[\frac{\partial \boldsymbol{U}^{\langle 0\rangle}}{\partial t} + \boldsymbol{U}^{\langle 0\rangle}\cdot\nabla\boldsymbol{U}^{\langle 0\rangle}\right] = -\nabla p^{\langle 0\rangle} + \nabla\cdot\mathscr{F}[\boldsymbol{G}^{\langle 0\rangle}(s)] + \rho\boldsymbol{f}^{\langle 0\rangle}(\boldsymbol{x},t)$$

and

$$\text{(IV.2)} \quad \nabla\cdot\boldsymbol{U}^{\langle 0\rangle}(\boldsymbol{x},t) = 0$$

hold in $\mathscr{V}(t)$, $\forall t\in R$ and

$$\text{(IV.3)} \quad \boldsymbol{U}^{\langle 0\rangle}(\boldsymbol{x},t) = \boldsymbol{q}^{\langle 0\rangle}(\boldsymbol{x},t), \qquad \boldsymbol{x}\in\partial\mathscr{V}(t),\ \forall t\in R.$$

This problem, like that in § III, is incompletely specified because the response functional $\mathscr{F}$ has been specified up to now at too great a level of generality. To get results free of overly restrictive assumptions about the form of $\mathscr{F}$, an inverse procedure is often used. In this procedure one specifies the motion and computes the data. The motion is chosen so that $\boldsymbol{G}^{\langle 0\rangle}$ is such that $\mathscr{F}[\boldsymbol{G}^{\langle 0\rangle}(s)]$ reduces to a simpler, potentially tractable form. It is then necessary to check that the assumed motion is compatible with equations of motion. For example,

(i) $\boldsymbol{U}^{\langle 0\rangle}(\boldsymbol{\chi}_t^{\langle 0\rangle}(\boldsymbol{x},\tau),\tau) = \boldsymbol{\Omega}\wedge\boldsymbol{\chi}_t^{\langle 0\rangle}$ describes the rigid motions of a body. For such motions, $\boldsymbol{G}^{\langle 0\rangle} = 0$ and $\mathscr{F}[0] = 0$. Then, if $\boldsymbol{f}^{\langle 0\rangle} = 0$,

$$\text{(IV.4)} \quad \rho\mathring{\boldsymbol{\Omega}}\wedge\boldsymbol{x} = -\nabla(p + \tfrac{1}{2}\rho|\boldsymbol{\Omega}\wedge\boldsymbol{x}|^2)$$

and (IV.4) is solvable if and only if $\mathring{\boldsymbol{\Omega}}\wedge\boldsymbol{x}$ is a gradient; that is, when

$$2\rho\mathring{\boldsymbol{\Omega}} = 0.$$

Hence, $\boldsymbol{U}^{\langle 0\rangle} = \boldsymbol{\Omega}\wedge\boldsymbol{\chi}_t^{\langle 0\rangle}$ *is compatible with the equations if and only if the rotation is steady.*

(ii) $\boldsymbol{U}^{\langle 0\rangle}(\boldsymbol{\chi}_t^{\langle 0\rangle}(\boldsymbol{\chi},\tau),\tau)$ is a viscometric motion. Then $\mathscr{F}[\boldsymbol{G}^{\langle 0\rangle}(s)]$ may be represented by three scalar functions of the rate of shearing. In some special cases the assumed kinematics is compatible with the equations of motion and data which is not contrived. In other cases, the assumed form of the motion is compatible with the equations only if inertia is neglected.

(iii) The motion is such that the Cauchy strain depends only on the Rivlin-Ericksen kinematic tensors $\boldsymbol{A}_1, \boldsymbol{A}_2, \ldots, \boldsymbol{A}_N$. For example, if $\boldsymbol{A}_{N+1} \equiv 0$ in $\mathscr{V}(t)$, then

$$\text{(IV.5)} \quad \boldsymbol{G}^{\langle 0\rangle}(s) = \sum_{n=1}^{N}\frac{(-s)^n}{n!}\boldsymbol{A}_n[\boldsymbol{U}^{\langle 0\rangle}(\boldsymbol{x},t)]$$

is a polynomial of degree $N$ in $s$ whose coefficients are Rivlin-Ericksen tensors $\boldsymbol{A}_n[\boldsymbol{U}^{\langle 0\rangle}(\boldsymbol{x},t)]$ and $\mathscr{F}[\boldsymbol{G}^{\langle 0\rangle}]$ is a constitutive equation for a fluid of complexity $N$ (see TRUESDELL & NOLL, 1965). To solve the inverse problem, it is necessary to find the data for which polynomial histories of the form (IV.5) correspond to solutions of (IV.1), (IV.2) and (IV.3) when $\mathscr{F}[\boldsymbol{G}^{\langle 0\rangle}]$ is the constitutive equation for a fluid of complexity $N$.

The motions found by the inverse method are insufficiently general because so much must be assumed to reduce $\mathscr{F}$ to a tractable form consistent with the equations of motion.

## V. Kinematics for the Perturbation of Special Solutions

It is necessary to describe the sequential procedure for solving the equations of motion when $\boldsymbol{f}(\boldsymbol{x}, t, \varepsilon)$ and $\boldsymbol{q}(\boldsymbol{x}, t, \varepsilon)$ are close to $\boldsymbol{f}^{\langle 0\rangle}(\boldsymbol{x}, t)$ and $\boldsymbol{q}^{\langle 0\rangle}(\boldsymbol{x}, t)$. When $\boldsymbol{f}$ and $\boldsymbol{q}$ are in permanent form, say, steady data or time periodic data, we demand that $\boldsymbol{f}$ and $\boldsymbol{q}$ be small perturbations of $\boldsymbol{f}^{\langle 0\rangle}$ and $\boldsymbol{q}^{\langle 0\rangle}$ for $-\infty<t<\infty$. For initial history problems, the data is perturbed for $t>0$ but the initial history $\boldsymbol{U}(\boldsymbol{\chi}_t(\boldsymbol{x}, \tau, \varepsilon), \tau, \varepsilon)$ is assumed to be close to the history $\boldsymbol{U}^{\langle 0\rangle}(\boldsymbol{\chi}_t^{\langle 0\rangle}(\boldsymbol{x}, \tau), \tau)$ which solves (IV.1), (IV.2) and (IV.3) when $\tau \leqq 0$. It is, of course, assumed that the functions with superscript $\langle 0\rangle$ solving (IV.1), (IV.2) and (IV.2) are all known.

We first relate

(V.1) $$\boldsymbol{\chi}_t(\boldsymbol{x}, \tau, \varepsilon)=\boldsymbol{\chi}_t^{\langle 0\rangle}(\boldsymbol{x}, \tau)+\varepsilon \boldsymbol{\chi}_t^{\langle 1\rangle}(\boldsymbol{x}, \tau)+\varepsilon^2 \boldsymbol{\chi}_t^{\langle 2\rangle}(\boldsymbol{x}, \tau)+\cdots$$

and

(V.2) $$\boldsymbol{U}(\boldsymbol{\chi}_t(\boldsymbol{x}, \tau, \varepsilon), \tau, \varepsilon) \equiv \tilde{\boldsymbol{U}}(\boldsymbol{x}, \tau, \varepsilon)=\tilde{\boldsymbol{U}}^{\langle 0\rangle}(\boldsymbol{x}, \tau)+\varepsilon \tilde{\boldsymbol{U}}^{\langle 1\rangle}(\boldsymbol{x}, \tau)+\varepsilon^2 \tilde{\boldsymbol{U}}^{\langle 2\rangle}(\boldsymbol{x}, \tau)+\cdots.$$

Since

(V.3) $$\frac{\partial \boldsymbol{\chi}_t(\boldsymbol{x}, \tau, \varepsilon)}{\partial t}=\tilde{\boldsymbol{U}}(\boldsymbol{x}, \tau, \varepsilon)=\boldsymbol{U}(\boldsymbol{\chi}_t(\boldsymbol{x}, \tau, \varepsilon), \tau, \varepsilon),$$

we have, assuming the particle label

(V.4) $$\boldsymbol{x}=\boldsymbol{\chi}_t(\boldsymbol{x}, t, \varepsilon)$$

is independent of $\varepsilon$, that for $n=0, 1, 2, \ldots$,

(V.5) $$\frac{\partial \boldsymbol{\chi}_t^{\langle n\rangle}(\boldsymbol{x}, \tau)}{\partial \tau}=\tilde{\boldsymbol{U}}^{\langle n\rangle}(\boldsymbol{x}, \tau).$$

Morever, using (V.1), (V.4) and (V.2), we find that

(V.6) $$\boldsymbol{\chi}_t^{\langle 0\rangle}(\boldsymbol{x}, t)=\boldsymbol{x},$$

(V.7) $$\boldsymbol{\chi}_t^{\langle n\rangle}(\boldsymbol{x}, t)=0$$

and

$$\tilde{\boldsymbol{U}}^{\langle n\rangle}(\boldsymbol{x}, t)=\boldsymbol{U}^{\langle n\rangle}(\boldsymbol{x}, t).$$

The function $\tilde{\boldsymbol{U}}(\boldsymbol{x}, \tau)$ is an auxiliary function used to facilitate our computation of particle paths.

To simplify notations, we define

(V.8) $$\boldsymbol{\chi}^{\langle n\rangle}(\tau) \equiv \boldsymbol{\chi}_t^{\langle n\rangle}(\boldsymbol{x}, \tau), \qquad n=1, 2, \ldots$$

and

(V.9) $$\boldsymbol{\xi}(\tau) \equiv \boldsymbol{\chi}_t^{\langle 0\rangle}(\boldsymbol{x}, \tau).$$

Then, using (V.3) and the chain rule, we find that

$$\boldsymbol{\xi}_{,\tau} = \tilde{\boldsymbol{U}}^{\langle 0\rangle}(\boldsymbol{x},\tau) = \boldsymbol{U}^{\langle 0\rangle}(\boldsymbol{\xi}(\tau),\tau),$$

$$\boldsymbol{\chi}^{\langle 1\rangle}_{,\tau} = \tilde{\boldsymbol{U}}^{\langle 1\rangle}(\boldsymbol{x},\tau) = \boldsymbol{U}^{\langle 1\rangle} + (\boldsymbol{\chi}^{\langle 1\rangle}\cdot\nabla_{\xi})\,\boldsymbol{U}^{\langle 0\rangle},$$

$$\boldsymbol{\chi}^{\langle 2\rangle}_{,\tau} = \tilde{\boldsymbol{U}}^{\langle 2\rangle}(x,\tau) = \boldsymbol{U}^{\langle 2\rangle} + (\boldsymbol{\chi}^{\langle 2\rangle}\cdot\nabla_{\xi})\,\boldsymbol{U}^{\langle 0\rangle} + (\boldsymbol{\chi}^{\langle 1\rangle}\cdot\nabla_{\xi})\,\boldsymbol{U}^{\langle 1\rangle} + \frac{1}{2}\chi_l^{\langle 1\rangle}\chi_j^{\langle 1\rangle}\frac{\partial^2\boldsymbol{U}^{\langle 0\rangle}}{\partial\xi_l\,\partial\xi_j},$$

$$\boldsymbol{\chi}^{\langle 3\rangle}_{,\tau} = \tilde{\boldsymbol{U}}^{\langle 3\rangle}(\boldsymbol{x},\tau) = \boldsymbol{U}^{\langle 3\rangle} + (\boldsymbol{\chi}^{\langle 3\rangle}\cdot\nabla_{\xi})\,\boldsymbol{U}^{\langle 0\rangle} + (\boldsymbol{\chi}^{\langle 2\rangle}\cdot\nabla_{\xi})\,\boldsymbol{U}^{\langle 1\rangle} + (\boldsymbol{\chi}^{\langle 1\rangle}\cdot\nabla_{\xi})\,\boldsymbol{U}^{\langle 2\rangle} + \frac{1}{2}\chi_l^{\langle 1\rangle}\chi_j^{\langle 1\rangle}\frac{\partial^2\boldsymbol{U}^{\langle 1\rangle}}{\partial\xi_l\,\partial\xi_j} + \frac{1}{2}\chi_l^{\langle 2\rangle}\chi_j^{\langle 1\rangle}\frac{\partial^2\boldsymbol{U}^{\langle 0\rangle}}{\partial\xi_l\,\partial\xi_j} + \frac{1}{6}\chi_l^{\langle 1\rangle}\chi_j^{\langle 1\rangle}\chi_k^{\langle 1\rangle}\frac{\partial^3\boldsymbol{U}^{\langle 0\rangle}}{\partial\xi_l\,\partial\xi_j\,\partial\xi_k},$$

(V.10)
$$\boldsymbol{\chi}^{\langle n\rangle}_{,\tau} = \tilde{\boldsymbol{U}}^{\langle n\rangle}(\boldsymbol{x},\tau) = \boldsymbol{U}^{\langle n\rangle} + (\boldsymbol{\chi}^{\langle n\rangle}\cdot\nabla_{\xi})\,\boldsymbol{U}^{\langle 0\rangle} + \text{other terms}$$

where

$$\boldsymbol{U}^{\langle n\rangle} = \boldsymbol{U}^{\langle n\rangle}(\boldsymbol{\xi}(\tau),\tau).$$

The functions $\boldsymbol{\chi}^{\langle n\rangle}(\tau)$ may be computed by integrating (V.10) subject to the conditions (V.6) and (V.7). These conditions also imply that

$$\tilde{\boldsymbol{U}}^{\langle n\rangle}(\boldsymbol{x},t) = \boldsymbol{U}^{\langle n\rangle}(\boldsymbol{x},t),$$

$$\boldsymbol{\xi}(\tau) - \boldsymbol{x} = \int_t^\tau \boldsymbol{U}^{\langle 0\rangle}(\boldsymbol{\xi}(\tau'),\tau')\,d\tau',$$

(V.11)
$$\boldsymbol{\chi}^{\langle n\rangle}(\tau) = \int_t^\tau \tilde{\boldsymbol{U}}^{\langle n\rangle}(x,\tau')\,d\tau', \qquad n \geqq 1.$$

Given $\boldsymbol{U}^{\langle l\rangle}(\boldsymbol{\xi}(\tau),\tau)$ for $l \leqq n$, we may compute $\boldsymbol{\chi}^{\langle n\rangle}(\tau)$ from (V.11).

We turn next to the computation of derivatives of the strain tensor. From the definition of the relative deformation gradient given in § II, we find, using (V.1) and (V.9), that

$$\boldsymbol{F}^{\langle n\rangle}(s) = \frac{1}{n!}\left.\frac{\partial^n \boldsymbol{F}(s,\varepsilon)}{\partial\varepsilon^n}\right|_{\varepsilon=0} = \nabla\boldsymbol{\chi}_t^{\langle n\rangle}(x,\tau)$$

where

$$\boldsymbol{F}^{\langle 0\rangle}(0) = 1 \quad \text{and} \quad \boldsymbol{F}^{\langle n\rangle}(0) = 0 \quad \text{for } n > 0.$$

Then

(V.12)
$$\boldsymbol{G}(s,\varepsilon) = \boldsymbol{G}^{\langle 0\rangle}(s) + \varepsilon\,\boldsymbol{G}^{\langle 1\rangle}(s) + \varepsilon^2\boldsymbol{G}^{\langle 2\rangle}(s) + \cdots$$

where

$$\boldsymbol{G}^{\langle 0\rangle}(s) = \boldsymbol{F}^{T\langle 0\rangle}(s)\cdot\boldsymbol{F}^{\langle 0\rangle}(s) - 1$$

and

(V.13)
$$\boldsymbol{G}^{\langle n\rangle}(s) = \sum_{l=0}^{n}\boldsymbol{F}^{T\langle l\rangle}(s)\cdot\boldsymbol{F}^{\langle n-l\rangle}(s).$$

## VI. Functional Derivatives of the Stress and the Equations Governing the Perturbation of Special Motions with Arbitrary Motions

Suppose that $G(s, \varepsilon)$ is the series given by (V.12). This series may be assumed to induce a functional expansion of the stress in powers of $\varepsilon$ (see JOSEPH, 1976; p. 197):

$$
\begin{aligned}
\mathscr{F}[G(s,\varepsilon)] = {} & \mathscr{F}[G^{\langle 0\rangle}] + \varepsilon \mathscr{F}_1[G^{\langle 0\rangle}|G^{\langle 1\rangle}] \\
& + \varepsilon^2\{\mathscr{F}_2[G^{\langle 0\rangle}|G^{\langle 1\rangle}, G^{\langle 1\rangle}] + \mathscr{F}_1[G^{\langle 0\rangle}|G^{\langle 2\rangle}]\} \\
& + \varepsilon^3\{\mathscr{F}_3[G^{\langle 0\rangle}|G^{\langle 1\rangle}, G^{\langle 1\rangle}, G^{\langle 1\rangle}] + 2\mathscr{F}_2[G^{\langle 0\rangle}|G^{\langle 1\rangle}, G^{\langle 2\rangle}] \\
& + \mathscr{F}_1[G^{\langle 0\rangle}|G^{\langle 3\rangle}]\} + O(\varepsilon^4).
\end{aligned} \tag{VI.1}
$$

Apart from a factorial, $\mathscr{F}_n$ is a functional derivative, typically a Fréchet derivative, evaluated on the history $G^{\langle 0\rangle}(s)$ of the special solution. The linear arguments of these derivatives, those following the vertical bar, are to be determined sequentially by solving the perturbation equations of motion which have yet to be specified. The functional derivatives given in (VI.1) are still too generally specified to be useful in the solution of problems. However, the first Fréchet derivative $\mathscr{F}_1[G^{\langle 0\rangle}|\cdot]$ may be assumed to be in integral form when $G(s, \varepsilon)$ lies in a $L_h^2(0, \infty)$ Hilbert space whose scalar product is defined by an integral with a weight $h(s)$, $h(s) \to 0$ as $s \to \infty$. Such a representation may be justified by appeal to the representation theorem of F. RIESZ.

Identifying independent powers of $\varepsilon$ in the expansion of (III.1)–(III.4), we may identify an ordered sequence of perturbation problems. The zero$^{\text{th}}$ order problem is defined by (IV.1), (IV.2) and (IV.3). At first order, we find that in $\mathscr{V}(t)$, $t>0$

$$
\begin{aligned}
\rho\left[\frac{\partial U^{\langle 1\rangle}}{\partial t} + (U^{\langle 0\rangle}\cdot\nabla)\,U^{\langle 1\rangle} + (U^{\langle 1\rangle}\cdot\nabla)\,U^{\langle 0\rangle}\right] \\
= -\nabla p^{\langle 1\rangle} + \mathscr{F}_1[G^{\langle 0\rangle}|G^{\langle 1\rangle}] + f^{\langle 1\rangle}(x, t),
\end{aligned} \tag{VI.2}
$$

$$\nabla\cdot U^{\langle 1\rangle}(x, t) = 0 \tag{VI.3}$$

and

$$\chi^{\langle 1\rangle}(\tau) = \int_t^\tau [U^{\langle 1\rangle} + (\chi^{\langle 1\rangle}\cdot\nabla)\,U^{\langle 0\rangle}]\,d\tau' \tag{VI.4}$$

where, under the integral in (VI.4), $U^{\langle n\rangle} = U^{\langle n\rangle}(\xi(\tau'), \tau')$ for $n=0$ and $n=1$ and $\chi^{\langle 1\rangle} = \chi_t^{\langle 1\rangle}(x, \tau')$. On the boundary $\partial\mathscr{V}(t)$, $t>0$,

$$U^{\langle 1\rangle}(x, t) = q^{\langle 1\rangle}(x, t). \tag{VI.5}$$

The history of the velocity

$$U^{\langle 1\rangle}(\xi(\tau), \tau) \quad \text{is prescribed in } \mathscr{V}(\tau),\ \tau \leqq 0. \tag{VI.6}$$

Since $F^{\langle 1\rangle}$ is the gradient of $\chi^{\langle 1\rangle}$, (VI.2), with

$$G^{\langle 1\rangle}(s) = F^{T\langle 0\rangle}(s)\cdot F^{\langle 1\rangle}(s) + F^{T\langle 1\rangle}(s)\cdot F^{\langle 0\rangle}(s), \tag{VI.7}$$

(VI.2), (VI.3) and (VI.4) may be viewed as seven linear equations in the seven unknown functions $\chi_t^{\langle 1\rangle}(x,\tau)$, $U^{\langle 1\rangle}(x,t)$ and $p^{\langle 1\rangle}(x,t)$.

A similar linear problem for $\chi^{\langle n\rangle}$, $U^{\langle n\rangle}$ and $p^{\langle n\rangle}$ ($n \geqq 2$) arises at higher orders. If these problems are solvable, they are sequentially solvable and the motion and strain history may be generated as power series. These linear perturbation problems, with $\mathscr{F}_1[G^{\langle 0\rangle}|\cdot]$ represented by an integral, are not too general for mathematical studies of existence and uniqueness.

## VII. Kinematics of Arbitrary Motions Perturbing Steady Rigid Rotations of a Simple Fluid

Now I am going to derive an algorithm for computing motions which perturb steady rigid rotations. I want solutions of (IV.1), (IV.2) and (IV.3) for which $G^{\langle 0\rangle}(s) \equiv 0$. Rigid body motions have $G^{\langle 0\rangle}(s) \equiv 0$ but, in general, such motions will not satisfy (IV.1) when $f^{\langle 0\rangle}(x,t)=0$ because $\mathring{\Omega} \wedge x$ is not conservative (see (IV.4)). I, therefore, set $\mathring{\Omega}=0$, $f^{\langle 0\rangle}(x,t)=0$ and put

$$U^{\langle 0\rangle}(\xi(\tau),\tau)=\Omega \wedge \xi(\tau) \tag{VII.1}$$

for $-\infty<\tau \leqq t$ at all points in $\mathscr{V}(\tau)$. Then $\mathscr{F}[G^{\langle 0\rangle}(s)] \equiv 0$ and

$$p^{\langle 0\rangle}(x,t)+\frac{\rho}{2}|\Omega \wedge x|^2=\text{const} \tag{VII.2}$$

at each $x \in \mathscr{V}(t)$ and at each and every instant $t>-\infty$. The path $\chi_t^{\langle 0\rangle}(x,\tau)=\xi(\tau)$ for $\tau \leqq t$ is obtained by integrating

$$\xi_{,\tau}=\Omega \wedge \xi(\tau), \qquad \xi(t)=x.$$

Without losing generality, we choose a fixed orthonormal basis $e_1$, $e_2$, $e_3$ such that

$$\Omega=e_3\,\Omega \tag{VII.3}$$

is a constant vector. Then the particle path is given by

$$\chi_t^{\langle 0\rangle}(x,\tau)=\xi(\tau)=Q(\Omega s)\cdot x \tag{VII.4}$$

where $Q(\Omega s)$ is the unique orthogonal tensor rotating the orthonormal basis $\hat{e}_1(\Omega s)$, $\hat{e}_2(\Omega s)$, $e_3$ into $e_1$, $e_2$, $e_3$; that is, $e_i=Q(\Omega s)\cdot \hat{e}_i(\Omega s)$. Relative to the fixed basis,

$$Q(\Omega s)=e_i\,Q_{ij}(\Omega s)\,e_j$$

where $Q_{ij}(\Omega s)=\hat{e}_1(\Omega s)\cdot e_j$; that is,

$$[Q_{ij}(\Omega s)]=\begin{pmatrix}\cos\Omega s & \sin\Omega s & 0\\ -\sin\Omega s & \cos\Omega s & 0\\ 0 & 0 & 1\end{pmatrix}. \tag{VII.5}$$

It follows from (VII.4) that

(VII.6) $$F^{\langle 0\rangle}=\nabla\chi_t^{\langle 0\rangle}(x,\tau)=Q(\Omega s).$$

Hence,

(VII.7) $$G^{\langle 0\rangle}(s)=Q^T(\Omega s)\cdot Q(\Omega s)-1=0$$

and $\mathscr{F}^{\langle 0\rangle}\equiv 0$.

The differential equations (V.10) for the perturbation coefficients $\chi_{,\tau}^{\langle n\rangle}=\tilde{U}^{\langle n\rangle}(x,\tau)$ giving the particle paths may now be expressed in terms of $\chi^{\langle l\rangle}=\chi_t^{\langle l\rangle}(x,\tau)$ and $U^{\langle l\rangle}=U^{\langle l\rangle}(\xi(\tau),\tau)$. Using the identity $(\chi^{\langle n\rangle}\cdot\nabla_\xi)U^{\langle 0\rangle}=(\chi^{\langle n\rangle}\cdot\nabla_\xi)(\Omega\wedge\xi)=\Omega\wedge\chi^{\langle n\rangle}$, we find that

(VII.8)
$$\begin{aligned}
&\tilde{U}^{\langle 0\rangle}-\Omega\wedge\xi=0,\\
&\tilde{U}^{\langle 1\rangle}-\Omega\wedge\chi^{\langle 1\rangle}=U^{\langle 1\rangle},\\
&\tilde{U}^{\langle 2\rangle}-\Omega\wedge\chi^{\langle 2\rangle}=U^{\langle 2\rangle}+(\chi^{\langle 1\rangle}\cdot\nabla_\xi)\,U^{\langle 1\rangle},\\
&\tilde{U}^{\langle 3\rangle}-\Omega\wedge\chi^{\langle 3\rangle}=U^{\langle 3\rangle}+(\chi^{\langle 2\rangle}\cdot\nabla_\xi)\,U^{\langle 1\rangle}+(\chi^{\langle 1\rangle}\cdot\nabla_\xi)\,U^{\langle 2\rangle}\\
&\qquad\qquad+\frac{1}{2}\chi_l^{\langle 1\rangle}\chi_j^{\langle 1\rangle}\frac{\partial^2 U^{\langle 1\rangle}}{\partial\xi_l\,\partial\xi_j}
\end{aligned}$$

and

$$\tilde{U}^{\langle n\rangle}-\Omega\wedge\chi^{\langle n\rangle}=U^{\langle n\rangle}+\text{other terms}.$$

It follows that

(VII.9)
$$\begin{aligned}
\tilde{U}(x,\tau,\varepsilon)-\Omega\wedge\chi_t(x,\tau,\varepsilon)&=V(\chi_t(x,\tau,\varepsilon);\tau,\varepsilon)\\
&=\varepsilon U^{\langle 1\rangle}(\xi(\tau),\tau)+\varepsilon^2[U^{\langle 2\rangle}+(\chi^{\langle 1\rangle}\cdot\nabla_\xi)\,U^{\langle 1\rangle}]+\cdots\\
&\quad+\varepsilon^n[U^{\langle n\rangle}+\text{other terms}]+\cdots.
\end{aligned}$$

I am going to argue, in § VIII, that the most natural measure of deformation for rotating simple fluids is the time derivative of the Cauchy strain

(VII.10) $$\mathbb{J}(s,\varepsilon)=-\frac{d}{ds}G(s,\varepsilon)=F_{,\tau}^T\cdot F+F^T\cdot F_{,\tau}=\nabla\tilde{U}^T\cdot F+F^T\cdot\nabla\tilde{U}.$$

We shall need the following expansion formula for

(VII.11) $$\mathbb{J}(s,\varepsilon)=\nabla V^T\cdot F+F^T\cdot\nabla V=\varepsilon\,\mathbb{J}^{\langle 1\rangle}+\varepsilon^2\mathbb{J}^{\langle 2\rangle}+\cdots$$

where the $\mathbb{J}^{\langle n\rangle}$ are defined in terms $Q(\Omega s)$, $F^{\langle l\rangle}$ and

(VII.12) $$L^{\langle n\rangle}=\nabla[\tilde{U}^{\langle n\rangle}(x,\tau)-\Omega\wedge\chi^{\langle n\rangle}],\qquad n\geqq 1.$$

Equations (VII.8) show that $L^{\langle n\rangle}$ depends on $U^{\langle l\rangle}(\xi(\tau),\tau)$ for $l\leqq n$ and on $\chi_t^l(x,\tau)$ for $l<n$. In fact,

(VII.13) $$\mathbb{J}^{\langle 1\rangle}=L^{T\langle 1\rangle}\cdot Q(\Omega s)+Q^T(\Omega s)\cdot L^{\langle 1\rangle}$$

and for $n \geqq 2$

(VII.14) $$\mathbb{J}^{\langle n\rangle} = \boldsymbol{L}^{T\langle n\rangle} \cdot \boldsymbol{Q}(\Omega s) + \boldsymbol{Q}^T(\Omega s) \cdot \boldsymbol{L}^{\langle n\rangle} + \sum_{l=1}^{n-1} (\boldsymbol{L}^{T\langle n-l\rangle} \cdot \boldsymbol{F}^{\langle l\rangle} + \boldsymbol{F}^{T\langle l\rangle} \cdot \boldsymbol{L}^{\langle n-l\rangle}).$$

To prove the expansion formula $(\mathrm{VII.11})_2$, we must first show that (VII.10) reduces to $(\mathrm{VII.11})_1$. This reduction is a direct consequence of the identity

$$[\nabla(\boldsymbol{\Omega} \wedge \boldsymbol{\chi}_t)^T \cdot \boldsymbol{F} + \boldsymbol{F}^T \cdot \nabla(\boldsymbol{\Omega} \wedge \boldsymbol{\chi}_t)]_{ij} = \frac{\partial(\boldsymbol{\Omega} \wedge \boldsymbol{\chi}_t)_l}{\partial x_i} \frac{\partial(\boldsymbol{\chi}_t)_l}{\partial x_j} + \frac{\partial(\boldsymbol{\Omega} \wedge \boldsymbol{\chi}_t)_l}{\partial x_j} \frac{\partial(\boldsymbol{\chi}_t)_l}{\partial x_i}$$
$$= \Omega_m \varepsilon_{lmn} \left[ \frac{\partial(\boldsymbol{\chi}_t)_n}{\partial x_i} \frac{\partial(\boldsymbol{\chi}_t)_l}{\partial x_j} + \frac{\partial(\boldsymbol{\chi}_t)_n}{\partial x_j} \frac{\partial(\boldsymbol{\chi}_t)_l}{\partial x_i} \right] = 0.$$

To obtain (VII.13) and (VII.14), we expand $\boldsymbol{F}(s, \varepsilon) = \boldsymbol{Q}(\Omega s) + \sum_1 \varepsilon^l \boldsymbol{F}^{\langle l\rangle}$ and $\varepsilon \boldsymbol{V}$, using (VII.9) and (VII.8), and collect the coefficients of independent powers of $\varepsilon$ in the induced expansion of (VII.11).

Some further transformations of the tensors $\mathbb{J}^{\langle n\rangle}(s)$ are used in the analysis. These transformations are motivated by the fact that the perturbation problems to be derived lead to the sequential determination of the velocity coefficients $\boldsymbol{U}^{\langle n\rangle}(\boldsymbol{\xi}(\tau), \tau)$ whose natural arguments are the components of the rotating vector $\boldsymbol{\chi}_t^{\langle 0\rangle}(\boldsymbol{x}, \tau) = \boldsymbol{\xi}(\tau)$. Noting now that when $\tau = t$, $\boldsymbol{\xi} = \boldsymbol{x}$ and div $\boldsymbol{U}^{\langle n\rangle}(\boldsymbol{x}, t) = 0$ in $\mathscr{V}(t)$, we find easily that

(VII.15) $$\operatorname{div}_\xi \boldsymbol{U}^{\langle n\rangle}(\boldsymbol{\xi}, \tau) = 0 \quad \text{in } \mathscr{V}(\tau), \ \tau \leqq t.$$

Noting next that

$$\nabla(\cdot) = \nabla_\xi(\cdot) \cdot \boldsymbol{Q}(\Omega s), \qquad \frac{\partial}{\partial x_i} = \frac{\partial}{\partial \xi_m} \frac{\partial \xi_m}{\partial x_i},$$

we find that

(VII.16) $$\boldsymbol{L}^{\langle l\rangle} = \nabla(\tilde{\boldsymbol{U}}^{\langle l\rangle} - \boldsymbol{\Omega} \wedge \boldsymbol{\chi}_t^{\langle l\rangle}) = \mathscr{L}^{\langle l\rangle} \cdot \boldsymbol{Q}(\Omega s)$$

and

(VII.17) $$\boldsymbol{F}^{\langle l\rangle} = \nabla \boldsymbol{\chi}^{\langle l\rangle} = \mathscr{f}^{\langle l\rangle} \cdot \boldsymbol{Q}(\Omega s)$$

where

$$\mathscr{L}^{\langle l\rangle} = \nabla_\xi[\tilde{\boldsymbol{U}} - \boldsymbol{\Omega} \wedge \boldsymbol{\chi}_t^{\langle l\rangle}] = \nabla_\xi[\boldsymbol{U}^{\langle l\rangle}(\boldsymbol{\xi}, \tau) + \cdots]$$

and

$$\mathscr{f}^{\langle l\rangle} = \nabla_\xi \boldsymbol{\chi}^{\langle l\rangle}.$$

Using (VII.8), we find that

$$\mathscr{L}^{\langle 1\rangle} = \nabla_\xi \boldsymbol{U}^{\langle 1\rangle}(\boldsymbol{\xi}, \tau)$$

and

$$\mathscr{L}^{\langle 2\rangle} = \nabla_\xi[\boldsymbol{U}^{\langle 2\rangle}(\boldsymbol{\xi}, \tau) + (\boldsymbol{\chi}^{\langle 1\rangle} \cdot \nabla_\xi) \boldsymbol{U}^{\langle 1\rangle}(\boldsymbol{\xi}, \tau)]$$
$$= \nabla_\xi \boldsymbol{U}^{\langle 2\rangle} + (\boldsymbol{\chi}^{\langle 1\rangle} \cdot \nabla_\xi) \mathscr{L}^{\langle 1\rangle} + \mathscr{L}^{\langle 1\rangle} \cdot \mathscr{f}^{\langle 1\rangle}.$$

Finally, setting

(VII.18) $$\boldsymbol{A}^{\langle l\rangle} = \nabla_\xi \boldsymbol{U}^{\langle l\rangle} + \nabla_\xi \dot{\boldsymbol{U}}^{\langle l\rangle T}, \qquad l \geqq 1,$$

we find that

(VII.19) $$\mathbb{J}^{\langle 1\rangle}=\boldsymbol{Q}^T(\Omega s)\cdot \boldsymbol{A}^{\langle 1\rangle}(s)\cdot \boldsymbol{Q}(\Omega s),$$

(VII.20) $$\mathbb{J}^{\langle 2\rangle}=\boldsymbol{Q}^T(\Omega s)\cdot (\boldsymbol{A}^{\langle 2\rangle}(s)+\boldsymbol{B}(s))\cdot \boldsymbol{Q}(\Omega s),$$

(VII.21) $$\mathbb{J}^{\langle l\rangle}=\boldsymbol{Q}^T(\Omega s)\cdot (\boldsymbol{A}^{\langle l\rangle}(s)+\text{l.o.t.})\cdot \boldsymbol{Q}(\Omega s),$$

where

(VII.22) $$\boldsymbol{B}(s)=(\boldsymbol{\chi}^{\langle 1\rangle}\cdot \nabla_\xi)\boldsymbol{A}^{\langle 1\rangle}+\boldsymbol{A}^{\langle 1\rangle}\cdot \not{f}^{\langle 1\rangle}+\not{f}^{T\langle 1\rangle}\cdot \boldsymbol{A}^{\langle 1\rangle}$$

and

l.o.t. = lower order terms.

## VIII. Canonical Forms for the Stress

My constitutive hypothesis is that the Fréchet derivatives of $\mathscr{F}[\boldsymbol{G}(s)]$ on the zero history can be represented by integrals.* I also assume that kernels in these integrals vanish at a rate sufficient to justify integrating by parts; for example,

(VIII.1) $$\mathscr{F}_1[0|\boldsymbol{G}(s)]=\int_0^\infty \frac{dG}{ds}\,\boldsymbol{G}(s)\,ds=\int_0^\infty G(s)\,\mathbb{J}(s)\,ds\equiv\tilde{\mathscr{F}}_1[0|\mathbb{J}(s)]$$

and

(VIII.2)
$$\begin{aligned}&\mathscr{F}_2[0|\boldsymbol{G}(s_1),\boldsymbol{G}(s_2)]\\&=\int_0^\infty\int_0^\infty\left[\frac{\partial^2\gamma(s_1,s_2)}{\partial s_1\partial s_2}\boldsymbol{G}(s_1)\cdot\boldsymbol{G}(s_2)+\frac{\partial^2\hat{\alpha}(s_1,s_2)}{\partial s_1\partial s_2}[\operatorname{tr}\boldsymbol{G}(s_1)]\,\boldsymbol{G}(s_2)\right]ds_1\,ds_2\\&=\int_0^\infty\int_0^\infty[\gamma(s_1,s_2)\mathbb{J}(s_1)\cdot\mathbb{J}(s_2)+\hat{\alpha}(s_1,s_2)[\operatorname{tr}\mathbb{J}(s_1)]\,\mathbb{J}(s_2)]\,ds_1\,ds_2\\&\equiv\tilde{\mathscr{F}}_2[0|\mathbb{J}(s_1),\mathbb{J}(s_2)].\end{aligned}$$

Explicit expressions for $\tilde{\mathscr{F}}_3$ and $\tilde{\mathscr{F}}_4$ are given by JOSEPH & BEAVERS (1977). This integration by parts allows us to introduce $\mathbb{J}(s)=-d\boldsymbol{G}(s)/ds$ as the fundamental measure of deformation and leads ultimately to a theory in which perturbation

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* GREEN & RIVLIN (1957) were the first to consider multiple integral representations of the stress. COLEMAN & NOLL (1961) studied the problem of integral representations of Noll's stress in a Hilbert space setting and introduced the notion of Fréchet expansions of the stress. They noted that Riesz's theorem justifies an integral representation for $\mathscr{F}_1[0|\boldsymbol{G}(s)]$ when $\boldsymbol{G}(s)$ is in a Hilbert space whose scalar product is defined in terms of integrals. They justified their Fréchet expansion through $\mathscr{F}_1$ and gave a formal argument for the second-order theory when the strains are small relative to some fixed configuration of the body. PIPKIN (1964) showed that COLEMAN & NOLL's second-order theory could be simplified when the material is incompressible. PIPKIN gave the isotropic forms of the stress through terms of order three. Isotropic forms through order four are given by JOSEPH & BEAVERS (1977). The isotropic forms for $\mathscr{F}_n$, $n>4$ may be written down by inspection (see Exercise 94.7 in JOSEPH's book (1976) on stability). The canonical forms for the stress and the perturbation equations of motion for perturbations of the rest state were given by JOSEPH (1976) and used to predict previously unknown Weissenberg effects (JOSEPH I & BEAVERS II, 1976) associated with torsional oscillations of a stirring rod in a simple fluid.

velocities are sequentially computed from four equations governing three components of velocity and the pressure as in an incompressible, Navier-Stokes fluid. Assuming that the stress $\mathscr{F}[\boldsymbol{G}(s)]$ admits a Fréchet expansion in integrals with good kernels, we get

$$\begin{aligned}\mathscr{F}[\boldsymbol{G}(s)] &= \mathscr{F}_1[0|\boldsymbol{G}(s)] + \mathscr{F}_2[0|\boldsymbol{G}(s_1), \boldsymbol{G}(s_2)] + \mathscr{F}_3[0|\boldsymbol{G}(s_1), \boldsymbol{G}(s_2), \boldsymbol{G}(s_3)] + \cdots \\ &= \tilde{\mathscr{F}}_1[0|\mathbb{J}(s)] + \tilde{\mathscr{F}}_2[0|\mathbb{J}(s_1), \mathbb{J}(s_2)] + \tilde{\mathscr{F}}_3[0|\mathbb{J}(s_1), \mathbb{J}(s_2), \mathbb{J}(s_3)] + \cdots \\ &= \tilde{\mathscr{F}}[\mathbb{J}(s)]. \end{aligned} \tag{VIII.3}$$

To obtain the canonical forms of the stress for the theory of rotating fluids, we identify independent coefficients in the series expansion of $\tilde{\mathscr{F}}[\mathbb{J}(s,\varepsilon)]$ induced by the expansion (VII.12) of $\mathbb{J}(s,\varepsilon)$. This leads us to

$$\begin{aligned}\tilde{\mathscr{F}}[\mathbb{J}(s,\varepsilon)] &= \varepsilon \tilde{\mathscr{F}}^{\langle 1\rangle} + \varepsilon^2 \tilde{\mathscr{F}}^{\langle 2\rangle} + \varepsilon^3 \tilde{\mathscr{F}}^{\langle 3\rangle} + \cdots \\ &= \varepsilon \tilde{\mathscr{F}}_1[0|\mathbb{J}^{\langle 1\rangle}(s)] + \varepsilon^2\{\tilde{\mathscr{F}}_1[0|\mathbb{J}^{\langle 2\rangle}(s)] + \tilde{\mathscr{F}}_2[0|\mathbb{J}^{\langle 1\rangle}(S_1), \mathbb{J}^{\langle 1\rangle}(s_2)]\} \\ &\quad + \varepsilon^3\{\tilde{\mathscr{F}}_1[0|\mathbb{J}^{\langle 3\rangle}(s)] + 2\tilde{\mathscr{F}}_2[0|\mathbb{J}^{\langle 1\rangle}(S_1), \mathbb{J}^{\langle 2\rangle}(s_2)] \\ &\quad + \tilde{\mathscr{F}}_3[0|\mathbb{J}^{\langle 1\rangle}(s_1), \mathbb{J}^{\langle 1\rangle}(s_2), \mathbb{J}^{\langle 1\rangle}(s_3)]\} \\ &\quad + O(\varepsilon^4)\end{aligned} \tag{VIII.4}$$

where $\mathbb{J}^{\langle 1\rangle}$, $\mathbb{J}^{\langle 2\rangle}$ and $\mathbb{J}^{\langle l\rangle}$ are given by (VII.19–21), $\tilde{\mathscr{F}}_1$ is given by (VIII.1) and

$$(\tilde{\mathscr{F}}_2[0|\mathbb{J}^{\langle 1\rangle}(s_1), \mathbb{J}^{\langle 1\rangle}(s_2)] = \int_0^\infty \int_0^\infty \gamma(s_1, s_2)\mathbb{J}^{\langle 1\rangle}(s_1)\cdot \mathbb{J}^{\langle 1\rangle}(s_2)\, ds_1\, ds_2. \tag{VIII.5}$$

The term proportional to $\hat{\alpha}(s_1, s_2)$ vanishes in the second-order approximation because $\operatorname{tr}[\mathbb{J}^{\langle 1\rangle}] = \operatorname{tr}[\boldsymbol{Q}^T(\Omega s)\cdot \boldsymbol{A}^{\langle 1\rangle}\cdot \boldsymbol{Q}(\Omega s)] = \operatorname{tr} \boldsymbol{A}^{\langle 1\rangle} = \operatorname{div}_\xi \boldsymbol{U}^{\langle 1\rangle}(\xi, \tau) = 0$.

The canonical forms of the stress for perturbations of steady rigid rotation are given through order two by

$$\begin{aligned}\tilde{\mathscr{F}}[\mathbb{J}(s,\varepsilon)] &= \varepsilon \int_0^\infty G(s)\, \boldsymbol{Q}^T(\Omega s)\cdot \boldsymbol{A}^{\langle 1\rangle}(s)\cdot \boldsymbol{Q}^T(\Omega s)\, ds \\ &\quad + \varepsilon^2 \left\{ \int_0^\infty G(s)\, \boldsymbol{Q}^T(\Omega s)\cdot [\boldsymbol{A}^{\langle 2\rangle}(s) + \boldsymbol{B}(s)]\cdot \boldsymbol{Q}(\Omega s)\, ds \right. \\ &\quad + \int_0^\infty \int_0^\infty \gamma(s_1, s_2)\, \boldsymbol{Q}^T(\Omega s_1)\cdot \boldsymbol{A}^{\langle 1\rangle}(s_1)\cdot \boldsymbol{Q}(\Omega s_1)\cdot \boldsymbol{Q}^T(\Omega s_2) \\ &\quad \cdot \boldsymbol{A}^{\langle 1\rangle}(s_2)\cdot \boldsymbol{Q}(\Omega s_2)\, ds_1\, ds_2 + O(\varepsilon^3)\end{aligned} \tag{VIII.6}$$

where $\boldsymbol{A}^{\langle l\rangle}(s)$ is defined by (VII.18) and $\boldsymbol{B}(s)$ by (VII.22). The higher-order stresses are not hard to derive. They are in the form

$$\tilde{\mathscr{F}}^{\langle n\rangle} = \int_0^\infty G(s)\, \boldsymbol{Q}^T(\Omega s)\cdot \boldsymbol{A}^{\langle n\rangle}\cdot \boldsymbol{Q}(\Omega s)\, ds + \text{lower-order terms}. \tag{VIII.7}$$

## IX. Canonical Forms of the Equations of Motion

After expanding in powers of $\varepsilon$, we find that

$$\rho\left[\frac{\partial \boldsymbol{U}^{\langle n\rangle}(\boldsymbol{x},t)}{\partial t}+\sum_{l=0}^{n}\left(\boldsymbol{U}^{\langle n-l\rangle}(\boldsymbol{x},t)\cdot\nabla\right)\boldsymbol{U}^{l}(\boldsymbol{x},t)\right] = -\nabla p^{\langle n\rangle}(\boldsymbol{x},t)+(\operatorname{div}\tilde{\mathscr{F}}^{\langle n\rangle})(\boldsymbol{x},t). \tag{IX.1}$$

When $n=0$, $\boldsymbol{U}^{\langle 0\rangle}(x;t)=\boldsymbol{\Omega}\wedge\boldsymbol{x}$, $\mathscr{F}^{\langle 0\rangle}=\partial\boldsymbol{U}^{\langle 0\rangle}/\partial t=0$, and $p^{\langle 0\rangle}+\frac{1}{2}\rho|\boldsymbol{\Omega}\wedge\boldsymbol{x}|^2$ is constant. *Now we shall demonstrate that*

$$\operatorname{div}\tilde{\mathscr{F}}^{\langle n\rangle}=\int_0^\infty G(s)\,\boldsymbol{Q}^T(\Omega s)\cdot\nabla_\xi^2\,\boldsymbol{U}^{\langle n\rangle}(\boldsymbol{\xi}(t-s),t-s)\,ds+\text{lower-order terms.} \tag{IX.2}$$

To establish (IX.2), we first show that for any tensor $\boldsymbol{M}(\boldsymbol{x},\tau)=\mathscr{M}(\boldsymbol{\xi},\tau)$, we have

$$\operatorname{div}[\boldsymbol{Q}^T(\Omega s)\cdot\boldsymbol{M}\cdot\boldsymbol{Q}(\Omega s)]=\boldsymbol{Q}^T(\Omega s)\cdot\operatorname{div}_\xi\mathscr{M}. \tag{IX.3}$$

Taking components in the fixed Cartesian basis, we find that

$$\begin{aligned}(\operatorname{div}\boldsymbol{Q}^T\cdot\boldsymbol{M}\cdot\boldsymbol{Q})_i&=\frac{\partial}{\partial x_j}(\boldsymbol{Q}^T\cdot\boldsymbol{M}\cdot\boldsymbol{Q})_{ij}=Q^T_{il}\left(\frac{\partial M_{ln}}{\partial x_j}\right)Q_{nj}\\&=Q^T_{il}\frac{\partial\mathscr{M}_{ln}}{\partial\xi_k}\frac{\partial\xi_k}{\partial x_j}Q_{nj}=Q^T_{il}\frac{\partial\mathscr{M}_{ln}}{\partial\xi_n}=(\boldsymbol{Q}^T\cdot\operatorname{div}_\xi\mathscr{M})_i\end{aligned}$$

where we have used $Q_{kj}=\partial\xi_k/\partial x_j$. Since $\partial U_i^{\langle n\rangle}(\xi,\tau)/\partial\xi_i=0$, we may also verify that

$$(\operatorname{div}_\xi\boldsymbol{A}^{\langle n\rangle})_i=\frac{\partial A_{ij}^{\langle n\rangle}}{\partial\xi_j}=\nabla_\xi^2\,U_i^{\langle n\rangle}(\boldsymbol{\xi},\tau). \tag{IX.4}$$

Combining (IX.3) and (IX.4) with (VIII.7), we prove (IX.2).

Equations (IX.1) may be expressed in terms of the components $U_i^{\langle n\rangle}$ of $\boldsymbol{U}^{\langle n\rangle}=\boldsymbol{e}_i U_i^{\langle n\rangle}$ relative to the fixed Cartesian basis $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$. To express (IX.2) in terms of $U_i^{\langle n\rangle}$, we note that

$$\boldsymbol{Q}^T(\Omega s)\cdot\boldsymbol{U}^{\langle n\rangle}=\boldsymbol{Q}^T(\Omega s)\cdot\boldsymbol{e}_i U_i^{\langle n\rangle}=\hat{\boldsymbol{e}}_i(s)\,U_i^{\langle n\rangle}. \tag{IX.5}$$

Equations (IV.1) and $\operatorname{div}\boldsymbol{U}^{\langle n\rangle}(\boldsymbol{x},t)=0$ hold identically in $\mathscr{V}(t)$ for $t>0$ and $\boldsymbol{U}^{\langle n\rangle}(\boldsymbol{x},t)$ is prescribed in $\mathscr{V}(t)$ when $t\leqq 0$. Boundary conditions, say $\boldsymbol{U}^{\langle n\rangle}(\boldsymbol{x},t)=\boldsymbol{q}^{\langle n\rangle}(\boldsymbol{x},t)$ for $\boldsymbol{x}\in\partial\mathscr{V}(t)$ and $t\geqq 0$, are also prescribed. Noting that $(\boldsymbol{U}^{\langle n\rangle}\cdot\nabla)\boldsymbol{U}^{\langle 0\rangle}=\boldsymbol{\Omega}\wedge\boldsymbol{U}^{\langle n\rangle}$, we may write the perturbation problems in sequence. When $n=1$,

$$\rho\left[\frac{\partial\boldsymbol{U}^{\langle 1\rangle}}{\partial t}+(\boldsymbol{\Omega}\wedge\boldsymbol{x})\cdot\nabla\boldsymbol{U}^{\langle 1\rangle}+\boldsymbol{\Omega}\wedge\boldsymbol{U}^{\langle 1\rangle}\right]+\nabla p^{\langle 1\rangle} -\int_0^\infty G(s)\,\boldsymbol{Q}^T(\Omega s)\cdot\nabla_\xi^2\,\boldsymbol{U}^{\langle 1\rangle}(\boldsymbol{\xi}(t-s),t-s)\,ds=0. \tag{IX.6}$$

The initial-history problem associated with (IX.6) determine $U^{\langle 1\rangle}(x,t)$ and $p^{\langle 1\rangle}(x,t)$. The solution is independent of the particle path $\chi^{\langle 1\rangle}(x,\tau)$ at first order. The path of the particle may be computed as an afterthought once $U^{\langle 1\rangle}(x,t)$ is known:

$$\chi^{\langle 1\rangle}_{,\tau}=\Omega\wedge\chi^{\langle 1\rangle}+U^{\langle 1\rangle}(\xi(\tau),\tau),\chi^{\langle 1\rangle}(x,t)=0. \tag{IX.7}$$

When $n=2$,

$$\begin{aligned}&\left[\frac{\partial U^{\langle 2\rangle}}{\partial t}+(\Omega\wedge x)\cdot\nabla U^{\langle 2\rangle}+\Omega\wedge U^{\langle 2\rangle}\right]+\nabla p^{\langle 2\rangle}\\ &\quad-\int_0^\infty G(s)\,Q^T(\Omega s)\cdot\nabla_\xi^2 U^{\langle 2\rangle}(\xi(t-s),t-s)\,ds=-\rho U^{\langle 1\rangle}\cdot\nabla U^{\langle 1\rangle}\\ &\quad+\int_0^\infty G(s)\,Q^T(\Omega s)\cdot\operatorname{div}_\xi B(s)\,ds\\ &\quad+\operatorname{div}\int_0^\infty\int_0^\infty \gamma(s_1,s_2)\,\mathbb{J}^{\langle 1\rangle}(s_1)\cdot\mathbb{J}^{\langle 1\rangle}(s_2)\,ds_1\,ds_2\end{aligned} \tag{IX.8}$$

where $\mathbb{J}^{\langle 1\rangle}$ is given by (VII.19) and $B(s)$ by (VII.22). The terms on the right of (IX.8) are known when $U^{\langle 1\rangle}$ and $\chi^{\langle 1\rangle}$ are known. Hence we may solve the initial-history problem associated with (IX.8) for $U^{\langle 2\rangle}(x,t)$ and $p^{\langle 2\rangle}(x,t)$. Then we can compute the path at second order:

$$\chi^{\langle 2\rangle}_{,\tau}=\Omega\wedge\chi^{\langle 2\rangle}+U^{\langle 2\rangle}(\xi(\tau),\tau)\;+(\chi^{\langle 1\rangle}\cdot\nabla_\xi)U^{\langle 1\rangle}(\xi(\tau),\tau),\chi^{\langle 2\rangle}(x,t)=0. \tag{IX.9}$$

It follows that at each order we may compute three velocity components $U^{\langle n\rangle}(x,t)$ and a pressure $p^{\langle n\rangle}(x,t)$ from an inhomogeneous, linear, initial-history problem associated with (IX.1) and $\operatorname{div} U^{\langle n\rangle}=0$. The particle path $\chi^{\langle n\rangle}$ appears as an auxiliary quantity which may be computed when $U^{\langle l\rangle}$, $l\leqq n$ is known. In other words, at each stage of the sequence, we solve four equations in four unknowns.

The mathematical problem defined by this perturbation sequence may be stated as follows: Given $f(x,t)$ in $\mathscr{V}(t)$ for $t>0$, $h(x,t)$ in $\mathscr{V}(t)$ for $t\leqq 0$ and $q(x,t)$ on $\partial\mathscr{V}(t)$ for $t>0$, find $a(x,t)$ and $\phi(x,t)$ such that $\operatorname{div} a(x,t)=0$ and $a(x,t)=h(x,t)$ in $\mathscr{V}(t)$ for $t\leqq 0$, $a(x,t)=q(x,t)$ for $x\in\partial\mathscr{V}(t)$, $t>0$ and

$$\rho\left\{\frac{\partial a}{\partial t}+(\Omega\wedge x)\cdot\nabla a+\Omega\wedge a\right\}+\nabla\phi-\int_0^\infty G(s)\,Q^T(\Omega s)\cdot\nabla_\xi^2\, a(\xi(t-s),t-s)\,ds=f(x,t).$$

It is known (SLEMROD, 1976) that under very mild conditions on the shear-relaxation modulus $G(s)$ the problem with $\Omega=0$ has a unique solution. The stability result proved in § XII suggests that if a solution of this problem exists when $\Omega\neq 0$, then it is unique.

Finally, I note that in limit $\Omega\to 0$ the theory of rotating simple fluids collapses into my previous theory of perturbations of the state of rest (JOSEPH, 1976).

## X. Equations of Motion in Cylindrical Coordinates

The canonical forms of stress and the equations of motion take a particularly simple form in the cylindrical coordinates suggested by the circular paths of particles at zero$^{\text{th}}$ order. Let $(x_1, x_2, x_3) = (r\cos\theta,\ r\sin\theta, z)$ denote the components of the particle at $\boldsymbol{x}$ at time $t$. Let $\boldsymbol{e}_r(\theta)$, $\boldsymbol{e}_\theta(\theta)$, $\boldsymbol{e}_3$ be an orthonormal basis in the cylindrical coordinate system $(r, \theta, z)$. The position of a rotating particle at time $\tau \leqq t$ is given by

$$\text{(X.1)}\qquad \begin{aligned} \boldsymbol{\chi}_t^{\langle 0\rangle}(\boldsymbol{x},\tau) &= \boldsymbol{\xi}(\tau) = \boldsymbol{Q}(\Omega s)\cdot\boldsymbol{x} = \boldsymbol{Q}^T(-\Omega s)\cdot\boldsymbol{x} = \hat{\boldsymbol{e}}_i(-\Omega s)\,x_i \\ &= r[\hat{\boldsymbol{e}}_1(-\Omega s)\cos\theta + \hat{\boldsymbol{e}}_2(-\Omega s)\sin\theta] + \boldsymbol{e}_3 z \\ &= r\,\boldsymbol{e}_r(\theta - \Omega s) + \boldsymbol{e}_3 z. \end{aligned}$$

The Cartesian components of $\boldsymbol{\xi}$ are $(\xi_1, \xi_2, \xi_3) = (r\cos(\theta - \Omega s),\ r\sin(\theta - \Omega s), z)$.

Suppose that the position of the particle $\boldsymbol{x}$ is in a general motion at the point $[\hat{r}, \hat{\theta}, \hat{z}] = [\hat{r}_t(r, \theta, z, \tau, \varepsilon),\ \hat{\theta}_t(r, \theta, z, \tau, \varepsilon),\ \hat{z}_t(r, \theta, z, \tau, \varepsilon)]$ when $\tau < t$. When $\tau = t$, $[\hat{r}, \hat{\theta}, \hat{z}] = [r, \theta, z]$. Introducing the natural orthonormal basis $(\boldsymbol{e}_r(\hat{\theta}), \boldsymbol{e}_\theta(\hat{\theta}), \boldsymbol{e}_3)$ at time $\tau$, we have

$$\boldsymbol{\chi}_t(\boldsymbol{x}, \tau, \varepsilon) = \boldsymbol{e}_r(\hat{\theta})\hat{r} + \boldsymbol{e}_3\hat{z}$$

and

$$\text{(X.2)}\qquad \frac{d\boldsymbol{\chi}_t}{d\tau} = \boldsymbol{e}_r(\hat{\theta})\frac{d\hat{r}}{d\tau} + \boldsymbol{e}_\theta(\theta)\hat{r}\frac{d\hat{\theta}}{dt} + \boldsymbol{e}_3\frac{d\hat{z}}{d\tau}.$$

Since

$$\boldsymbol{\Omega} \wedge \boldsymbol{\chi}_t(\boldsymbol{x}, \tau, \varepsilon) = \boldsymbol{e}_\theta(\hat{\theta})\hat{r}\Omega,$$

we find, using (VII.9), that

$$\text{(X.3)}\qquad \frac{d\boldsymbol{\chi}_t}{d\tau} = \boldsymbol{e}_\theta(\hat{\theta})\hat{r}\Omega + \boldsymbol{V}(\boldsymbol{\chi}_t(\boldsymbol{x}, \tau, \varepsilon), \tau, \varepsilon)$$

where $\boldsymbol{V} = 0$ when $\varepsilon = 0$. Setting

$$\boldsymbol{V} = \boldsymbol{e}_r(\hat{\theta})\, U_r(\hat{r}, \hat{\theta}, \hat{z}, \tau, \varepsilon) + \boldsymbol{e}_\theta(\hat{\theta})\, U_\theta(\hat{r}, \hat{\theta}, \hat{z}, \tau, \varepsilon) + \boldsymbol{e}_3\, U_z(\hat{r}, \hat{\theta}, \hat{z}, \tau, \varepsilon),$$

we find, by comparing (X.2) and (X.3), that

$$\text{(X.3)}\qquad \frac{d\hat{r}}{d\tau} = U_r(\hat{r}, \hat{\theta}, \hat{z}, \tau, \varepsilon),$$

$$\text{(X.4)}\qquad \hat{r}\frac{d\hat{\theta}}{d\tau} = \hat{r}\Omega + U_\theta(\hat{r}, \hat{\theta}, \hat{z}, \tau, \varepsilon),$$

and

$$\text{(X.5)}\qquad \frac{d\hat{z}}{d\tau} = U_z(\hat{r}, \hat{\theta}, \hat{z}, \tau, \varepsilon).$$

Since $U_r$, $U_\theta$ and $U_z$ vanish when $\varepsilon = 0$, and $(\hat{r}, \hat{\theta}, \hat{z}) = (r, \theta, z)$ when $\tau = t$, we have

$$\text{(X.6)}\qquad \begin{aligned} r^{\langle 0\rangle} &= r, \\ \theta^{\langle 0\rangle} &= \theta - \Omega s, \end{aligned}$$

and $z^{\langle 0\rangle} = z$.

Then

$$\hat{r}\frac{d(\hat{\theta}-\theta^{\langle 0\rangle})}{d\tau}=U_\theta(\hat{r},\hat{\theta},\hat{z},\tau,\varepsilon).$$

Expanding in powers of $\varepsilon$, we may obtain differential equations for the perturbed path functions in cylindrical coordinates. For example, at order one,

$$\frac{dr^{\langle 1\rangle}(\tau)}{d\tau}=U_r^{\langle 1\rangle}(r,\theta-\Omega s,z,\tau),\qquad r^{\langle 1\rangle}(t)=0,$$

(X.7)
$$r\frac{d\theta^{\langle 1\rangle}(\tau)}{d\tau}=U_\theta^{\langle 1\rangle}(r,\theta-\Omega s,z,\tau),\qquad \theta^{\langle 1\rangle}(t)=0,$$

$$\frac{dz^{\langle 1\rangle}(\tau)}{d\tau}=U_z^{\langle 1\rangle}(r,\theta-\Omega s,z,\tau),\qquad z^{\langle 1\rangle}(t)=0;$$

and, at order two,

$$\frac{dr^{\langle 2\rangle}(\tau)}{d\tau}=U_r^{\langle 2\rangle}+(r^{\langle 1\rangle}\partial_r+\theta^{\langle 1\rangle}\partial_\theta+z^{\langle 1\rangle}\partial_z)U_r^{\langle 1\rangle},\qquad r^{\langle 2\rangle}(t)=0;$$

(X.8)
$$r\frac{d\theta^{\langle 2\rangle}(\tau)}{d\tau}+r^{\langle 1\rangle}\frac{d\theta^{\langle 1\rangle}}{d\tau}=U_\theta^{\langle 2\rangle}+(r^{\langle 1\rangle}\partial_r+\theta^{\langle 1\rangle}\partial_\theta+z^{\langle 1\rangle}\partial_z)U_\theta^{\langle 1\rangle},$$

$$\theta^{\langle 2\rangle}(t)=0,$$

$$\frac{dz^{\langle 2\rangle}(\tau)}{d\tau}=U_z^{\langle 2\rangle}+(r^{\langle 1\rangle}\partial_r+\theta^{\langle 1\rangle}\partial_\theta+z^{\langle 1\rangle}\partial_z)U_z^{\langle 1\rangle},\qquad z^{\langle 2\rangle}(t)=0$$

where $\boldsymbol{U}^{\langle 2\rangle}$ is evaluated at $(r,\theta-\Omega s,z,\tau)$.

We turn next to the computation of stresses in cylindrical coordinates. We shall need to consider components of vectors and tensors relative to the orthonormal basis $(\boldsymbol{e}_1(\theta),\boldsymbol{e}_2(\theta),\boldsymbol{e}_3)=(\boldsymbol{e}_r(\theta),\boldsymbol{e}_\theta(\theta),\boldsymbol{e}_3)$. This basis has the property that when $\theta=0$, $(\boldsymbol{e}_r(0),\boldsymbol{e}_\theta(0),\boldsymbol{e}_3(0))$ coincides with the fixed basis $(\boldsymbol{e}_1,\boldsymbol{e}_2,\boldsymbol{e}_3)$. The following conventions are then adopted for components: $v_i\langle\theta\rangle\sim(v_1\langle\theta\rangle,v_2\langle\theta\rangle,v_3\langle\theta\rangle)=(v_r\langle\theta\rangle,v_\theta\langle\theta\rangle,v_z\langle\theta\rangle)$ are the components $\boldsymbol{v}$ relative to the basis $e_i(\theta)$. We use three different orthonormal bases:

$$\boldsymbol{v}=\boldsymbol{e}_l(0)\,v_l\langle 0\rangle=\boldsymbol{e}_l(\theta-\Omega s)\,v_l\langle\theta-\Omega s\rangle=\boldsymbol{e}_l(\theta)\,v_l\langle\theta\rangle.$$

In an identical notation, we represent any tensor $\boldsymbol{D}$ as

$$\begin{aligned}D&=\boldsymbol{e}_i(0)\,\boldsymbol{e}_j(0)D_{ij}\langle 0\rangle=\boldsymbol{e}_i(\theta-\Omega s)\,\boldsymbol{e}_j(\theta-\Omega s)D_{ij}\langle\theta-\Omega s\rangle\\&=\boldsymbol{e}_i(\theta)\,\boldsymbol{e}_j(\theta)D_{ij}\langle\theta\rangle\end{aligned}$$

where, for example,

$$D_{12}\langle\theta-\Omega s\rangle=D_{r\theta}\langle\theta-\Omega s\rangle=\boldsymbol{e}_1(\theta-\Omega s)\cdot\boldsymbol{D}\cdot\boldsymbol{e}_2(\theta-\Omega s).$$

The orthogonal tensor $\boldsymbol{Q}(\theta)$ is like $\boldsymbol{Q}(\Omega s)$,

(X.11)
$$\boldsymbol{e}_i(\theta)=\boldsymbol{Q}^T(\theta)\cdot\boldsymbol{e}_i(0)$$

where

$$e_i(0)\cdot Q(\theta)\cdot e_j(0)=Q_{ij}(\theta)=e_i(\theta)\cdot e_j(0). \tag{X.12}$$

To calculate components in the basis $(e_r(\theta), e_\theta(\theta), e_3)$, we prove a computational lemma:

(X.13) *Suppose that* $v=Q(\theta-\Omega s)\cdot u$. *Then* $v_k\langle 0\rangle=u_k\langle\theta-\Omega s\rangle$; *that is,* $(v_1\langle 0\rangle, v_2\langle 0\rangle, v_3\langle 0\rangle)=(u_r\langle\theta-\Omega s\rangle, u_\theta\langle\theta-\Omega s\rangle, u_z\langle\theta-\Omega s\rangle)$.

$$Q^T(\theta)\cdot Q(\theta-\Omega s)\cdot u=e_l(\theta)\,u_l\langle\theta-\Omega s\rangle. \tag{X.14}$$

(X.15) *If* $C=Q(\theta-\Omega s)\cdot D\cdot Q^T(\theta-\Omega s)$, *then* $C_{km}\langle 0\rangle=D_{km}\langle\theta-\Omega s\rangle$.

$$Q^T(\theta)\cdot Q(\theta-\Omega s)\cdot D\cdot Q^T(\theta-\Omega s)\cdot Q(\theta)=e_l(\theta)\,e_k(\theta)D_{lk}\langle\theta-\Omega s\rangle. \tag{X.16}$$

To prove (X.15), we set $D=e_i(0)\,e_j(0)D_{ij}\langle 0\rangle$. Then

$$\begin{aligned}&e_k(0)\cdot Q(\theta-\Omega s)\cdot e_i(0)\cdot e_j(0)D_{ij}\langle 0\rangle\cdot Q^T(\theta-\Omega s)\cdot e_m(0)\\&\quad=Q_{ki}(\theta-\Omega s)Q_{mj}(\theta-\Omega s)D_{ij}\langle 0\rangle=D_{km}\langle\theta-\Omega s\rangle.\end{aligned}$$

(X.16) is true because

$$Q^T(\theta)\cdot C\cdot Q(\theta)=Q^T(\theta)\cdot e_i(0)\,C_{ij}\langle 0\rangle\,e_j(0)\cdot Q(\theta)=e_i(\theta)\,e_j(\theta)D_{ij}\langle\theta-\Omega s\rangle.$$

From matrix multiplication and addition formulas for trigonometric functions, we derive

$$Q^T(\Omega s)=Q^T(\theta)\cdot Q(\theta-\Omega s). \tag{X.17}$$

Application of (X.16) and (X.17) to the integrands in the integrals (VIII.6) which give the stress gives

$$\begin{aligned}&\mathbb{J}^{\langle 1\rangle}(s)=Q^T(\Omega s)\cdot A^{\langle 1\rangle}(s)\cdot Q(\Omega s)=e_l(\theta)\,e_k(\theta)\hat{A}^{\langle 1\rangle}_{lk}(s),\\&\mathbb{J}^{\langle 1\rangle}(s_1)\cdot\mathbb{J}^{\langle 1\rangle}(s_2)=e_l(\theta)\,e_k(\theta)\hat{A}^{\langle 1\rangle}_{lm}(s_1)\hat{A}^{\langle 1\rangle}_{mk}(s_2),\\&\mathbb{J}^{\langle 2\rangle}(s)=Q^T(\Omega s)\cdot[A^{\langle 2\rangle}(s)+B(s)]\cdot Q(\Omega s)\\&\qquad=e_l(\theta)\,e_k(\theta)[\hat{A}^{\langle 2\rangle}_{lk}(s)+\hat{B}_{lk}(s)]\end{aligned} \tag{X.18}$$

where, to simplify the notation, we have used a roof to designate components in the basis $e_i(\theta-\Omega s)$; e.g., $\hat{A}_{lk}(s)=A_{lk}(s)\langle\theta-\Omega s\rangle=e_l(\theta-\Omega s)\cdot A(s)\cdot e_k(\theta-\Omega s)$,

$$\begin{aligned}e_l(\theta)\,e_k(\theta)\hat{B}_{lk}(s)=&(\chi^{\langle 1\rangle}\cdot\nabla_\xi)\,e_l(\theta)\,e_k(\theta)\hat{A}^{\langle 1\rangle}_{lk}(s)\\&+e_l(\theta)\,e_k(\theta)[\hat{A}^{\langle 1\rangle}_{lm}(s)\hat{f}^{\langle 1\rangle}_{mk}(s)+\hat{A}^{\langle 1\rangle}_{km}(s)\hat{f}^{\langle 1\rangle}_{ml}(s)]\end{aligned}$$

and

$$\begin{aligned}(\chi^{\langle 1\rangle}\cdot\nabla_\xi)\,e_l(\theta)\,e_k(\theta)\hat{A}^{\langle 1\rangle}_{lk}=\theta^{\langle 1\rangle}\{&2(e_\theta e_\theta-e_r e_r)\hat{A}^{\langle 1\rangle}_{r\theta}\\&+(e_\theta e_r+e_r e_\theta)(\hat{A}^{\langle 1\rangle}_{rr}-\hat{A}^{\langle 1\rangle}_{\theta\theta})+(e_\theta e_z+e_z e_\theta)\hat{A}^{\langle 1\rangle}_{zr}\\&-(e_z e_r+e_r e_z)\hat{A}^{\langle 1\rangle}_{z\theta}\}+e_l(\theta)\,e_k(\theta)(\chi^{\langle 1\rangle}\cdot\nabla_\xi)\hat{A}_{lk}\end{aligned}$$

where $(e_r, e_\theta)=(e_r(\theta), e_\theta(\theta))$.

Since $U^{\langle n\rangle}(\xi(\tau),\tau)=e_i(\theta-\Omega s)U_i^{\langle n\rangle}(r,\theta-\Omega s,z,\tau)$, we find that $U_i^{\langle n\rangle}\langle\theta-\Omega s\rangle=\hat{U}_i^{\langle n\rangle}=U_i^{\langle n\rangle}(r,\theta-\Omega s,z,\tau)$; that is, $(\hat{U}_1^{\langle n\rangle},\hat{U}_2^{\langle n\rangle},\hat{U}_3^{\langle n\rangle})=(U_r^{\langle n\rangle},U_\theta^{\langle n\rangle},U_z^{\langle n\rangle})$ $(r,\theta-\Omega s,z,\tau)$. Similary, the components of $A^{\langle n\rangle}$ and $f^{\langle 1\rangle}=\nabla_\xi\xi^{\langle 1\rangle}$ in the basis $e_i(\theta-\Omega s)$ are

$$\begin{pmatrix}\hat{A}_{rr}^{\langle n\rangle} & \hat{A}_{r\theta}^{\langle n\rangle} & \hat{A}_{rz}^{\langle n\rangle}\\ A_{\theta r}^{\langle n\rangle} & \hat{A}_{\theta\theta}^{\langle n\rangle} & \hat{A}_{\theta z}^{\langle n\rangle}\\ \hat{A}_{rz}^{\langle n\rangle} & \hat{A}_{\theta z}^{\langle n\rangle} & \hat{A}_{zz}^{\langle n\rangle}\end{pmatrix}$$

$$=\begin{pmatrix}2\partial_r\hat{U}_r^{\langle n\rangle} & \frac{1}{r}\partial_\theta\hat{U}_r^{\langle n\rangle}+r\partial_r\frac{\hat{U}_\theta^{\langle n\rangle}}{r} & \partial_z\hat{U}_r^{\langle n\rangle}+\partial_r\hat{U}_z^{\langle n\rangle}\\ \frac{1}{r}\partial_\theta\hat{U}_r^{\langle n\rangle}+r\partial_r\frac{\hat{U}_\theta^{\langle n\rangle}}{r} & 2\left(\frac{1}{r}\partial_\theta\hat{U}_\theta^{\langle n\rangle}+\frac{\hat{U}_r^{\langle n\rangle}}{r}\right) & \partial_z\hat{U}_\theta^{\langle n\rangle}+\frac{1}{r}\partial_\theta\hat{U}_z^{\langle n\rangle}\\ \partial_z\hat{U}_r^{\langle n\rangle}+\partial_r\hat{U}_z^{\langle n\rangle} & \partial_z\hat{U}_\theta^{\langle n\rangle}+\frac{1}{r}\partial_\theta\hat{U}_z^{\langle n\rangle} & 2\partial_z\hat{U}_z^{\langle n\rangle}\end{pmatrix}$$

and

$$\begin{pmatrix}\hat{f}_{rr}^{\langle n\rangle} & \hat{f}_{r\theta}^{\langle n\rangle} & \hat{f}_{rz}^{\langle n\rangle}\\ \hat{f}_{\theta r}^{\langle n\rangle} & \hat{f}_{\theta\theta}^{\langle n\rangle} & \hat{f}_{\theta z}^{\langle n\rangle}\\ \hat{f}_{zr}^{\langle n\rangle} & \hat{f}_{z\theta}^{\langle n\rangle} & \hat{f}_{zz}^{\langle n\rangle}\end{pmatrix}\begin{pmatrix}\partial_r r^{\langle n\rangle} & \frac{1}{r}\partial_\theta r^{\langle n\rangle}-\theta^{\langle n\rangle} & \partial_z r^{\langle n\rangle}\\ \partial_r(r\,\theta^{\langle n\rangle}) & \partial_\theta\theta^{\langle n\rangle}+r^{\langle n\rangle}/r & r\,\partial_z\theta^{\langle n\rangle}\\ \partial_r z^{\langle n\rangle} & \frac{1}{r}\partial_\theta z^{\langle n\rangle} & \partial_z z^{\langle n\rangle}\end{pmatrix}.$$

We next list the equations of motion for the perturbation fields:

$$\text{(X.19)}\qquad \partial_r U_r^{\langle n\rangle}+\frac{1}{r}(U_r^{\langle n\rangle}+\partial^\theta U_\theta^{\langle n\rangle})+\partial_z U_z^{\langle n\rangle}=0,\qquad n=1,2,\ldots$$

where $U_r^{\langle n\rangle}=U_r^{\langle n\rangle}(r,\theta,z,t)$, *etc.*,

$$\rho\begin{pmatrix}\delta_t & -2\Omega & 0\\ 2\Omega & \delta_t & 0\\ 0 & 0 & \delta_t\end{pmatrix}\begin{pmatrix}U_r^{\langle n\rangle}(r,\theta,z,t)\\ U_\theta^{\langle n\rangle}(r,\theta,z,t)\\ U_z^{\langle n\rangle}(r,\theta,z,t)\end{pmatrix}+\begin{pmatrix}\partial_r\\ \frac{1}{r}\partial_\theta\\ \partial_z\end{pmatrix}p^{\langle n\rangle}(r,\theta,z,t)$$

$$\text{(X.20)}\qquad -\int_0^\infty G(s)\begin{pmatrix}\nabla^2-\frac{1}{r^2} & -\frac{2}{r^2}\partial_\theta & 0\\ \frac{2}{r^2}\partial_\theta & \nabla^2-\frac{1}{r^2} & 0\\ 0 & 0 & \nabla^2\end{pmatrix}\begin{pmatrix}U_r^{\langle n\rangle}(r,\theta-\Omega s,z,t-s)\\ U_\theta^{\langle n\rangle}(r,\theta-\Omega s,z,t-s)\\ U_z^{\langle n\rangle}(r,\theta-\Omega s,z,t-s)\end{pmatrix}ds$$

$$=\begin{pmatrix}f_r^{\langle n\rangle}(r,\theta,z,t)\\ f_\theta^{\langle n\rangle}(r,\theta,z,t)\\ f_z^{\langle n\rangle}(r,\theta,z,t)\end{pmatrix}$$

where $\delta_t = \partial_t + \Omega\partial_\theta$ is a material derivative following the rigid motion, $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial^2 z^2}$,

$$f_r^{\langle 1\rangle} = f_\theta^{\langle 1\rangle} = f_z^{\langle 1\rangle} = 0,$$

(X.21)
$$\begin{pmatrix} f_r^{\langle 2\rangle} \\ f_\theta^{\langle 2\rangle} \\ f_z^{\langle 2\rangle} \end{pmatrix} = -\left(U_r^{\langle 1\rangle}\partial_r + \frac{U_\theta^{\langle 1\rangle}}{r}\partial_\theta + U_z^{\langle 1\rangle}\partial_z\right)\begin{pmatrix} U_r^{\langle 1\rangle} \\ U_\theta^{\langle 1\rangle} \\ U_z^{\langle 1\rangle} \end{pmatrix}$$
$$+\frac{U_\theta^{\langle 1\rangle}}{r}\begin{pmatrix} U_\theta^{\langle 1\rangle} \\ -U_r^{\langle 1\rangle} \\ 0 \end{pmatrix} + \begin{pmatrix} \partial_r M_{rr} + \frac{1}{r}\partial_\theta M_{r\theta} + \partial_z M_{rz} + \frac{1}{r}(M_{rr} - M_{\theta\theta}) \\ \partial_r M_{\theta r} + \frac{1}{r}\partial_\theta M_{\theta\theta} + \partial_z M_{\theta z} + \frac{1}{r}(M_{r\theta} + M_{\theta r}) \\ \partial_r M_{zr} + \frac{1}{r}\partial_\theta M_{z\theta} + \partial_z M_{zz} + \frac{1}{r}M_{zr} \end{pmatrix},$$

$$M_{ij} = \int_0^\infty G(s)\hat{B}_{ij}(s)\,ds + \int_0^\infty\int_0^\infty \gamma(s_1, s_2)\hat{A}_{il}^{\langle 1\rangle}(s_1)\hat{A}_{lj}^{\langle 1\rangle}(s_2)\,ds_1\,ds_2$$

and

$$M_{12} = M_{r\theta}, \text{ etc.}$$

The motion at first order is driven by the initial-history and boundary conditions; there are no inhomogeneous terms. Equations (X.19) and (X.20) are four equations for the unknown functions $U_r^{\langle 1\rangle}$, $U_\theta^{\langle 1\rangle}$, $U_z^{\langle 1\rangle}$, $p^{\langle 1\rangle}$. The particle paths at first order may be computed from (X.7) when these functions are known. The inhomogeneous terms in (X.20) with $n=2$ may be formed when the velocity and paths at first order are known. Then we may compute the velocity and pressure at second order. The particle paths at second order then follows by integration of (X.8) and so on.

The following special cases are of interest:

(i) $\Omega = 0$: In this case we are perturbing the rest state, and the perturbation equations of motion reduce to those derived by JOSEPH (1976).
(ii) Steady flow: In this case the functions are independent of the time argument in the last place and $\partial_t = 0$. The functions in the integrals which give the stress depend on the time $s$ through the argument $\theta - \Omega s$.
(iii) Axisymmetric flow: In this case the functions are independent of the angular variable in the second place, $\partial_\theta = 0$. We shall consider this case in more detail in § XI.
(iv) Steady axisymmetric flow. In this case velocity components are independent of time and the particle paths depend on $t$ only through $s = t - \tau$. The theory of rotating fluids may then be expressed in terms of the stresses for "fluids of grade $n$". Equation (X.20) may be written as

$$2\rho\Omega\begin{pmatrix}-U_\theta^{\langle n\rangle}\\ U_r^{\langle n\rangle}\\ 0\end{pmatrix}+\begin{pmatrix}\partial_r\\ 0\\ \partial_z\end{pmatrix}p^{\langle n\rangle}(r,z)$$

$$-\mu\begin{pmatrix}\nabla^2-\frac{1}{r^2} & 0 & 0\\ 0 & \nabla^2-\frac{1}{r^2} & 0\\ 0 & 0 & \nabla^2\end{pmatrix}\cdot\begin{pmatrix}U_r^{\langle n\rangle}(r,z)\\ U_\theta^{\langle n\rangle}(r,z)\\ U_z^{\langle n\rangle}(r,z)\end{pmatrix}=\begin{pmatrix}f_r^{\langle n\rangle}(r,z)\\ f^{\langle n\rangle}(r,z)\\ f_z^{\langle n\rangle}(r,z)\end{pmatrix}$$

where

(X.22) $$\mu=\int^{\infty} G(s)\,ds.$$

When $n=1$, $\boldsymbol{f}^{\langle 1\rangle}=0$ and (X.7) yields

$$\begin{pmatrix}r^{\langle 1\rangle}\\ r\,\theta^{\langle 1\rangle}\\ z^{\langle 1\rangle}\end{pmatrix}=-\begin{pmatrix}U_r^{\langle 1\rangle}(r,z)\\ U_\theta^{\langle 1\rangle}(r,z)\\ U_z^{\langle 1\rangle}(r,z)\end{pmatrix}s.$$

Then $\boldsymbol{f}^{\langle 2\rangle}$ is as before with $\partial_\theta=0$ and

$$M_{ij}=\alpha_2\hat{A}_{2,ij}+\alpha_1\hat{A}_{il}\hat{A}_{lj}$$

where $\boldsymbol{A}_2$ is the second Rivlin-Ericksen tensor and (COLEMAN & MARKOVITZ, 1964)

(X.23) $$\alpha_2=-\int_0^{\infty} s\,G(s)\,ds$$

and

(X.24) $$\alpha_1=\int_0^{\infty}\int_0^{\infty}\gamma(s_1,s_2)\,ds_1\,ds_2.$$

Proceeding in the same way, we can show that the problem at third order can be framed in terms of the constants defining fluids of grade 3 and so on.

## XI. Fourier Decomposition of the Equations of Motion

Since solutions must be single-valued functions of position, we may without loss of generality assume periodicity, with period $2\pi$, in $\theta$. Then

(XI.1) $$\begin{pmatrix}\boldsymbol{U}^{\langle n\rangle}(r,\theta,z,t)\\ \boldsymbol{f}^{\langle n\rangle}(r,\theta,z,t)\\ P^{\langle n\rangle}(r,\theta,z,t)\end{pmatrix}=\sum_{-\infty}^{\infty}e^{il\theta}\begin{pmatrix}\boldsymbol{u}(r,z,t)\\ \boldsymbol{f}(r,z,t)\\ p(r,z,t)\end{pmatrix}$$

where $l=0$ is included in the summation and real-valued solutions are assumed; $u_{-l}=\bar{u}_l$, $f_{-l}=\bar{f}_l$ where the overbar designates conjugate.

The representation (XI.1) may not be convenient when $\partial\mathscr{V}(t)$ has no special symmetry. But if the container is axisymmetric with, say, a boundary equation in the form $r=r(z)$, then, on the boundary, $u_l(r(z), z, t)$ is independent of $\theta$ and the Fourier components of the solutions may be related directly to the Fourier components of the prescribed boundary data.

The equations satisfied by the Fourier components are

$$\partial_r u_l + \frac{1}{r}(u_l + i l v_l) + \partial_z \omega_l = 0 \tag{XI.2}$$

and

$$\rho \begin{pmatrix} \delta_t & -2\Omega & 0 \\ 2\Omega & \delta_t & 0 \\ 0 & 0 & \delta_t \end{pmatrix} \begin{pmatrix} u_l(r,z,t) \\ v_l(r,z,t) \\ \omega_l(r,z,t) \end{pmatrix} + \begin{pmatrix} \partial_r \\ i\frac{l}{r} \\ \partial_z \end{pmatrix} p_l(r,z,t)$$

$$- \int^{\infty} G(s)\, e^{-il\Omega s} \begin{pmatrix} \nabla^2 - \frac{1}{r^2}(l^2+1) & -\frac{2il}{r^2} & 0 \\ \frac{2il}{r^2} & \nabla^2 - \frac{1}{r}(l^2+1) & 0 \\ 0 & 0 & \nabla^2 - \frac{l^2}{r^2} \end{pmatrix} \begin{pmatrix} u_l(r,z,t-s) \\ v_l(r,z,t-s) \\ w_l(r,z,t-s) \end{pmatrix} ds \tag{XI.3}$$

$$= \begin{pmatrix} f_{r\,l}(r,z,t) \\ f_{\theta\, l}(r,z,t) \\ f_{z\,l}(r,z;t) \end{pmatrix}$$

where $\nabla^2 = \partial_r^2 + \frac{1}{r}\partial_r + \partial_z^2$ and $\delta_t = \partial_t + i l \Omega$. The problem of axisymmetric flow mentioned in § X corresponds to the case in which the Fourier coefficients with $l \neq 0$ are null.

A further simplification of the equations may be obtained for flows which are periodic in time with period $2\pi/\omega$. For such flows

$$\begin{pmatrix} \mathbf{u}_l(r,z,t) \\ p_l(r,z,t) \\ f_l(r,z,t) \end{pmatrix} = \sum_{-\infty}^{\infty} e^{im\omega t} \begin{pmatrix} \mathbf{u}_{lm}(r,z) \\ p_{lm}(r,z) \\ f_{lm}(r,z) \end{pmatrix} \tag{XI.4}$$

and

$$\rho \begin{pmatrix} i(\omega m + \Omega l) & -2\Omega & 0 \\ 2\Omega & i(\omega m + \Omega l) & 0 \\ 0 & 0 & i(\omega m + \Omega l) \end{pmatrix} \begin{pmatrix} u_{lm}(r,z) \\ v_{lm}(r,z) \\ w_{lm}(r,z) \end{pmatrix} + \begin{pmatrix} \partial_r \\ i\frac{l}{r} \\ \partial_z \end{pmatrix} p_{lm}(r,z) \tag{XI.5}$$

$$-\eta^*(\omega m + \Omega l) \begin{pmatrix} \nabla^2 - \frac{1}{r^2}(l^2+1) & -2i\frac{l}{r^2} & 0 \\ 2i\frac{l}{r^2} & \nabla^2 - \frac{1}{r^2}(l^2+1) & 0 \\ 0 & 0 & \nabla^2 - \frac{l^2}{r^2} \end{pmatrix} \begin{pmatrix} u_{lm}(r,z) \\ v_{lm}(r,z) \\ w_{lm}(r,z) \end{pmatrix} \tag{XI.6}$$

$$= \begin{pmatrix} f_{r\,lm}(r,z) \\ f_{\theta\, lm}(r,z) \\ f_{z\,lm}(r,z) \end{pmatrix}$$

where

$$\eta^*(\omega m+\Omega l)=\int_0^\infty G(s)\,e^{-i(\omega m+\Omega l)s}\,ds. \tag{XI.7}$$

Steady flows appear as the special case of time-periodic flows in which all Fourier coefficients with $m\neq 0$ are null. For steady flow we put $m=0$ in (XI.6) and (XI.7).

Rheologists frequently work with the "complex viscosity"

$$\eta^*(\Omega)=\int_0^\infty G(s)\,e^{-i\Omega s}\,ds \tag{XI.8}$$

which arises from (XI.7) when $m=0$ and $l=1$. Of course, the "complex viscosity" is not a material parameter since it depends on the data of the problem which selects the values $l$ and $m$.

Equation (XI.6) was derived by ABBOTT & WALTERS (1970) for the special case in which $m=0$, $l=1$ and $\boldsymbol{f}=0$. They used an entirely different method to derive their result.

## XII. Linearized Stability of Rotating Simple Fluids

Suppose $\boldsymbol{U}_2$ and $\boldsymbol{U}_1=\boldsymbol{\Omega}\wedge\boldsymbol{x}$ satisfy the same initial-history problem (III.1)–(III.4). Let $\boldsymbol{u}=\boldsymbol{U}_1-\boldsymbol{U}_2$ and $p=P_1-P_2$ and suppose that $\boldsymbol{u}$ and $\nabla\boldsymbol{u}$ are small. Then, retaining only those terms which are linear in $\boldsymbol{u}$, we arrive at the variational equations for the stability of $\boldsymbol{\Omega}\wedge\boldsymbol{x}$. The polar components $(u, v, w)$ of $\boldsymbol{u}$ satisfy (X.19) and (X.20) with $\boldsymbol{u}=\boldsymbol{U}^{\langle n\rangle}$, $p=p^{\langle n\rangle}$ and $\boldsymbol{f}^{\langle n\rangle}=0$, $\boldsymbol{u}=0$ on $\partial\mathscr{V}$ and $\boldsymbol{u}$ may be assumed to have a nontrivial history in $t<0$. We are assuming now that $\mathscr{V}(t)$ is independent of $t$; that is, $\mathscr{V}$ rotates about an axis of symmetry.

Consider solutions of the variational equations in the form $(u, v, w, p)(r,\theta,z,t)=e^{-\sigma t}(u, v, w, p)(r,\theta,z)$ where $\sigma=\xi+i\omega$ is a complex number. These solutions satisfy the *spectral problem* for the variational equations governing the linearized stability of $\boldsymbol{\Omega}\wedge\boldsymbol{x}$:

$$\begin{aligned}&\rho\begin{pmatrix}-\sigma+\Omega\partial_\theta & -2\Omega & 0\\ 2\Omega & -\sigma+\Omega\partial_\theta & 0\\ 0 & 0 & -\sigma+\Omega\partial_\theta\end{pmatrix}\begin{pmatrix}u(r,\theta,z)\\ v(r,\theta,z)\\ w(r,\theta,z)\end{pmatrix}+\begin{pmatrix}\partial_r\\ \frac{1}{r}\partial_\theta\\ \partial_z\end{pmatrix}p(r,\theta,z)\\ &-\int_0^\infty G(s)\,e^{\sigma s}\begin{pmatrix}\nabla^2-\frac{1}{r^2} & -\frac{2}{r^2}\partial_\theta & 0\\ \frac{2}{r}\partial_\theta & \nabla^2-\frac{1}{r^2} & 0\\ 0 & 0 & \nabla^2\end{pmatrix}\begin{pmatrix}u(r,\theta-\Omega s,z)\\ v(r,\theta-\Omega s,z)\\ w(r,\theta-\Omega s,z)\end{pmatrix}ds=0.\end{aligned} \tag{XII.1}$$

The relation between the spectral problem and true nonlinear stability is assumed to be as follows: Suppose $\{\sigma\}$ is the set of all eigenvalues for all solutions of (XII.1) and the continuity equation which vanish on the boundary $\partial\mathscr{V}$ of $\mathscr{V}$. The flow $\boldsymbol{\Omega}\wedge\boldsymbol{x}$ is said to be stable to small disturbances (conditionally stable) if $\xi>0$ for all $\sigma$ in the set $\{\sigma\}$ and is said to be unstable if there exist values $\sigma$ for which $\xi<0$. No theorem of conditional stability has been proved for simple fluids. Such a theorem must necessarily state (i) that the stability of $\boldsymbol{\Omega}\wedge\boldsymbol{x}$ to small disturbances can be studied in the linearized approximation, and (ii) that the spectral problem governs the stability of the linearized equations. SLEMROD (1977) has shown (ii) holds when $\Omega=0$ and $G(s)$ ultimately decays exponentially with a sufficiently large decay rate. But no results are known for (i).

Assuming now the validity of the spectral problem in the study of stability of $\boldsymbol{\Omega}\wedge\boldsymbol{x}$, we effect a Fourier decomposition of the solution as in (XI.4) and find that

$$\rho\begin{pmatrix}-\sigma+il\Omega & -2\Omega & 0\\ 2\Omega & -\sigma+il\Omega & 0\\ 0 & 0 & -\sigma+il\Omega\end{pmatrix}\begin{pmatrix}u_l(r,z)\\ v_l(r,z)\\ w_l(r,z)\end{pmatrix}+\begin{pmatrix}\partial_r\\ il r\\ \partial_z\end{pmatrix}p_l(r,z)$$

(XII.2)

$$=\eta^*(i\sigma+\Omega l)\begin{pmatrix}\nabla^2-\frac{1}{r^2}(l^2+1) & -\frac{2il}{r^2} & 0\\ \frac{2il}{r^2} & \nabla^2-\frac{1}{r^2}(l^2+1) & 0\\ 0 & 0 & \nabla^2-\frac{l^2}{r^2}\end{pmatrix}\begin{pmatrix}u_l(r,z)\\ v_l(r,z)\\ w_l(r,z)\end{pmatrix}$$

where

(XII.3) $$\partial_r u_l+\frac{1}{r}(u_l+ilv_l)+\partial_z w_l=0$$

and $u_l(r,z)$, $v_l(r,z)$ and $w_l(r,z)$ vanish on $r=r(z)$.

We next derive an energy integral for $\sigma$. We define

$$\langle\cdot\rangle=\iint\cdot\, r\,dr\,dz$$

and show that

$$\langle(\bar{u}_l,\bar{v}_l,\bar{w}_l)\,(\text{XII.2})\rangle=0$$

implies

(XII.4) $$\rho(\xi+i\omega-il\Omega)\langle|\boldsymbol{u}_l|^2\rangle-4i\rho\,\Omega\,\operatorname{im}\langle u_l\bar{v}_l\rangle=\eta^*\mathscr{D}_l$$

where $\eta^*=\eta^*(i\xi+l\Omega-\omega)$, $|\boldsymbol{u}_l|^2=|u_l|^2+|v_l|^2+|w_l|^2$ and

$$\mathscr{D}_l=\left\langle\left|\frac{\partial\boldsymbol{u}_l}{\partial r}\right|^2+\left|\frac{\partial\boldsymbol{u}_l}{\partial z}\right|^2+\frac{l^2}{r^2}|w_l|^2+\frac{1}{r^2}(|lu_l+iv_l|^2+|lv_l-iu_l|^2)\right\rangle.$$

The real part of (XII.4) is given by

(XII.5) $$\frac{\rho\xi}{re(\eta^*)}=\frac{\mathscr{D}_l}{\langle|\boldsymbol{u}_l|^2\rangle}\geqq\lambda$$

where

(XII.6) $$\lambda=\min_H\frac{\mathscr{D}_l}{\langle|u_l|^2\rangle}>0,$$

and

$$H=\{\boldsymbol{u}_l\,|\,\boldsymbol{u}_l(r(z),z)=0,\ \boldsymbol{u}_l \text{ satisfies (XII.3)}\}.$$

*Suppose that* $\mathcal{G}_\xi(s)=G(s)\,e^{\xi s}$ *is a positive decreasing function which vanishes as* $s\to\infty$ *and that* $|\mathcal{G}'_\xi(s)|$ *is decreasing for* $0\leqq s<\infty$. *Then all solutions of* (XII.5) *have* $\xi>0$.

**Proof.** Let $\phi=\omega-l\Omega$. Then

$$\eta^*=\int_0^\infty \mathcal{G}_\xi(s)\,[\cos\phi\, s-i\sin\phi\, s]\,ds$$

and

$$\rho\,\xi\geqq\lambda\int_0^\infty \mathcal{G}_\xi(s)\cos\phi\, s\,ds=\frac{\lambda}{\phi}\int_0^\infty |\mathcal{G}'_\xi(s)|\sin\phi\, s\,ds.$$

Since $|\mathcal{G}'_\xi(s)|$ is decreasing and the integrand $|\mathcal{G}'_\xi(s)|\dfrac{\sin\phi\, s}{\phi}>0$ when $s$ is small, it is positive when $s<\pi/\phi$. The integrand changes sign at each zero of $\sin\phi\, s$ and the contribution to the value of the integral on each interval is of decreasing magnitude. The positive contributions are therefore larger than the negative ones.

For each eigenfunction of (XII.2), the ratio $\mathcal{D}_l/\langle|\boldsymbol{u}_l|^2\rangle=\lambda_l$ is a fixed positive number and $\operatorname{im}\langle u_l\bar{v}_l\rangle/\langle|\boldsymbol{u}_l|^2\rangle=\gamma_l$ is a fixed real number in the interval $(-\frac{1}{2},\frac{1}{2})$. Allowed values of $\xi$ are then given by the real roots $\xi>0$ of

$$\rho\,\xi=\lambda_l\int^\infty G(s)\,e^{\xi s}\cos(\omega-l\Omega)\,s\,ds=\lambda_l\,f(\xi) \tag{XII.7}$$

subject to the condition that $\omega$ satisfies the equation

$$-\rho(\omega-l\Omega-4\Omega\gamma_l)=\lambda_l\int_0^\infty G(s)\,e^{\xi s}\sin(\omega-l\Omega)\,s\,ds. \tag{XII.8}$$

If we suppose that there are roots $\xi>0$ and that the criterion of stability $\xi>0$ implies asymptotic stability of the solution of the linearized equations, then solutions of the initial-history and boundary-value problems arising in the perturbation sequence are unique. The difference between two such solutions having the same boundary values and initial histories must then decay to zero.

## XIII. Equations of Motion in Coordinate Systems with Axial Symmetry

A particle in steady rotation about a fixed axis changes its position at a fixed elevation and radius by changing its azimuthal position $\theta^{\langle 0\rangle}=\theta-\Omega s$ relative to its present position $\theta$. The radius and elevation of such a particle may be described in a variety of ways; for example, by spherical coordinates using the polar radius and angle or, more generally, in terms of any set of orthogonal coordinates $\zeta$, $\theta$, $\eta$ obtained from $r$, $\theta$, $z$ by the $\theta$ independent invertible transformation

$$r=r(\zeta,\eta),$$
$$z=z(\zeta,\eta).$$

In this orthogonal system, we may introduce a fixed basis $\tilde{e}_i(\zeta,\theta,\eta) \sim (\tilde{e}_1(\zeta,\theta,\eta), \tilde{e}_2(\zeta,\theta,\eta), \tilde{e}_3(\zeta,\theta,\eta)) \equiv (e_\zeta(\zeta,\theta,\eta), e_\theta(\theta), e_\eta(\zeta,\theta,\eta))$. In the plane $\theta=$ constant, the unit vector $e_\theta(\theta)$ is independent of $\zeta$ and $\eta$. A tilde overbar is also used to designate components of vectors and tensors in the basis $\tilde{e}_i(\zeta,\theta,\eta)$; for example,

$$U(x,t) = \tilde{e}_i(\zeta,\theta,\eta)\, \tilde{U}_i(\zeta,\theta,\eta,t) = e_\zeta U_\zeta + e_\theta U_\theta + e_\eta U_\eta.$$

A similar notation is used for tensors:

$$A(x,t) = \tilde{e}_i(\zeta,\theta,\eta)\, \tilde{e}_j(\zeta,\theta,\eta)\, \tilde{A}_{ij}(\zeta,\theta,\eta,t)$$

where, for example $\tilde{A}_{12} = A_{\zeta\theta}(\zeta,\theta,\eta,t)$. When $\zeta=r$ and $\eta=z$ we may delete the tilde overbar.

Decomposing the vectors and tensors mentioned in (X.14) and (X.18) into tilde overbar components, we get

$$Q^T(\Omega s)\cdot U^{\langle n\rangle}(\xi(t-s),t-s) = \tilde{e}_l(\zeta,\theta,\eta)\, \tilde{U}_l^{\langle n\rangle}(\zeta,\theta-\Omega s,\eta,t-s)$$

and

$$\mathbb{J}^{\langle 1\rangle}(s) = \tilde{e}_l(\zeta,\theta,\eta)\, \tilde{e}_k(\zeta,\theta,\eta)\, \hat{\tilde{A}}_{lk}^{\langle 1\rangle}(s)$$

where

$$\hat{\tilde{A}}_{lk} = \tilde{e}_l(\zeta,\theta-\Omega s,\eta)\cdot A\cdot \tilde{e}_k(\zeta,\theta-\Omega s,\eta).$$

In the same way,

$$\mathbb{J}^{\langle 2\rangle}(s) = \tilde{e}_l(\zeta,\theta,\eta)\, \tilde{e}_k(\zeta,\theta,\eta)\, [\hat{\tilde{A}}_{lk}^{\langle 2\rangle}(s) + \hat{\tilde{B}}_{lk}(s)].$$

The equations (X.3) relating particle paths to the velocity may be written as

$$\begin{aligned}\frac{d\chi_t(x,\tau,\varepsilon)}{d\tau} &= e_\theta(\hat{\theta})\,[\hat{r}(\hat{\zeta}\ \hat{\eta})\,\Omega + U_\theta(\hat{\zeta},\hat{\theta},\hat{\eta},\tau,\varepsilon)] \\ &\quad + e_\zeta(\hat{\zeta},\hat{\theta},\hat{\eta})\, U_\zeta(\hat{\zeta},\hat{\theta},\hat{\eta},\tau,\varepsilon) \\ &\quad + e_\eta(\hat{\zeta},\hat{\theta},\hat{\eta},\tau,\varepsilon).\end{aligned}$$

The perturbation equations for pathlines follow from expanding the component equations

(XIII.1)

$$\hat{r}(\hat{\zeta}(\zeta,\theta,\eta,\tau,\varepsilon), \hat{\eta}(\zeta,\theta,\eta,\tau,\varepsilon))\frac{d\hat{\theta}-\theta^{\langle 0\rangle}}{d\tau} = U_\theta(\hat{\zeta},\hat{\theta},\hat{\eta},\tau,\varepsilon),$$

$$e_\zeta(\hat{\zeta},\hat{\theta},\hat{\eta})\cdot\frac{d\chi_t}{d\tau} = U_\zeta(\hat{\zeta},\hat{\theta},\hat{\eta},\tau,\varepsilon),$$

and

$$e_\eta(\hat{\zeta},\hat{\theta},\hat{\eta})\cdot\frac{d\chi_t}{d\tau} = U_\eta(\hat{\zeta},\hat{\theta},\hat{\eta},\tau,\varepsilon).$$

The perturbation equations of motion (X.20) in $(\zeta,\theta,\eta)$ may be written as

(XIII.2) $$\rho(\delta_t + \boldsymbol{\Omega}\wedge)\, U^{\langle n\rangle} + \nabla p^{\langle n\rangle} - \int_0^\infty G(s)\,\nabla^2\tilde{U}^{\langle n\rangle}(\zeta,\theta-\Omega s,\eta,t-s)\,ds = f^{\langle n\rangle}$$

where $\operatorname{div} U^{\langle n\rangle}=0$,

$$U^{\langle n\rangle} = e_\zeta(\zeta,\theta,\eta)\, U_\zeta(\zeta,\theta,\eta,t) + e_\theta(\theta)\, U_\theta(\zeta,\theta,\eta,t) + e_\eta(\zeta,\theta,\eta)\, U_\eta(\zeta,\theta,\eta,t)$$

and

$$\tilde{U}^{\langle n\rangle} = e_\zeta(\zeta,\theta,\eta)\, U_\zeta^{\langle n\rangle}(\zeta,\theta-\Omega s,\eta,t-s) + e_\theta(\theta)\, U_\theta^{\langle n\rangle}(\zeta,\theta-\Omega s,\eta,t-s)$$
$$+ e_\eta(\zeta,\theta,\eta)\, U_\eta^{\langle n\rangle}(\zeta,\theta-\Omega s,\eta,t-s).$$

The following comparison rule holds: Consider the perturbation equations for the stress, motion and pathlines perturbing the steady rotation of a simple fluid written with respect to an orthogonal coordinate system with axial symmetry $(\zeta,\theta,\eta)$. When $\Omega=0$ these equations coincide with those governing the perturbation of the rest state (JOSEPH, 1976). To obtain the equation with $\Omega\neq 0$ from the simpler ones with $\boldsymbol{\Omega}=0$,

1.) Add $\boldsymbol{\Omega}\wedge\boldsymbol{x}$ to the velocity and $\frac{1}{2}\rho|\boldsymbol{\Omega}\wedge\boldsymbol{x}|^2$ to the pressure.
2.) Use (XIII.1) to compute the angular displacement $\hat{\theta}(\zeta,\theta,\eta,\tau,\varepsilon)$ of a particle. (In the rest state $\theta^{\langle 0\rangle}=\theta$.)
3.) Replace $\theta$ with $\theta-\Omega s$ in the perturbation velocity components which appear in the integrals giving the stress.
4.) Replace time derivatives $\partial_t$ in the inertial terms of the perturbation equations of motion with $\delta_t+\boldsymbol{\Omega}\wedge$ where $\delta_t=\partial_t+\Omega\partial_\theta$.

## XIV. The Dominance of Inertia and Remarks About Ekman Layers When $\Omega$ is Large

Sufficiently small deviations from states of rigid rotation are governed by (X.19) and (X.20) with $n=1$. Though the restriction $n=1$ generally implies that the deviation $\varepsilon$ from rigid rotation is small, the values of $\Omega$ for which (X.20) holds are unrestricted. In the theory of rotating Newtonian fluids (see GREENSPAN, 1969), the first order equations (the linear theory) are the main ones studied and the results of these studies are most interesting when $\Omega$ is large. When $\Omega$ is large, the terms which arise from extra stress cannot balance the large contributions from inertia except in regions of high shear. Neglecting terms arising from the extra stress, we get a geostrophic balance between terms proportional to $\Omega$ and the pressure gradient whose main consequence is an inertial stiffening (the Proudman-Taylor theorem) in which vertical variations of velocity are suppressed, $\partial_z \boldsymbol{U}^{\langle 1\rangle}=0$. In general, velocity fields which give a balance between inertial terms proportional to $\Omega$ and the pressure gradient cannot satisfy the boundary conditions; the transition from the interior values of the velocity in the geostrophic balance to the stated values at the boundary take place in a narrow shear layer (the Ekman layer) whose size is $O(\sqrt{\mu/\Omega\rho})$.

The same types of hydrodynamics can characterize rotating simple fluids when $\varepsilon$ is small and $\Omega$ is large. But new features of the hydrodynamics of simple fluids in almost rigid motions also appear.

We shall now agree that the linear theory of almost rigid rotations of a simple fluid is defined by (X.19), (X.20) with $n=1$ along with prescribed values of $\boldsymbol{U}^{\langle 1\rangle}(r,\theta,z,t)$ on the boundary for $t\in R$ and in the interior for $t<0$. We write (X.20) in vector form as in (XIII.2) using cylindrical coordinates $(\zeta,\theta,\eta)=(r,\theta,z)$:

$$\text{(XIV.1)} \qquad \rho(\delta_t+\boldsymbol{\Omega}\wedge)\,\boldsymbol{U}^{\langle 1\rangle}+\nabla p^{\langle 1\rangle}-\nabla^2\int^{\infty} G(s)\,\tilde{\boldsymbol{U}}^{\langle 1\rangle}(s)\,ds=0$$

where $U^{\langle 1\rangle}=e_l(\theta)U_l^{\langle 1\rangle}(r,\theta,z,t)$, $\tilde{U}^{\langle 1\rangle}(s)=e_l(\theta)U_l^{\langle 1\rangle}(r,\theta-\Omega s,z,t-s)$ and $\operatorname{div} U^{\langle 1\rangle}=0$. The Navier-Stokes problem may be obtained formally from (XIV.1) by putting $G(s)=\mu\,\delta(s)$ where $\mu$ is given by (X.22) and $\delta(s)$ is Dirac's function. In the Navier-Stokes problem, it is usual to write the equations for an observer rotating with angular velocity $\boldsymbol{\Omega}=e_3\,d\theta(t)/dt$. Then $e_l(\theta(t))$ depends on $t$ and the accelerations seen by a rotating observer are given by $(\partial_t+\boldsymbol{\Omega}\wedge)\,e_l(\theta(t))\,U_l^{\langle 1\rangle}=e_l(\theta)\,\partial_t U_l^{\langle 1\rangle}+2\boldsymbol{\Omega}\wedge U^{\langle 1\rangle}$. When $\Omega$ is large, the geostrophic balance is in the form $2\rho\,\boldsymbol{\Omega}\wedge U^{\langle 1\rangle}+\nabla p^{\langle 1\rangle}=0$, and this, together with $\operatorname{div} U^{\langle 1\rangle}=0$ implies $\partial U^{\langle 1\rangle}/\partial z=0$. This type of interior balance can also be postulated for solutions of (XIV.1).

In the Navier-Stokes theory, it is known that when $\Omega$ is large and $\varepsilon$ is small the inertial balance is broken only in narrow Ekman layers where the velocity is reduced from the values taken in the geostrophic balance to the values prescribed at the boundary. In the Ekman layer itself, approximations of a type familiar in boundary layer analysis apply and lead to simplified problems which can be solved in general circumstances.

The same kind of boundary layer analysis can sometimes be applied to viscoelastic fluids. This is obviously true of axisymmetric motions which are steady in a fixed basis or in a rotating basis because, in this case, the linear theory of rotating simple fluids is exactly the same as the linear theory of rotating Navier-Stokes fluids. For viscoelastic fluids the structure of boundary layers, when they exist, can depend in a sensitive way on the azimuthal periodicity of the boundary data. If we suppose that the boundary data is steady relative to the fixed basis $e_l(\theta)$, then the Fourier components of the steady solution satisfy (XII.2) with $\sigma=0$. When $l\neq 0$, we find by integrating by parts that

$$\text{(XIV.2)}\qquad \eta^*(l\Omega)=\int_0^\infty G(s)\,e^{-il\Omega s}\,ds=\frac{G(0)}{il\Omega}+\frac{G'(0)}{l^2\Omega^2}+O(\Omega^{-3})$$

and when $\Omega=0$, $\eta^*=\mu$. When $l\Omega$ is large, the elasticity of the fluid is important and when $l\Omega\to 0$, the fluid is viscous and not elastic. This interesting property of the "complex viscosity" has a big effect on the boundary layer in the orthogonal rheometer, which is treated in § XV. In this rheometer the boundary data has a first mode azimuthal periodicity and the usual boundary layer assumptions lead to a boundary thickness of $O(\sqrt{G(0)/\rho\,\Omega^2})$. The ratio of the boundary layer thickness of a viscoelastic fluid in the orthogonal rheometer to that in a Navier-Stokes fluid is $O(\sqrt{G(0)/\Omega\,\mu})$. The thickness of the boundary layer in the viscoelastic fluid is ever so much smaller.

## XV. The Maxwell Orthogonal Rheometer

A simple fluid fills the space between parallel disks which rotate with the same angular velocity around different centers. Let $a$ be the distance between centers and $h$ the distance between the disks (see Fig. 1). In the idealized mathematical problem the radii or the disks are infinite. A cylindrical coordinate system is centered on the bottom disk midway on the line between the centers. Let $y$ be along the line of centers and count $\theta$ increasing in the direction of arrows called $\Omega$ in Fig. 1.

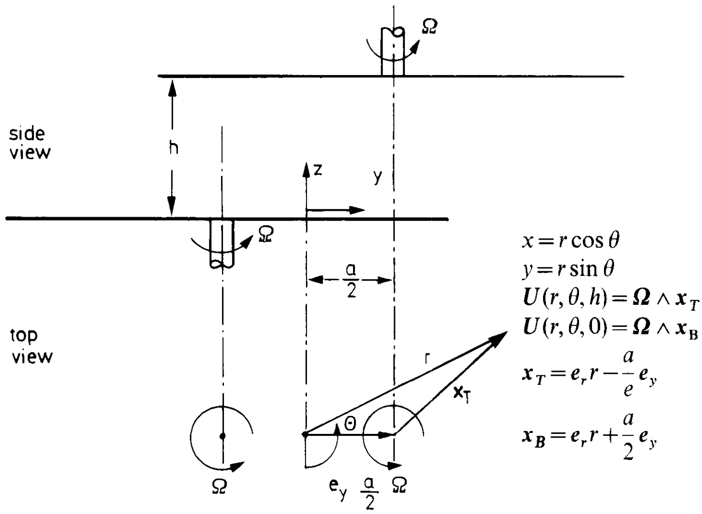


Fig. 1. The Orthogonal Rheometer

The Maxwell orthogonal rheometer which is used in laboratories is like the idealized configuration shown in Fig. 1 except that bounding surfaces are disks of finite radius rather than infinite planes. For this reason, the orthogonal rheometer is sometimes called eccentric rotating disks. There is a fairly extensive literature devoted to analysis of the idealized problem and to experiments with eccentric disks. The literature through 1975 is reviewed in HUILGOL's book (1975) on viscoelastic fluids. More recent results are reviewed in the paper by DAVIS & MACOSKO (1974); their paper gives fairly complete experimental results for cases in which the effects of the inertial terms are not important. The excellent analysis of ABBOTT & WALTERS (1970), correct through first order in $\varepsilon$, was the first to include the effects of inertia. GOLDSTEIN & SCHOWALTER (1975) carried out an analysis of the problem for a Bird-Carreau fluid through terms of order three. Their analysis is based on a good guess of the form of $\boldsymbol{G}(s,\varepsilon)$ in which the motion is always in circles around the line of centers on a to-be-determined curve joining the centers of rotation on the rotating plates.

All previous authors have based their analysis on some guess of the form of the strain history. None of these authors has noticed the inertial stiffening and formation of Ekman layers when $\Omega$ is large.

The boundary conditions for the rheometer are given in Fig. 1. Let $\varepsilon = a/h$. Then

(XV.1)
$$[U_r^{\langle 1\rangle}(r,\theta,0), U_\theta^{\langle 1\rangle}(r,\theta,0), U_r^{\langle 1\rangle}(r,\theta,h), U_\theta^{\langle 1\rangle}(r,\theta,h)] = \frac{h}{2}\Omega[-\cos\theta, \sin\theta, \cos\theta, -\sin\theta]$$

and

$$[U_z^{\langle 1\rangle}, \boldsymbol{U}^{\langle n\rangle}] = [0,0] \quad \text{on } z=0 \text{ and } z=h \quad \text{when } n>1.$$

The form of the boundary conditions suggests that we look for steady solutions of (X.20) with $n=1$ possessing a first mode azimuthal variation. Such solutions correspond to the Fourier modes $m=0$, $l=1$ of (XI.6). Setting $(u_{10}, v_{10}, w_{10}, p_{10}) = (u, v, w, p)$, we find that (XV.1) is satisfied if

(XV.2) $$[u(r,0), v(r,0), u(r,h), v(r,h)] = \frac{h\Omega}{4}[-1, -i, 1, i]$$

and $w(r,0) = w(r,h) = 0$. The solution of (XI.6) subject to (XI.2) has $w \equiv 0 \equiv p$ and is independent of $r$ so that (XI.2) becomes

(XV.3) $$u + iv = 0$$

with

(XV.4) $$u'' - \alpha^2 u = 0, \qquad \alpha^2 = -i\Omega\rho/\eta^*(\Omega).$$

(XV.5) $$u(z) = \frac{h\Omega}{4\sinh\alpha h}\{\sinh\alpha z + \sinh\alpha(z-h)\}.$$

Hence

(XV.6) $$U_r^{\langle 1\rangle}(z,\theta) = u(z)\,e^{i\theta} + \bar{u}\,e^{-i\theta}$$

and

(XV.7) $$U_\theta^{\langle 1\rangle}(z,\theta) = v(z)\,e^{i\theta} + \bar{v}(z)\,e^{-i\theta}.$$

The solution just given is the same as the one given by ABBOTT & WALTERS (1970). To obtain their solution they assumed the form of the pathlines through order one and used Oldroyd's method to relate the perturbed displacement functions $\hat{r}$, $\hat{\theta}$, $\hat{z}$ to $u(z)$ and $v(z)$. The present method does not require that the perturbed displacement be given; instead, having found $\boldsymbol{U}^{\langle 1\rangle}$ from a completely deterministic problem, we may compute the displacement functions from (X.7):

(XV.8) $$r^{\langle 1\rangle} = \frac{i}{\Omega}(1 - e^{-i\Omega s})\,u(z)\,e^{i\theta} + \text{conjugate}$$

and

(XV.9) $$r\theta^{\langle 1\rangle} = \frac{i}{\Omega}(1 - e^{-i\Omega s})\,v(z)\,e^{i\theta} + \text{conjugate}.$$

ABBOTT & WALTERS (1970) solved the problem of this section for a Newtonian fluid exactly, without perturbations. They and subsequent authors seem not to have mentioned that the exact solution for a Newtonian fluid is exactly the same as the perturbation solution for a viscoelastic fluid through first order with $G(s) = \mu\delta(s)$ where $\delta(s)$ is Dirac's delta function. For this choice of $G(s)$, $\eta^*(\Omega) = \mu$ and $\alpha = (1-i)\sqrt{\Omega/2\nu}$.

In all the studies of this problem which I have seen, consideration is given to limit $\alpha h \to 0$. In this limit

(XV.10) $$u(z) \to \frac{\Omega}{4}(2z - h)$$

is an exact solution corresponding to $\Omega \to 0$ and it satisfies the first-order equations and boundary conditions with inertia neglected. In fact, the constitutive equation does not enter in (XV.10). It is a universal form for the velocity, true for every fluid, one of the flows PIPKIN calls controllable. The experiments which I have seen so far support the idea that in the rheometers used in laboratories the actual fluid mechanics is well-approximated by (XV.10). It is necessary to add, however, that (XV.10) arises only in a special limit. The situation is drastically different when $\Omega$ is large.

Asymptotically, we find, using (XIV.2), that $\eta^*(\Omega)=G(0)/i\Omega+\frac{G'(0)}{\Omega^2}$ and $\alpha^2=\frac{\rho\Omega^2}{G(0)}\left[1+\frac{iG'(0)}{\Omega G(0)}\right]^{-1}$. It then follows from (XV.5) that for each fixed $z$ in the open interval $(0,h)$, $u(z,\Omega)\to\frac{h\Omega}{4}e^{-\Omega(h-z)\sqrt{\rho/G(0)}}$; hence, $u$ and $v$ vanish as $\Omega\to\infty$. This means that when $\Omega$ is large, $\boldsymbol{U}^{\langle 1\rangle}$ vanishes and $\boldsymbol{U}=\boldsymbol{U}^{\langle 0\rangle}=\boldsymbol{\Omega}\wedge\boldsymbol{x}$ is a rigid rotation. In the interior flow, we can neglect the stresses but not inertia. We also get a boundary layer, but since $\phi_1=u_1+iv_1=0$, there is no Ekman flux and we cannot introduce $\phi_1$ as the dependent variable in the boundary layer. Instead, we are led to $d^2u/d\zeta^2-u=0$, $u(0)=-h\Omega/4$, $u(\infty)=0$ and $u=-\frac{h\Omega}{4}e^{-\zeta}$ where $\zeta=\left(\Omega-\frac{iG'(0)}{G(0)}\right)\sqrt{\frac{\rho}{G(0)}}\,z$ and $\delta_1=\sqrt{G(0)/\rho\Omega^2}$. The same boundary layer solution can be obtained exactly in the limit $\Omega\to\infty$, from the exact solution (XV.5).

It would be of interest, and is not too difficult, to carry out a higher-order analysis of the orthogonal rheometer based on the general constitutive expressions developed in this paper. Considerations of symmetry based on changes in the sign of $\varepsilon$ (of $a$) lead to the conclusion that the velocity components must be given by a series of odd powers of $\varepsilon$, whilst the pressure and normal stresses proceed in even powers.

To compute the solution at second order in $\varepsilon$, we first need to compute the forcing vector $\boldsymbol{f}^{\langle 2\rangle}$ on the right of (X.20). This vector is determined by (X.21) through $\boldsymbol{A}^{\langle 1\rangle}$ and $\boldsymbol{B}$. We find that all of the components $\hat{A}_{ij}$ of $A^{\langle 1\rangle}$ are zero except

$$\hat{A}_{rz}=u'(z)\,e^{i(\theta-\Omega s)}+\text{conjugate}.$$

All the components $\hat{\gamma}_{ij}$ of $\gamma$ are zero except

$$\hat{\gamma}_{rz}=\partial_z r^{\langle 1\rangle}=\frac{i}{\Omega}(e^{i\Omega s}-1)\,u'(z)\,e^{i(\theta-\Omega s)}+\text{conjugate}.$$

It follows that

$$\begin{aligned}\boldsymbol{e}_l(\theta)\,\boldsymbol{e}_k(\theta)\hat{B}_{lk}&=\theta^{\langle 1\rangle}(\boldsymbol{e}_\theta\boldsymbol{e}_3+\boldsymbol{e}_3\boldsymbol{e}_\theta)(\hat{A}_{rz}+\partial_\theta\hat{A}_{\theta z})\\&\quad+\theta^{\langle 1\rangle}(\boldsymbol{e}_r\boldsymbol{e}_3+\boldsymbol{e}_3\boldsymbol{e}_r)(\partial_\theta\hat{A}_{rz}-\hat{A}_{\theta z})+2\boldsymbol{e}_3\boldsymbol{e}_3(\hat{\gamma}_{rz}\hat{A}_{rz}+\hat{\gamma}_{\theta z}\hat{A}_{\theta z})\\&=-\boldsymbol{e}_3\boldsymbol{e}_3\frac{4\sin\Omega s}{\Omega}(|u'|^2+|v'|^2),\end{aligned}$$

$$\begin{aligned}\boldsymbol{e}_l(\theta)\,\boldsymbol{e}_k(\theta)\hat{A}^{\langle 1\rangle}_{lm}(s_1)\hat{A}^{\langle 1\rangle}_{mk}(s_2)&=\boldsymbol{e}_r\boldsymbol{e}_r\,\partial_z U_r^{\langle 1\rangle}(s_1)\,\partial_z U_r^{\langle 1\rangle}(s_2)\\&\quad+\boldsymbol{e}_r\boldsymbol{e}_\theta\,\partial_z U_r^{\langle 1\rangle}(s_1)\,\partial_z U_\theta^{\langle 1\rangle}(s_2)+\boldsymbol{e}_\theta\boldsymbol{e}_r\,\partial_z U_r^{\langle 1\rangle}(s_2)\,\partial_z U_\theta^{\langle 1\rangle}(s_1)\\&\quad+\boldsymbol{e}_\theta\boldsymbol{e}_\theta\,\partial_z U_\theta^{\langle 1\rangle}(s_1)\,\partial_z U_\theta^{\langle 1\rangle}(s_2)+\boldsymbol{e}_3\boldsymbol{e}_3[\partial_z U_r^{\langle 1\rangle}(s_1)\,\partial_z U_r^{\langle 1\rangle}(s_2)\\&\quad+\partial_z U_\theta^{\langle 1\rangle}(s_1)\,\partial_z U_\theta^{\langle 1\rangle}(s_2)],\end{aligned}$$

$$\operatorname{div}\{e_l(\theta)\,e_k(\theta)\,A^{\langle 1\rangle}_{lm}(s_1)\,A^{\langle 1\rangle}_{mk}(s_2)\} = \operatorname{div}\{e_3 e_3\, 2\cos\Omega(s_1-s_2)(|u|'^2+|v'|^2).$$

Hence, when $n=2$, (X.20) may be written as

$$\left(\Omega\frac{\partial}{\partial\theta}+\boldsymbol{\Omega}\wedge\right)\left(e_l(\theta)\,U_l^{\langle 2\rangle}(r,\theta,z)\right)-\nabla^2 e_l(\theta)\int_0^\infty G(s)\,U_l^{\langle 2\rangle}(r,\theta-\Omega s,z)\,ds$$
$$=-\nabla\left\{p^{\langle 2\rangle}+(|u'|^2+|v'|^2)\left(\int_0^\infty 4G(s)\frac{\sin\Omega s}{s}\right.\right.$$
$$\left.\left.-2\int_0^\infty\int_0^\infty \gamma(s_1,s_2)\cos\Omega(s_1-s_2)\,ds_1\,ds_2\right)\right\}$$

where $U^{\langle 2\rangle}$ vanishes on $z=0$ and $z=h$. Hence $U^{\langle 2\rangle}\equiv 0$ but $p^{\langle 2\rangle}\neq 0$.

It goes almost without saying that the same type of analysis, based on the equations of § XIII written for spherical coordinates, can be developed for the flow between concentric spheres rotating about different axes. This problem was studied, using other methods, by WALTERS (1970).

This work was supported by the U.S. Army Research Office and under NSF grant 19047. I thank PRAKASH DIPIT for checking the equations in this paper.

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Dept. of Aerospace Eng. Mechanics
University of Minnesota
Minneapolis

*(Received July 18, 1977)*