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THE CONVERGENCE OF BIORTHOGONAL SERIES FOR BIHARMONIC AND STOKES FLOW EDGE PROBLEMS. PART I

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Abstract. Sufficient conditions are established for the convergence of the biorthogonal series solving edge problems which arise in elasticity and in Stokes flow in cavities. These conditions greatly improve those stated in the excellent work of R. C. T. Smith (1952).

R. C. T. Smith (1952) established an algorithm for solving the biharmonic problem governing the bending of a semi-infinite strip clamped on the long edges when arbitrary displacements and couples (the data) are prescribed on the short edge. The solution is expressed as a biorthogonal series and the given data is expanded into a biorthogonal series. Smith also established conditions on the data sufficient to guarantee the convergence of the biorthogonal series. Smith's conditions are too restrictive for applications. In this note I am going to establish that much less restricted conditions suffice to guarantee convergence. These conditions arise naturally in problems involving free surfaces shaped by Stokes flows in trenches.

Consider the biharmonic edge problem

(1a) $$\nabla^4\psi=\left(\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial y^2}\right)^2\psi=0 \quad \text{in}\,[-1<t<1,\,-\infty<y\leqq 0]$$

where

(1b) $$\psi=\frac{\partial\psi}{\partial t}=0 \quad \text{when } t=\pm 1$$

and, on $y=0$, the data

(1c) $$[f(t),\,g(t)]=\left[\frac{\partial^2\psi}{\partial y^2},\,\frac{\partial^2\psi}{\partial t^2}\right]$$

is prescribed. Problem (1a, b, c) was solved by Smith (1952).

In Smith's problem ψ is the normal deflection of the plate which is clamped on the long ends. $f(t)$ and $g(t)$ are arbitrary independent functions of t determined, for example, if the displacements of the short edge and the couples about this edge are known. Though $g(t)$ is an arbitrary function, (1b) and (1c) are compatible if and only if

(2) $$\int_{-1}^{1} g(t)\,dt=\int_{-1}^{1} tg(t)\,dt=0.$$

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The conditions (2) do not enter into my proof. Their significance is discussed at the conclusion of this paper.

The elasticity problem studied by Smith has also been studied by Johnson and Little (1965). The method used by the authors named last are different from but equivalent to those used by Smith. Unlike Smith, Johnson and Little carried out numerical computations. They showed good convergence using numerical criteria.[1]

Problem (1) also arises in fluid mechanics. In the fluids problem, ψ is the stream function and $f(t)$ and $g(t)$ are arbitrary functions of t determined, for example, if the stream function and shear stresses are prescribed on the short side.

Joseph and Fosdick (1973) and Joseph and Sturges (1975) used Smith's method to solve plane Stokes flow problems in a deep trench. These problems arise in the study of the free surface on the liquid on the top of the trench when the state of rest is perturbed by steady motion. Convergence of Smith's series is demonstrated mathematically in the two papers just mentioned but the central importance of (21) and (26) was not recognized. The formulation of Smith's problem used here is essentially that given by Joseph and Sturges (1975). Numerical computations consistent with the estimates of this paper are given by Joseph and Sturges.

It is an interesting fact that biorthogonal series which are similar to those studied in this paper can be formed for a large number of Stokes flow problems in regions other than strips and for Stokes flow operators which do not reduce to biharmonic problems. As examples we call attention to the study of Joseph (1974) of the free surface on a liquid between the parallel circular disks of a torsion flow viscometer, the study of the free surface on a liquid filling a wedge shaped cavity heated from its side (Liu and Joseph (1977)) the study of the same problem when the cavity is an annulus between two cylinders (Yoo and Joseph (1977)) and the study of axisymmetric flow in conical corners (Liu and Joseph (1977)). Criteria for convergence similar to those to be specified in Theorems 1 and 2 can be established for the biorthogonal series solving all of these problems.

With these preliminary motivating remarks aside, we turn now to a description of Smith's solution to the biharmonic problem (1). This solution is given by a series of biharmonic eigenfunctions (Papkovich-Fadle functions). There are even eigenfunctions,

$$\begin{bmatrix}\phi_1^{(n)}(t)\\ \phi_2^{(n)}(t)\end{bmatrix} = \begin{pmatrix} S_n \sin S_n \cos S_n t - S_n t \cos S_n \sin S_n t \\ -(S_n \sin S_n + 2\cos S_n)\cos S_n t + S_n t \cos S_n \sin S_n t \end{pmatrix}, \tag{3}$$

[1] A referee for this journal has pointed out that many other (less convenient) methods have been used to treat the elasticity problem. He notes that Fourier series and Fourier integral expansions are used in the paper of Iyenger (1960). Weiner–Hopf methods are used by Buchwald and Doran (1965). Eigenfunction expansions (not the natural eigenfunctions) with reorthogonalization are used by Gaydon and Shepherd (1964). There are also approximate solutions which may perhaps converge to the correct solution; e.g. Mendelson and Roberts (1965) use collocation methods and Benthem (1963) uses Laplace transforms and Fourier series. Finally there are approximate solutions which do not converge to the correct solution; e.g. Horvay (1953), (1960).

and odd eigenfunctions,

$$\begin{bmatrix}\hat{\phi}_1^{(n)}(t)\\ \hat{\phi}_1^{(n)}(t)\end{bmatrix} = \begin{pmatrix} P_n \cos P_n \sin P_n t - P_n t \sin P_n \cos P_n t \\ -(P_n \cos P_n - 2\sin P_n)\sin P_n t + P_n t \sin P_n \cos P_n t \end{pmatrix}, \tag{4}$$

where

$$\phi_1^{(n)}(\pm 1) = \hat{\phi}_1^{(n)}(\pm 1) = 0.$$

The functions

$$e^{S_n y}\phi_1^{(n)}(t)$$

and

$$e^{P_n y}\hat{\phi}_1^{(n)}(t)$$

are biharmonic. They satisfy the side-wall boundary conditions (1b) if

$$\frac{d\phi_1^{(n)}(\pm 1)}{dt} = \frac{d\hat{\phi}_1^{(n)}(\pm 1)}{dt} = 0. \tag{5}$$

Condition (5) generates eigenvalues S_n, $n = 1, 2, \dots$, which are the first-quadrant complex roots of

$$\sin 2S + 2S = 0, \tag{6}$$

and eigenvalues P_n, $n = 1, 2, \cdots$, which are the first-quadrant complex roots of

$$\sin 2P - 2P = 0. \tag{7}$$

S_n and P_n are numbered in a sequence corresponding to the magnitude of their real parts. The functions $\phi_1^{(n)}(t)$ are used when $f(t)$ and $g(t)$ are even functions and the $\phi_2^{(n)}(t)$ are used when $f(t)$ and $g(t)$ are odd functions. When mixed data is prescribed, the data is resolved into even and odd parts; the equations for these parts are solved separately and the solutions are superposed.

Until further notice there will be no need to distinguish between even and odd functions. We shall use the notation corresponding to even functions, but the analysis applies equally to the odd functions.

The series solving (1) may be written in the following form (see Joseph and Sturges (1975)):

$$\psi(y, t) = \sum_{-\infty}^{\infty} c_n \exp(S_n y)\phi_1^{(n)}(t)/S_n^2, \tag{8}$$

where

$$S_{-n} = \bar{S}_n,$$

$$\phi_1^{(-n)}(t) = \bar{\phi}_1^{(n)}(t),$$

and, if f and g are real-valued,

$$c_{-n} = \bar{c}_n.$$

The overbar denotes complex conjugation. Smith's basic contribution was an algorithm for the computation of c_n from a series representation of the data

$$\begin{pmatrix} f(t) \\ g(t) \end{pmatrix} = \sum_{-\infty}^{\infty} c_n \begin{pmatrix} \phi_1^{(n)}(t) \\ \phi_2^{(n)}(t) \end{pmatrix}. \tag{9}$$

We shall discuss this algorithm next.

To determine the constants c_n, we introduce the data vectors

$$\mathbf{f} = \mathbf{e}_1 f + \mathbf{e}_2 g,$$

the eigenvector

$$\boldsymbol{\phi}^{(n)} = \mathbf{e}_1 \phi_1^{(n)} + \mathbf{e}_2 \phi_2^{(n)}$$

and the adjoint eigenvector

$$\boldsymbol{\psi} = \mathbf{e}_1 \psi_1^{(n)} + \mathbf{e}_2 \psi_2^{(n)},$$

where $\mathbf{e}_1$, $\mathbf{e}_2$ are orthonormal base vectors. The eigenvector $\boldsymbol{\phi}^{(n)}(t)$ satisfies the differential equation

$$\frac{d^2 \boldsymbol{\phi}^{(n)}}{dt^2} + S^2 \mathbf{A} \cdot \boldsymbol{\phi}^{(n)} = 0, \tag{10a}$$

where

$$\mathbf{A} = -\mathbf{e}_1\mathbf{e}_2 + \mathbf{e}_2\mathbf{e}_1 + 2\mathbf{e}_2\mathbf{e}_2$$

and

$$\boldsymbol{\phi}^{(n)} \cdot \mathbf{e}_1 = \frac{d\boldsymbol{\phi}^{(n)}}{dt} \cdot \mathbf{e}_1 = 0 \quad \text{at } t = \pm 1. \tag{10b}$$

The components of the dyad $\mathbf{A}$ may be obtained from the following rule: $A_{ij} = \mathbf{e}_i \cdot \mathbf{A} \cdot \mathbf{e}_j$, $i, j = 1, 2$. In matrix notation

$$(\mathbf{A}) = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}.$$

The adjoint eigenvector $\boldsymbol{\psi}^{(n)}(t)$ satisfies

$$\frac{d^2 \boldsymbol{\psi}^{(n)}}{dt^2} + S^2 \mathbf{A}^T \cdot \boldsymbol{\psi}^{(n)} = 0, \tag{11a}$$

where

$$\mathbf{A}^T = -\mathbf{e}_2\mathbf{e}_1 + \mathbf{e}_1\mathbf{e}_2 + 2\mathbf{e}_2\mathbf{e}_2$$

and

$$\boldsymbol{\psi}^{(n)} \cdot \mathbf{e}_2 = \frac{d\boldsymbol{\psi}^{(n)}}{dt} \cdot \mathbf{e}_2 = 0 \quad \text{at } t = \pm 1. \tag{11b}$$

By elimination, we find that $\psi_2^{(n)}(t)$ and $\phi_1^{(n)}(t)$ both satisfy the reduced biharmonic

$$\frac{d^4F}{dt^4} + 2S^2\frac{d^2F}{dt^2} + S^4F = 0, \tag{12a}$$

where

$$F(\pm 1) = F'(\pm 1) = 0. \tag{12b}$$

We may therefore put

$$\psi_2^{(n)}(t) = \phi_1^{(n)}(t) \tag{13}$$

and $\psi_1^{(n)}(t)$ may be determined directly from the second component of (11a)

$$\frac{d^2\psi_2^{(n)}}{dt^2} + S_n^2(2\psi_2^{(n)} - \psi_1^{(n)}) = 0. \tag{14a}$$

The first component of (11a) is

$$\frac{d^2\psi_1^{(n)}}{dt^2} + S_n^2\psi_2^{(n)} = 0. \tag{14b}$$

An easy computation leads to

$$\psi_1^{(n)} = (S_n \sin S_n - 2\cos S_n)\cos S_n t - S_n t \cos S_n \sin S_n t$$

for even functions, and

$$\hat{\psi}_1^{(n)} = (P_n \cos P_n + 2\sin P_n)\sin P_n t - P_n t \sin P_n \cos P_n t$$

for odd functions.

The biorthogonality condition follows in a standard way from (10a, b) and (11a, b):

$$\int_{-1}^{1} \boldsymbol{\psi}^{(m)} \cdot \mathbf{A} \cdot \boldsymbol{\phi}^{(n)}\, dt = k_n \delta_{nm}, \tag{15}$$

where for even functions

$$k_n = -4\cos^4 S_n$$

and for odd functions

$$k_n = -4\sin^4 P_n.$$

Expressing the series (9) in vector form

$$\mathbf{f}(t) = \sum_{-\infty}^{\infty} c_n\, \boldsymbol{\phi}^{(n)}(t), \tag{16}$$

we find, using (15), that

$$\int_{-1}^{1} \boldsymbol{\psi}^{(m)} \cdot \mathbf{A} \cdot \mathbf{f}\, dt = c_n k_n. \tag{17}$$

Equation (17) determines the coefficients

$$c_n = \frac{1}{k_n} \int_{-1}^{1} \{(2\psi_2^{(n)} - \psi_1^{(n)})g + \psi_2^{(n)} f\}\, dt. \tag{18}$$

Equations (8) and (18) define a formal solution of the biharmonic edge problem. The formal solution is the solution if the series on the right of (16) converges to $\mathbf{f}(t)$. Smith (1952) notes that the large n representations of the solution are such that if $f(t)$ and $g(t)$ are merely of bounded variation, then the series on the left of (16) may diverge. The asymptotic forms of the solution are

$$2S_n \to (2n - \tfrac{1}{2})\pi + i \log (4n-1)\pi,$$

$$2P_n \to (2n + \tfrac{1}{2})\pi + i \log (4n+1)\pi,$$

$$\sin S_n t = \frac{i}{2}[(4n-1)\pi]^{t/2}\, e^{-i(n-1/4)\pi t} + O(n^{-t/2}), \qquad 0<t<1,$$

and

$$\sin P_n t = \tfrac{1}{2}[(4n+1)\pi]^{t/2}\, e^{-i(n+1/4)\pi t} + O(n^{-t/2}), \qquad 0<t<1.$$

Hence when n is large

$$\begin{gathered} S_n = O(n), \quad P_n = O(n), \quad k_n = O(n^2), \\ \boldsymbol{\phi}^{(n)} = O(n^{(3+|t|)/2}, \qquad \boldsymbol{\psi}^{(n)} = O(n^{(3+|t|)/2}), \end{gathered} \tag{19}$$

and assuming bounded variation of $f(t)$ and $g(t)$,

$$c_n = O(n^{-1})$$

and

$$c_n \boldsymbol{\phi}^{(n)} = O(n).$$

Smith argued that the divergence of (16) need not necessarily affect the practical value of the solution since (8) converges rapidly for any $y>0$, however small (see Smith (1952, p. 231)). For uniform convergence, stronger conditions are required. Smith proved convergence when

$$f(\pm 1) = f'(\pm 1) = g(\pm 1) = g'(\pm 1) = 0 \tag{20}$$

and f'' and g'' are of bounded variation. He notes that the details of his calculation make it unlikely that the conditions (20) can be much relaxed (p. 237).

The conditions (20) are not satisfied by the data which is given in most applications. Fortunately Smith's conjecture is incorrect and the conditions (20) can be considerably relaxed.

THEOREM 1. *Suppose $g(t)$ satisfying (2) and $f(t)$ are twice continuously differentiable and four-times piecewise differentiable with a finite number of bounded jumps when $-1<t<1$. Suppose further that*

$$f(\pm 1)=f'(\pm 1)=0 \tag{21}$$

with no further restrictions on $g(t)$. Then

$$c_n=\frac{-1}{k_nS_n^2}\int_{-1}^{1}[\psi_2^{(n)}g''(t)+\psi_1^{(n)}f''(t)]\,dt. \tag{22}$$

When n is large,

$$c_n=O(1/n^4) \tag{23}$$

and for each t, $-1\leqq t\leqq 1$, the series (16) may be majorized a convergent numerical series

$$c\sum_{n=1} 1/n^{(5-|t|)/2}, \qquad -1\leqq t\leqq 1,$$

where c is a constant independent of n.

Remark 1. The crucial hypothesis is (21). This condition will always hold if (1b) holds identically on the half-closed interval $-\infty<y\leqq 0$. Therefore (21) may be considered normal for nice problems.

Remark 2. The series (8) converges very rapidly even when $y=0$. When $y=0$ and n is large, a typical term of (8) is of

$$O(1/n^{(9-|t|)/2})$$

Proof. To establish (22) use (14a) and (14b) to introduce second derivatives of $\psi_2^{(n)}(t)$ and $\psi_1^{(n)}(t)$ into (18). Then integrate by parts using (11b) and (21).

Comparing the formula for $\psi_1^{(n)}(t)$ under (14b) with $\phi_1^{(n)}(t)=\psi_2^{(n)}(t)$, we note that (22) may be written as

$$c_n=-\frac{1}{k_nS_n^2}\int_{-1}^{1}\{(g+f)''\phi_1^{(n)}(t)-2\cos S_n\cos S_ntf''\}\,dt. \tag{24}$$

To complete the proof of (23) we need that the integral in (24) if of $O(1)$ when n is large. The largest term in the integrand is $\phi_1^{(n)}(t)=O(n^{(3+|t|)/2})$. We now compute

$$\begin{aligned} I&=\int_{-1}^{1}(g+f)''\phi_1^{(n)}(t)\,dt\\ &=S_n\left[\sin S_n\int_{-1}^{1}(g+f)''\cos S_nt\,dt\right.\\ &\qquad\left.-\cos S_n\int_{-1}^{1}t(g+f)''\sin S_nt\,dt\right] \qquad (25)\\ &=2(g+f)''-\sin S_n\int_{-1}^{1}(g+f)'''\sin S_nt\,dt\\ &\qquad-\cos S_n\int_{-1}^{1}(t(g+f)'')'\cos S_nt, \end{aligned}$$

where we have used the fact that f and g are even functions of t. A similar reduction holds when f and g are odd. One more integration by parts of the last term in (24) and the last two terms in (25) shows that

$$c_n = \frac{1}{k_n S_n^2} O(1) \rightarrow O(1/n^4)$$

establishing (23). The last estimate of the convergence theorem follows easily from (23) and (14).

Theorem 1 establishes conditions for convergence of the right side of (9) but not necessarily to the left side. It is apparent from (9) that to have convergence at the endpoints $t = \pm 1$ it is necessary that $f(t)$ and $f'(t)$ vanish at the endpoints because $\phi_1^{(n)}$ and $\phi_1'^{(n)}$ vanish at the endpoints. The series (9) will converge, however, even at the endpoints where it cannot converge to $f'(t)$ when the following conditions hold.

THEOREM 2. *Suppose $g(t)$ satisfying (2) and $f(t)$ are continuously differentiable and three-times piecewise differentiable with a finite number of bounded jumps when $-1 < t < 1$. Suppose further that*

$$f(\pm 1) = 0 \tag{26}$$

with no further restrictions on $f(t)$ or $g(t)$. Then

$$c_n = \frac{1}{k_n s_n^2} \int_{-1}^{1} (\psi_2'^{(n)} g'(t) + \psi_1'^{(n)} f'(t))\, dt. \tag{27}$$

When n is large

$$c_n = O(1/n^3), \tag{28}$$

and for each t, $-1 < t < 1$, the series (16) may be majorized by a convergent numerical series

$$C \sum_{n=1} 1/n^{(3-|t|)/2}, \tag{29}$$

where C is a constant independent of n.

Proof. Equation (27) is established in the same way as (22) except that the integration by parts is carried out once instead of twice. Comparing the formula for $\psi_1^{(n)}(t)$ under (14b) with $\phi_1^{(n)}(t) = \psi_2^{(n)}(t)$, we note that (27) may be written as

$$C_n = \frac{1}{k_n S_n^2} \int_{-1}^{1} ((g+f)'\phi_1'^{(n)} + 2f'S_n \cos S_n \sin S_n t)\, dt. \tag{30}$$

To complete the proof of (28) we need that the integral in (30) is of $O(n)$ when n is large. The largest term in the integrand is $\phi_1'^{(n)}(t) = O(n^{(5+|t|)/2})$. We now compute

$$I = \int_{-1}^{1} (g+f)'\phi_1'^{(n)}(t)\, dt = -S_n^2 \sin S_n \int_{-1}^{1} (g+f)' \sin S_n t\, dt$$
$$- S_n^2 \cos S_n \int_{-1}^{1} (g+f)' t \cos S_n t - S_n \cos S_n \int_{-1}^{1} (g+f)' \sin S_n t\, dt$$

and find, after integrating once by parts, that the term of highest order $2S_n \sin S_n \cos S_n (g+f)'|_{t=1}$ subtracts out of the resulting expression, leaving

$$
\begin{aligned}
-I = {} & S_n \sin S_n \int_{-1}^{1} (g+f)'' \cos S_n t \, dt \\
& + S_n \cos S_n \cdot \int_{-1}^{1} (t(g+f)')' \sin S_n t \, dt \\
& + \cos S_n \int_{-1}^{1} (g+f)'' \cos S_n t \, dt + 2 \cos^2 S_n (g'+f')_{t=1}.
\end{aligned} \tag{31}
$$

One more integration by parts of the last term in (30) and of the integrals in (30) and (31) shows that $I = O(n)$ when n is large and

$$C_n = \frac{1}{k_n S_n^2} O(S_n) \rightarrow O(1/n^3).$$

The rest of the proof of Theorem 2 follows along the lines of the proof of Theorem 1.

Theorems 1 and 2 specify conditions which guarantee convergence of the series on the right of (9). These theorems do not guarantee that the series converge to the prescribed edge data. In the absence of theorems of completeness it is necessary to verify (as in [8], [10], [11] and [14]) that the functions to which the series converge are the prescribed ones.

In considering the completeness of the eigenfunction expansions (9) it is necessary to stress the vectorial character of the expansions. A completeness theorem for the expansion must accomodate data in the form

$$\begin{pmatrix} f(t) \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ g(t) \end{pmatrix} \tag{32a, b}$$

In fact, the solution ψ corresponding to (9) is a superposition of solutions ψ belonging to the separate data (32a) and (32b). Recent work of D. Joseph and L. Sturges (Part II, to appear) indicates that when both conditions $f(\pm 1) = 0$ and $f'(\pm 1)$ are dropped $C_n = O(1/n^2)$ and the convergence of (9) is conditional. Their results suggest that arbitrary even data of the form (32a) may be expanded as in (9) and that the corresponding biharmonic ψ given by (8) satisfies the side-wall boundary conditions (1b) when $y < 0$. Identical considerations apply to arbitrary data of the form (32a).

We close with brief discussion of the role of the compatibility conditions (2) in theory of biorthogonal eigenfunction expansions. Two points merit attention:

(i) The conditions (2) do not enter into the proof of convergence.

(ii) The vector

$$\boldsymbol{\phi}^{(0)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{33}$$

belonging to the eigenvalue $S = 0$ is an eigenvector of (10a, b) and

$$\boldsymbol{\psi}^{(0)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad k_0 = -2.$$

Similarly

$$\hat{\boldsymbol{\phi}}^{(0)} = \begin{pmatrix} 0 \\ t \end{pmatrix}, \tag{34}$$

is an odd eigenvector of the same problem with eigenvalue $P = 0$ and

$$\hat{\boldsymbol{\psi}}^{(0)} = \begin{pmatrix} t \\ 0 \end{pmatrix}, \qquad \hat{k}_0 = -\tfrac{2}{3}.$$

Smith excluded eigenfunctions with zero eigenvalues because they are not compatible with (2). If (2) is suspended, then for even data,

$$\begin{pmatrix} 0 \\ g(t) \end{pmatrix} = C_0 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \sum_{-\infty}^{\infty} C_n \begin{pmatrix} \phi_1^{(n)} \\ \phi_2^{(n)} \end{pmatrix}, \tag{35}$$

where, using (18), (33) and (1c), we find that

$$C_0 = \frac{1}{k_0} \int_{-1}^{1} -g(t)\, dt = \frac{1}{2} \int_{-1}^{1} \frac{\partial^2 \psi}{\partial t^2}\, dt = \psi'(1)$$

and

$$\psi = \tfrac{1}{2}\psi'(1)(t^2 - 1) + \sum_{-\infty}^{\infty} C_n \exp(S_n y) \phi_1^{(n)}(t)/S_n^2.$$

For odd data

$$\begin{pmatrix} 0 \\ g(t) \end{pmatrix} = C_0 \begin{pmatrix} 0 \\ t \end{pmatrix} + \sum_{-\infty}^{\infty} C_n \begin{pmatrix} \hat{\phi}_1^{(n)} \\ \hat{\phi}_2^{(n)} \end{pmatrix}, \tag{36}$$

where, using the analogues for odd functions of (18), (33) and (1c), we find that

$$C_0 = \frac{1}{k_0} \int_{-1}^{1} -tg(t)\, dt = \frac{3}{2} \int_{-1}^{1} t \frac{\partial^2 \psi}{\partial t^2}\, dt = -3\psi(1)$$

and

$$\psi = \tfrac{1}{2}\psi(1)(3t - t^3) + \sum_{-\infty}^{\infty} C_n \exp(P_n y) \hat{\phi}_1^{(r)}(t)/P_n^2.$$

When (2) holds it is possible to formally expand arbitrary edge data in biharmonic eigenvectors which satisfy the side-wall boundary conditions. When (2) is suspended, it is necessary to supplement these eigenvectors with a y-independent eigenvector belonging to the eigenvalue zero which satisfies only one of two side-wall boundary conditions.

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