PERTURBATIONS OF THE REST STATE OF A SIMPLE FLUID: THE WEISSENBERG EFFECT INDUCED BY TORSIONAL OSCILLATION OF A ROD

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ABSTRACT

A rod of small diameter (2a) is partially immersed in a vat of simple fluid. The rod is set into torsional oscillation with an angular frequency Ω equal to $\varepsilon \sin \omega t$. An analysis of this problem based on a newly developed theory of perturbations of the rest state with unsteady motion (1) is constructed through terms of $O(\varepsilon^2)$. The analysis predicts a mean climb plus an oscillating component with an oscillation frequency 2ω. Preliminary experiments carried out jointly with G. S. Beavers seem to verify the predictions of analysis. The mean climb dominates the whole motion and the 2ω oscillation is barely visible.

INTRODUCTION

One of the best known and most striking effects in the flow of viscoelastic fluids is the Weissenberg effect. When a rod of small diameter is partially immersed in a viscoelastic fluid and rotated with a steady angular frequency, the fluid will climb up to the rod. This phenomenon has only recently come to be understood by analysis (2, 3, 4). The analysis is based on the notion of domain perturbations and it leads rationally to approximations for the stress in a simple fluid by expressions for the stress in fluids of grade N (5). The analysis predicts properties of the Weissenberg effect which were previously unknown; for example, the fluid will climb up a rod of small diameter but sink near a rod of large diameter. Measurements of the climb are in good enough agreement with experiments (3, 4) to use the rotating rod as a viscometer for measuring a characterizing constant for the second order fluid and the dependence of this constant on the temperature.

The aim of this communication is to show how the ideas used in (2) and (5) can be generalized to the problem in which the state of rest is perturbed with a time-periodic motion rather than with a steady motion. To solve this problem it was necessary to derive the canonical forms of the stress tensor for fluids of the integral type. A complete exposition of this new theory can be found in (1). Here we shall summarize some of the main results of the application of the new theory to the problem of climbing on a rod undergoing torsional oscillations. Attention is directed both to the theory and to the preliminary results of experiments carried out in collaboration with G. S. Beavers. A fuller account of these matters will be given in a forthcoming publication (8).

MATHEMATICAL FORMULATION

The governing boundary value problem is completely stated in *Fig. 1*. The expression

$$\mathcal{S}[G(s)] = \int_0^\infty G(s)\, F_t(t-s)\, A_1(s)\, F_t(t-s)\,ds + \int_0^\infty\int_0^\infty \gamma(s_1,s_2)[F_t^T(t-s_1)\, A_1(s_1)\, F_t(t-s_1)] \cdot [F_t^T(t-s_2)\, A_1(s_2)\, F_t(t-s_2)]ds_1 ds_2 \quad (1)$$

is sufficiently general for perturbations of the rest state through terms of order ε^2. The symbols used in eq. (1) follow conventions established (6): G(s) is the shear relaxation modulus, $G(s)$ is the Cauchy tensor minus the unit tensor, $F_t(\tau)$ is the relative deformation, $A_1(s)$ is the first Rivlin-Ericksen tensor (twice the rate of strain) and $\gamma(s_1, s_2) = \gamma(s_2, s_1)$ is a material function. When the analysis is carried to third order it is necessary to replace eq. (1) with a stress tensor appropriate to third order fluids of integral type. Redundant terms of order higher than 2 must be purged from eq. (1).

THE METHOD OF SOLUTION

We imagine that the rod is and always was undergoing torsional oscillations. Then the solution of the problem in *Fig. 1* is a function of ε. We differentiate the equations once with respect to ε at $\varepsilon=0$ and find that

$$\rho\frac{\partial U^{<1>}}{\partial t} = \int_0^\infty G(s)\,\nabla^2 U^{<1>}(x,t-s), \qquad \nabla\cdot U^{<1>} = 0 \quad (2)$$

where $U^{<1>} = \delta U(x,t,\varepsilon)/\delta\varepsilon$ evaluated at $\varepsilon=0$ and, at the rod surface $r=a$, $U^{<1>} = e_\theta\, a \sin \omega t$. An exact approximate solution of eq. (2) may be given in terms of Bessel functions. A good approximate solution, using ideas introduced in (3), may be obtained in the form

$$U^{<1>} = a(a/r)^{\Lambda_r} \sin[\omega t + \Lambda_i \log(a/r)] \quad (3)$$

where

$$\Lambda_r = re\sqrt{1+a^2\Lambda^2}$$

$$\Lambda_i = im\sqrt{1+a^2\Lambda^2}$$

and

$$\Lambda^2 = i\rho\omega/\int_0^\infty G(s)\exp(-i\omega s)ds.$$

At second order: $U^{<2>}(a,z,t) = 0$, $\operatorname{div} U^{<2>} = 0$

$$\rho\frac{\partial U^{<2>}}{\partial t} + \nabla\Phi^{<2>} - \nabla^2\int_0 G(s)\, U^{<2>}(x,t-s)ds = -\rho U^{<1>}\cdot\nabla U^{<1>} + \nabla\cdot\int_0^\infty G(s)[X_t^{<1>}\cdot\nabla A(s) + (A(s)\nabla X_t^{<1>} + \text{transpose})] + \nabla\cdot\int_0^\infty\int_0^\infty \gamma(s_1,s_2)\, A(s_1)\cdot A(s_2)ds_1 ds_2 \quad (4)$$

where

$$A(s) = \nabla U^{<1>} + \nabla U^{<1>T}$$

and

$$X_t^{<1>} = \frac{\partial}{\partial\varepsilon} X_t(x,t-s)\Big|_{\varepsilon=0} = \int_t^{t-s} U^{<1>}(x,t-s')ds'.$$

$X_t(x,t-s)$ is the relative position vector of the particle which is presently at $x = X_t(x,t)$ and $(\cdot)^{<2>} \equiv 1/2(\delta^2(\cdot)/\delta\varepsilon^2)_{\varepsilon=0}$. The free surface conditions at 2nd order are as follows: The vanishing of the shear stress becomes

$$\int_0^\infty G(s)\left[\frac{\partial U^{<2>}}{\partial z}(r,z,t-s) + \frac{\partial W^{<2>}}{\partial r}(r,z,t-s)\right]ds = 0.$$

The kinematic condition becomes

$$W^{<2>} = \partial h^{<2>}/\partial t\,.$$

The jump condition for the normal stress becomes

$$-\phi^{<2>} + 2\int G(s)\frac{\partial W^{<2>}}{\partial z}(r,z,t-s)ds + \rho g h^{<2>} =$$

$$= \frac{\sigma}{r}\frac{\partial}{\partial r}\left(r\frac{\partial h}{\partial r}\right). \tag{5}$$

The problem at 2nd order splits into a mean part and to a part which oscillates with a frequency of 2ω. The problem of the mean climb may be completely solved. There is no mean motion. The second derivative of the mean head is found by integration of eq. (4) and the mean height rise at 2nd order is found by integrating eq. (5) under the conditions that the average of $h^{<2>}$ over an oscillation period has a horizontal angle of contact at $r = a$ and tends to zero as $r \to \infty$.
The mean height rise at the wall is then obtained in the form

$$\overline{h^{<2>}}(a,z,\varepsilon^2) = \frac{a\varepsilon^2}{\sqrt{\rho g\sigma}} \cdot \frac{|\Lambda+1|^2\hat{\beta}_\Lambda}{(\Lambda_r+1)[2(\Lambda_r+1)+\lambda]} - \frac{\rho a^2}{2\Lambda_r(2\Lambda_r+\lambda)} \tag{6}$$

where

$$\hat{\beta}_\Lambda = -(2\Lambda_r+1)\int_0^\infty G(s)\frac{\sin\omega s}{\omega}ds$$

$$+ (\Lambda_r+1)\int_0^\infty \gamma(s_1,s_2)\cos\omega(s_1-s_2)ds_1 ds_2\,. \tag{7}$$

CONCLUSIONS AND PRELIMINARY EXPERIMENTS

If $G(s)$ and $\gamma(s_1,s_2)$ are assumed to be in exponential form, the amplitude and time constants may be characterized as follows (7):

$$G(s) = -\frac{\mu^2}{\alpha_1}\exp[(\mu/\alpha_1)s] \tag{8}$$

where μ is the shear viscosity,

$$\gamma(s_1,s_2) = \alpha_2 K^2 \exp[-K(s_1+s_2)] \tag{9}$$

and μ, α_1 and α_2 are the constants of the fluid of second grade and may be presumed to be known from experiments. Only K is unknown. Assuming the validity of eqs. (8) and (9), *any* simple fluid of integral type is completely characterized in *any* motion when the constants (μ, α_1, α_2, K) are known. We may determine K from experiments on torsional oscillations of the rod. Using eqs. (8)
and (9) we find that

$$a^2\Lambda^2 = [i\mathcal{R}_\omega + \tilde{\alpha}\mathcal{R}_\omega^2] \tag{10}$$

and

$$\hat{\beta}_\Lambda = \frac{(2\Lambda_r+1)\alpha_1}{1+(\tilde{\alpha}\mathcal{R}_\omega)^2} + \frac{(\Lambda_r+1)\alpha_2}{1+(\omega/K)^2} \tag{11}$$

where

$$\mathcal{R}_\omega = a^2\omega/\nu\,, \qquad \nu = \mu/\rho.$$

Preliminary experiments of G. S. Beavers appear to be in good agreement with predictions arising from eq. (6). In the experiments, the mean climb dominates the whole motion and the 2ω oscillation is barely visible.

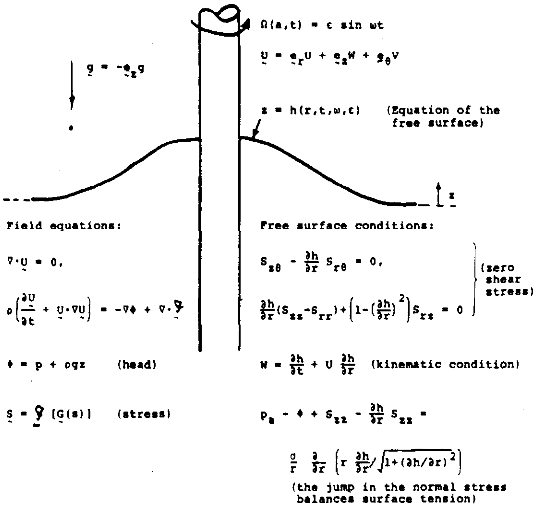

Fig. 1. Governing equations for unsteady rod climbing. The constants $[\rho, g, p_a, \sigma]$ = *[density, gravity, atmospheric pressure, surface tension].*

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