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The Free Surface on a Simple Fluid Between Cylinders Undergoing Torsional Oscillations. Part III: Oscillating Planes

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1. Introduction

In Part I (JOSEPH, 1976A) of this paper in three parts, a recently developed algorithm (JOSEPH, 1976B) for computing the motions of a simple fluid of integral type which perturb the state of rest was applied to the problem of finding the shape of the free surface on a simple fluid between cylinders undergoing torsional oscillations. The analysis of this problem was carried out through terms of second order in the amplitude ε. When the cylinder radii are arbitrary, the solutions may be expressed in terms of Bessel functions but the resulting expressions are cumbersome. Simpler solutions are possible when the cylinders are infinitely far apart and when they are close together. In the first case, we are considering the change of elevation of the free surface on a sea of fluid around a rod undergoing torsional oscillations. In the second case, we are considering the change in elevation of the free surface on a fluid between oscillating parallel planes which oscillate with a velocity proportional to $\varepsilon \sin \omega t$. The first case was studied in Part I and the second case, here in Part III. The predictions following from the analysis of Part I were tested in the experiments reported in Part II (BEAVERS, 1976). The analysis of Part I could be criticized because the main formulas for the mean rise of height were based on an approximation, however good, and though a unique solution giving the secondary motion exists, it was not given. In the second limiting case, oscillating parallel planes, approximations are unnecessary and exact expressions are obtained for the mean rise in height, the oscillatory secondary flow and the oscillating change in the elevation of the free surface.

Four results of analysis appear to us to deserve early mention:

(i) The predictions of the response of all simple fluids of integral type in small-amplitude motions which perturb the rest state are completely determined when two material functions $G(s)$ (the shear relaxation modulus) and $\gamma(s_1, s_2) = \gamma(s_2, s_1)$ (the quadratic relaxation modulus) are given. Various methods of determining $G(s)$ and $\gamma(s_1, s_2)$ from experiments can be evaluated using this fact. For example, the same $G(s)$ and $\gamma(s_1, s_2)$ should be determined by comparing theory and experiment for oscillating rods (Part I) and oscillating planes (Part III).

(ii) The mean rise in height of the fluid between oscillating planes is proportional to an unsteady equivalent $\hat{N}_2(\omega)$ of the second normal stress. As $\omega \to 0$, $\hat{N}_2(\omega) = 2\alpha_1 + \alpha_2$ where $2\alpha_1 + \alpha_2$ is the limiting value of the ratio of the second normal stress upon shear rate squared. Observations of the elevation changes on the oscillating free surfaces give rheological data which includes information about the hard-to-measure second normal stress.

(iii) The mean rise between parallel planes is interesting because *no rise occurs* when planes are in steady motion and the shear rate is constant and uniform over the whole field of flow. In contrast, there is an appreciable steady rise on a rod rotating with even a small, steady, angular velocity. This contrast actually extends to the oscillating planes because the rise between planes is an order of magnitude smaller than the corresponding rise on the oscillating rod (compare H given by Fig. 8 of Part I and Fig. 6 of Part III). The big mean rise on the rod dominates the whole rise, but between oscillating planes the mean rise is much smaller and steady and unsteady changes in elevation can be equally important.

(iv) The solution of the fourth-order partial differential equation which governs the motion at second order is of a type which arises in the vibrations of elastic plates. We believe the nice eigenfunction solution which we construct is new and of potentially wide application.

Before proceeding with the analysis, we list some of the symbols used:

$(x, y, z), (\boldsymbol{e}_x, \boldsymbol{e}_y, \boldsymbol{e}_z)$	Rectangular Cartesian coordinates, coordinate base vectors.
$\boldsymbol{U}, (U, V, W)$	Velocity, components of velocity.
$\omega, T, \rho, \mu, p_a, \boldsymbol{g} = -\boldsymbol{e}_z\, g, \zeta$	Frequency, surface tension, density, viscosity, atmospheric pressure, gravity, and reciprocal of the capillary radius.
$V(d, y, t, \varepsilon) = \varepsilon \sin \omega t$	Definition of ε.
ΔY	Maximum displacement of the oscillating wall at $x = d$ (3.1).
$G(s), \gamma(s_1, s_2), \eta(\omega), \Lambda^2(\omega)$	Shear relaxation modulus, quadratic relaxation modulus, "complex viscosity" (4.4), (4.3).
$F(X, \omega), M(\omega)$	Functions defined by (5.2f).
$\hat{N}_2(\omega), \mathcal{N}(\omega)$	Second normal stress function for time periodic flows (6.2), (6.5).
$\left.\begin{array}{l}\Gamma^2 = d^2 \Lambda^2(2\omega),\\ \gamma^2 = d^2 \Lambda^2(\omega)\end{array}\right\}$	Non-dimensional functions (Section 8).
S_n, μ_n, P_n, ν_n	Eigenvalues (8.4), (8.5).

$\phi_1^{(n)}, \phi_2^{(n)}, \psi_1^{(n)}, \psi_2^{(n)}$ $\hat{\phi}_1^{(n)}, \hat{\phi}_2^{(n)}, \hat{\psi}_1^{(n)}, \hat{\psi}_2^{(n)}$	Eigenfunctions (8.3), (8.12), (8.13), (8.14), (8.3), (8.16), (8.17), (8.18).
$\boldsymbol{\phi}^{(n)}, \boldsymbol{\psi}^{(n)}, \boldsymbol{T}(S_n, \Gamma)$	(8.10).
$k_n, \hat{k}_n$	(8.19), (8.20).
$H(x)=\frac{\bar{h}(x)}{(\Delta Y)^2}$	Normalized mean rise in height (6.7).
$\mathscr{H}=\frac{h(x,t,\varepsilon)-\bar{h}(x,\varepsilon)}{(\Delta Y)^2}$	Normalized oscillatory part of the rise in elevation of the free surface (8.26).

2. Mathematical Formulation

The free surface problem studied in this paper is sketched in Fig. 1. The region occupied by the incompressible simple fluid is designated as

$$\mathscr{V}_\varepsilon=[x, y, z: \ -d<x<d, z<h(x,t,\varepsilon), -\infty<y<\infty].$$

Equations governing the motion of the fluid in $\mathscr{V}_\varepsilon$ are

$$\operatorname{div} \boldsymbol{U}=0 \tag{2.1 a}$$

and

$$\rho\left[\frac{\partial \boldsymbol{U}}{\partial t}+\boldsymbol{U}\cdot\nabla\boldsymbol{U}\right]=\nabla\Phi+\nabla\cdot\boldsymbol{S} \tag{2.1 b}$$

where ρ is density, $\boldsymbol{U}$ is velocity, $\Phi=p(x,z)-p_a+\rho g z$ is the head, p_a is atmospheric pressure, p is the constitutively indeterminate part of the reaction pressure, g is

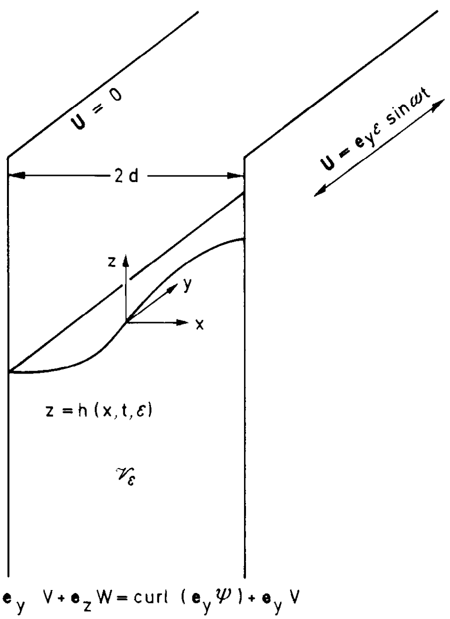

Fig. 1. Notation and definition of symbols used in the study of the free surface on a simple fluid between oscillating planes.

the acceleration of gravity and $\boldsymbol{S}=\boldsymbol{T}+p\mathbf{1}$ where $\boldsymbol{T}$ is the stress tensor. The fluid velocity is prescribed at solid boundaries:

$$\boldsymbol{U}=0 \quad \text{at } x=-d, \tag{2.1 c}$$

$$\boldsymbol{U}=\boldsymbol{e}_y\,\varepsilon\sin\omega t. \tag{2.1 d}$$

The following conditions hold on the free surface $z=h(x,t,\varepsilon)$:

a) The free surface is a material surface

$$w=h_{,t}+u h_{,x}. \tag{2.1 e}$$

b) The free surface is free of shear stresses

$$S_{zy}-h_{,x}S_{xy}=0, \tag{2.1 f}$$

$$(S_{zz}-S_{xx})h_{,x}+(1-h_{,x}^2)S_{xz}=0. \tag{2.1 g}$$

c) The jump in the normal stress is balanced by surface tension

$$S_{zz}-h_{,x}S_{zx}-\Phi=T\left(\frac{h_{,x}}{\sqrt{1+h_{,x}^2}}\right)_{,x}-\rho g h. \tag{2.1 h}$$

Equations (2.1 f, g, h) assume that the viscous tractions of air on liquid are negligible.

In addition, we require that the total volume of liquid be fixed and independent of ε

$$\frac{1}{2d}\int_{-d}^{d} h(x,t,\varepsilon)\,dx=0. \tag{2.1 k}$$

The solution is bounded and independent of z as $z\to-\infty$.

To complete our formulation of governing equations, it is necessary to specify some conditions on the line of contact $h(\pm d,t,\varepsilon)$. To specify these conditions we note the solution which we construct may be decomposed into a mean part and a periodic part. The mean rise in height is defined as

$$\bar{h}(x,\varepsilon)=\frac{\omega}{2\pi}\int_0^{2\pi/\omega} h(x,t,\varepsilon)\,dt. \tag{2.2}$$

We require that the mean angle of contact between the liquid and solid walls should vanish,

$$\bar{h}_{,x}(\pm d,\varepsilon)=0. \tag{2.3}$$

The periodic part of the solution oscillates around the mean line of contact $\bar{h}(\pm d,\varepsilon)$. It follows that

$$h(\pm d,t,\varepsilon)-\bar{h}(\pm d,\varepsilon)=0. \tag{2.4}$$

3. Perturbation of the Rest State

The solution of the problem (2) can be constructed in a series of powers of the amplitude ε. The amplitude ε is related to given and easily measured data by the formula

$$\varepsilon = \frac{\omega \Delta Y}{2} \tag{3.1}$$

where

$$\Delta Y = Y_m - Y(0)$$

is the maximum displacement of the oscillating wall at $x=d$. Equation (3.1) follows from the equation

$$\frac{dY}{dt} = \varepsilon \sin \omega t$$

governing the motion of the oscillating wall.

The solution is expanded into powers of ε; the y component V of velocity and shear stresses S_{xy} and S_{zy} are odd functions of ε; and Φ, U, W, the other components of $\boldsymbol{S}$ and $h(x, t, \varepsilon)$ are even functions of ε. When $\varepsilon=0$, the fluid is at rest and the free surface is flat $h(x, t, 0) \equiv 0$ in $\mathscr{V}_0$. The formation of the boundary value problems which govern the coefficients in the series expansion of the solution was described in Part I and, in more detail, in JOSEPH's book (1976) on stability. The principal problems in the theory of perturbation of the rest state with arbitrary motions concern the monitoring of the history of the motion and the reduction of the stress tensor to canonical form. A brief summary of the algorithms for computing histories sequentially by perturbations is given below. The reduced form of the stress tensors will be assumed to be known. Detailed derivations of algorithms for treating both problems can be found in the two works cited above.

The history of the motion is monitored through the history of the relative position vector

$$\boldsymbol{\chi}_t(\boldsymbol{x}, \tau, \varepsilon) = \boldsymbol{x} + \sum_{n=1} \boldsymbol{\chi}^{\langle n \rangle}(\boldsymbol{x}, \tau)\varepsilon^n \tag{3.2}$$

where

$$\boldsymbol{\chi}^{\langle n \rangle}(\boldsymbol{x}, \tau) = 0.$$

The relative position vector gives the position at time $\tau < t$ of the particle $\boldsymbol{X}$ which is presently at $\boldsymbol{x}$. The particle labels are assigned in the rest state $\varepsilon = 0$. Hence,

$$\boldsymbol{\chi}_t(\boldsymbol{x}, \tau, \varepsilon) = \boldsymbol{\xi}\left(\boldsymbol{X}(\boldsymbol{x}, t, \varepsilon), \tau, \varepsilon\right)$$

where $\boldsymbol{\xi}(\boldsymbol{X}, \tau, \varepsilon)$ is the position of the particle which is at $\boldsymbol{X} = \boldsymbol{\xi}(\boldsymbol{X}, \tau, 0)$ when $\varepsilon = 0$. Of course, $\boldsymbol{\xi}(\boldsymbol{X}, \tau, 0)$ is independent of τ. The function $\boldsymbol{X}(\boldsymbol{x}, t, \varepsilon)$ is obtained by solving the equation $\boldsymbol{x} = \boldsymbol{\xi}(\boldsymbol{X}, t, \varepsilon)$ for $\boldsymbol{X}$. In the expansion (3.2), $\boldsymbol{x}$ and ε are considered independent. Then $\boldsymbol{X} = \boldsymbol{X}(\varepsilon)$. In free surface problems, the domain $\mathscr{V}_\varepsilon$ depends on ε. It is useful to map $\mathscr{V}_\varepsilon(\boldsymbol{x})$ into the configuration $\mathscr{V}_0(\boldsymbol{X})$ of the rest state. The material mapping $\boldsymbol{x} = \boldsymbol{\xi}(\boldsymbol{X}, t, \varepsilon)$ may also be regarded as an invertible domain mapping. When $\boldsymbol{X}$ and ε are independent, $\boldsymbol{x} = \boldsymbol{\xi}(\boldsymbol{X}, t, \varepsilon)$ depends on ε. The history of the relative position vector may be expressed in the rest coordinates by forming the expansion in powers of ε of the functions $\boldsymbol{\chi}^{\langle n \rangle}\left(\boldsymbol{\xi}(\boldsymbol{X}, t, \varepsilon), \tau\right)$ which appear in the last term of (3.2). This leads to

$$\begin{aligned}\boldsymbol{\chi}_t\left(\boldsymbol{\xi}(\boldsymbol{X}, t, \varepsilon), \tau, \varepsilon\right) - \boldsymbol{x} = & \; ((\boldsymbol{\xi}^{[1]}))\,\varepsilon \\ & + \left(((\boldsymbol{\xi}^{[2]})) - \boldsymbol{\xi}^{[1]}(\boldsymbol{X}, t) \cdot \nabla_{\boldsymbol{X}} \boldsymbol{\chi}^{\langle 1 \rangle}\right)\varepsilon^2 + \cdots\end{aligned}$$

where

$$((\xi^{[n]})) = \xi^{[n]}(X, \tau) - \xi^{[n]}(X, t)$$

and

$$\xi^{[n]} = \frac{1}{n!} \frac{\partial^n \xi}{\partial \varepsilon^n}(X, \tau, \varepsilon)\bigg|_{\varepsilon=0} \quad \text{holding } X \text{ fixed}.$$

In the computational algorithm for perturbations of the rest state, we first compute $U^{\langle 1 \rangle}(X, \tau)$ as the solution of a well-posed boundary value problem. We may then compute

$$((\xi^{[1]})) = \int_t^\tau U^{\langle 1 \rangle}(X, \tau')\, d\tau'.$$

Given $U^{\langle 1 \rangle}$ and $((\xi^{[1]}))$, it is possible to solve another boundary value problem for $U^{\langle 2 \rangle}(X, \tau)$. Then we can calculate

$$((\xi^{[2]})) = \int_t^\tau [U^{\langle 2 \rangle}(X, \tau') + \xi^{[1]}(X, \tau') \cdot \nabla_X U^{\langle 1 \rangle}(X, \tau')]\, d\tau'.$$

In this way, we may generate the functions $U^{\langle n \rangle}$ and $((\xi^{[n]}))$ sequentially as in other well-behaved perturbation theories.

In the work to follow we shall compute the perturbation through order two. The perturbation problems are all posed in the domain $\mathscr{V}_0(X)$ of the rest state. We shall find that

$$V(x, z, t, \varepsilon) = V^{\langle 1 \rangle}(X, t)\varepsilon + O(\varepsilon^3),$$

$$\Psi(x, z, t, \varepsilon) = \Psi^{\langle 2 \rangle}(X, Z, t)\varepsilon^2 + O(\varepsilon^4),$$

$$\Phi(x, z, t, \varepsilon) = \Phi^{\langle 2 \rangle}(X, Z, t)\varepsilon^2 + O(\varepsilon^4),$$

and

$$h(x, t, \varepsilon) = h^{\langle 2 \rangle}(X, t)\varepsilon^2 + O(\varepsilon^4).$$

The functions on the left are defined in $\mathscr{V}_\varepsilon(x)$, the functions on the right in $\mathscr{V}_0(X)$ and X and x are related through the mapping $x = \xi(X, t, \varepsilon)$. We shall find (see (5.3)) that $\Psi^{\langle 2 \rangle}$, $\Phi^{\langle 2 \rangle}$ and $h^{\langle 2 \rangle}$ may be split into a mean part which is independent of Z, and a time-periodic part. We show that there is no mean motion and we compute the mean rise and the time-periodic motion and change in elevation.

4. First Order

Following the derivation of the first-order problems given in Section 5 of Part I, we find that

$$\rho \frac{\partial V^{\langle 1 \rangle}}{\partial t} = \nabla_X^2 \int_0^\infty G(s)\, V^{\langle 1 \rangle}(X, t-s)\, ds, \tag{4.1 a}$$

$$V^{\langle 1 \rangle}(-d, t) = 0, \tag{4.1 b}$$

and

$$V^{\langle 1 \rangle}(d, t) = \sin \omega t. \tag{4.1 c}$$

The solution of (4.1 a, b, c) is given by

$$V^{\langle 1 \rangle}(X, t) = v(X)e^{i\omega t} + \bar{v}(X)e^{-i\omega t} \tag{4.2}$$

where the overbar designates complex conjugation,

$$v(X)=\frac{\sinh[(X+d)\Lambda(\omega)]}{2i\sinh[2d\Lambda(\omega)]},\qquad(4.3)$$

$$\Lambda^2(\omega)=i\rho\,\omega/\eta(\omega)$$

and

$$\eta(\omega)=\int_0^\infty G(s)e^{-i\omega s}ds\qquad(4.4)$$

where $G(s)$ is the shear-relaxation modulus and $\eta(\omega)$ is the "complex viscosity". The solution given by (4.2), (4.3) and (4.4) may be regarded as a superposition of waves; with

$$\Lambda(\omega)=\Lambda_r(\omega)+i\Lambda_i(\omega),$$

we may write (4.2) as

$$\begin{aligned}V^{\langle 1\rangle}(X,t)=\frac{1}{4|\sinh 2\Lambda d|}\{&e^{\Lambda_r(X+3d)}\sin[\omega t+\Lambda_i(X-d)]\\&+e^{-\Lambda_r(X+3d)}\sin[\omega t-\Lambda_i(X-d)]\\&+e^{-\Lambda_r(X-d)}\sin[\omega t-\Lambda_i(X+3d)]\\&+e^{\Lambda_r(X-d)}\sin[\omega t+\Lambda_i(X+3d)]\}.\end{aligned}\qquad(4.5)$$

The particle paths at first order are given by

$$((\boldsymbol{\xi}^{[1\,1]}))=\boldsymbol{e}_Y\int_t^\tau V^{\langle 1\rangle}(X,\tau')\,d\tau'=\boldsymbol{e}_Y((\mathscr{Y}^{[1\,1]})),$$

$$((\mathscr{Y}^{[1\,1]}))=(e^{i\omega(t-s)}-e^{i\omega t})\frac{v(X)}{i\omega}-(e^{-i\omega(t-s)}-e^{-i\omega t})\frac{\bar v(X)}{i\omega}.$$

The following quantities, defined on the first-order solution, appear in the inhomogeneous terms of the governing equation at second order:

$$\boldsymbol{A}(s)=\nabla_{\boldsymbol{X}}\boldsymbol{U}^{\langle 1\rangle}+(\nabla_{\boldsymbol{X}}\boldsymbol{U}^{\langle 1\rangle})^T=(\boldsymbol{e}_X\boldsymbol{e}_Y+\boldsymbol{e}_Y\boldsymbol{e}_X)V^{\langle 1\rangle}_{,X},$$

$$\boldsymbol{U}^{\langle 1\rangle}\cdot\nabla_{\boldsymbol{X}}\boldsymbol{U}^{\langle 1\rangle}=0,$$

$$\nabla_{\boldsymbol{X}}((\boldsymbol{\xi}^{[1\,1]}))=\boldsymbol{e}_Y\boldsymbol{e}_X((\mathscr{Y}^{[1\,1]}))_{,X},$$

$$((\boldsymbol{\xi}^{[1\,1]}))\cdot\nabla_{\boldsymbol{X}}\boldsymbol{A}=((\mathscr{Y}^{[1\,1]}))\boldsymbol{A}_{,Y}=0,$$

$$\boldsymbol{A}\cdot\nabla_{\boldsymbol{X}}((\boldsymbol{\xi}^{[1\,1]}))+(\boldsymbol{A}\cdot\nabla_{\boldsymbol{X}}((\boldsymbol{\xi}^{[1\,1]})))^T=2\boldsymbol{e}_X\boldsymbol{e}_X V^{\langle 1\rangle}_{,X}((\mathscr{Y}^{[1\,1]}))_{,X}$$

and

$$\boldsymbol{A}(s_1)\cdot\boldsymbol{A}(s_2)=(\boldsymbol{e}_X\boldsymbol{e}_X+\boldsymbol{e}_Y\boldsymbol{e}_Y)V^{\langle 1\rangle}_{,X}(X,t-s_1)V^{\langle 1\rangle}_{,X}(X,t-s_2).$$

5. Second Order

Following the procedures of Section 6 of Part I, we find that

$$\operatorname{div}\boldsymbol{U}^{\langle 2\rangle}=0,\qquad(5.1)$$

$$\rho U^{\langle 2\rangle}_{,t} + \nabla_X \Phi^{\langle 2\rangle} - \nabla_X^2 \int_0^\infty G(s)\, U^{\langle 2\rangle}(X, t-s)\, ds$$

$$= \nabla_X \cdot \int_0^\infty G(s) \left[((\xi^{[1]})) \cdot \nabla_X A(s) + A(s) \cdot \nabla_X((\xi^{[1]})) + (A(s) \cdot \nabla_X((\xi^{[1]})))^T\right] ds$$

$$+ \nabla_X \cdot \int_0^\infty \int_0^\infty \gamma(s_1, s_2)\, A(s_1) \cdot A(s_2)\, ds_1\, ds_2 - \rho U^{\langle 1\rangle} \cdot \nabla_X U^{\langle 1\rangle} \tag{5.2a}$$

$$= e_X \left\{ 2 \int_0^\infty G(s) \left[V^{\langle 1\rangle}_{,X}(X, t-s)\, ((\mathscr{Y}^{[1]}))_{,X}\right]_{,X} ds + \int_0^\infty \int_0^\infty \gamma(s_1, s_2)\, V^{\langle 1\rangle}_{,X}(X, t-s_1)\, V^{\langle 1\rangle}_{,X}(X, t-s_2)\, ds_1\, ds_2 \right\}$$

where $\gamma(s_1, s_2)$ is called the quadratic shear relaxation modulus. At the side walls

$$U^{\langle 2\rangle}(\pm d, Z, t) = 0 \tag{5.2b}$$

On the free surface, $Z=0$,

$$S^{\langle 2\rangle}_{XZ} = \int_0^\infty G(s) \left[U^{\langle 2\rangle}_{,Z}(X, Z, t-s) + W^{\langle 2\rangle}_{,X}(X, Z, t-s)\right] ds = 0, \tag{5.2c}$$

$$W^{\langle 2\rangle} = h^{\langle 2\rangle}_{,t}, \tag{5.2d}$$

and

$$-\Phi^{\langle 2\rangle} + 2 \int_0^\infty G(s)\, W^{\langle 1\rangle}_{,Z}(X, Z, t-s)\, ds + \rho g h^{\langle 2\rangle} = T h^{\langle 2\rangle}_{,XX}. \tag{5.2e}$$

Since $U^{\langle 2\rangle} = e_X U^{\langle 2\rangle} + e_Z W^{\langle 2\rangle}$, it is possible to eliminate (5.1) by introducing the stream function Ψ,

$$U^{\langle 2\rangle} = \operatorname{curl}(e_Y \Psi) = (-\Psi_{,Z}, 0, \Psi_{,X}) = \nabla_X \Psi \wedge e_Y,$$

satisfying

$$\rho[\operatorname{curl}(e_Y \Psi)]_{,t} - \int_0^\infty G(s)\, \nabla_X^2 \operatorname{curl}(e_Y \Psi)\, ds + \nabla_X \Phi^{\langle 2\rangle}$$

$$= e_X\, 2(|v_{,X}|^2)_{,X} \left\{ -2 \int_0^\infty G(s) \frac{\sin \omega s}{\omega}\, ds + \int_0^\infty \int_0^\infty \gamma(s_1, s_2) \cos \omega(s_1 - s_2)\, ds_1\, ds_2 \right\} \tag{5.2f}$$

$$+ e_X\left[F(X, \omega)\, e^{2i\omega t} + \bar{F}(X, \omega)\, e^{-2i\omega t}\right]$$

where

$$F(X, \omega) = M(\omega)\, (v^2_{,X})_{,X},$$

$$M(\omega) = \frac{2}{i\omega}(\eta(2\omega) - \eta(\omega)) + \int_0^\infty \int_0^\infty \gamma(s_1, s_2)\, e^{-i\omega(s_1+s_2)}\, ds_1\, ds_2$$

and

$$(v^2_{,X})_{,X} = -\frac{\Lambda^3(\omega) \sinh[2(X+d)\Lambda(\omega)]}{4 \sinh^2[2d\Lambda(\omega)]}.$$

Solutions at second order can be decomposed into a mean part and a time-periodic part:

$$\begin{aligned}\Psi^{\langle 2\rangle}&=\tfrac{1}{2}\overline{\overline{\psi}}+\tfrac{1}{4}\psi_1\,e^{2i\omega t}+\tfrac{1}{4}\overline{\psi}_1\,e^{-2i\omega t},\\ \Phi^{\langle 2\rangle}&=\tfrac{1}{2}\overline{\overline{\phi}}+\tfrac{1}{4}\phi_1\,e^{2i\omega t}+\tfrac{1}{4}\overline{\phi}_1\,e^{-2i\omega t},\\ h^{\langle 2\rangle}&=\tfrac{1}{2}\overline{\overline{\hbar}}+\tfrac{1}{4}\hbar_1\,e^{2i\omega t}+\tfrac{1}{4}\overline{\hbar}_1\,e^{-2i\omega t}\end{aligned}\tag{5.3}$$

where a single overbar denotes complex conjugation and the double overbar denotes the time average over a cycle of period $2\pi/\omega$ (see (2.2)).

6. The Mean Motion and Height Rise at Second Order

Substituting (5.3) into (5.2) we find that

$$\nabla_X^4\overline{\overline{\psi}}=0 \quad \text{in } \mathscr{V}_0,$$

$$\overline{\overline{\psi}}=\overline{\overline{\psi}}_X \quad \text{at } X=\pm d$$

and

$$\overline{\overline{\psi}}_{,ZZ}-\overline{\overline{\psi}}_{,XX}=\overline{\overline{\psi}}_{,X}=0 \quad \text{on } Z=0.$$

It follows that

$$\overline{\overline{\psi}}\equiv 0 \quad \text{in } \mathscr{V}_0 \tag{6.1}$$

so that the mean forcing terms in (5.2a) may be balanced by the mean head. Introducing now the second normal stress function for time-periodic flows

$$\hat{N}_2(\omega)\equiv -2\int_0^\infty G(s)\frac{\sin\omega s}{\omega}\,ds+\int_0^\infty\int_0^\infty \gamma(s_1,s_2)\cos\omega(s_1-s_2)\,ds_1\,ds_2, \tag{6.2}$$

we find that

$$\tfrac{1}{2}\overline{\overline{\phi}}_{,X}=2(|v_{,X}|^2)_{,X}\,\hat{N}_2(\omega) \quad \text{in } \mathscr{V}_0(X), \tag{6.3a}$$

$$-\overline{\overline{\phi}}+\rho g\overline{\overline{\hbar}}=T\overline{\overline{\hbar}}_{,XX} \quad \text{on } Z=0, \tag{6.3b}$$

$$\overline{\overline{\hbar}}_{,X}=0 \quad \text{at } X=\pm d \tag{6.3c}$$

and

$$\frac{1}{2d}\int_{-d}^{d}\overline{\overline{\hbar}}\,dX=0. \tag{6.3d}$$

It follows that the mean pressures and the mean rise in height are proportional to $\hat{N}_2(\omega)$. When $\omega\to 0$ this function may be identified with the second normal stress $N_2(\kappa^2)$ for viscometric flow with shear rate κ:

$$\lim_{\omega\to 0}\hat{N}_2(\omega)=2\alpha_1+\alpha_2=\lim_{\kappa\to 0}N_2(\kappa^2)/\kappa^2. \tag{6.4}$$

Equations (6.3b, c, d) imply that the mean value of $\overline{\overline{\phi}}(X)$ must vanish. It then follows from (6.3a) that

$$\begin{aligned}\overline{\overline{\phi}}&=2\mathscr{N}(\omega)\left\{|\cosh\Lambda(X+d)|^2-\int_{-d}^{d}|\cosh\Lambda(X+d)|^2\,dX\right\}\\ &=\mathscr{N}(\omega)\left\{\cosh 2\Lambda_r(X+d)-\int_{-d}^{d}\cosh 2\Lambda_r(X+d)\,dX\right.\\ &\qquad\left.+\cos 2\Lambda_i(X+d)-\int_{-d}^{d}\cos 2\Lambda_i(X+d)\,dX\right\}\end{aligned}\tag{6.5}$$

where

$$\mathcal{N}(\omega)=\frac{\hat{N}_2(\omega)\,|A|^2}{2\,|\sinh 2A\,d|}$$

and

$$A=A(\omega).$$

Using (6.5), we may integrate (6.3b) twice. The solution which satisfies (6.3c) and (6.3d) is found in the form

$$\begin{aligned}\bar{h}(X)=\frac{\mathcal{N}(\omega)}{T}\Bigg\{&\frac{\cosh 2A_r(X+d)}{\zeta^2-4A_r^2}+\frac{\cos 2A_i(X+d)}{\zeta^2+4A_i^2}\\&+\frac{2\cosh\zeta(X+d)}{\zeta\sinh 2\zeta d}\left[\frac{A_i\sin 4A_i d}{\zeta^2+4A_i^2}-\frac{A_r\sinh 4A_r d}{\zeta^2-4A_r^2}\right]\\&-\frac{1}{\zeta^2}\left[\frac{\sinh 4A_r d}{2A_r}+\frac{\sin 4A_i d}{4A_i}\right]\Bigg\}\end{aligned}\tag{6.6}$$

where

$$\zeta^2=\frac{\rho g}{T}.$$

The normalized mean discrepancy of rise in height,

$$\begin{aligned}H(d)-H(-d)&=\frac{\bar{h}(d)-\bar{h}(-d)}{(\Delta Y)^2}\sim\frac{[\bar{h}(d)-\bar{h}(-d)]\,\omega^2}{8}\\&=\frac{\mathcal{N}(\omega)}{T}\Bigg\{\frac{[\cosh 4A_r d]-1}{\zeta^2-4A_r^2}+\frac{[\cos 4A_i d]-1}{\zeta^2+4A_i^2}\\&\quad+\frac{[2\cosh 2\zeta d]-2}{\zeta\sinh 2\zeta d}\left(\frac{A_i\sin 4A_i d}{\zeta^2+4A_i^2}-\frac{A_r\sinh 4A_r d}{\zeta^2-4A_r^2}\right)\Bigg\}\frac{\omega^2}{8},\end{aligned}\tag{6.7}$$

should be easily measurable in experiments.

7. The Edge Problem Governing the Time-periodic Secondary Motion

To obtain the equations satisfied by ψ_1, ϕ_1, and h_1 we substitute (5.3) into (5.2) and set to zero the coefficients of $e^{2i\omega t}$. For example, from (5.3) and (5.2f) we find that

$$2i\rho\omega\,\mathrm{curl}(e_Y\psi_1)-\eta(2\omega)\,\nabla_X^2\,\mathrm{curl}(e_Y\psi_1)+\nabla_X\phi_1=e_X F(X,\omega).\tag{7.1}$$

The inhomogeneous term in (7.1) is irrotational and the curl of (7.1) leads to

$$-2i\rho\omega\,\nabla_X^2\psi_1+\eta(2\omega)\,\nabla_X^4\psi_1=0\qquad\text{in }\mathscr{V}_0.\tag{7.2a}$$

Combining (5.3) with (5.2), we find that

$$\psi_1=\psi_{1,X}=0\qquad\text{at }X=\pm d\tag{7.2b}$$

and, on $Z=0$,

$$\eta(2\omega)(\psi_{1,XX}-\psi_{1,ZZ})=0, \tag{7.2c}$$

$$\psi_{1,X}=2i\omega \hbar_1, \tag{7.2d}$$

$$-\phi_1+2\eta(2\omega)\psi_{1,XZ}+\rho g \hbar_1 = T\hbar_{1,XX}. \tag{7.2e}$$

A further reduction of the conditions on $Z=0$ is useful. First, $\hbar_1$ is eliminated from (7.2e), using (7.2d). To eliminate ϕ_1, we differentiate (7.2e) with respect to X; then we replace $\phi_{1,X}$ using the expression

$$\phi_{1,X}=F(X,\omega)-\eta(2\omega)\nabla_X^2\psi_{,Z}+2i\rho\omega\Psi_{,Z}$$

for the X component of (7.1). This leads to

$$\eta(2\omega)[3\psi_{1,XXZ}+\psi_{1,ZZZ}]-2i\rho\omega\psi_{1,Z}+\frac{\rho g\psi_{1,XX}}{2\omega i}-\frac{T}{2i\omega}\psi_{1,XXXX}=F(X,\omega). \tag{7.2f}$$

It is convenient for computations to eliminate ψ_{XX} and ψ_{XXXX}, using (7.2a) on $Z=0$ and (7.2c), from (7.2f). This gives

$$\begin{aligned}\frac{M(\omega)\Lambda^3(\omega)\sinh 2\Lambda(\omega)(X+d)}{4\sinh^2 2d\Lambda(\omega)}&=\eta(2\omega)[3\psi_{1,XXZ}+\psi_{1,ZZZ}]\\ -2i\rho\omega\psi_{1,Z}&+\left[\frac{\rho g}{2i\omega}-\frac{T\Lambda^2(2\omega)}{3i\omega}\right]\psi_{1,ZZ}+\frac{T}{6i\omega}\psi_{1,ZZZZ}.\end{aligned} \tag{7.2g}$$

Equations (7.2a, b, c, g) govern the motion of the fluid at second order.

8. Solution of the Edge Problem

The problem defined in Section 7 is an edge problem of the biharmonic type and it may be solved by biorthogonal series of the type used by JOSEPH (1974, 1976 C), JOSEPH & STURGES (1975) and JOSEPH & STURGES (1977). The equations governing the edge problem, expressed in dimensionless coordinates $(x,z)=(X/d, Z/d)$ with $\psi\equiv\psi_1$, $\Gamma^2=d^2\Lambda^2(2\omega)$ and $\gamma^2=d^2\Lambda^2(\omega)$, are listed below:

$$\nabla^4\psi-\Gamma^2\nabla^2\psi=0 \qquad \text{in } \mathscr{V}_0, \tag{8.1a}$$

$$\psi(\pm 1,z)=\psi_{,x}(\pm 1,z)=0, \tag{8.1b}$$

$$\psi_{,xx}-\psi_{,zz}=0 \qquad \text{on } z=0, \tag{8.2a}$$

$$\begin{aligned}&\eta(2\omega)[3\psi_{,xxz}+\psi_{,zzz}]-2i\rho\omega d^2\psi_{,z}+\frac{T}{6i\omega d}\psi_{,zzzz}+\frac{T}{i\omega}\left[\frac{\zeta^2 d}{2}-\frac{\Gamma^2}{3d}\right]\psi_{,zz}\\ &=\frac{M(\omega)\gamma^3\sinh 2\gamma(x+1)}{4\sinh^2 2\gamma} \qquad \text{on } z=0.\end{aligned} \tag{8.2b}$$

Each term of the series

$$\psi\sim\sum_{n=-\infty}^{\infty}\left[\frac{C_n\phi_1^{(n)}(x)}{S_n^2}e^{S_n z}+\frac{D_n\hat{\phi}_1^{(n)}(x)}{P_n^2}e^{P_n z}\right] \tag{8.3}$$

where

$$\phi_1^{(n)}(x) = S_n^2 [\cos \mu_n \cos S_n x - \cos S_n \cos \mu_n x],$$
$$\mu_n = \sqrt{S_n^2 - \Gamma^2}$$

and

$$\hat{\phi}_1^{(n)}(x) = P_n^2 [\sin \nu_n \sin P_n x - \sin P_n \sin \nu_n x],$$
$$\nu_n = \sqrt{P_n^2 - \Gamma^2}$$

satisfies (8.1 a) and vanishes at $x = \pm 1$. $\phi_1^{(n)}$ is an even function of x and $\hat{\phi}_1^{(n)}$ is an odd function of x. S_n are the eigenvalues of $\phi_1^{(n)}(x)$ chosen so that $\phi_{1,x}^{(n)}(\pm 1) = 0$; this definition implies that $S = S_n$ appear as the complex roots of

$$\frac{\sin(S+\mu)}{S+\mu} = -\frac{\sin(S-\mu)}{S-\mu}, \quad \mu = \sqrt{S^2 - \Gamma^2}. \tag{8.4}$$

P_n are the eigenvalues of $\hat{\phi}_1^{(n)}(x)$ chosen so that $\hat{\phi}_{1,x}^{(n)}(\pm 1) = 0$; hence, $P = P_n$ are the complex roots of

$$\frac{\sin(P+\nu)}{P+\nu} = \frac{\sin(P-\nu)}{P-\nu}, \quad \nu = \sqrt{P^2 - \Gamma^2}. \tag{8.5}$$

Bounded solutions of the form (8.3) must have eigenvalues with positive real parts. Hence, for problems in the semi-infinite strip $\mathscr{V}_0$, we need the first and fourth quadrant roots of (8.4) and (8.5). The eigenvalues are numbered according to the size of their real part. For example,

$$S_n = x_n + i\, y_n, \quad x_1 < x_2 < x_3 < \cdots \quad \text{and} \quad x_{-1} > x_{-2} > x_{-3} > \cdots.$$

The magnitude of the imaginary part of the eigenvalues increases with the real part but at a slower rate; the modulus and real part of the eigenvalues increase with n. Since Γ is complex, $S_n \neq \bar{S}_{-n}$.

When $\Gamma = 0$, $\mu = S$ and $\nu = P$. In the limit $\Gamma \to 0$, (8.1 a) reduces to the biharmonic and $\phi_1^{(n)}(x)$ and $\hat{\phi}_1^{(n)}(x)$ become biharmonic eigenfunctions of Papkovich-Fadle type whose eigenvalues satisfy (see JOSEPH & STURGES, 1975)

$$\sin 2S = -2S, \quad \sin 2P = 2P. \tag{8.6}$$

When $\Gamma = 0$, $S_n = \bar{S}_{-n}$ and $P_n = \bar{P}_{-n}$ and when n is large

$$S_n = (n - \tfrac{1}{4})\pi + \frac{i}{2}\log(4n-1)\pi, \quad P_n = (n + \tfrac{1}{4})\pi + \frac{i}{2}\log(4n+1)\pi. \tag{8.7}$$

The same asymptotic distribution of eigenvalues holds for (8.4) and (8.5) because as S_n and P_n get large, $\mu_n \to S_n$ and $\nu_n \to P_n$.

(8.3) is our candidate for the solution of (8.1) and (8.2). It remains to be shown that C_n and D_n can be selected to satisfy (8.2). For this selection we need adjoint eigenfunctions and a biorthogonality condition. These are derived below.

Consider the even eigenfunctions $\phi_1^{(n)}(x)$ and note that

$$\psi_{,zz} = \sum_n \phi_1^{(n)}(x)\, e^{S_n z}$$

and

$$\psi_{,xx}=\sum_n \phi_2^{(n)}(x)\, e^{S_n z}.$$

Since $\psi_{,xxzz}=\psi_{,zzxx}$, we have

$$\phi^{(n)}_{1,xx}=S_n^2\,\phi_2^{(n)} \tag{8.8}$$

and, since $\nabla^4\psi=\Gamma^2\nabla^2\psi$, we have

$$\phi^{(n)}_{2,xx}+S_n^2(2\phi_2^{(n)}+\phi_1^{(n)})-\mu_n^2(\phi_2^{(n)}+\phi_1^{(n)})=0 \tag{8.9}$$

where $\phi_1^{(n)}(\pm 1)=\phi^{(n)}_{1,x}(\pm 1)=0$. We may write (8.8) and (8.9) as a vector equation for the vector $\boldsymbol{\phi}^{(n)}$ with components $\phi_1^{(n)}$ and $\phi_2^{(n)}$:

$$\boldsymbol{\phi}^{(n)}_{,xx}+\boldsymbol{T}(S_n,\Gamma)\cdot\boldsymbol{\phi}^{(n)}=0, \qquad \phi_1^{(n)}(\pm 1)=\phi_{1,x}(\pm 1)=0 \tag{8.10}$$

where the matrix of tensor $\boldsymbol{T}(S,T)$ is given by

$$\begin{pmatrix} 0 & -S^2 \\ S^2-\Gamma^2 & 2S^2-\Gamma^2 \end{pmatrix}.$$

The adjoint eigenvalue problem for (8.10) in $L^2[-1,1]$ is obtained from the bilinear concomitant

$$\int_{-1}^{1} \boldsymbol{\psi}^{(n)}\cdot[\boldsymbol{\phi}^{(n)}_{,xx}+\boldsymbol{T}(S_n,\Gamma)\cdot\boldsymbol{\phi}^{(n)}]\,dx$$
$$=\psi_2^{(n)}\phi^{(n)}_{2,x}-\psi^{(n)}_{2,x}\phi_2^{(n)}]_{-1}^{1}+\int_{-1}^{1}\boldsymbol{\phi}^{(n)}\cdot[\boldsymbol{\psi}^{(n)}_{,xx}+\boldsymbol{T}^T(S_n,\Gamma)\cdot\boldsymbol{\psi}^{(n)}]\,dx.$$

It follows that the problem adjoint to (8.10) is

$$\boldsymbol{\psi}^{(n)}_{,xx}+\boldsymbol{T}^T(S_n,\Gamma)\cdot\boldsymbol{\psi}^{(n)}=0, \qquad \psi_2^{(n)}(\pm 1)=\psi^{(n)}_{2,x}(\pm 1)=0 \tag{8.11}$$

where $\boldsymbol{T}^T(S_n,\Gamma)$ is the transpose of $\boldsymbol{T}(S_n,\Gamma)$. We find that

$$\phi_2^{(n)}=-\phi_1^{(n)}-\Gamma^2\cos S_n\cos\mu_n x, \tag{8.12}$$

$$\psi_1^{(n)}=\phi_1^{(n)}-\Gamma^2\cos\mu_n\cos S_n x, \tag{8.13}$$

and

$$\psi_2^{(n)}=\phi_1^{(n)}. \tag{8.14}$$

The biorthogonality conditions now follow in the usual way. From (8.10) and (8.11) we find that

$$\begin{aligned} 0&=\int_{-1}^{1}[\boldsymbol{\psi}^{(m)}\cdot\boldsymbol{T}(S_n,\Gamma)\cdot\boldsymbol{\phi}^{(n)}-\boldsymbol{\phi}^{(n)}\cdot\boldsymbol{T}(S_m,\Gamma)\cdot\boldsymbol{\psi}^{(m)}]\,dx \\ &=\int_{-1}^{1}\boldsymbol{\psi}^{(m)}\cdot\{\boldsymbol{T}(S_n,\Gamma)-\boldsymbol{T}^T(S_m,\Gamma)\}\cdot\boldsymbol{\phi}^{(n)}\,dx \\ &=(S_n^2-S_m^2)\int_{-1}^{1}\boldsymbol{\psi}^{(m)}\cdot\boldsymbol{A}\cdot\boldsymbol{\phi}^{(n)}\,dx \end{aligned} \tag{8.15}$$

where

$$[\boldsymbol{A}]=\begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}.$$

The same type of biorthogonality condition (8.15) holds for odd eigenfunctions with eigenvalues P_n and

$$\hat{\phi}_2^{(n)} = -\hat{\phi}_1^{(n)} - \Gamma^2 \sin P_n \sin v_n x, \tag{8.16}$$

and

$$\hat{\psi}_1^{(n)} = \hat{\phi}_1^{(n)} - \Gamma^2 \sin v_n \sin P_n x \tag{8.17}$$

$$\hat{\psi}_2^{(n)} = \hat{\phi}_1^{(n)}. \tag{8.18}$$

Summarizing, we have found that

$$\int_{-1}^{1} [\psi_1^{(m)}, \psi_2^{(m)}] \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} \phi_1^{(n)} \\ \phi_2^{(n)} \end{pmatrix} dx = \delta_{nm} k_n \tag{8.19}$$

where

$$k_n = \Gamma^2 \left\{ S_n^2 [\cos^2 S_n - \cos^2 \mu_n] + \frac{\Gamma^2 S_n}{\mu_n^2} \sin S_n \cos S_n \cos^2 \mu_n \right\}$$

and

$$\int_{-1}^{1} [\hat{\psi}_1^{(m)}, \hat{\psi}_2^{(m)}] \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} \hat{\phi}_1^{(n)} \\ \hat{\phi}_2^{(n)} \end{pmatrix} dx = \delta_{nm} \hat{k}_n \tag{8.20}$$

where

$$\hat{k}_n = \Gamma^2 \left\{ P_n^2 [\sin^2 P_n - \sin^2 v_n] - \frac{\Gamma^2 P_n}{v_n^2} \sin P_n \cos P_n \sin^2 v_n \right\}.$$

We turn now to the computation of the coefficients C_n and D_n in the expression (8.3). In preparation for the application of the biorthogonality conditions, we rewrite (8.2a) and (8.2b) as a vector equation:

$$\begin{aligned}
&\frac{T}{i\omega}\left[\frac{\zeta^2 d}{2} - \frac{\Gamma^2}{3d}\right] \sum \left\{ C_n \begin{pmatrix} \phi_1^{(n)} \\ \phi_2^{(n)} \end{pmatrix} + D_n \begin{pmatrix} \hat{\phi}_1^{(n)} \\ \hat{\phi}_2^{(n)} \end{pmatrix} - C_n \begin{pmatrix} 0 \\ \phi_1^{(n)} \end{pmatrix} - D_n \begin{pmatrix} 0 \\ \hat{\phi}_1^{(n)} \end{pmatrix} \right\} \\
&\quad + \eta(2\omega) \sum C_n S_n \begin{pmatrix} \phi_1^{(n)} + 3\phi_2^{(n)} \\ 0 \end{pmatrix} + D_n P_n \begin{pmatrix} \hat{\phi}_1^{(n)} + 3\hat{\phi}_2^{(n)} \\ 0 \end{pmatrix} \\
&\quad - 2 i \rho \omega^2 d^2 \sum C_n/S_n \begin{pmatrix} \phi_1^{(n)} \\ 0 \end{pmatrix} + D_n/P_n \begin{pmatrix} \hat{\phi}_1^{(n)} \\ 0 \end{pmatrix} \\
&\quad + \frac{T}{6 i \omega d} \sum C_n S_n^2 \begin{pmatrix} \phi_1^{(n)} \\ 0 \end{pmatrix} + D_n P_n^2 \begin{pmatrix} \hat{\phi}_1^{(n)} \\ 0 \end{pmatrix} \\
&\quad = M(\omega) \frac{\gamma^3 \sinh 2\gamma(x+1)}{4 \sinh^2 2\gamma} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.
\end{aligned} \tag{8.21}$$

The biorthogonality conditions (8.19) and (8.20) are now applied to (8.21). For the even functions we get

$$\begin{aligned}
Q(\phi^{(n)}, \psi^{(l)}, S_n, k_l) &\equiv \frac{T}{i\omega}\left[\frac{\zeta^2 d}{2} - \frac{\Gamma^2}{3d}\right] \left\{ C_l k_l - \sum_{-\infty}^{\infty} C_n C_{nl} \right\} \\
&\quad + \eta(2\omega) \sum_{-\infty}^{\infty} C_n S_n B_{nl} + \sum_{-\infty}^{\infty} C_n \left[\frac{T S_n^2}{6 i \omega d} - \frac{2 i \rho \omega^2 d}{S_n}\right] A_{nl} \\
&= \frac{M(\omega)\gamma^3}{4 \sinh^2 2\gamma} \int_{-1}^{1} \psi_2^{(l)} \sinh 2\gamma(x+1)\, dx
\end{aligned} \tag{8.22}$$

where for $n \neq l$

$$A_{nl} = \int_{-1}^{1} \psi_2^{(l)} \phi_1^{(n)} dx = 2 S_l^2 S_n^2 \cos\mu_l \cos\mu_n \left\{ \frac{2}{S_l^2 - S_n^2} + \frac{1}{S_n^2 - \mu_l^2} - \frac{1}{S_l^2 - \mu_n^2} \right\}$$
$$\{S_l \sin S_l \cos S_n - S_n \cos S_l \sin S_n\},$$

$$B_{nl} = \int_{-1}^{1} \psi_2^{(l)} [\phi_1^{(n)} + 3\phi_2^{(n)}] dx = 2 S_l^2 \cos\mu_l \cos\mu_n$$
$$\left\{ \frac{3\mu_n^2 - S_n^2}{S_l^2 - \mu_n^2} - \frac{2S_n^2}{S_n^2 - \mu_l^2} - \frac{S_n^2 + 3\mu_n^2}{S_l^2 - S_n^2} \right\} \{S_l \sin S_l \cos S_n - S_n \cos S_l \sin S_n\},$$

$$C_{nl} = \int_{-1}^{1} \phi_1^{(n)} [2\psi_2^{(l)} - \psi_1^{(l)}] dx = 2 S_n^2 \cos\mu_l \cos\mu_n$$
$$\left\{ \frac{2S_l^2 + \Gamma^2}{S_l^2 - S_n^2} + \frac{S_l^2}{S_n^2 - \mu_l^2} - \frac{S_l^2 + \Gamma^2}{S_l^2 - \mu_n^2} \right\} \{S_l \sin S_l \cos S_n - S_n \cos S_l \sin S_n\},$$

$$\int_{-1}^{1} \psi_2^{(l)} \sinh 2\gamma(x+1) dx = S_l^2 \cos\mu_l \left\{ \frac{1}{4\gamma^2 + S_l^2} - \frac{1}{4\gamma^2 + \mu_l^2} \right\}$$
$$\{2\gamma \cos S_l [\cosh 4\gamma - 1] + S_l \sin S_l \sinh 4\gamma\},$$

and when $n = l$,

$$A_{nn} = \int_{-1}^{1} \psi_2^{(n)} \phi_1^{(n)} dx$$
$$= S_n^4 \left\{ \cos^2\mu_n + \cos^2 S_n + \frac{S_n + \mu_n}{S_n \mu_n} \sin S_n \cos S_n \cos^2\mu_n \right\},$$

$$B_{nn} = \int_{-1}^{1} \psi_2^{(n)} [\phi_1^{(n)} + 3\phi_2^{(n)}] dx$$
$$= S_n^2 \left\{ (S_n^2 - 3\mu_n^2) \cos^2 S_n - 2 S_n^2 \cos^2\mu_n + S_n \frac{S_n - 6\mu_n}{\mu_n} \sin S_n \cos S_n \cos^2\mu_n \right\},$$

$$C_{nn} = \int_{-1}^{1} \phi_1^{(n)} [2\psi_2^{(n)} - \psi_1^{(n)}] dx$$
$$= S_n^4 \cos^2 S_n + S_n^2 (S_n^2 + \Gamma^2) \cos^2\mu_n + \frac{S_n (2 S_n^4 - \Gamma^4)}{\mu_n^2} \sin S_n \cos S_n \cos^2\mu_n.$$

For the odd functions, we get

$$Q(\hat{\phi}^{(n)}, \hat{\psi}^{(l)}, P_n, \hat{k}_l) = 0 \tag{8.23}$$

where for $n \neq l$,

$$A_{nl} = \int_{-1}^{1} \hat{\psi}_2^{(l)} \hat{\phi}_1^{(n)} dx = 2 P_l^2 P_n^2 \sin\nu_l \sin\nu_n \left\{ \frac{2}{P_l^2 - P_n^2} + \frac{1}{P_n^2 - \nu_l^2} - \frac{1}{P_l^2 - \nu_n^2} \right\}$$
$$\{P_n \sin P_l \cos P_n - P_l \cos P_l \sin P_n\},$$

$$B_{nl} = \int_{-1}^{1} \hat{\psi}_2^{(l)} [\hat{\phi}_1^{(n)} + 3\hat{\phi}_2^{(n)\prime}]\,dx = 2P_l^2 \sin\nu_l \sin\nu_n \left\{\frac{3\nu_n^2 - P_n^2}{P_l^2 - \nu_n^2} - \frac{2P_n^2}{P_n^2 - \nu_l^2} - \frac{P_n^2 + 3\mu_n^2}{P_l^2 - P_n^2}\right\}$$
$$\{P_n \sin P_l \cos P_n - P_l \cos P_l \sin P_n\},$$

$$C_{nl} = \int_{-1}^{1} \hat{\phi}_1^{(n)} [2\hat{\psi}_2^{(l)} - \hat{\psi}_1^{(l)\prime}]\,dx = 2P_n^2 \sin\nu_l \sin\nu_n \left\{\frac{2P_l^2 + \Gamma^2}{P_l^2 - P_n^2} + \frac{P_l^2}{P_n^2 - \nu_l^2} - \frac{P_l^2 + \Gamma^2}{P_l^2 - \nu_n^2}\right\}$$
$$\{P_n \sin P_l \cos P_n - P_l \cos P_l \sin P_n\},$$

$$\int_{-1}^{1} \hat{\psi}_2^{(l)} \sinh 2\gamma(x+1)\,dx = P_l^2 \sin\nu_l \left\{\frac{1}{4\gamma^2 + P_l^2} - \frac{1}{4\gamma^2 + \nu_l^2}\right\}$$
$$\{2\gamma \sin P_l [\cosh 4\gamma - 1] - P_l \cos P_l \sinh 4\gamma\},$$

and when $n = l$,

$$A_{nn} = \int_{-1}^{1} \hat{\psi}_2^{(n)} \hat{\phi}_1^{(n)}\,dx = P_n^4 \left\{\sin^2 P_n + \sin^2 \nu_n - \frac{P_n^2 + \nu_n^2}{P_n \nu_n^2} \sin P_n \cos P_n \sin^2 \nu_n\right\},$$

$$B_{nn} = \int_{-1}^{1} \hat{\psi}_2^{(n)} [\hat{\phi}_1^{(n)} + 3\hat{\phi}_2^{(n)\prime}]\,dx$$
$$= P_n^2 \left\{(P_n^2 - 3\nu_n^2)\sin^2 P_n - 2P_n^2 \sin^2 \nu_n + P_n \frac{6\nu_n^2 - P_n^2}{\nu_n^2} \sin P_n \cos P_n \sin^2 \nu_n\right\},$$

$$C_{nn} = \int_{-1}^{1} \hat{\phi}_1^{(n)} [2\hat{\psi}_2^{(n)} - \hat{\psi}_1^{(n)\prime}]\,dx$$
$$= P_n^4 \sin^2 P_n + P_n^2 \nu_n^2 \sin^2 \nu_n - \frac{P_n(2P_n^4 - \Gamma^4)}{\nu_n^2} \sin P_n \cos P_n \sin^2 \nu_n.$$

Equations (8.22) and (8.23) are of the form

$$C_l k_l + \sum_{-\infty}^{\infty} C_n D_{nl} = F_l.$$

This infinite set of algebraic equations can be solved by the method of reduction (KANTOROVICH & KRYLOV, 1958, p. 30). The truncated system

$$C_l^{(N)} k_l + \sum_{-N}^{N} C_n^{(N)} D_{nl} = F_l \qquad (l = \pm 1, \pm 2, \ldots, \pm N)$$

is solved for the $2N$ coefficients $C_l^{(N)}$ for increasing N. Numerical tests indicate that $C_l^{(N)} \to C_l$, but a proof of convergence has yet to be given.

Given $C_l^{(N)}$ for large N, it remains to verify that the truncation of the series on the left of (8.21) actually approximates the data vector given on the right of (8.21). This is done in Figure 7. We note that the series on the left of (8.21) *cannot* converge to the vector-valued function on the right of (8.21) when $x = \pm 1$ because the boundary values of $\phi_1^{(n)}$ and $\phi_2^{(n)}$ are incompatible with the boundary values of the function prescribed on the right. In the canonical edge problem, this incompatibility of the edge data and eigenfunctions at the boundary implies that the conver-

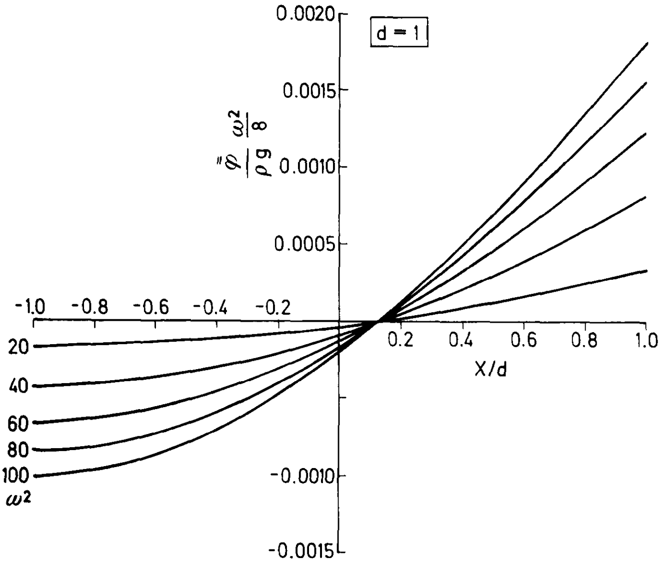

Fig. 2. Mean pressure distribution. The mean rise in height (neglecting surface tension) is proportional to $\bar{\phi}\omega^2$ (Equation 6.5). When $d=1$, the maximum and minimum mean pressure are on the walls.

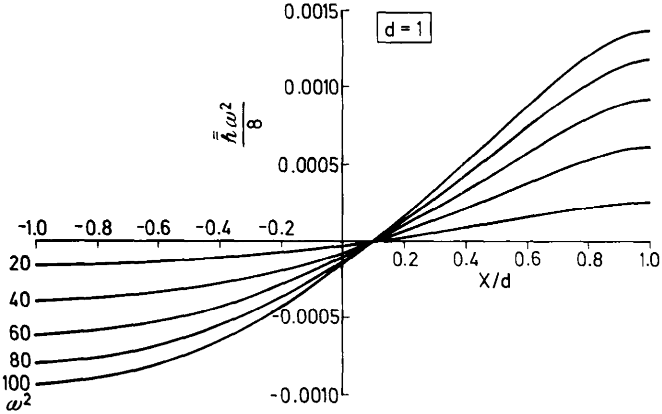

Fig. 3. Mean rise in height when the effects of surface tension are included (Equation 6.6). The mean rise in height and the mean pressure distribution have a similar variation over the channel except near the walls. In this figure and the others, the variation in mean rise in height may be regarded as due to the variation in the mean pressure. When $d=1$, the maximum and minimum mean rise in height are on the walls.

gence of the series is conditional and not absolute (Joseph & Sturges, 1977). In fact the numerical work shows that the convergence of (8.21) is conditional.

Given C_n, the stream function $\psi=\psi_1$ is completely determined by (8.3) and

$$h_1=\frac{1}{2i\omega}\psi_{,x}=\frac{1}{2i\omega d}\sum_{n=-\infty}^{\infty}\frac{C_n}{S_n^2}\phi^{(n)}_{1,x}+\frac{D_n}{P_n^2}\hat{\phi}^{(n)}_{1,x}. \tag{8.24}$$

The real-valued, time-periodic part of the stream function is given by

$$\tfrac{1}{16}[\psi(x,z)e^{2i\omega t}+\bar{\psi}(x,z)e^{-2i\omega t}]\,\omega^2(\Delta Y)^2 \tag{8.25}$$

and the real-valued, time-periodic part of the rise in height is given by

$$h(x,t,\varepsilon)-\bar{h}(x,\varepsilon)\sim\tfrac{1}{16}[\hbar_1(x)e^{2i\omega t}+\bar{\hbar}_1(x)e^{-2i\omega t}]\,\omega^2(\Delta Y)^2. \tag{8.26}$$

9. Quantitative Predictions of the Theory

It is not possible to obtain quantitative results from our formulas without further assumptions about the shear relaxation modulus $G(s)$ and the quadratic shear relaxation modulus $\gamma(s_1,s_2)$. In Parts I and II of this paper, these functions were approximated by what was termed a generalized (N, M) Maxwell model. In this model, $G_N(s)\sim G(s)$ is a $2N$ parameter exponential function and $\gamma_M(s_1,s_2)$ is

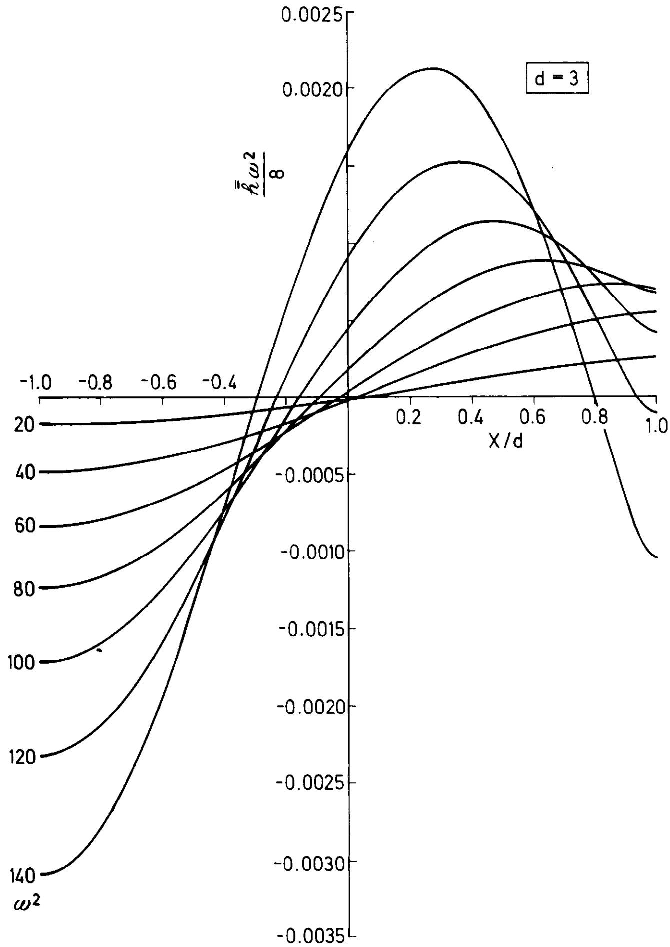

Fig. 4. Mean rise in height (Equation 6.6). When $d=3$, the maximum rise in height shifts to the interior as the frequency ω of oscillation is increased. This shift is due to an identical shift in the mean pressure.

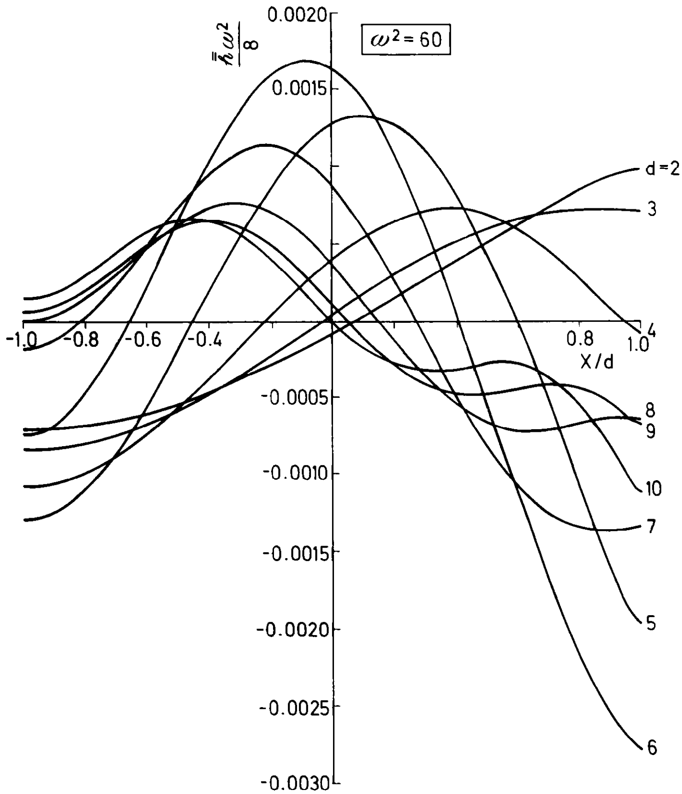

Fig. 5. The mean rise in height (Equation 6.6) as a function of d. The position of the maximum mean rise in height (and the maximum mean pressure) is a sensitive function of the gap width.

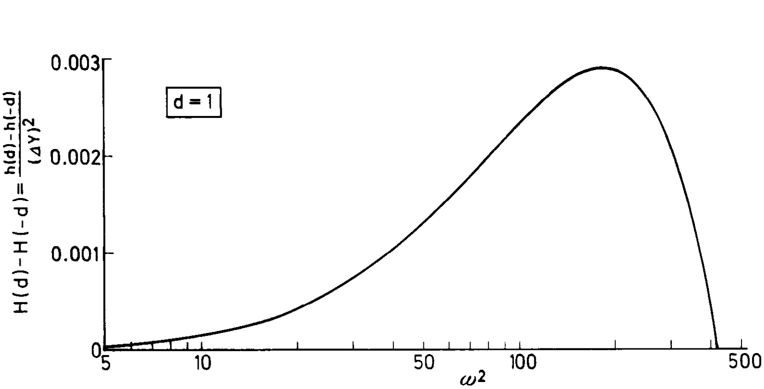

Fig. 6. The normalized mean rise in height as a function of the frequency ω of oscillation for the (1, 2) fluid described in Section 9. The (1, 2) approximation is expected to be valid when ω^2 is not too large. Comparison of this figure with Figure 8 of Part II shows that the mean climb between oscillating planes is an order of magnitude less than the corresponding climb on the oscillating rod.

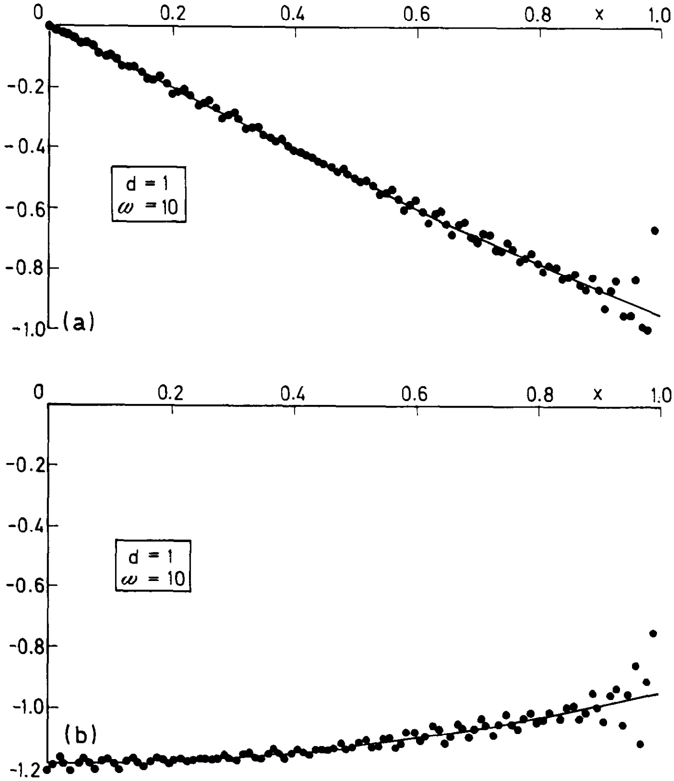

Fig. 7. Convergence of the series on the left of (8.21) to the data vector on the right. The data vector has been decomposed into its odd and even parts shown as a solid line in (a) and (b), respectively. The dots give the values of the series with coefficients D_n in (a) and C_n in (b) after 100 terms.

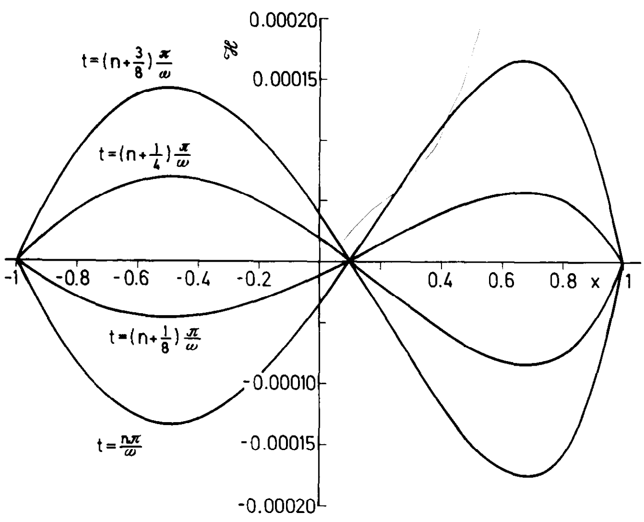

Fig. 8. The oscillatory part of the change of elevation of the free surface (Equation 8.24) with time as a parameter when $d = 1$ and $\omega = 5$. Comparison of this figure with Figure 3 shows that the oscillatory part of the change of elevation and the mean rise in height are equally important in the computation of the total change in elevation.

a $2M$ parameter exponential function. The values of the parameters are fixed by fitting the mean-rise curve for small amplitudes (in the present case, the mean-rise curves are given in Figures 3–6) with the experimentally determined universal rise curve. The fitting procedure has the desirable property that the number of parameters needed to fit the universal rise curve is apparently an increasing function of the oscillation frequency ω. In Part II it was shown that the (1, 2) fluid could be fit to the universal rise curve for TL-227 when $\omega^2 < 450$.

We now adopt the hypothesis that the (1, 2) model describes TL-227 in all motions which perturb the rest state when the amplitude is sufficiently small. In the present case, the hypothesis adopted implies that the rise curves computed in this section are a quantitative prediction of the rise which would be observed in TL-227 when $\omega^2 < a^2$, where $a^2 \simeq 450$.

In the (N, M) fluid

$$G_N(S) = \frac{-\mu^2}{\alpha_1} \sum_1^N \frac{a_n}{b_n} e^{(\mu/\alpha_1)(a_n/b_n)s},$$

$$\gamma_M(s_1, s_2) = \alpha_2 \sum_1^M C_n k_n^2 e^{-k_n(s_1+s_2)}$$

where μ is the shear viscosity at zero shear, α_1 and α_2 are the constants in the fluid of second grade and a_n, b_n, c_n and k_n are other constants. When $(N, M) = (1, 2)$,

$$G_1(s) = \frac{-\mu^2}{\alpha_1} e^{(\mu s/\alpha_1)},$$

$$\gamma_2(s_1, s_2) = \alpha_2 [c_1 k_1^2 e^{-k_1(s_1+s_2)} + (1-c_1) k_2^2 e^{-k_2(s_1+s_2)}],$$

$$\eta(\omega) = \frac{\mu}{1 - i\dfrac{\alpha_1}{\rho}\dfrac{\omega}{\nu}},$$

$$\Lambda^2(\omega) = \frac{\alpha_1}{\rho}\left(\frac{\omega}{\nu}\right)^2 + i\frac{\omega}{\nu},$$

$$\hat{N}_2 = \frac{2\alpha_1}{1 + \alpha_1^2 \omega^2/\mu^2} + \alpha_2 \left(\frac{c_1}{1 + \left(\dfrac{\omega}{k_1}\right)^2} + \frac{1 - c_1}{1 + \left(\dfrac{\omega}{k_2}\right)^2} \right),$$

$$\mathcal{N}(\omega) = \frac{\hat{N}_2(\omega)\,|\Lambda(\omega)|^2}{2\,|\sinh 2\Lambda(\omega) d|}.$$

For TL-227

$\rho = 0.896$ gm/cm^3,
$T = 30.5$ dyne/cm,
$\mu = 200$ poise,
$\alpha_1 = -50$ gm/cm,
$\alpha_2 = 85.9$ gm/cm,
$c_1 = 0.9735$,
$k_1^2 = 14.50$,
$k_2^2 = 307.0$.

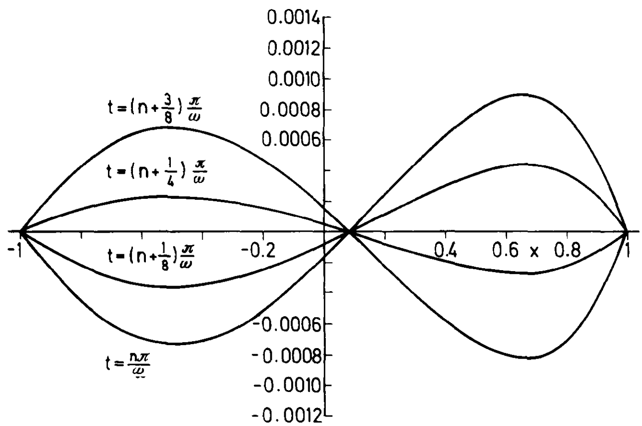

Fig. 9. The oscillatory part of the change in elevation of the free surface (Equation 8.24) with time as a parameter when $d=1$ and $\omega=10$.

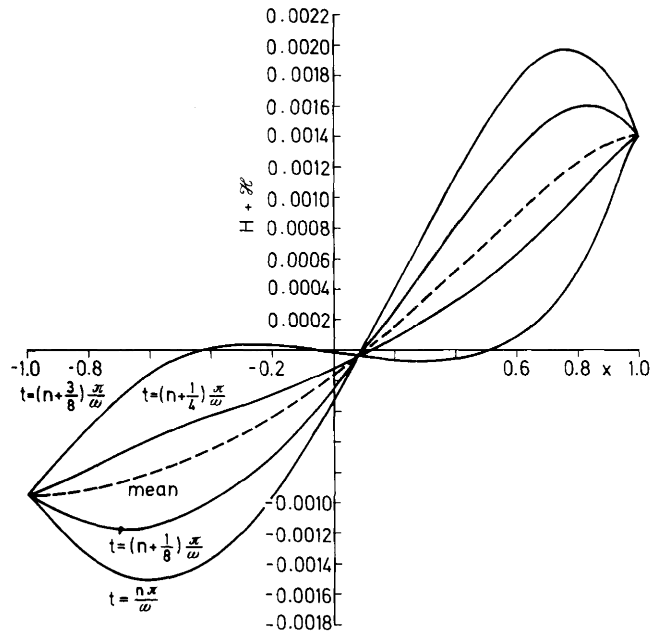

Fig. 10. The total change in elevation of the free surface with time as a parameter when $d=1$ and $\omega=10$. This figure is a superposition of Figures 3 and 9.

Figures 2–10 have been computed for the (1, 2) model of TL-227. The interpretation of the results summarized in the figures are given in the figure captions.

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