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Bifurcation and Stability of nT-Periodic Solutions Branching from T-Periodic Solutions at Points of Resonance

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I. Introduction

We shall study solutions which bifurcate from forced, T-periodic solutions of evolution equations of the Navier-Stokes type. Our principal interest is in subharmonic bifurcating solutions, nT-periodic solutions with $n \geqq 1$. Other authors (SACKER, 1964; RUELLE & TAKENS, 1971; FENICHEL, 1975) who have studied the bifurcation of T-periodic solutions have focused their attention on proving bifurcation into a torus. Their studies exclude the nT-periodic solutions (with $n = 1, 2, 3, 4$) which are our principal interest.

We suppose that a forced T-periodic solution of an evolution equation in Banach space, which depends on a parameter μ, loses stability as μ crosses zero from negative to positive values. The loss of stability of the T-periodic solution is associated with a Floquet multiplier $\lambda(\mu)$ of the linearized evolution operator. $\lambda(\mu)$ leaves the unit disc in the complex plane strictly as μ is increased past criticality ($\mu=0$). It is known from the work of SACKER (1964) and FENICHEL (1975) that if at criticality $\lambda^n(0)\neq 1$, $n=1,2,3,4$, then the T-periodic solution bifurcates into an invariant torus. RUELLE & TAKENS (1971), independently, have constructed an efficient proof of bifurcation into a torus, but their proof excludes $n=5$ as well as $n=1, 2, 3, 4$. MARSDEN & MCCRACKEN (1976) and IOOSS (1975) have extended the results of RUELLE & TAKENS to partial differential equations.

The values of n for which $\lambda^n(0)=1$, the roots of unity, are called points of resonance. We shall study bifurcation into $n\,T$-periodic solutions (same n) at points of resonance under the hypothesis that $\lambda(0)$ is a simple eigenvalue of the linearized evolution operator. We show that a single one-parameter family of T-periodic solutions bifurcates on both sides of criticality when $n=1$, and that the branch of the solution on the subcritical side ($\mu<0$) is unstable and the branch on the supercritical side ($\mu>0$) is stable. When $n=2$ a single one-parameter family of $2T$-periodic solutions bifurcates on one (or the other) side of criticality. If the bifurcation is subcritical, the $2T$-periodic family is unstable; if the bifurcation is supercritical, the $2T$-periodic family is stable. When $n=3$ a single one-parameter family of $3T$-periodic solutions bifurcates and is unstable on both sides of criticality. When $n=4$ we find three possibilities: (i) one unstable, one-parameter family of $4T$-periodic solutions bifurcates on each side of criticality; (ii) two one-parameter families of $4T$-periodic solutions bifurcate on the same side of criticality and at least one of these two is unstable; (iii) no $4T$-periodic solutions bifurcate. When $n\geqq 5$ there are, in general, no bifurcating $n\,T$-periodic solutions.

It remains an open problem whether or not an invariant torus exists when $n=4$ and $4T$-periodic solutions do not bifurcate.

The existence of T-periodic bifurcation when $n=1$ has been proved by MARKMAN (1971, 1972) but no analysis of stability is given. Existence and stability results for T-periodic bifurcation ($n=1$) have been given by JOSEPH (1973) and by IOOSS (1974b). YIH & LI (1972) have given results of a numerical study of the spectral problem for T-periodic convection in a fluid layer heated from below. Their results appear to show that the hypotheses for $2T$-periodic bifurcation are satisfied for certain values of the parameters. SACKER (1964) has given some results for resonant cases in $\mathbb{R}^n$ but they are incomplete. IOOSS's study (1974b) of the general problem of this paper is also incomplete because his method of analysis did not facilitate the calculation of certain integrals which were incorrectly assumed not to vanish. This weakness in the previous theory of IOOSS is rectified in Section VII.

The organization of this paper is given in the table of contents. The main theorems are proved using analytic perturbation theory in Sections IV and V. The analytic method leads to explicit expressions for power series representations of the bifurcating solutions. A more geometric method, using the central manifold theorem, is given in Section VI. This method yields all of the results previously proved by analytic perturbation theory and is better adapted to autonomous problems like those studied by IOOSS (1977). In Section VI.5 we show how the method of RUELLE & TAKENS may be modified to show the existence of an invariant torus when $n=5$.

II. Notations, Spaces and Linear Operators

II.1. Notations

$I_0 \subset \mathbb{R}$	An open real interval containing zero.
$\mu \in I_0$	A bifurcation parameter. In hydrodynamics $\mu = \nu - \nu_c$ where ν is the viscosity and ν_c the critical viscosity for the linear stability of the basic periodic flow.
$C_0 \subset \mathbb{C}$	A complex neighborhood of I_0.
$\mathbb{N}^* = \mathbb{N} \setminus \{0\}$	Positive integers.
H	A Hilbert space with scalar product $(u, v)_H = \overline{(v, u)_H}$.
$H_T = L^2(T; H)$	A Hilbert space obtained by completing continuous, T-periodic functions taking values in H, under the norm $[\cdot,\cdot]_{H_T}^{\frac{1}{2}}$, where the scalar product is defined by $[u, v]_{H_T} = \frac{1}{T} \int_0^T (u(t), v(t))_H \, dt.$ In a convenient notation, we shall write $[u, v]_T \equiv [u, v]_{H_T}$.
$\overset{(j)}{u}; u_{,t}; \overset{(0)}{u}$	$\frac{d^j u}{dt^j}$ $(j \geqq 0)$; $\frac{du}{dt} = \overset{(1)}{u} = u_{,t}$; $\overset{(0)}{u} = u$.

$H_T^m(H) = \{u;\ \overset{(j)}{u} \in H_T,\ j = 0, 1, \ldots, m\}$

A Hilbert space with scalar product

$$[u, v]_{H_T^m} = \frac{1}{T} \sum_{l=0}^{m} \int_0^T \left(\overset{(l)}{u}(t), \overset{(l)}{v}(t)\right)_H dt.$$

$\mathscr{B}_1, \mathscr{B}_2$	Banach spaces.
$\mathscr{L}[\mathscr{B}_1; \mathscr{B}_2]$	Banach space of bounded linear operators from $\mathscr{B}_1$ to $\mathscr{B}_2$.
$\mathscr{L}(\mathscr{B}_1) \equiv \mathscr{L}[\mathscr{B}_1; \mathscr{B}_1]$.	
$\mathscr{V}(q)$	A neighborhood of q in the corresponding space of q.

II.2. The Linear Operator $L(t, \mu)$

Consider a linear operator in H, $L(t, \mu)$, which is T-periodic with respect to t:

(II.1) $L(t, \mu) = A(\mu) + B(t, \mu) = A(\mu) + B(t + T, \mu) = L(t + T, \mu)$ where $A(\mu)$ and $B(t, \mu)$ are linear operators with the following properties:

i) $\{A(\mu);\ \mu \in C_0\}$ is a holomorphic family of linear operators in H which KATO (1966) calls type (A). The domain $D(A)$ of $A(\mu)$ is independent of μ. $D(A)$ is a Hilbert space with natural scalar product

(II.2) $$(u, v)_{D(A)} = (A(\mu) u, A(\mu) v)_H + (u, v)_H.$$

Norms corresponding to different values of $\mu \in C_0$ are uniformly equivalent. $-A(\mu)$ is the infinitesimal generator of a holomorphic semi-group $e^{-A(\mu)t}$ in H, when t lies in a sector, with wedge angle independent of μ, containing the real semi-axis $t > 0$.

ii) There is a Hilbert space K such that $D(A) \hookrightarrow K \hookrightarrow H$. For each $T>0$, there are constants $C>0$ and $\alpha<1$ such that $\forall \mu \in C_0$

$$\|e^{-A(\mu)t}\|_{\mathscr{L}[K;D(A)]} \leqq C t^{-\alpha}, \qquad t\in(0,T]. \tag{II.3}$$

iii) The map $(t,\mu) \mapsto B(t,\mu)$ from $\mathbb{R}\times\mathbb{C}$ into $\mathscr{L}[D(A);K]$ is analytic and bounded, for $\mu\in C_0$.

iv) The imbedding $D(A) \hookrightarrow H$ is compact.

v) $L(t,\mu)$ has a maximal adjoint in H, $L^*(t,\mu)$ with domain $D(A^*)$ where $A^*(\mu)$ is the adjoint of $A(\mu)$ in H.*

vi) The operators $A(\mu)$ and $B(t,\mu)$ map real vectors into real vectors when $\mu\in I_0$.

II.3. The Linear Operator $J(\mu)$

For all $u\in D^m(J)=H_T^{m+1}(H)\cap H_T^m(D(A))$ we define

$$[J(\mu)u](t)=u_{,t}(t)+L(t,\mu)\,u(t). \tag{II.4}$$

$J(\mu)$ is unbounded in $H_T^m(H)$, but it is a bounded linear operator from $D^m(J)$ into $H_T^m(H)$.

Lemma 1. *The family $\{J(\mu);\mu\in C_0\}$ of linear operators is holomorphic of type (A) in $H_T^m(H)$. The domain $D^m(J)$ is independent of μ and J has a compact resolvent. The adjoint $J^*(\mu)$ of $J(\mu)$ in H_T has domain $D(J^*)=H_T^1(H)\cap H_T^0[D(A^*)]$ and is defined by*

$$[J^*(\mu)u](t)=-u_{,t}(t)+L^*(t,\mu)\,u(t). \tag{II.5}$$

Corollary. *The operators $J(\mu)$, $\mu\in C_0$, are Fredholm operators with a zero index: the dimension of the null space equals the codimension of the range which is closed in $H_T^m(H)$.*

Proof. We first want to prove that for a sufficiently large $\chi>0$, there is a unique $u\in D^m(J)$ satisfying

$$J(\mu)u+\chi u=w\in H_T^m(H). \tag{II.6}$$

Moreover, there is a constant C independent of $\mu\in C_0$ such that

$$\|u\|_{D^m(J)} \leqq C\,\|w\|_{H_T^m}. \tag{II.7}$$

If (II.6) and (II.7) are assumed for the moment, it follows that $J(\mu)$ is closed in $H_T^m(H)$ and, since $D^m(J)$ is independent of $\mu\in C_0$, $J(\cdot)$ is an holomorphic family of type (A). Moreover, according to assumption (iv) of Section II.2, the imbedding $D^m(J)\hookrightarrow H_T^m(H)$ is compact for all $m\geqq 0$; hence, the resolvent of $J(\mu)$ is compact. This also proves that $J(\mu)$ is a Fredholm operator with an index equal to zero [see KATO (1966)]. To prove (II.6) and (II.7) we first define the linear operator $K(\mu)$ such that $J(\mu)=K(\mu)+B(\cdot,\mu)$ where

$$[K(\mu)u](t)=u_{,t}(t)+A(\mu)\,u(t), \qquad u\in D^m(J).$$

* For the Navier-Stokes system, we have $D(A)=D(A^*)$.

For a sufficiently large χ, the operator $K(\mu)+\chi$ is invertible. To see this, we write

(II.8) $$(K(\mu)+\chi)u = g \in H_T^m(H)$$

where

$$\begin{pmatrix} g(t) \\ u(t) \end{pmatrix} = \sum_{n\in\mathbb{Z}} \exp\left(2\pi in\frac{t}{T}\right)\begin{pmatrix} g_n \\ u_n \end{pmatrix}.$$

The Fourier series converge in $H_T^m(H)\times D^m(J)$. Hence,

(II.9) $$[A(\mu)+\chi+2\pi in/T]u_n = g_n \in H, \forall n\in\mathbb{Z}.$$

By the classical properties of the generators of holomorphic semi-groups (see, for example, KATO (1966)) we know that, for a sufficiently large χ, there is a constant C_1 independent of $\mu\in C_0$ and $n\in\mathbb{Z}$ such that

(II.10) $$\|(A(\mu)+\chi+2\pi in/T)^{-1}\|_{\mathscr{L}(H)} \leqq C_1/(|n|+\chi)$$

and

(II.11) $$\|(A(\mu)+\chi+2\pi in/T)^{-1}\|_{\mathscr{L}[H;D(A)]} \leqq C_1.$$

Equations (II.9), (II.10) and (II.11) imply that $\forall n\in\mathbb{Z}$

(II.12) $$\|u_n\|_{D(A)}^2 + (1+n^2)\|u_n\|_H^2 \leqq 2C_1^2\|g_n\|_H^2.$$

Hence $(K(\mu)+\chi)^{-1}\in\mathscr{L}[H_T^m(H);D^m(J)]$ when χ is large enough.

We now consider the inversion of (II.6) which, in view of the invertibility of $K(\mu)+\chi$, may be written as

(II.13) $$u+[K(\mu)+\chi]^{-1}B(\cdot,\mu)u = [K(\mu)+\chi]^{-1}w.$$

From assumption (iii) of Section (II.2), there is a constant $C_2>0$ independent of $\mu\in C_0$ such that

$$\|B(\cdot,\mu)\|_{\mathscr{L}[D^m(J);H_T^m(K)]} \leqq C_2.$$

It remains to show that the norm of the linear operator $[K(\mu)+\chi]^{-1}$ from $H_T^m(K)$ into $D^m(J)$ can be taken as small as we want when χ is sufficiently large. In fact, by assumption (ii) of Section (II.2), we can use an estimate of IOOSS (1972, equation (2.10)), which says that there is a constant $C_3>0$ independent of $n\in\mathbb{Z}$ and $\mu\in C_0$ such that

$$\|(A(\mu)+\chi+2\pi in/T)^{-1}\|_{\mathscr{L}[K;D(A)]} \leqq \frac{C_3}{1+(|n|+\chi)^{1-\alpha}},$$

and obviously

$$|n|\,\|(A(\mu)+\chi+2\pi in/T)^{-1}\|_{\mathscr{L}[K;H]} \leqq \frac{C_3}{1+(|n|+\chi)^{1-\alpha}},$$

where $1-\alpha>0$. It follows that, when χ is large enough,

$$\|(K(\mu)+\chi)^{-1}B(\cdot,\mu)\|_{\mathscr{L}[D^m(J)]} \leqq \tfrac{1}{2}.$$

Hence (II.13) is invertible and

$$u=[1+(K(\mu)+\chi)^{-1}B(\cdot,\mu)]^{-1}[K(\mu)+\chi]^{-1}w$$

is the unique solution of (II.6). The estimate (II.7) follows from the boundedness of $[K(\mu)+\chi]^{-1}$ from $H_T^m(H)$ into $D^m(J)$ and the boundedness of $[1+(K(\mu)+\chi)^{-1}B(\cdot,\mu)]^{-1}$ in $D^m(J)$.

To complete the proof of the lemma, we must define the adjoint of $J(\mu)$ in H_T. Consider the operator $J^*(\mu)$ defined in Lemma 1. The following identity holds for $u\in D(J)$ and $v\in D(J^*)$:

$$[J(\mu)u,v]_{H_T}=\frac{1}{T}\int_0^T(u_{,t}+L(t,\mu)\,u(t),v(t))_H\,dt$$
$$=\frac{1}{T}\int_0^T(u(t),-v_{,t}(t)+L^*(t,\mu)\,v(t))_H\,dt=[u,J^*(\mu)v]_{H_T}.$$

Reasoning once again as we did for $J(\mu)$, we find that $J^*(\mu)$ is closed in H_T and has a resolvent $(J^*(\mu)+\chi)^{-1}$ in $\mathscr{L}[H_T;D(J^*)]$ when χ is sufficiently large. Hence $J^*(\mu)$ is the maximal adjoint in H_T of the operator $J(\mu)$. This proves Lemma 1 and its corollary.

In our study of bifurcation and stability, we shall make assumptions about some eigenvalues of the operator $J(\mu)$. Since $J(\mu)$ has a compact resolvent, any σ in its spectrum is an eigenvalue of finite multiplicity. The spectrum is the same in H_T or in $H_T^m(H)$ because of the regularizing effect of the resolvent.

Let us now consider the initial value problem

$$u_{,t}+L(t,\mu)\,u(t)=0,\qquad u(0)=u_0\in D(A).$$

By our assumptions in Section II.2, and IOOSS (1975), we can define a monodromy operator $S_\mu(T)$, which is the map $u_0\mapsto u(T)$ in $D(A)$. We know that $S_\mu(T)$ is a linear compact operator in $D(A)$, which generalizes the classical monodromy matrix introduced in the study of ordinary differential equations (see, for example, Section 7 of JOSEPH (1976)). It is shown by IOOSS (1977) that each eigenvalue σ of $J(\mu)$ corresponds to an eigenvalue $e^{-\sigma T}$ of $S_\mu(T)$ with the same index and the same multiplicity. The eigenvalues $e^{-\sigma T}$ of $S_\mu(T)$ are the Floquet multipliers, and the $-\sigma$ are the Floquet exponents.

III. The Nonlinear Evolution Problem

We turn now to the study of bifurcation of the solution $u\equiv 0$ of the evolution equation

(III.1) $$u_{,t}+L(t,\mu)u+N(t,\mu,u)=0$$

where

(III.2) $$(t,\mu,u)\mapsto N(t,\mu,u)=N(t+T,\mu,u)$$

is an analytic map from $\mathbb{R}\times C_0\times D(A)$ into K. Moreover, for all small $u\in\mathscr{V}(0)\subset D(A)$, we have

$$\|N(t,\mu,u)\|_K^2\leqq C\,\|u\|_{D(A)}^2$$

where C is a constant independent of $u \in C_0$. The map (III.2) can be extended analytically (see IOOSS, 1974a) into a map from $C_0 \times H_T^1(D(A))$ into $H_T^1(K)$.

III.1. Simplifications of the Evolution Equation

Several simplifications which do not restrict the generality of our analysis may be introduced at this point. First, the dependence of N on t enters into the analysis in a passive way. Results to be proved when N is independent of t hold equally when N is T-periodic. It is convenient to suppress the dependence of N on t and to set

$$N(t, \mu, u) = N(\mu, u).$$

The solution $u \equiv 0$ bifurcates when $\mu = 0$. Hence in our local analysis it suffices to expand into a Taylor series in (μ, u):

$$N(\mu, u) = \sum_{\substack{\gamma, l \\ l \geqq 2}} \mu^{\gamma} N^{(\gamma, l)}(u, \dots, u) \tag{III.3}$$

where the $N^{(\gamma, l)}$ are bounded l-linear operators, symmetric with respect to all their arguments, from $H_T^1[D(A)]$ into $H_T^1(K) \hookrightarrow H_T^1(H)$. The terms of order $|\mu^2| \, \|u\|^2$ and $\|u\|^4$ can already be omitted. In the same spirit, following the assumptions (i) and (iii) of Section (II.2), we can write

$$L(t, \mu) = L_0(t) + \mu L_1(t) + \mu^2 L_2(t) + \cdots, \tag{III.4}$$

where the L_j are analytic in t, taking values in $\mathscr{L}[D(A); H]$. Retaining then only those terms which can enter into our local analysis we may, without losing generality, replace (III.1) with a slightly simpler equation

$$u_{,t} + [L_0(t) + \mu L_1(t) + \mu^2 L_2(t)] u + N(u, u) + \mu N_1(u, u) + M(u, u, u) = 0 \tag{III.5}$$

where $N \equiv N^{(0,2)}$, $N_1 \equiv N^{(1,2)}$, $M \equiv N^{(0,3)}$. In our local analysis $u = \varepsilon v$ and $\mu = \mu_1 \varepsilon + \mu_2 \varepsilon^2 + \cdots$ where $|\varepsilon|$ is the order of magnitude of the amplitude of the bifurcated flow. Thus the terms $\mu^2 L_2(t) u$, $\mu N_1(u, u)$ and $M(u, u, u)$ enter into the analysis first at order $O(\varepsilon^3)$. The main results of local analysis are found at $O(\varepsilon^2)$ when $\mu_1 \neq 0$ and at $O(\varepsilon^3)$ when $\mu_1 = 0$. In both cases the terms arising from $\mu^2 L_2(t) u$ and $\mu N_1(u, u)$ are of higher order and, without loss of generality, these terms may also be dropped from (III.5). The simplified evolution problem which replaces (III.1) may be written as

$$u_{,t} + L_0(t) u + \mu L_1(t) u + N(u, u) + M(u, u, u) = 0, \tag{III.6}$$

where N and M are symmetric in their arguments.

III.2. Spectral Assumptions

The operator which arises from linearizing (III.6) around $u = 0$ is now denoted by

$$J(\mu) = J_0 + \mu L_1, \qquad J_0 \equiv J(0). \tag{III.7}$$

It is known (Iudovich, 1970) that if, for all eigenvalues $\sigma(\mu)=\xi(\mu)+i\omega(\mu)$ of $J(\mu)$ we have $\xi(\mu)>0$, then all the Floquet multipliers are of modulus less than one, and $u\equiv 0$ is Liapunov stable. We assume that when $\mu<0$ all the eigenvalues $\sigma(\mu)$ of $J(\mu)$ have positive real parts $\xi(\mu)>0$. As μ is increased past zero, the real part of some of the eigenvalues change sign; $\xi(\mu)<0$ as μ becomes >0. In this case $u\equiv 0$ loses stability. We shall assume that the loss of stability is strict: $\xi_1=\xi,_{\mu}(0)<0$, $\xi(0)=0$. At criticality, $\sigma_0=i\omega_0$, and the multiplier $\lambda_0=e^{-i\omega_0 T}$ is of modulus one. The eigenvalues $\sigma=i\omega_0+2\pi ik/T$, $k\in\mathbb{Z}$ correspond to one and the same multiplier.

To motivate the hypotheses we shall need to make about λ_0 (or ω_0) recall that we are looking for $nT(n\geqq 1)$ periodic bifurcation of T-periodic solutions. The bifurcating solutions are constructed from the Floquet representation $e^{-i\omega_0 t}\zeta(t)$, $\zeta(t)=\zeta(t+T)$, of eigenvectors on the null space of J_0. This representation is nT periodic if and only if

$$e^{-i\omega_0(t+nT)}\zeta(t+nT)=e^{-i\omega_0 t}\zeta(t).$$

Hence $e^{-i\omega_0 nT}=1$ and $\omega_0 nT=2\pi m_1$, $m_1\in\mathbb{Z}^*$. We can choose m so that $m_1=kn+m$, $0\leqq m<n$. It follows that

$$\omega_0=2\pi(r+k)/T \quad \text{where } k\in\mathbb{Z}, \quad \text{and} \quad 0\leqq r=m/n<1.$$

Therefore the Floquet exponents on the imaginary axis at criticality are $-\sigma_p=-2\pi i(r+p)/T$, $p\in\mathbb{Z}$. The corresponding multiplier is $\lambda_0=e^{-2i\pi r}$, a root of unity of order n.

We may now lay down our *spectral assumptions:*

(H.1) Define $r\in\mathbb{Q}$ where $0\leqq r=m/n<1$. We assume that $\sigma_0=2\pi ir/T$ is a simple eigenvalue of J_0. Then $\lambda_0=e^{-2\pi ir}$ is a simple eigenvalue of $S_0(T)$.

(H.2) When $r=0$ or $\frac{1}{2}$, we assume that the numbers $\sigma_0+2k\pi i/T$, $k\in\mathbb{Z}$, are the only eigenvalues of J_0 on the imaginary axis; then $\lambda_0=1$ or -1 is the only eigenvalue of $S_0(T)$ on the unit circle.

When $r\neq 0$ or $\frac{1}{2}$, we assume that the numbers $\pm\sigma_0+2k\pi i/T$, $k\in\mathbb{Z}$ are the only eigenvalues of J_0 on the imaginary axis; then λ_0 and $\bar{\lambda}_0$ are the only eigenvalues of $S_0(T)$ on the unit circle.

Remark. The eigenvalues λ_0 and $\bar{\lambda}_0$ are roots of unity: $\lambda_0^n=\bar{\lambda}_0^n=1$. For $r\neq 0$ or $\frac{1}{2}$, we shall consider, for example, the cases $n=3$ with $m=1$ or 2, $n=4$ with $m=1$ or 3, etc.....

III.3. Fredholm Alternatives

We understand ζ to be the eigenvector of J_0 belonging to the eigenvalue σ_0, and ζ^* to be the eigenvector of J_0^* belonging to $\bar{\sigma}_0$. We choose a normalization such that

(III.8) $$[\zeta,\zeta^*]_T=1.$$

Moreover, ζ and ζ^* satisfy, respectively,

(III.9) $$\frac{-2\pi ir}{T}\zeta+J_0\zeta=0, \qquad \zeta\in H_T^m[D(A)]\,\forall m,$$

and

(III.10) $$\frac{2\pi i r}{T}\zeta^* + J_0^* \zeta^* = 0, \qquad \zeta^* \in H_T^m[D(A^*)]\,\forall m.$$

(III.9) and (III.10) imply that $[\zeta, \bar{\zeta}^*]_T = 0$ when $r \neq 0$. The regularity of the eigenvectors ζ and ζ^* is a consequence of smoothing by the resolvents of J_0 and J_0^*.

Lemma 2. *Let* (H.1) *and* (H.2) *hold and suppose that*

(III.11) $$\frac{2\pi i r}{T} u - J_0 u = f \in H_T^m(H).$$

Then there exists a unique $u \in D^m(J)$ *solving* (III.11) *and such that* $[u, \zeta^*]_T = 0$, *if and only if*

(III.12) $$[f, \zeta^*]_T = 0.$$

Proof. This lemma is a direct consequence of the corollary of Lemma 1.

(H.1) implies when $\mu \in \mathscr{V}(0)$, there is a unique simple eigenvalue of $J(\mu)$ near σ_0. We call it $\sigma(\mu) = \xi(\mu) + i\,\omega(\mu)$. It is well known that σ is analytic and that the first derivative of σ with respect to μ

(III.13) $$\sigma_1 = \sigma_{,\mu}(0) = \xi_1 + i\,\omega_1$$

may be obtained as

(III.14) $$\sigma_1 = [L_1 \zeta, \zeta^*]_T,$$

by perturbing the equation $-\sigma u + J(\mu) u = 0$ at $\mu = 0$, using (III.7). The assumption that the null solution loses stability strictly at $\mu = 0$ may be expressed as

(H.3) $$\xi_1 = \operatorname{Re}[L_1 \zeta, \zeta^*]_T < 0.$$

To construct nT-periodic bifurcating solutions when $n > 1$ $(r \neq 0)$, we shall need an operator $\mathbb{J}(\mu)$ which is like $J(\mu)$ except that $D^m(\mathbb{J})$ consists of nT-periodic functions and $D^m(\mathbb{J}) \supset D^m(J)$. Thus

$$D^m(\mathbb{J}) = H_{nT}^{m+1}(H) \cap H_{nT}^m[D(A)]$$

is the domain of the operator

(III.15) $$\mathbb{J}(\mu) = \frac{d}{dt} + L_0(t) + \mu L_1(t)$$

in the space $H_{nT}^m(H)$. When $n = 1$ $(r = 0)$, $\mathbb{J} = J$, and $\mathbb{J}(0) = \mathbb{J}_0$.

The spectrum of $\mathbb{J}(\mu)$ corresponds, in the usual way, to the spectrum of $S_\mu(nT)$, and since $S_0(nT) = [S_0(T)]^n$, 1 is the only eigenvalue of $S_0(nT)$ on the unit circle. When $n = 2$ $(r = \frac{1}{2})$, this eigenvalue is simple; and when $n \geqq 3$, this eigenvalue is *double*. Unit eigenvalues of $S_0(nT)$ correspond to zero eigenvalues of $\mathbb{J}_0$. The nT-periodic vector

(III.16) $$Z(t)=e^{\frac{-2\pi i r t}{T}}\zeta(t)\in D^m(\mathbb{J}), \quad \forall m\in\mathbb{N}$$

is obviously an eigenvector of $\mathbb{J}_0$, $\mathbb{J}_0 Z=0$. For $r=\frac{1}{2}$ it is the only such vector and we can choose it *real*. For $n\geqq 3$ there are two eigenvectors, Z and $\bar{Z}$, of $\mathbb{J}_0$.

Lemma 3. *Let* (H.1) *and* (H.2) *hold at criticality. Then when*

(III.17) $n=1$, *zero is a simple eigenvalue of* $\mathbb{J}_0$, *and the eigenvector is real:* $Z=\zeta=\bar{\zeta}=\bar{Z}$; *when*

(III.18) $n=2$, *zero is a simple eigenvalue of* $\mathbb{J}_0$, *and the eigenvector is real;* $t\mapsto Z(t)=e^{\frac{-\pi i t}{T}}\zeta(t)=e^{\frac{\pi i t}{T}}\bar{\zeta}(t)=\bar{Z}(t)$; *when*

(III.19) $n>2$, *zero is a double eigenvalue of* $\mathbb{J}_0$, *and the invariant subspace is generated by the eigenvectors* $\{Z,\bar{Z}\}$ *where* $Z(t)=e^{\frac{-2\pi i r t}{T}}\zeta(t)$, $J_0\zeta=\frac{2\pi i r}{T}\zeta$.

We may define the adjoint of $\mathbb{J}(\mu)$ in H_{nT} as we did for $J(\mu)$. The domain of $\mathbb{J}(\mu)$ in H_{nT} is $D(\mathbb{J}^*)=H^1_{nT}(H)\cap H_{nT}[D(A^*)]$ and

$$\mathbb{J}_0^*=-\frac{d}{dt}+L_0^*.$$

By Lemma 3, zero is an eigenvalue of $\mathbb{J}_0^*$ and the corresponding eigenvectors are Z^* and; if $n>2$, $\bar{Z}^*$ where

(III.20) $$t\mapsto Z^*(t)=e^{\frac{-2\pi i r t}{T}}\zeta^*(t)\in D^m(\mathbb{J}^*), \quad \forall m\in\mathbb{N},$$

we find, from a direct calculation using (III.8), that

(III.21) $$[Z,Z^*]_{nT}=[\zeta,\zeta^*]_T=1,$$

and, when $n=1$ or 2, we can suppose that Z^* is real. (It suffices to choose $\zeta^*(0)$ real.) When $n>2$ we have

(III.22) $$[Z,\bar{Z}^*]_{nT}=\frac{1}{nT}\int_0^{nT} e^{\frac{-4\pi i m t}{nT}}(\zeta(t),\bar{\zeta}^*(t))_H\,dt$$

with $(\zeta(t),\bar{\zeta}^*(t))_H=\sum_{k\in\mathbb{Z}} a_k e^{\frac{2i\pi k t}{T}}$, in $L^2(0,nT)$. But $nk-2m$ can never be zero because m is prime with n and $n>2$ (2 cannot be a multiple of n); hence,

(III.23) $$[Z,\bar{Z}^*]_{nT}=0=[\bar{Z},Z^*]_{nT}.$$

We can now define the projection $\mathbb{P}_0$, commuting with $\mathbb{J}_0$, relative to the isolated eigenvalue zero:

(III.24) $$\forall u\in H_{nT}, \quad \mathbb{P}_0 u=[u,Z^*]_{nT}Z+[u,\bar{Z}^*]_{nT}\bar{Z}$$

when $n>2$, and

(III.25) $$\forall u\in H_{nT}, \quad \mathbb{P}_0 u=[u,Z^*]_{nT}Z, \quad \text{when } n=1 \text{ or } 2.$$

The regularity of Z is such that $\mathbb{P}_0\in\mathscr{L}[H_{nT};D^m(\mathbb{J})]$, $\forall m\in\mathbb{N}$.

To study perturbations of the eigenvalue zero of the operator $\mathbb{J}(\mu)$, when $\mu \in \mathscr{V}(0)$, we distinguish cases for which $n=1$ or 2 and $n>2$. When $n=1$ or 2, zero is a simple isolated eigenvalue of $\mathbb{J}_0$, and $\sigma(\mu)-2\pi m i/nT$ is the only eigenvalue of $\mathbb{J}(\mu)$ near zero for $\mu \in \mathscr{V}(0)$. (Here we have used the fact that $\exp[-\sigma(\mu) n T]$ is the only simple eigenvalue of $S_\mu(nT)$ near 1.) Hence

(III.26) $$\sigma_1 = [L_1 Z, Z^*]_{nT} = [L_1 \zeta, \zeta^*]_T \quad \text{for } n=1 \text{ or } 2.$$

In the case $n>2$, zero is a double, *semi-simple* isolated eigenvalue of $\mathbb{J}_0$ and $\sigma(\mu) -2\pi m i/nT$ and $\bar{\sigma}(\mu)+2\pi m i/nT$ are the only eigenvalues of $\mathbb{J}(\mu)$ near zero for $\mu \in \mathscr{V}(0)$. From the theory of perturbations (see KATO (1966)) we know that σ_1 and $\bar{\sigma}_1$ are the eigenvalues of the two-dimensional linear operator

(III.27) $$\mathbb{P}_0 L_1 \mathbb{P}_0$$

In fact, this operator is diagonal in the basis $\{Z, \bar{Z}\}$ because

(III.28) $$[L_1 Z, Z^*]_{nT} = \sigma_1, \qquad [L_1 Z, \bar{Z}^*]_{nT} = 0.$$

It follows that the matrix (III.27) is given by $\begin{pmatrix} \sigma_1 & 0 \\ 0 & \bar{\sigma}_1 \end{pmatrix}$.

We close this section with the following solvability lemma:

Lemma 4. *Let* (H.1) *and* (H.2) *hold, and suppose that*

(III.29) $$\mathbb{J}_0 u = f \in H^m_{nT}(H).$$

Then there is a unique $u \in D^m(\mathbb{J})$ *such that* $\mathbb{P}_0 u = 0$, *if and only if* $\mathbb{P}_0 f = 0$.

The proof of Lemma 4 follows as a direct consequence of the Fredholm character of $\mathbb{J}_0$ and the definitions (III.24) and (III.25) of $\mathbb{P}_0$.

IV. Conditions for the Existence of nT-periodic Bifurcating Solutions

Now we shall seek nT-periodic bifurcating solutions $t \mapsto u(t, \varepsilon)$ of (III.6) of amplitude ε. The amplitude ε may be defined in various equivalent ways consistent with the requirement that $v(t,\varepsilon) = u(t,\varepsilon)/\varepsilon$ is bounded when $\varepsilon \to 0$. We find it convenient to define ε by the requirement

(IV.1) $$[u, Z^*]_{nT} = \varepsilon\, e^{i\phi(\varepsilon)}.$$

This definition is consistent with the fact that the principal part of any bifurcated solution lies in the eigenspace of the linearized operator relative to the eigenvalue zero.

Introducing $u = \varepsilon v$, we combine (III.6) and (III.15) and find that v satisfies

(IV.2) $$\mathbb{J}_0 v + \mu L_1 v + \varepsilon N(v,v) + \varepsilon^2 M(v,v,v) = 0,$$

where $v \in D^1(\mathbb{J}) = H^2_{nT}(H) \cap H^1_{nT}[D(A)]$. Since

(IV.3) $$v(t,\varepsilon) = v(t+nT, \varepsilon),$$

and (IV.2) is defined in $H^1_{nT}(H)$, the solutions $v(\cdot, \varepsilon)$ will be continuous and nT-periodic in $D(A)$.

We already noted in Lemma 3 that nT-periodic bifurcating solutions with $n=1$ or 2 were special; for these values of n, and no others, zero is a simple eigenvalue of $\mathbb{J}_0$ and Z and Z^* are real-valued. Hence when $n=1$ or 2, we can assume that $\phi(\varepsilon)=0$; the amplitude of the solution is $|\varepsilon|$ and ε may have either sign.

The methods of analysis of bifurcation from a simple eigenvalue with $n=1$ (Markman, 1971, 1972; Joseph, 1973; Iooss, 1974b) or $n=2$ (Iooss, 1974b) are identical to those used in steady problems (Sattinger, 1972; Iooss, 1974a; Joseph, 1976). Without repeating justifications of this by now conventional type of analysis, we note here that the calculations given in § V show that when $n=1$ or 2, a "single solution", analytic in ε bifurcates:

$$\begin{pmatrix} u(t,\varepsilon) \\ \mu(\varepsilon) \end{pmatrix} = \sum_{l \geqq 1} \varepsilon^l \begin{pmatrix} u_l(t) \\ \mu_l \end{pmatrix} \tag{IV.4}$$

where $u_1(t)=Z(t)$.

When $n=1$,

$$\mu_1 \sigma_1 + [N(Z,Z), Z^*]_T = 0$$

and the bifurcation is two-sided. When $n=2$, we find, using (III.18), that

$$-\mu_1 \sigma_1 = [N(Z,Z), Z^*]_{2T} = [e^{-i\pi t/T} N(\zeta,\zeta), \zeta^*]_{2T} = 0. \tag{IV.5}$$

We then get a unique $\tilde{u}_1$, with $[\tilde{u}_1, Z^*]_{2T}=0$, using Lemma 4, in the form

$$\tilde{u}_1 = -\mathbb{J}_0^{-1} N(Z,Z)$$

where $\mathbb{J}_0^{-1}$ is the pseudo-inverse of $\mathbb{J}_0$ in $(\mathbb{1}-P_0) H_{2T}$. Now

$$\mu_2 \sigma_1 + [2N(Z,\tilde{u}_1) + M(Z,Z,Z), Z^*]_{2T} = 0, \tag{IV.6}$$

and it is not hard to prove that $\mu_{2l+1}=0$, $l \in \mathbb{N}$ (see § V).

For other values of $n \in \mathbb{N} \setminus \{0,1,2\}$, Lemma 4 justifies the decomposition of $v(t,\varepsilon)$, solving (IV.2) and (IV.3), into a part on the null space of $\mathbb{J}_0$ and a part on the natural supplementary space (Liapunov-Schmidt method). Thus

$$v(t,\varepsilon) = a(\varepsilon) Z(t) + \bar{a}(\varepsilon) \bar{Z}(t) + \varepsilon w(t,\varepsilon)$$

where $[w, Z^*]_{nT} = [w, \bar{Z}^*]_{nT} = 0$. Moreover, following (IV.1) we have $a(\varepsilon)=e^{i\phi(\varepsilon)}$. We may, therefore, write the decomposition as

$$v(t,\varepsilon) = \tilde{v}(t,\varepsilon) + \varepsilon w(t,\varepsilon) \tag{IV.7}$$

where (see III.19)

$$\tilde{v}(t,\varepsilon) = Z(t) e^{i\phi(\varepsilon)} + \bar{Z}(t) e^{-i\phi(\varepsilon)} = \sum_{\nu=\pm 1} e^{i\nu\phi(\varepsilon)} e^{-i\nu\theta t} \zeta_\nu \tag{IV.8}$$

and

$$\theta = 2\pi r/T, \qquad \zeta_1 = \zeta, \qquad \zeta_{-1} = \bar{\zeta}. \tag{IV.9}$$

Then setting

$$\text{(IV.10)} \qquad \mu(\varepsilon)=\varepsilon\tilde{\mu}(\varepsilon)$$

where $\tilde{\mu}(0)=\mu_1$, we split (IV.2), using Lemma 4. Thus,

$$\text{(IV.11)} \qquad \mathbb{P}_0[\tilde{\mu} L_1 v+N(v,v)+\varepsilon M(v,v,v)]=0,$$

and

$$\text{(IV.12)} \qquad w+\mathbb{J}_0^{-1}[\tilde{\mu} L_1 v+N(v,v)+\varepsilon M(v,v,v)]=0,$$

where $\mathbb{J}_0^{-1}$ is the pseudo-inverse in $(\mathbb{1}-\mathbb{P}_0)H_{nT}$ of the Fredholm operator $\mathbb{J}_0$.
We now need the following

Corollary of Lemma 4. *Let* (H.1) *and* (H.2) *hold, and suppose that*

$$\text{(IV.13)} \qquad t\mapsto(\mathbb{J}_0 u)(t)=f(t)\,e^{ki\theta t}\in H^m_{nT}(H),$$

where $\theta=2\pi r/T$, $f\in H^m_T(H)$ *and* $k\in\mathbb{Z}$. *Then there is a unique* $u\in D^m(\mathbb{J})$ *such that* $\mathbb{P}_0 u=0$ *if and only if* f *is orthogonal to the kernel of* $J_0^*-k\,i\,\theta$ *in* H_T. *Moreover,*

$$t\mapsto u(t)=g(t)\,e^{ki\theta t}\in D^m(\mathbb{J})$$

where

$$\text{(IV.14)} \qquad g\in D^m(J), \quad \textit{and} \quad g=(J_0+k\,i\,\theta)^{-1}f$$

where $(J_0+k\,i\,\theta)^{-1}$ *is the inverse or the pseudo-inverse of* $(J_0+k\,i\,\theta)$ *in* $[\mathrm{Ker}(J_0^*-k\,i\,\theta)]^\perp$.

Proof. According to Lemma 4, it suffices to show that the condition that $f e^{ki\theta t}$, $f\in H_T$ is orthogonal to Z^* and $\bar{Z}^*$ in H_{nT} is equivalent to the condition that $f\in H_T$ is orthogonal to the kernel of $J_0^*-k\,i\,\theta$. In fact, if this is true, we have, obviously, with $u(t)=g(t)\,e^{ki\theta t}$, that

$$(\mathbb{J}_0 u)(t)=\{[(k\,i\,\theta+J_0)\,g](t)\}\,e^{ki\theta t}=f(t)\,e^{ki\theta t},$$

and we know that the solution is unique. Now, by (III.20), the condition that $f e^{ki\theta t}$ is orthogonal to Z^* and $\bar{Z}^*$ is equivalent to the identities

$$\text{(IV.15)} \qquad \int_0^{nT}(f(t),\zeta^*(t))_H\,e^{(k+1)i\theta t}\,dt=\int_0^{nT}(f(t),\bar{\zeta}^*(t))_H\,e^{(k-1)i\theta t}\,dt=0.$$

Because of the T-periodicity of the factors, (IV.15) is satisfied $\forall f\in H_T$ if $(k+1)m/n$ and $(k-1)m/n$ are not integers. But because of (H.1) and (H.2), the eigenvalues of J_0^* on the imaginary axis are

$$\pm i\theta+2p\pi i/T=2\pi i[p\pm m/n]/T, \qquad p\in\mathbb{Z}.$$

It is clear that if $(k\pm1)m/n\notin\mathbb{Z}$, then $i\,k\,\theta=\dfrac{2\pi i}{T}k\dfrac{m}{n}$ is not an eigenvalue of J_0^*.

However, if one of the numbers $(k+1)\,m/n$ or $(k-1)\,m/n$ is an integer, then $i\,k\,\theta$ is an eigenvalue of J_0^*, *simple* if there is only one such number, and *double* if both numbers are integers.

Hence, the corollary is proved when $(k+1)\,m/n$, and $(k-1)\,m/n$ are not integers. Now let us assume $(k+1)\,m/n=p$; then,

$$\zeta^* e^{\frac{-i2\pi pt}{T}} \in \operatorname{Ker}(J_0^* - k\,i\,\theta)$$

where

$$\left[(J_0^* - k\,i\,\theta)\,\zeta^* e^{\frac{-i2\pi pt}{T}}\right](t) = [i\,2\pi\,p/T - k\,i\,\theta - i\,\theta]\,\zeta^*(t)\,e^{\frac{-i2\pi pt}{T}} = 0.$$

In the same way, if $(k-1)\,m/n = p' \in \mathbb{Z}$, then

$$\bar{\zeta}^* e^{\frac{-i2\pi p't}{T}} \operatorname{Ker}(J_0^* - k\,i\,\theta).$$

To prove the corollary it remains to remark that (IV.15) is equivalent to the statement that f in H_T is orthogonal to

$$\zeta^* e^{-(k+1)i\theta t} = \zeta^* e^{\frac{-i2\pi pt}{T}} \quad \text{and} \quad \bar{\zeta}^* e^{-(k-1)i\theta t} = \bar{\zeta}^* e^{\frac{-2\pi i p' t}{T}}$$

when these vectors are in H_T; this implies that $f \in [\operatorname{Ker}(J_0^* - k\,i\,\theta)]^{\perp}$.

We next consider (IV.12) multiplied by ε^2 and obtain an expression $\mathscr{F}(\varepsilon^2 w, \varepsilon\,\tilde{\mu}, \varepsilon\, e^{-i(\theta t - \phi)}, \ \varepsilon\, e^{i(\theta t - \phi)}) = 0$ which is analytic in its arguments, with coefficients in $D^1(J)$. Applying the implicit function theorem (in the analytic case) and the corollary of Lemma 4, we find that

$$\text{(IV.16)} \qquad t \mapsto \varepsilon^2 w(t) = \sum_{\substack{k+l \geq 1 \\ p+k+l \geq 2}}^{\infty} w_{pkl}(t)\,\tilde{\mu}^p\,\varepsilon^{p+k+l}\,e^{i(l-k)(\theta t - \phi)}.$$

(IV.16) converges in $D^1(\mathbb{J})$, and $w_{pkl} \in D^1(J)$. The coefficient of ε^2 on the right of (IV.16) is given by

$$\text{(IV.17)} \qquad \begin{aligned} w_{020} &= -(J_0 - 2i\,\theta)^{-1} N(\zeta, \zeta), \\ w_{011} &= -2J_0^{-1} N(\zeta, \bar{\zeta}), \\ w_{002} &= -(J_0 + 2i\,\theta)^{-1} N(\bar{\zeta}, \bar{\zeta}), \\ w_{110} &= -(J_0 - i\,\theta)^{-1}(L_1\zeta - \sigma_1\zeta), \\ w_{101} &= -(J_0 + i\,\theta)^{-1}(L_1\bar{\zeta} - \bar{\sigma}_1\bar{\zeta}) \end{aligned}$$

where all the inverse operators are understood to project first on the subspace where they act. Replacing w by (IV.16) in the equation (IV.11), we obtain a system of two complex equations which are conjugate because v is real-valued. It is, therefore, sufficient to consider one complex equation given by

$$\text{(IV.18)} \qquad \begin{aligned} [\{\tilde{\mu}\,L_1(\tilde{v} + \varepsilon\,w) &+ N(\tilde{v} + \varepsilon\,w, \tilde{v} + \varepsilon\,w) \\ &+ \varepsilon\,M(\tilde{v} + \varepsilon\,w, \tilde{v} + \varepsilon\,w, \tilde{v} + \varepsilon\,w)\}, Z^*]_{nT} = 0. \end{aligned}$$

The bifurcation equation (IV.18) may be written as

(IV.19) $$\int_0^{nT} \sum_{\substack{p+k+l\geq 2\\ k+l\geq 1}} \tilde{\mu}^p \varepsilon^{p+k+l-2} e^{i(l-k)(\theta t-\phi)} \left(f_{pkl}(t), \zeta^*(t)\right)_H e^{i\theta t}\, dt = 0.$$

The only non-zero terms in (IV.19) are those for which $(l-k+1)\,m/n$ is an integer. Since m is prime with n, the values of l and k which give non-vanishing terms are

(IV.20) $$l-k=nq-1, \quad \text{with } q\in\mathbb{Z}.$$

This leads to an equation of bifurcation in $\mathbb{C}$ of the following type:

(IV.21) $$\sum \tilde{\mu}^p \varepsilon^{p+2k+nq-3} e^{i(1-nq)\phi} \alpha_{pkq}=0$$

where the summation is over values $k\in\mathbb{N}$, $p\in\mathbb{N}$, $q\in\mathbb{Z}$, and

$$2k+nq\geqq 2, \qquad p+2k+nq\geqq 3.$$

The principal part of (IV.18) is, for $n\geqq 3$,

(IV.22) $$\tilde{\mu}\, e^{i\phi}\sigma_1 + e^{-2i\phi}[N(\bar{\zeta},\bar{\zeta})\, e^{3i\theta t}, \zeta^*]_{nT} + O(\varepsilon)=0.$$

Consider now the special case when $n=3$. (When $n>3$, the scalar product in (IV.22) disappears, and $\tilde{\mu}=O(\varepsilon)$; hence, $\mu_1=0$.) The following theorem of bifurcation in the case $n=3$ may now be stated.

Theorem 1. *Let* (H.1), (H.2) *and* (H.3) *hold with* $n=3$; i.e., $m=1$ *or* 2. *We note that*

(IV.23) $$\lambda_1=[e^{3i\theta t} N(\bar{\zeta},\bar{\zeta}), \zeta^*]_T, \qquad \theta=2\pi m/3T,$$

and $\mu_1=|\lambda_1/\sigma_1|$ *assuming that* $\lambda_1\neq 0$. *Then there is a unique nontrivial* $3T$*-periodic solution of* (III.6) *bifurcating for* μ *near zero. The bifurcation is two-sided and the solution is globally invariant under the translation* $t\to t+T$. *Moreover, the principal part of the solution is*

(IV.24)
$$u(t,\varepsilon)=\varepsilon(e^{i\phi(\varepsilon)}\zeta(t)\, e^{-i\theta t}+e^{-i\phi(\varepsilon)}\bar{\zeta}(t)\, e^{i\theta t})+O(\varepsilon^2),$$
$$\mu(\varepsilon)=\varepsilon\,\mu_1+O(\varepsilon^2),$$
$$\phi(\varepsilon)=\frac{1}{3}\arg\left(-\frac{\lambda_1}{\sigma_1}\right)+2k\pi/3+O(\varepsilon), \qquad k=0,1,2,$$

where u, ϕ, μ *are analytic in* ε *in a neighborhood of zero, and* $k=0,1,2$ *corresponds to translations of the origin in* t: $0, T, 2T$ *if* $m=1$ *and* $0, 2T, T$ *if* $m=2$.

Proof. Equation (IV.22) may be written as

(IV.25) $$\tilde{\mu}\,\sigma_1+\lambda_1 e^{-3i\phi}+O(\varepsilon)=0.$$

Hence $\tilde{\mu}(0)=\mu_1$, where λ_1 is defined by (IV.23) and $\phi(0)$, is defined by (V.24). Now we can solve (IV.21) using the implicit function theorem with respect to the variables $\tilde{\mu}$ and $3\phi \bmod (2\pi)$. (Put $n=3$ in (IV.21), and divide the resulting expression by $e^{i\phi}$.) Equation (IV.25) may be written as $\mathscr{F}(\tilde{\mu},3\phi,\varepsilon)\equiv(\mathscr{F}_r+i\,\mathscr{F}_i)(\tilde{\mu},3\phi,\varepsilon)=0$. Then $\mathscr{F}(\mu_1,3\phi_0,0)=0$, and

$$\frac{\partial(\mathscr{F}_r,\mathscr{F}_i)}{\partial(\tilde{\mu},\phi)}(\mu_1,3\phi_0,0)=\begin{pmatrix}\xi_1 & -3\omega_1\mu_1\\ \omega_1 & 3\xi_1\mu_1\end{pmatrix}$$

is invertible. Hence $\tilde{\mu}$ and 3ϕ are analytic functions of ε near zero. We let $\phi(\varepsilon)\mapsto\phi(\varepsilon)+2\pi/3$ and keep the same $\tilde{\mu}(\varepsilon)$. Then using (IV.16), we find that the solution $u(t,\varepsilon)\mapsto u(t-T,\varepsilon)$ if $m=1$, or $u(t,\varepsilon)\mapsto u(t+T,\varepsilon)$ if $m=2$. If we recall that the bifurcated solutions are $3T$-periodic, the last statement in the theorem, hence the whole theorem, is proved.

We turn now to the values $n\geqq 4$ and show, using (IV.21), that $\tilde{\mu}(0)=0$. Hence, we may set $\tilde{\mu}=\varepsilon\tilde{\tilde{\mu}}(\varepsilon)$ where $\tilde{\tilde{\mu}}(0)=\mu_2$. Then dividing (IV.21) by ε ($\varepsilon=0$ is the trivial solution), we obtain

$$\sum \tilde{\tilde{\mu}}^p \varepsilon^{2p+2k+nq-4} e^{i(1-nq)\phi} \alpha_{pkq}=0 \tag{IV.26}$$

where the summation is over $k\in\mathbb{N}$, $p\in\mathbb{N}$, $q\in\mathbb{Z}$, and $2k+nq\geqq 2$, $2p+2k+nq\geqq 4$. The principal part of (IV.26) is in the form

$$\Big[\sum_{k+l=2} 2N(\tilde{v}, w_{0kl})\, e^{i(l-k)(\theta t-\phi)} + M(\tilde{v},\tilde{v},\tilde{v}), \zeta^* e^{-i\theta t}\Big]_{nT}$$

where w_{0kl} is defined by (IV.17). This leads us to consider the special case $n=4$, for which (IV.26) takes the form

$$\tilde{\tilde{\mu}} e^{i\phi}\sigma_1 + \lambda_2 e^{i\phi} + \lambda_3 e^{-3i\phi} + O(\varepsilon^2)=0 \tag{IV.27}$$

where

$$\lambda_2 = 2[N(\zeta, w_{011}), \zeta^*]_T + 3[M(\zeta,\zeta,\bar{\zeta}), \zeta^*]_T + 2[N(\bar{\zeta}, w_{020}), \zeta^*]_T$$

and

$$\lambda_3 = 2[N(\bar{\zeta}, w_{002})\, e^{4i\theta t}, \zeta^*]_T + [M(\bar{\zeta},\bar{\zeta},\bar{\zeta})\, e^{4i\theta t}, \zeta^*]_T.$$

In the case $n\geqq 5$, (IV.26) has the form

$$\tilde{\tilde{\mu}} e^{i\phi}\sigma_1 + \lambda_2 e^{i\phi} + O(\varepsilon)=0, \tag{IV.28}$$

with the same λ_2 as in (IV.27). In general, (IV.28) is not solvable because $\tilde{\tilde{\mu}}$ is real-valued and, in general, $\operatorname{Im}(\lambda_2/\sigma_1)\neq 0$. In the special case in which $\mu_2=-\lambda_2/\sigma_1$ is real, the principal part of ϕ is to be determined by the consideration of higher-order terms. We summarize the implications of these observations in

Theorem 2. *Let* (H.1), (H.2) *and* (H.3) *hold with* $n\geqq 5$, *the coefficients* σ_1 *and* λ_2 *being defined by* (III.14) *and* (IV.27). *Then if* $\operatorname{Im}(\lambda_2/\sigma_1)\neq 0$, *there is no small amplitude, nT-periodic bifurcated solution of* (III.6) *when* $|\mu|$ *is small.*

Now we consider the case $n=4$. We have to solve (IV.27) for $\tilde{\tilde{\mu}}$ and ϕ as functions of ε. After dividing (IV.27) by $e^{i\phi}$, we obtain an equation in the form

$$\tilde{\tilde{\mu}}\sigma_1 + \lambda_2 + \lambda_3 e^{-4i\phi} + \sum \tilde{\tilde{\mu}}^p \varepsilon^{2p+2k+4(q-1)} e^{-4iq\phi} \alpha_{pkq}=0 \tag{IV.29}$$

where the summation is on $k\in\mathbb{N}$, $p\in\mathbb{N}$, $q\in\mathbb{Z}$, and $k+2q\geqq 1$, $p+k+2q\geqq 3$. This means that we have an equation in $\mathbb{C}$, of the form $\mathscr{F}(\tilde{\tilde{\mu}}, 4\phi, \varepsilon^2)=0$, $\mathscr{F}$ being analytic. For the principal part of (IV.29), we have

$$e^{-4i\phi_0} = -(\mu_2+\lambda_2/\sigma_1)/(\lambda_3/\sigma_1), \tag{IV.30}$$

which leads to the following condition for μ_2:

$$\left|\frac{\lambda_3}{\sigma_1}\right| = \left|\mu_2 + \frac{\lambda_2}{\sigma_1}\right|. \tag{IV.31}$$

Real values of μ_2, solving (IV.31), exist if and only if

$$\left|\frac{\lambda_3}{\sigma_1}\right| \geqq \left|\mathrm{Im}\left(\frac{\lambda_2}{\sigma_1}\right)\right|. \tag{IV.32}$$

If (IV.32) holds, then

$$\mu_2 = -\mathrm{Re}\left(\frac{\lambda_2}{\sigma_1}\right) \pm \left[\left(\frac{\lambda_3}{\sigma_1}\right)^2 - \left(\mathrm{Im}\left(\frac{\lambda_2}{\sigma_1}\right)\right)^2\right]^{\frac{1}{2}}. \tag{IV.33}$$

We denote the two different solutions by $\mu_2^{(1)}$ and $\mu_2^{(2)}$ when inequality holds in (IV.32). In this case, (IV.30) gives two different values for $4\phi_0 \bmod 2\pi$. Moreover, when (IV.32) holds we have

$$\mathscr{F}(\mu_2, 4\phi_0, 0) = 0.$$

To use the implicit function theorem, we calculate

$$\begin{aligned}\frac{\partial(\mathscr{F}_r, \mathscr{F}_i)}{\partial(\tilde{\tilde{\mu}}, \phi)}(\mu_2, 4\phi_0, 0) &= \begin{pmatrix}\xi_1 & 4\,\mathrm{Im}(\lambda_3 e^{-4i\phi_0}) \\ \omega_1 & -4\,\mathrm{Re}(\lambda_3 e^{-4i\phi_0})\end{pmatrix} \\ &= \begin{pmatrix}\xi_1 & -4(\omega_1\mu_2 + \mathrm{Im}\lambda_2) \\ \omega_1 & 4(\xi_1\mu_2 + \mathrm{Re}\lambda_2)\end{pmatrix}.\end{aligned} \tag{IV.34}$$

The determinant of the matrix (IV.34) is

$$4|\sigma_1|^2[\mu_2 + \mathrm{Re}(\lambda_2/\sigma_1)], \tag{IV.35}$$

which does not vanish when the inequality (IV.32) is strict. The foregoing results form the basis for the following theorem of bifurcation for $n=4$.

Theorem 3. *Let* (H.1), (H.2) *and* (H.3) *hold with $n=4$, the coefficients σ_1, λ_2 and λ_3 being defined by* (III.14) *and* (IV.27). *Then if $|\mathrm{Im}(\lambda_2/\sigma_1)| > |\lambda_3/\sigma_1|$, there is no small amplitude, 4T-periodic bifurcated solution of* (III.6), *for μ near zero. If $|\lambda_3/\sigma_1| > |\mathrm{Im}(\lambda_2/\sigma_1)|$, two nontrivial 4T-periodic solutions of* (III.6) *bifurcate, each on one side of criticality. If $|\lambda_2| < |\lambda_3|$, one solution exists only for $\mu \geqq 0$; the other exists only for $\mu \leqq 0$. If $|\lambda_2| > |\lambda_3|$ the two-solutions bifurcate on the same side of $\mu=0$, $\mu \geqq 0$ if* $\mathrm{Re}(\lambda_2/\sigma_1) < 0$, *$\mu \leqq 0$ if* $\mathrm{Re}(\lambda_2\sigma_1) > 0$. *The principal part of the bifurcating solutions are given by*

$$\begin{aligned}&u^{(j)}(t,\varepsilon) = \varepsilon(e^{i\phi^{(j)}(\varepsilon^2)}\zeta(t)e^{-i\theta t} + e^{-i\phi^{(j)}(\varepsilon^2)}\bar{\zeta}(t)e^{i\theta t}) + O(\varepsilon^2), \\ &\mu^{(j)}(\varepsilon^2) = \varepsilon^2\mu_2^{(j)} + O(\varepsilon^4), \qquad \mu_2^{(1)} = 0 \quad \text{if } |\lambda_2| = |\lambda_3|, \\ &\phi^{(j)}(\varepsilon^2) = \tfrac{1}{4}\arg[-\lambda_3/(\sigma_1\mu_2 + \lambda_2)] + k\pi/2 + O(\varepsilon^2).\end{aligned} \tag{IV.36}$$

Here $\theta = m\pi/2T$, $m=1$ or 3, $j=1$ and 2. The values $k=0,1,2,3$ correspond to translations of t through period T: 0, T, $2T$, $3T$ if $m=1$, 0, $3T$, $2T$, T if $m=3$. The functions $\mu^{(j)}$ and $\phi^{(j)}$ are analytic in ε^2, and $u^{(j)}$ is analytic in ε with values in $D^1(\mathbb{J})$.

Proof. When the inequality in (IV.32) is strict, the implicit function theorem gives, for each choice of $\mu_2^{(j)}(j=1,2)$, unique functions of ε^2, $4\phi^{(j)}(\varepsilon^2) \bmod 2\pi$ and $\mu^{(j)}(\varepsilon^2)$. In other words, there are two distinct pairs of solutions $(\tilde{\mu}, 4\phi)$ of (IV.29). To verify the statement about the period T translations of t, we first note, using (IV.16), that under the transformation $\phi \mapsto \phi+\pi/2$ for a fixed value of μ, we have $u(t,\varepsilon) \mapsto u(t-T,\varepsilon)$, if $m=1$, or $u(t,\varepsilon) \mapsto u(t+T,\varepsilon)$ if $m=3$. In the same way by use of (IV.16), the transformation $\phi \mapsto \phi+\pi$, for the same μ, induces the transformation $u(t,\varepsilon) \mapsto u(t-2T,\varepsilon)=u(t,-\varepsilon)$ for both $m=1$ and $m=3$. This completes the proof of Theorem 3.

V. Properties of the Bifurcating Solutions and their Stability

V.1. Explicit Calculation of the Bifurcated Solutions

$n\,T$-periodic solutions which bifurcate from T-periodic ones satisfy (IV.1), (IV.2), (IV.4) and are analytic in ε for ε near zero. The power series for the bifurcating solutions

(V.1) $$\begin{pmatrix} v(t,\varepsilon) \\ \mu(\varepsilon) \end{pmatrix} = \sum_{n=0} \varepsilon^n \begin{pmatrix} v_n(t) \\ \mu_n \end{pmatrix}, \qquad \mu(0)=0$$

can be explicitly calculated ($u=\varepsilon v$). Perturbation problems for the Taylor coefficients may be obtained from (IV.2) and (V.1) by identification:

(V.2) $$\mathbb{J}_0 v_0 = 0, \qquad \mathbb{J}_0 v_1 + \mu_1 L_1 v_0 + N(v_0, v_0) = 0,$$

(V.3) $$\mathbb{J}_0 v_r + \sum_{l=1}^{r} \mu_l L_1 v_{r-l} + \sum_{l=0}^{r-1} N(v_l, v_{r-l-1}) + \sum_{k+l=0}^{r-2} M(v_l, v_k, v_{r-l-k-2}) = 0, \qquad r \geqq 2.$$

Lemma 4 states that (V.2) and (V.3) are sovable if

(V.4) $$\mu_1 [L_1 v_0, Z^*]_{nT} + [N(v_0, v_0), Z^*]_{nT} = 0$$

and

(V.5) $$\mu_r [L_1 v_0, Z^*]_{nT} + \left[\sum_{l=1}^{r-1} \mu_l L_1 v_{r-l} + \sum_{l=0}^{r-1} N(v_l, v_{r-l-1}) + \sum_{k+l=0}^{r-2} M(v_l, v_k, v_{r-l-k-2}), Z^* \right]_{nT} = 0.$$

When $n=1$ or $n=2$ zero is a simple eigenvalue of $\mathbb{J}_0$, v_0 is in the null space, and $v_n (n>0)$ is in the supplementary space $[v_n, Z^*]_{nT}=0$. In this case (V.4) and (V.5) give μ_1 and μ_r uniquely in terms of quantities of lower order, and subsequent coefficients may be computed sequentially. We note that the splitting of the solution which corresponds to (IV.7) is here given by

(V.6) $$v(t,\varepsilon) = Z(t) + \varepsilon w(t,\varepsilon).$$

It follows that the corollary of Lemma 4 applies and that (IV.16) is valid. Now, following the path leading to (IV.21), we find a bifurcation equation of the following type:

(V.7) $$\sum \tilde{\mu}^p \varepsilon^{p+2k+nq-3} \alpha_{pkq} = 0$$

where the summation is over $k \in \mathbb{N}$, $p \in \mathbb{N}$, $q \in \mathbb{Z}$, $2k+nq \geqq 2$, $p+2k+nq \geqq 3$ ($n=1$ or 2). The principal part of (V.7) is now (see (IV.18))

(V.8) $$\tilde{\mu}\sigma_1 + [N(Z,Z), Z^*]_{nT} + O(\varepsilon) = 0.$$

When $n=2$, (V.8) reduces to

(V.9) $$[N(Z,Z), Z^*]_{2T} = [e^{-i(\pi/T)t} N(\zeta,\zeta), \zeta^*]_{2T} = 0.$$

It follows that $\mu_1 = 0$ and $\tilde{\mu} = \varepsilon \tilde{\tilde{\mu}}$ in (V.7):

(V.10) $$\sum \tilde{\tilde{\mu}}^p \varepsilon^{2(p+k+q-2)} \alpha_{pkq} = 0.$$

The summation in (V.10) is as defined under (V.7) but now $p+k+q \geqq 2$. (V.10) determines a unique function $\tilde{\tilde{\mu}}(\varepsilon^2)$. Now change the sign of ε and observe that $\mu = \varepsilon^2 \tilde{\tilde{\mu}}(\varepsilon^2)$ is unchanged. Using (IV.16), we find that the bifurcated $2T$-periodic solution $u(t,\varepsilon)$ is transformed under the change of sign into $u(t+T,\varepsilon)$; in fact, $\varepsilon^{k+l} e^{i(l-k)\theta t}$ gives $(-1)^{k+l} \varepsilon^{k+l} e^{i(l-k)\theta t}$ and

$$e^{i(l-k)\theta(t+T)} = e^{i(l-k)\theta t} e^{i(l-k)\pi} = (-1)^{l-k} e^{i(l-k)\theta t}.$$

Hence

$$(-\varepsilon)^{k+l} e^{i(l-k)\theta t} = \varepsilon^{k+l} e^{i(l-k)\theta(t+T)}.$$

Our results about the bifurcation of T-periodic and $2T$-periodic solutions are summarized under

Theorem 4. *Let* (H.1), (H.2) *and* (H.3) *hold with* $n=1$ *or* 2. *Then there is a unique nontrivial* nT*-periodic bifurcated solution of* (III.6). *When* $n=1$ *the bifurcation is, in general, two-sided, whereas in the case* $n=2$ *it is one-sided. The principal part of the bifurcated solution is*

(V.11) $$\begin{aligned} u(t,\varepsilon) &= \varepsilon \zeta(t) e^{-i\theta t} + O(\varepsilon^2), \\ \mu(\varepsilon) &= \varepsilon \mu_1 + O(\varepsilon^2) \quad \text{when } n=1, \\ \mu(\varepsilon) &= \varepsilon^2 \mu_2 + O(\varepsilon^4) \quad \text{when } n=2 \end{aligned}$$

where $\theta = 0$ *if* $n=1$, $\theta = \pi/T$ *if* $n=2$. *Moreover, in the case* $n=2$, μ *is an analytic function of* ε^2 *and the change* $\varepsilon \mapsto -\varepsilon$ *corresponds to a translation of the time from* t *to* $t+T$.

We turn now to the calculation of the nT-periodic solution for $n \geqq 3$. We have

(V.12) $$v_0 = a_0 Z + \bar{a}_0 \bar{Z}, \qquad a_0 = e^{i\phi_0},$$

and

$$v_n = a_n Z + \bar{a}_n \bar{Z} + w_{n-1}$$

where

$$a_n = \frac{1}{n!} \frac{d^n e^{i\phi(\varepsilon)}}{d\varepsilon^n}\bigg|_{\varepsilon=0} = i\phi_n e^{i\phi_0} + b_n e^{i\phi_0} \tag{V.13}$$

and b_n depends on ϕ_l, $l<n$. w_n satisfies

$$[w_n, Z^*]_{nT} = 0 \quad \text{and} \quad \mathbb{J}_0 v_n = \mathbb{J}_0 w_{n-1}.$$

Using (V.12), we may eliminate v_l entirely from (V.2), (V.3), (V.4) and from the solvability equation (V.5). Since Z^* is a complex vector when $n \geqq 3$, (V.5) is a complex equation and it determines the values of μ_l and ϕ_l. More precisely, when $n=3$ and $\lambda_1 (\neq 0)$ is given by (IV.23), (V.5) may be written as

$$\mu_r \sigma_1 a_0 + i\mu_1 \sigma_1 a_0 \phi_{r-1} - 2i\bar{a}_0^2 \phi_{r-1} \lambda_1 + \text{terms independent of } \mu_r \text{ and } \phi_{r-1} = 0. \tag{V.14}$$

Since $\mu_1 = |\lambda_1/\sigma_1| \neq 0$, (V.14) determines μ_r and ϕ_{r-1}.

When $n=4$, (V.5) gives

$$\begin{aligned} &\mu_r \sigma_1 a_0 + i\mu_2 \sigma_1 a_0 \phi_{r-2} + i a_0 \phi_{r-2} \{2[N(w_0, Z), Z^*]_{4T} \\ &\quad + 3[M(v_0, v_0, Z), Z^*]_{4T} - 4[N(v_0, \mathbb{J}_0^{-1} N(v_0, Z)], Z^*]_{4T}\} \\ &\quad - i\bar{a}_0 \phi_{r-2} \{2[N(w_0, \bar{Z}), Z^*]_{4T} + 3[M(v_0, v_0, \bar{Z}), Z^*]_{4T} \\ &\quad - 4[N(v_0, \mathbb{J}_0^{-1} N(v_0, \bar{Z})), Z^*]_{4T}\} + \text{terms independent of} \\ &\quad \mu_r \text{ and } \phi_{r-2} = 0. \end{aligned} \tag{V.15}$$

Taking account of the definitions of λ_2 and λ_3 given by (IV.27), we may write (V.15) as

$$\mu_r \sigma_1 a_0 + i\mu_2 \sigma_1 a_0 \phi_{r-2} + i a_0 \lambda_2 \phi_{r-2} - i\bar{a}_0^3 3\lambda_3 \phi_{r-2} + \cdots = 0. \tag{V.16}$$

Hence

$$\mu_r + 4i\phi_{r-2}\left[\mu_2 + \frac{\lambda_2}{\sigma_1}\right] + \cdots = 0,$$

and since $\mu_2 + \operatorname{Re}\left(\frac{\lambda_2}{\sigma_1}\right) \neq 0$ (see IV.35), we can solve (V.16) for μ_r and ϕ_{r-2}.

Remark. The bifurcating nT-periodic solutions are not only locally analytic in ε but they are analytic in t, $t \in \mathbb{R}$ (see Lemma 1 of IOOSS (1977)).

V.2. Stability of the Bifurcated Solutions

We turn next to the problem of stability of the nT-periodic bifurcating solutions ($n=1,2,3,4$). We consider solutions of the evolution problem (III.6) in the form $u(t,\varepsilon) + \hat{y}(t)$ where $u(t,\varepsilon) = \varepsilon v(t,\varepsilon)$ is the nT-periodic bifurcating solution which satisfies (IV.1), (IV.2), (IV.3) and

$$\hat{y}_{,t} + L_0 \hat{y} + \mu L_1 \hat{y} + 2N(u, \hat{y}) + 3M(u, u, \hat{y}) + R(u, \hat{y}) = 0. \tag{V.17}$$

Here $R(u, \hat{y})$ is at least quadratic in $\hat{y}$, is analytic in its arguments and is of the type described in § III.1. The stability of the zero solution of (V.17) can therefore be studied by the spectral analysis of the linear operator $\mathscr{J}(\varepsilon)$ which is defined as follows: $D[\mathscr{J}(\varepsilon)] = D(\mathbb{J}_0)$ and $\forall y \in D(\mathbb{J}_0)$

(V.18) $$\mathscr{J}(\varepsilon) y = \mathbb{J}_0 y + \mu L_1 y + 2N(u, y) + 3M(u, u, y), \qquad \mu = \mu(\varepsilon), \quad u = u(\cdot, \varepsilon),$$

where

$$\mathscr{J}(0) = \mathbb{J}_0.$$

The family of operators $\{\mathscr{J}(\varepsilon); \varepsilon \in \mathscr{V}(0)\}$ is like the family $J(\mu)$ whose properties are summarized in Lemma 1. Using the methods used to prove that lemma, we can show that $\mathscr{J}(\varepsilon)$ has a compact resolvent and a pure point spectrum of eigenvalues of finite multiplicity. It follows from arguments analogous to those given in § III.2 that the stability of $u(\cdot, \varepsilon)$ is determined by the sign of the real part of the eigenvalues γ ($-\gamma$ is a Floquet exponent). By assumption, $\mathbb{J}_0 = \mathscr{J}(0)$ has no eigenvalue with a negative real part, so that we may confine our attention to the eigenvalues $\gamma(\varepsilon)$ for which $\gamma(0) = 0$.

The stability analysis is easiest when $n = 1$ or 2. In these cases zero is a simple eigenvalue of $\mathscr{J}(0)$, and the perturbation theory gives an analytic eigenvalue $\gamma(\varepsilon)$ with an analytic eigenvector $y(\varepsilon)$. The following *factorization theorem holds:*

(V.19) $$y(\varepsilon) = b(\varepsilon)[u_{,\varepsilon}(\varepsilon) + \mu_{,\varepsilon}(\varepsilon) g(\varepsilon)]$$

and

(V.20) $$\gamma(\varepsilon) = \mu_{,\varepsilon}(\varepsilon) \hat{\gamma}(\varepsilon)$$

where $b(\varepsilon)$ *is a normalizing factor,* $g(\cdot, \varepsilon) \equiv g(\varepsilon) \in D(\mathbb{J}_0)$ *is a* nT*-periodic function and* $\hat{\gamma}(\varepsilon)$ *and* $g(\varepsilon)$ *satisfy* (V.24) *below. When* ε *is small*

(V.21) $$\hat{\gamma}(\varepsilon) = -\sigma_1 \varepsilon + O(\varepsilon^p), \qquad p = 2 \text{ when } n = 1 \text{ and } p = 3 \text{ when } n = 2.$$

The proof of this factorization theorem follows along the lines laid out by JOSEPH (1977). Combining (V.19) and (V.20) in the equation

(V.22) $$\mathscr{J}(\varepsilon) y(\varepsilon) = \gamma(\varepsilon) y(\varepsilon),$$

we find that

(V.23) $$\mathscr{J}(\varepsilon)[u_{,\varepsilon}(\varepsilon) + \mu_{,\varepsilon}(\varepsilon) g(\varepsilon)] = \mu_{,\varepsilon}(\varepsilon) \hat{\gamma}(\varepsilon)[u_{,\varepsilon}(\varepsilon) + \mu_{,\varepsilon}(\varepsilon) g(\varepsilon)].$$

On the other hand, $u_{,\varepsilon}(\varepsilon)$ satisfies

$$\mathscr{J}(\varepsilon) u_{,\varepsilon}(\varepsilon) + \mu_{,\varepsilon}(\varepsilon) L_1 u(\varepsilon) = 0$$

by a direct differentiation of (III.6). After elimination of $\mathscr{J}(\varepsilon) u_{,\varepsilon}(\varepsilon)$ in (V.23), we find that $\mu_{,\varepsilon}$ is a common factor in all terms and

(V.24) $$\mathscr{J}(\varepsilon) g(\varepsilon) - L_1 u(\varepsilon) - \hat{\gamma}(\varepsilon) u_{,\varepsilon}(\varepsilon) - \hat{\gamma}(\varepsilon) \mu_{,\varepsilon}(\varepsilon) g(\varepsilon) = 0.$$

Since $\mathscr{J}(\varepsilon)$ is a Fredholm operator, and since $\gamma(\varepsilon)$ is a simple eigenvalue of $\mathscr{J}(\varepsilon)$, (V.24) is solvable if and only if $L_1 u(\varepsilon)+\hat{\gamma}(\varepsilon)\, u,_\varepsilon(\varepsilon)$ is orthogonal to the eigenvector in the null space of the operator adjoint to $-\gamma(\varepsilon)+\mathscr{J}(\varepsilon)$ in H_{nT}. This orthogonality condition determines $\hat{\gamma}(\varepsilon)$. g($\varepsilon$) is then obtained by using the pseudo-inverse of $\mathscr{J}(\varepsilon) -\hat{\gamma}(\varepsilon)\mu,_\varepsilon(\varepsilon)$. Since $u(\varepsilon)$ and $\mu(\varepsilon)$ are analytic for $\varepsilon\in\mathscr{V}(0)$, we find $\hat{\gamma}(\varepsilon)=\hat{\gamma}_0+\varepsilon\hat{\gamma}_1+\cdots$ with $\hat{\gamma}_0=0$ and

$$[L_1 v_0+\hat{\gamma}_1 v_0, Z^*]_{nT}=0;$$

that is, $\hat{\gamma}_1=-\sigma_1$. We omit the proof of the statement that $\hat{\gamma}(\varepsilon)=-\varepsilon\sigma_1+O(\varepsilon^3)$ when $n=2$. It follows from considerations like those leading to Theorem 4.

The factorizations (V.19) and (V.20) actually hold all along the branch $\mu(\varepsilon)$ when this branch exists and is regular in ε. Locally the theorem shows that the bifurcating solutions are stable when $\varepsilon\mu,_\varepsilon(\varepsilon)\sigma_1<0$ and unstable when $\varepsilon\mu,_\varepsilon(\varepsilon)\sigma_1>0$. Since by (H.3), $\sigma_1<0$, we have instability when $\varepsilon\mu,_\varepsilon(\varepsilon)<0$; that is, *subcritical solutions are unstable and supercritical solutions are stable when $n=1$ or 2 and $\varepsilon\in\mathscr{V}(0)$.*

We turn now to the study of the stability of the $3T$-periodic and $4T$-periodic bifurcating solutions. In these cases zero is a double, semi-simple eigenvalue of $\mathscr{J}(0)$ and we cannot assume *a priori* the analyticity of the eigenvalues and eigenvectors for $\varepsilon\in\mathscr{V}(0)$. To study this problem, we follow the theory developed by Kato (1966). We first develop the operator $\mathscr{J}(\varepsilon)$ which is analytic for $\varepsilon\in\mathscr{V}(0)$; thus

$$\mathscr{J}(\varepsilon)=\mathbb{J}_0+\varepsilon\mathscr{J}_1+\varepsilon^2\mathscr{J}_2+O(\varepsilon^3) \tag{V.25}$$

in $\mathscr{L}[D(\mathbb{J}); H_{nT}]$. We next consider the projection $\mathbb{P}_0$, defined by (III.24), on the null space of $\mathbb{J}_0$. The semi-simplicity of the eigenvalue zero of $\mathbb{J}_0$ is enough to insure that the eigenvalues $\gamma(\varepsilon)\in\mathscr{V}(0)$ of $\mathscr{J}(\varepsilon)$ are of the form

$$\gamma_i(\varepsilon)=\varepsilon\gamma_i+o(\varepsilon), \qquad i=1 \quad \text{or} \quad 2 \tag{V.26}$$

where the γ_i are the eigenvalues of the two-dimensional operator $\mathbb{P}_0 L_1\mathbb{P}_0$. By (V.1) we have

$$\mathscr{J}_1=\mu_1 L_1+2N(v_0,\cdot) \tag{V.27}$$

and

$$\mathscr{J}_2=\mu_2 L_1+2N(v_1,\cdot)+3M(v_0,v_0,\cdot). \tag{V.28}$$

Now by (III.28) we know that the matrix $\mathbb{P}_0 L_1\mathbb{P}_0$ in the basis $\{Z,\bar{Z}\}$ of $\mathbb{P}_0 H_{nT}$ is given by

$$\mathbb{P}_0 L_1\mathbb{P}_0=\begin{pmatrix}\sigma_1 & 0\\ 0 & \bar{\sigma}_1\end{pmatrix}.$$

Moreover, the projection $2\mathbb{P}_0 N(v_0,\cdot)\mathbb{P}_0$ is easily calculated by use of (IV.23) when $n=3$. When $n=4$, this projection is zero (see (IV.22)). Hence, when $n=3$,

$$\mathbb{P}_0\mathscr{J}_1\mathbb{P}_0=\begin{pmatrix}\mu_1\sigma_1 & 2e^{-i\phi_0}\lambda_1\\ 2e^{i\phi_0}\bar{\lambda}_1 & \mu_1\bar{\sigma}_1\end{pmatrix}, \tag{V.29}$$

where ϕ_0, μ_1, λ_1 are defined in Theorem 1. When $n=4$, $\mu_1=0$ and $\mathbb{P}_0\mathscr{J}_1\mathbb{P}_0=0$ is obviously semi-simple. Hence the eigenvalues are of the form

$$\gamma_i(\varepsilon)=\varepsilon^2\gamma_i'+o(\varepsilon^2), \qquad i=1,2. \tag{V.30}$$

Here the γ_i' are the eigenvalues of a two-dimensional operator which will be defined below.

Returning now to the case $n=3$, we obtain

$$\gamma_1+\gamma_2=2\mu_1\operatorname{Re}\sigma_1=2\mu_1\xi_1<0$$

and

$$\gamma_1\gamma_2=\mu_1^2|\sigma_1|^2-4|\lambda_1|^2=-3|\lambda_1|^2<0.$$

We have, therefore, found two distinct eigenvalues of $\mathbb{P}_0\mathscr{J}_1\mathbb{P}_0$. This leads to the existence of two small eigenvalues $\gamma_i(\varepsilon)$ of $\mathscr{J}(\varepsilon)$, $i=1$ and 2, which are analytic in ε, and two associated eigenvectors. From the formulas for $\gamma_1+\gamma_2$ and $\gamma_1\gamma_2$ just given, we find that the two small eigenvalues $\gamma_i(\varepsilon)$ of $\mathscr{J}(\varepsilon)$ are real and of opposite sign for $\varepsilon\in\mathscr{V}(0)\setminus\{0\}$. It follows that the $3T$-periodic bifurcating solution is *unstable on both sides* of $\mu=0$ (criticality).

We next consider $n=4$ and determine the γ_i' in (V.30). Introducing Q_0 (formerly $\mathbb{J}_0^{-1}$) for pseudo-inverse of $\mathbb{J}_0$ in H_{4T}, we have

$$Q_0\mathbb{J}_0\subset\mathbb{J}_0Q_0=\mathbb{1}-\mathbb{P}_0.$$

This operator is precisely defined by Lemma 4. $\mathbb{P}(\varepsilon)$ is the projection operator, defined classically by a Dunford integral, which commutes with $\mathscr{J}(\varepsilon)$, associated with the part of the spectrum near zero. $\mathbb{P}(\varepsilon)$ may be decomposed as follows (KATO, 1966):

$$\mathbb{P}(\varepsilon)=\mathbb{P}_0+\varepsilon\mathbb{P}_1+O(\varepsilon^2) \tag{V.31}$$

where $\mathbb{P}$ is analytic near zero, in $\mathscr{L}[H_{4T};D^m(\mathbb{J})]$, $\forall m\in\mathbb{N}$. In the semi-simple case (our case) $\mathbb{P}_1$ in (V.31) is given by

$$\mathbb{P}_1=-\mathbb{P}_0\mathscr{J}_1Q_0-Q_0\mathscr{J}_1\mathbb{P}_0. \tag{V.32}$$

Moreover, the eigenvalues of $\mathscr{J}(\varepsilon)$ near zero are also the eigenvalues of the following operator of rank 2:

$$\mathbb{P}(\varepsilon)\mathscr{J}(\varepsilon)\mathbb{P}(\varepsilon)=\varepsilon^2[\mathbb{P}_0\mathscr{J}_2\mathbb{P}_0-\mathbb{P}_0\mathscr{J}_1Q_0\mathscr{J}_1\mathbb{P}_0]+O(\varepsilon^3) \tag{V.33}$$

where the coefficient of ε^2 in (V.33) has been simplified by use of

$$\mathbb{P}_0\mathbb{J}_0\subset\mathbb{J}_0\mathbb{P}_0=0 \quad\text{and}\quad \mathbb{P}_0\mathscr{J}_1\mathbb{P}_0=0.$$

It follows that the γ_i' are the eigenvalues of the two-dimensional operator

$$\mathbb{P}_0\mathscr{J}_2\mathbb{P}_0-\mathbb{P}_0\mathscr{J}_1Q_0\mathscr{J}_1\mathbb{P}_0 \tag{V.34}$$

where $\mathscr{J}_1$ and $\mathscr{J}_2$ are defined by (V.27) and (V.28) evaluated with

$$v_0=e^{i\phi_0}Z+e^{-i\phi_0}\bar{Z}$$

and

$$v_1 = -Q_0 N(v_0, v_0) = \sum_{k+l=2} w_{0kl} e^{i(l-k)[\theta t - \phi_0]},$$

as in (IV.17). Inserting these definitions into (V.34), we reduce our problem to the study of the eigenvalues of the operator

(V.35) $$\mu_2 \mathbb{P}_0 L_1 \mathbb{P}_0 - 2\mathbb{P}_0 N[Q_0 N(v_0, v_0), \cdot]\mathbb{P}_0 + 3\mathbb{P}_0 M(v_0, v_0, \cdot)\mathbb{P}_0 - 4\mathbb{P}_0 N(v_0, \cdot) Q_0 N(v_0, \cdot)\mathbb{P}_0.$$

In the basis $\{Z, \bar{Z}\}$ of $\mathbb{P}_0 H_{4T}$, the eigenvalues of (V.35) may be computed as eigenvalues of the matrix

(V.36) $$\begin{pmatrix} \mu_2\sigma_1 + 2\lambda_2 & \lambda_2 e^{2i\phi_0} + 3\lambda_3 e^{-2i\phi_0} \\ \bar{\lambda}_2 e^{-2i\phi_0} + 3\bar{\lambda}_3 e^{2i\phi_0} & \mu_2\bar{\sigma}_1 + 2\bar{\lambda}_2 \end{pmatrix}$$

where $\mu_2\sigma_1 + \lambda_2 + \lambda_3 e^{-4i\phi_0} = 0$ and λ_1 and λ_2 are given under (IV.27). The eigenvalues γ_i' of (V.36) satisfy

(V.37) $$\gamma_1' + \gamma_2' = 2(\mu_2 \xi_1 + 2 \operatorname{Re} \lambda_2)$$

and

(V.38) $$\begin{aligned} \gamma_1' \gamma_2' &= |\mu_2\sigma_1 + 2\lambda_2|^2 - |\lambda_2 + 3\lambda_3 e^{-4i\phi_0}|^2 \\ &= |\sigma_1|^2 \left[\left| \mu_2 + \frac{2\lambda_2}{\sigma_1} \right|^2 - \left| 3\mu_2 + \frac{2\lambda_2}{\sigma_1} \right|^2 \right] \\ &= -8\mu_2 |\sigma_1|^2 \left[\mu_2 + \operatorname{Re}\left(\frac{\lambda_2}{\sigma_1}\right) \right]. \end{aligned}$$

If $|\lambda_2| < |\lambda_3|$, we know from Theorem 3 that $\mu_2^{(1)} \cdot \mu_2^{(2)} < 0$ and

$$\mu_1^{(1)} + \operatorname{Re}\left(\frac{\lambda_2}{\sigma_1}\right) = -\left[\mu_1^{(2)} + \operatorname{Re}\left(\frac{\lambda_2}{\sigma_1}\right) \right].$$

Hence, $\gamma_1' \cdot \gamma_2' < 0$ for each of the two bifurcated solutions. This means that the two $4T$-periodic bifurcating solutions are unstable. On the other hand, if $|\lambda_2| > |\lambda_3|$ and $|\operatorname{Im}(\lambda_2/\sigma_1)| < |\lambda_3/\sigma_1|$, then $\mu_2^{(1)}\mu_2^{(2)} > 0$, and $\gamma_1'\gamma_2'$ is negative for one of the two bifurcating solutions. For the other solution, $\gamma_1'\gamma_2' > 0$ and stability is determined by the sign of $\mu_2\xi_1 + 2\operatorname{Re}\lambda_2$ (stable if >0, unstable if <0).

We conclude this chapter with a summary of results proved about the stability of nT-periodic bifurcating solutions to small disturbances.

Theorem 5. *Suppose that the assumptions* (H.1), (H.2) *and* (H.3) *hold.*

(i) *When $n=1$ a single, one-parameter (ε) family of T-periodic solutions of* (III.6) *bifurcate on both sides of criticality (Theorem* 4). *When $n=2$ a single one-parameter (ε) family of $2T$-periodic solutions of* (III.6) *bifurcate on one side of criticality. Supercritical ($\mu(\varepsilon)>0$) bifurcating solutions are stable; subcritical ($\mu(\varepsilon)<0$) bifurcating solutions are unstable.*

(ii) *When $n=3$ a single, one parameter (ε) family of $3T$-periodic solutions of* (III.6) *bifurcates and is unstable on both sides of criticality (Theorem* 1).

(iii) *When* $n=4$ *and* $|\lambda_3/\sigma_1|>|\mathrm{Im}(\lambda_2/\sigma_1)|$, λ_2 *and* λ_3 *being defined by* (IV.27), *two one-parameter* (ε) *families of* 4 *T-periodic solutions of* (III.6) *bifurcate. If* $|\lambda_2|<|\lambda_3|$, *one of the two bifurcating solutions bifurcates on the subcritical side* ($\mu(\varepsilon^2)>0$) *and the other on the supercritical side* ($\mu(\varepsilon^2)>0$) *and both solutions are unstable. If* $|\lambda_2|>|\lambda_3|$, *the two solutions bifurcate on the same side of criticality and at least one of the two is unstable; the stability of the other solution depends on the details of the problem.*

(iv) *When* $n\geqq 5$ *and* $\mathrm{Im}(\lambda_2/\sigma_1)\neq 0$, λ_2 *being defined by* (IV.27), *there is no small amplitude n T-periodic solution of* (III.6) *near criticality* (*Theorem* 2).

VI. *n T*-periodic Bifurcation and the Center Manifold Theorem

VI.1. Formulation of the Problem

In this chapter we shall prove all the previous theorems about nT-periodic bifurcation and stability by a geometric method, using the center manifold theorem. We have already remarked that the evolution problem (III.6) suffices, without loss of generality, for the local analysis of (III.1) near the bifurcation point. Both problems satisfy the assumptions necessary to use the work of IOOSS (1975). In particular, we may define the Poincaré map

$$u_0 \overset{\Phi_\mu}{\longmapsto} \mathscr{U}(u_0,\mu,T) \tag{VI.1}$$

from $D(A)$ into itself, where $(u_0,\mu)\longmapsto\Phi_\mu(u_0)$ is analytic $D(A)\times\mathbb{C}\to D(A)$ for $(u_0,\mu)\in\mathscr{V}(0)$, and where, by definition, $t\longmapsto\mathscr{U}(u_0,\mu,t)$ is the unique solution, continuous in $D(A)$ of (III.1) or (III.6), such that $u(0)=u_0$. The Fréchet derivative at $u_0=0$ of Φ_μ is the monodromy operator $S_\mu(T)$ the eigenvalues of which are the Floquet multipliers (see the end of Section II). The spectral assumptions (H.1), (H.2), (H.3) in § III.2 and § III.3, state that the fixed point $u_0=0$ of Φ_μ is attractive for $\mu<0$ and repulsive for $\mu>0$, $\mu\in\mathscr{V}(0)$. The total multiplicity of the multipliers of moduli one for $S_0(T)$ is one or two. Other eigenvalues of $S_0(T)$ are all of moduli less than one. Now appeal to the "Center Manifold Theorem" (see LANFORD (1973)) as in the work of IOOSS (1975) delivers a one-dimensional or two-dimensional center manifold M_μ in a neighborhood of zero in $D(A)$, when $\mu\in\mathscr{V}(0)$, which is C^k for any arbitrarily fixed $k\in\mathbb{N}^*$. The manifold M_μ is locally invariant and attracting for Φ_μ and can be written as

$$X=G(Y,\mu). \tag{VI.2}$$

Here $Y\in P_0D(A)$, $X\in(\mathbb{1}-P_0)D(A)$ and P_0 is the invariant projection, associated with multipliers of modulus one, commuting with $S_0(T)$. $G'_Y(0,0)=0$ and $G(0,\mu)=0$. The property of local invariance allows one to reduce the problem of bifurcation of periodic flow to a finite-dimensional problem of dimension equal to that of $P_0D(A)$; that is, of dimension one or two. In this one-dimensional or two-dimensional space we seek nT-periodic solutions as fixed points of the projection of the Poincaré map:

$$Y\longmapsto\phi_\mu(Y)=P_0\Phi_\mu[Y+G(Y,\mu)], \qquad Y\in P_0D(A). \tag{VI.3}$$

We have to find the fixed points, not zero, of the map $u_0\longmapsto\Phi_\mu^n(u_0)$ (the same n as in nT). In fact, it is sufficient to study fixed points, not zero, of the n^{th} iterate ϕ_μ^n in

$P_0 D(A)$ of the projection of Poincaré map. This simplification is a consequence of the fact that since $u_0 \in M_\mu$, $u_0 = X + Y$ with $X = G(Y, \mu)$. It follows that an nT-periodic bifurcating solution corresponds to a fixed point of order n of ϕ_μ.

VI.2. Reduction of the Poincaré Map

Our analysis begins with a derivation of representations for the projection P_0.

Lemma 5. *The following representations hold* $\forall u \in H$:

$$n=1 \text{ or } 2, \qquad P_0 u = (u, \zeta^*(0))_H \zeta(0);$$
$$n \geqq 3, \qquad P_0 u = (u, \zeta^*(0))_H \zeta(0) + (u, \bar{\zeta}^*(0))_H \bar{\zeta}(0).$$

Proof. (III.9) implies that

$$\frac{d}{dt}\zeta(t) + L_0(t)\zeta(t) - (2\pi i r/T)\zeta(t) = 0, \qquad \zeta \in H_T^m[D(A)].$$

We may rewrite this differential equation as

$$\frac{d}{dt}[e^{-2\pi irt/T}\zeta(t)] + L_0(t)[e^{-2\pi irt/T}\zeta(t)] = 0.$$

Referring now to the text under (II.7), we may identify this equation as the one satisfied in the initial-value problem for the monodromy operator $S_0(T)$. Hence

$$e^{-2\pi irt/T}\zeta(t) = S_0(t)\zeta(0)$$

and

(VI.4) $$S_0(T)\zeta(0) = \lambda_0 \zeta(0).$$

By (III.10) we also have

$$\frac{d}{dt}\zeta^*(t) - L_0^*(t)\zeta^*(t) - (2\pi i r/T)\zeta^*(t) = 0, \qquad \zeta^* \in H_T^m[D(A^*)].$$

Consider next the properties of the function

$$f(t) = (S_0(t)u, \zeta^*(t))_H, \qquad \text{for } u \in D(A).$$

We find that

$$f'(t) = (-L_0(t)S_0(t)u, \zeta^*(t))_H + \left(S_0(t)u, \frac{d}{dt}\zeta^*(t)\right)_H$$
$$= (S_0(t)u, (2\pi i r/T)\zeta^*(t))_H = -(2\pi i r/T) f(t).$$

Hence

$$(S_0(t)u, \zeta^*(t))_H = e^{-2\pi irt/T}(u, \zeta^*(0))_H \,\forall u \in D(A)$$

and

$$([S_0(T) - \lambda_0]u, \zeta^*(0))_H = 0 \,\forall u \in D(A).$$

For technical reasons, we follow IOOSS (1975) and assume that it is possible to define a continuous extension of $S_\mu(T)$ in H. Such a continuous extension can be defined, for example, in Navier-Stokes problems. Given the extension of $S_\mu(T)$, the last identity shows that

$$S_0^*(T)\,\zeta^*(0)=\bar{\lambda}_0\,\zeta^*(0). \tag{VI.5}$$

Moreover, this identity also implies that

$$\big(S_0(t)\,\zeta(0),\zeta^*(t)\big)_H=e^{-2\pi irt/T}\big(\zeta(0),\zeta^*(0)\big)_H.$$

Hence the scalar product

$$\big(\zeta(t),\zeta^*(t)\big)_H=\big(\zeta(0),\zeta^*(0)\big)_H,$$

is independent of t and $\big(\zeta(0),\zeta^*(0)\big)_H=1$ because $[\zeta,\zeta^*]_T=1$. This completes the proof of **Lemma 5**.

We turn now to a study of the behavior, near the fixed point zero, of the map ϕ_μ in $P_0 D(A)$. The equation

$$\frac{\partial\phi_\mu}{\partial Y}(0)=P_0\,S_\mu(T)\,P_0+P_0\,S_\mu(T)\,G_Y'(0,\mu) \tag{VI.6}$$

follows from the definition of ϕ_μ (see (VI.3)) and the properties of G. The matrix of this linear operator in $P_0 D(A)$ in the basis $\{\zeta(0),\bar{\zeta}(0)\}$ can be easily obtained. For the case $n=1$ or 2, we have only to consider a matrix of one element defined by the scalar product

$$\big(S_\mu(T)\,\zeta(0)+S_\mu(T)\,G_Y'(0,\mu)\,\zeta(0),\zeta^*(0)\big)_H \tag{VI.7}$$

whereas for $n\geqq 3$, we have a 2×2 matrix.

Lemma 6. *When $n=1$ or 2, we have, in the basis $\zeta(0)$,*

$$\frac{\partial\phi_\mu}{\partial Y}(0)=\lambda_0(1-\mu\,\sigma_1\,T)+O(\mu^2); \tag{VI.8}$$

whereas, when $n\geqq 3$ we have, in the basis $\{\zeta(0),\bar{\zeta}(0)\}$,

$$\frac{\partial\phi_\mu}{\partial Y}(0)=\begin{pmatrix}\lambda_0(1-\mu\,\sigma_1\,T)+O(\mu^2) & O(\mu)\\ O(\mu) & \bar{\lambda}_0(1-\mu\,\bar{\sigma}_1\,T)+O(\mu^2)\end{pmatrix}. \tag{VI.9}$$

Proof. Since $\sigma=\sigma_0+\mu\,\sigma_1+O(\mu^2)$ is a simple eigenvalue of $J(\mu)$,

$$e^{-\sigma T}=e^{-\sigma_0 T}(1-\mu\,\sigma_1\,T)+O(\mu^2)$$

is a simple eigenvalue of $S_\mu(T)$, near the simple eigenvalue $e^{-\sigma_0 T}=\lambda_0$. Moreover, we also know (IOOSS, 1975) that $\mu\mapsto S_\mu(T)$ is analytic, for $\mu\in\mathscr{V}(0)$, in $\mathscr{L}[D(A)]$. It follows that

$$\big(S_\mu(T)\,\zeta(0),\zeta^*(0)\big)_H=\big(S_0(T)\,\zeta(0),\zeta^*(0)\big)_H+\mu\big(S_1\,\zeta(0),\zeta^*(0)\big)_H+O(\mu^2).$$

But

$$\big(S_1\,\zeta(0),\zeta^*(0)\big)_H=\frac{d}{d\mu}\big(e^{-\sigma(\mu)T}\big)\big|_{\mu=0},$$

so that

$$(S_\mu(T)\,\zeta(0), \zeta^*(0))_H = \lambda_0[1-\mu\sigma_1 T] + O(\mu^2).$$

In the same way, we find that

$$(S_\mu(T)\,\zeta(0), \bar{\zeta}^*(0))_H = O(\mu).$$

Now

$$P_0 S_\mu(T)\, G'_Y(0,\mu) = P_0 S_0(T)\, G'_Y(0,\mu) + \mu P_0 S_1 G'_Y(0,\mu) + O(\mu^2).$$

But $P_0 G'_Y(0,\mu)=0$ because $G'_Y(0,\mu)$ acts in $(\mathbb{1}-P_0)D(A)$. Moreover, $\|G'_Y(0,\mu)\| = O(\mu)$ because $G'_Y(0,0)=0$. It is then clear that

$$P_0 S_\mu(T)\, G'_Y(0,\mu) = O(\mu^2) \quad \text{in} \quad \mathscr{L}[P_0 D(A)],$$

and Lemma 6 is proved.

We next reduce the map ϕ_μ in $P_0 D(A)$ into a more explicit form. Suppose $n=1$ or 2 and let

$$\text{(VI.10)} \qquad \lambda(\mu) = \frac{\partial \phi_\mu}{\partial Y}(0), \qquad Y = y\,\zeta(0), \qquad \psi_\mu(y) = (\phi_\mu(Y), \zeta_0^*)_H.$$

Then

$$\text{(VI.11)} \qquad \psi_\mu(y) = \lambda(\mu)\, y + A_2(\mu)\, y^2 + O(|y|^3)$$

where A_2 is regular for $\mu \in \mathscr{V}(0)$ and $\lambda(\mu) = \lambda_0(1-\mu\sigma_1 T) + O(\mu^2)$. Suppose $n \geqq 3$ and let

$$\text{(VI.12)} \qquad Y = z\,\zeta(0) + \overline{z\,\zeta}(0), \qquad \psi_\mu(z) = (\phi_\mu(Y), \zeta^*(0))_H.$$

Then

$$\text{(VI.13)} \qquad \psi_\mu(z) = \lambda_1(\mu)\, z + \mu\lambda_2(\mu)\,\bar{z} + A_2(\mu, z) + O(|z|^3)$$

where A_2 is quadratic in $(z, \bar{z})$, $\lambda_1(\mu) = \lambda_0(1-\mu\sigma_1 T) + O(\mu^2)$ and λ_1, λ_2 and A_2 are regular for $\mu \in \mathscr{V}(0)$. The maps ψ_μ are regular enough to allow use of the implicit function theorem. Therefore, we have arrived at forms for the map ϕ_μ in $P_0 D(A)$ analogous to those used in the work of IOOSS (1975) (equation (V.9)). When $n=1$ or 2, ψ_μ is a map in $\mathbb{R}$, whereas when $n \geqq 3$, ψ_μ acts in $\mathbb{C}$ and zero is always a fixed point of ψ_μ.

VI.3. Bifurcation and Stability of nT-periodic Solutions for n=1 or 2

For $n=1$, we seek nontrivial fixed points of ψ_μ in $\mathbb{R}$, where ψ_μ is defined by (VI.14) and (VI.11) and $\lambda_0 = 1$. At a fixed point we have

$$\text{(VI.14)} \qquad y = \psi_\mu(y).$$

It follows now from (VI.14) and (VI.11) that

$$\text{(VI.15)} \qquad [\lambda(\mu) - 1]\, y + A_2(\mu)\, y^2 + O(|y|^3) = 0.$$

This is a classical bifurcation problem in one dimension. Set $\lambda(\mu)=1-\mu\xi_1 T+O(\mu^2)$ and

$$A_2(\mu)\,y^2=a\,y^2+O(|\mu|\,|y|^2).$$

If $a\neq 0$, we obtain a unique nontrivial fixed point

$$y(\mu)=\mu(T\xi_1/a)+O(\mu^2).$$

This is a two-sided bifurcation (*cf.* Theorem 4). In any case, we can parametrize the solution in the form $\mu=\mu(y)$, because $\xi_1\neq 0$. The T-periodic bifurcating flow is then given by the function

$$t\mapsto\mathscr{U}[Y+G(Y,\mu),\mu,t]\quad\text{where}\quad Y=y\zeta(0). \tag{VI.16}$$

To study the stability of the bifurcating T-periodic flow, we consider the attractivity of the new fixed point of ψ_μ. Let us change coordinates in $\mathbb{R}$:

$$y=y(\mu)+y'.$$

The new map is y' is

$$\psi'_\mu(y')=\lambda'(\mu)\,y'+O(|y'|^2),$$

where, by direct calculation,

$$\begin{aligned}\lambda'(\mu)&=1-\frac{\partial\psi_\mu}{\partial\mu}(y(\mu))\cdot\frac{d\mu}{dy(\mu)}\\ &=1+[\xi_1 T y(\mu)+O(|y(\mu)|^2]\,\frac{d\mu}{dy(\mu)}.\end{aligned}\tag{VI.17}$$

Because $\xi_1<0$, the new fixed point is attractive if $y(\mu)\dfrac{d\mu}{dy}>0$, repulsive if $y(\mu)\dfrac{d\mu}{dy}<0$. This is exactly the result we obtained from (V.20); the supercritical solution is stable and the subcritical solution is unstable.

When $n=2$, we may put $\lambda_0=-1$ in ψ_μ defined by (VI.11). The only iterates of ψ_μ which can have fixed points bifurcating from zero are of type ψ_μ^{2k}. We seek the fixed points of ψ_μ^2 for y near to zero. We next eliminate quadratic terms in ψ_μ by a suitable change of coordinates:

$$y'=y+\alpha(\mu)\,y^2.$$

The new map ψ'_μ is then

$$\psi'_\mu(y')=\lambda(\mu)\,y'+A_2(\mu)\,y'^2+\alpha(\mu)\,[\lambda^2(\mu)-\lambda(\mu)]\,y'^2+O(|y'|^3).$$

Since $\lambda^2(\mu)-\lambda(\mu)\neq 0$ when $\mu\in\mathscr{V}(0)$, we can choose $\alpha(\mu)$ so that all quadratic terms vanish in $\psi'_\mu(y')$. Hence, without loss of generality, we may write

$$\psi_\mu(y)=\lambda(\mu)\,y+A_3(\mu)\,y^3+O(|y|^4), \tag{VI.18}$$

where A_3 is regular in μ for μ near zero. In fact, if $A_3(0)=0$, we can prove that there is a change of coordinates in $\mathbb{R}$ leading from ψ_μ to a ψ'_μ with no third-order or fourth-order terms, and this result can be iterated for A_{2k+1} if $A_{2k+1}(0)=0$. Let us assume here, however, that $A_3(0)=b\neq 0$. Then

$$\psi_\mu(y)=\lambda(\mu)\,y+b\,y^3+O(|\mu|\;|y|^3+|y|^4)$$

and

$$\psi_\mu^2(y)=\psi_\mu[\psi_\mu(y)]=\lambda^2(\mu)\,y+2\lambda_0\,b\,y^3+O(|\mu|\;|y|^3+|y|^4) \tag{VI.19}$$

where

$$\lambda^2(\mu)=1-2\mu\,\xi_1\,T+O(\mu^2).$$

The bifurcated fixed points satisfy

$$y=\psi_\mu^2(y), \tag{VI.20}$$

which is an equation of type (VI.15) in $\mathbb{R}$. But here, in any case, the bifurcation is one-sided. Combining (VI.19) and (VI.20), we find when $b\neq 0$ that

$$\mu(y)=(\lambda_0\,b/\xi_1\,T)\,y^2+O(y^3); \tag{VI.21}$$

there are therefore two fixed points of the map ψ_μ^2. These two fixed points correspond to *a single fixed point of order two for* ψ_μ because if $y(\mu)$ satisfies (VI.20) then $\psi_\mu(y(\mu))=\psi_\mu(\psi_\mu^2\,y(\mu))=\psi_\mu^2(\psi_\mu(y(\mu)))$ is also a solution of (VI.20). In this way, we recover the results of Theorem 4 about the bifurcation of $2T$-periodic solutions.

The stability of the $2T$-periodic flow may be determined by calculations which are nearly identical to those used previously in the case $n=1$; we linearize (VI.19) for small disturbances y' of $y(\mu)$ and obtain again (VI.17) with $2\xi_1$ replacing ξ_1. In this way we recover the results of Theorem 5 about the stability of the $2T$-periodic bifurcating solutions.

VI.4. Bifurcation and Stability of nT-periodic Solutions for $n\geqq 3$

When $n\geqq 3$, the map ψ_μ is in $\mathbb{C}$ and is defined by (VI.13). It is clear that $z=0$ is an isolated fixed point of all iterates of ψ_μ of order p when p is not such that $\lambda_1^p(0)=1$. This means that the iterates ψ_μ^{nk} are the only candidates for bifurcated fixed points near $z=0$. We therefore commence our study with a search for the fixed points of ψ_μ^n. We first simplify the expression of ψ_μ by changing coordinates in $\mathbb{C}$:

$$z'=z+\mu\,\beta(\mu)\,\bar{z}. \tag{VI.22}$$

It is easy to choose a regular β to remove the term in ψ_μ which is linear in $\bar{z}$. Then after changing variables and suppressing the primes,

$$\psi_\mu(z)=\lambda(\mu)\,z+A_2(\mu,z)+O(|z|^3) \tag{VI.23}$$

where

$$\lambda(\mu)=\lambda_0(1-\mu\,\sigma_1\,T)+O(\mu^2).$$

With a second change of coordinates

$$z' = z + \gamma_2(\mu, z),$$

where γ_2 is quadratic in $(z, \bar{z})$, we suppress all quadratic terms in (VI.23). This suppression is always possible (see LANFORD, 1973) if $n \neq 3$ $(\lambda_0^3 \neq 1)$. When $n = 3$, we find the reduced form

(VI.24)
$$\psi_\mu(z) = \lambda(\mu)\, z + \alpha_0(\mu)\, \bar{z}^2 + A_3(\mu, z) + O(|z|^4),$$

where α_0 is regular in μ near zero, and A_3 contains all third order terms in $(z, \bar{z})$ and is regular in μ. When $n \geqq 4$ the reduced form of ψ_μ is given by (VI.24) without $\alpha_0(\mu)\, \bar{z}^2$. A third change of coordinates is now introduced to suppress third order terms. Thus

$$z' = z + \gamma_3(\mu, z)$$

where γ_3 is homogeneous of third order in $(z, \bar{z})$ and regular in $\mu \in \mathscr{V}(0)$. It is known that if $n = 4$ we can choose γ_3 so that

(VI.25)
$$\psi_\mu(z) = \lambda(\mu)\, z + \alpha_1(\mu)\, z^2\, \bar{z} + \alpha_2(\mu)\, \bar{z}^3 + O(|z|^4),$$

where α_1 and α_2 are regular in μ. When $n \geqq 5$, we can choose γ_3 so that (VI.25) holds without the term $\alpha_2(\mu)\, \bar{z}^3$:

(VI.26)
$$\psi_\mu(z) = \lambda(\mu)\, z + \alpha_1(\mu)\, z^2\, \bar{z} + O(|z|^4).$$

We turn now to a calculation of the iterates ψ_μ^n on the reduced forms (VI.24), (VI.25), (VI.26). When $n = 3$, we find by iterating (VI.24) that

(VI.27)
$$\psi_\mu^3(z) = \lambda^3(\mu)\, z + 3\alpha_0\, \bar{\lambda}_0\, \bar{z}^2 + O(|\mu|\, |z|^2 + |z|^3)$$

where $\alpha_0 = \alpha_0(0)$ and $\lambda^3(\mu) = 1 - 3\mu\, \sigma_1\, T + O(\mu^2)$. When $n = 4$, we find by iterating (VI.25) that

(VI.28)
$$\psi_\mu^4(z) = \lambda^4(\mu)\, z + 4\alpha_1\, \bar{\lambda}_0\, z^2\, \bar{z} + 4\alpha_2\, \bar{\lambda}_0\, \bar{z}^3 + O(|\mu|\, |z|^3 + |z|^4)$$

where

$$\alpha_1 = \alpha_1(0), \qquad \alpha_2 = \alpha_2(0) \quad \text{and} \quad \lambda^4(\mu) = 1 - 4\mu\, \sigma_1\, T + O(\mu^2).$$

When $n \geqq 5$, we find by iterating (VI.26) that

(VI.29)
$$\psi_\mu^n(z) = \lambda^n(\mu)\, z + n\, \alpha_1\, \bar{\lambda}_0\, z^2\, \bar{z} + O(|\mu|\, |z|^3 + |z|^4),$$

where

$$\lambda^n(\mu) = 1 - n\, \mu\, \sigma_1\, T + O(\mu^2).$$

Consider the case $n \geqq 5$. The fixed points of ψ_μ^n bifurcating from zero in $\mathbb{C}$ satisfy

(VI.30)
$$z = \psi_\mu^n(z),$$

where $\psi_\mu^{(n)}(z)$ is given by (VI.29); that is

(VI.31) $$-n\mu\sigma_1 Tz + n\alpha_1\bar{\lambda}_0 z^2\bar{z} + O(|\mu|^2|z| + |\mu|\,|z|^3 + |z|^4) = 0.$$

Introducing $z=\varepsilon e^{i\phi}$ into (VI.31), we find, after dividing by $\varepsilon e^{i\phi}$, that

(VI.32) $$-\mu\sigma_1 T + \alpha_1\bar{\lambda}_0\varepsilon^2 + O(|\mu|^2 + |\mu|\,\varepsilon^2 + \varepsilon^3) = 0,$$

where ϕ appears in the terms of higher order. This problem is similar to the one defined by equation (IV.28) and here, as there, if

$$\operatorname{Im}(\alpha_1\bar{\lambda}_0/\sigma_1) \neq 0,$$

then there are no functions $\phi(\varepsilon)$ and $\mu(\varepsilon)$, regular for $\varepsilon\in\mathscr{V}(0)$ with $\mu(0)=0$. This is exactly the result of Theorem 2.

When $n=3$ the fixed point equation (VI.30) may be written

(VI.33) $$-\mu\sigma_1 Tz + \alpha_0\bar{\lambda}_0\bar{z}^2 + O(|\mu|^2|z| + |\mu|\,|z|^2 + |z|^3) = 0.$$

We again set $z=\varepsilon e^{i\phi}$ in (VI.33) and divide the resulting equation by $\varepsilon e^{i\phi}$. This leads to

(VI.34) $$-\mu\sigma_1 T + \alpha_0\bar{\lambda}_0\varepsilon e^{-3i\phi} + O(|\mu|^2 + |\mu|\,\varepsilon + \varepsilon^2) = 0.$$

This equation is the analogue of (IV.25). If $\alpha_0 \neq 0$, which is the analogue of the condition $\lambda_1 \neq 0$ of the Theorem 1, there are three solutions $(\mu(\varepsilon), \phi(\varepsilon))$ of (VI.34), corresponding to *one fixed point of ψ_μ of order three.* (If $z(\mu)$ is a solution of (VI.33), then $\psi_\mu[z(\mu)]$ and $\psi_\mu^2[z(\mu)]$ are also solutions of (VI.33).) This is exactly the result of Theorem 1.

The stability of the fixed point $z(\mu)$ of ψ_μ^3 may be studied by perturbing the fixed point. Setting

$$z = z(\mu) + z'$$

we find that the perturbation of map ψ_μ^3 may be written as

(VI.35) $$z' \mapsto [1 - 3\mu T\sigma_1 + O(\mu^2)]\,z' + [6\alpha_0\bar{\lambda}_0\bar{z}(\mu) + O(\mu^2)]\,\bar{z}' + O(|z'|^2).$$

The eigenvalues of the linearized operator near zero are in the form $1+\mu\tilde{\sigma}_i + o(\mu)$ where $\tilde{\sigma}_i$, $i=1,2$ are the eigenvalues of the matrix

(VI.36) $$3\begin{pmatrix} -T\sigma_1 & 2\alpha_0\bar{\lambda}_0\bar{z}_1 \\ 2\bar{\alpha}_0\lambda_0 z_1 & -T\bar{\sigma}_1 \end{pmatrix}$$

and $z(\mu) = \mu z_1 + O(\mu^2)$. Repeating now the arguments used to discuss the eigenvalues of (V.29), we find that $\tilde{\sigma}_1\cdot\tilde{\sigma}_2 < 0$. Hence, for $\mu\in\mathscr{V}(0)\setminus\{0\}$, there is always one eigenvalue of the linearization of the map (VI.35) which is of modulus greater than 1. It follows again, as in Theorem 5, that the $3T$-periodic bifurcating solution is unstable on both sides of criticality.

When $n=4$, the fixed point equation (VI.30) may be written as

(VI.37) $$-\mu\sigma_1 Tz + \alpha_1\bar{\lambda}_0 z^2\bar{z} + \alpha_2\bar{\lambda}_0\bar{z}^3 + O(|\mu|\,|z|^3 + |z|^4 + |\mu|^2|z|) = 0.$$

Again, with $z=\varepsilon e^{i\phi}$, we obtain after simplification

$$\text{(VI.38)} \qquad -\mu\sigma_1 T+\alpha_1\bar{\lambda}_0\varepsilon^2+\alpha_2\bar{\lambda}_0\varepsilon^2 e^{-4i\phi}+O(|\mu|^2+|\mu|\varepsilon^2+\varepsilon^3)=0.$$

This equation is the same as the principal part of (IV.27). We obtain non-trivial bifurcating fixed points only if

$$\text{(VI.39)} \qquad \left|\frac{\alpha_2}{\sigma_1}\right| \geqq \left|\operatorname{Im}\left(\frac{\alpha_1\bar{\lambda}_0}{\sigma_1}\right)\right|.$$

If the inequality is strict, we can use the implicit function theorem, as in (IV.29), to establish the existence of eight fixed points $(\mu(\varepsilon), \phi(\varepsilon))$ of ψ_μ^4. These eight fixed points correspond to two fixed points of order four for the map ψ_μ. (When $z(\mu)$ is a solution of (VI.37), then $\psi_\mu[z(\mu)]$, $\psi_\mu^2[z(\mu)]$, $\psi_\mu^3[z(\mu)]$ are also solutions of (VI.37).) In this way, then, we obtain all the results of Theorem 3. To study the stability of these fixed points of order four for the map ψ_μ, we perturb the fixed points of ψ_μ^4 setting

$$z=z(\varepsilon)+z' \quad \text{in } \mathbb{C} \text{ and } \mu(\varepsilon)=\mu_2\varepsilon^2+O(\varepsilon^3).$$

This leads to the mapping

$$\text{(VI.40)} \qquad \begin{aligned} z' \mapsto & [1-4\mu_2 T\sigma_1\varepsilon^2+8\alpha_1\bar{\lambda}_0\varepsilon^2+O(\varepsilon^3)]\, z' \\ & +[4\alpha_1\bar{\lambda}_0\varepsilon^2 e^{2i\phi}+12\alpha_2\bar{\lambda}_0\varepsilon^2 e^{-2i\phi}+O(\varepsilon^3)]\,\bar{z}'+O(|z'|^2). \end{aligned}$$

The eigenvalues of the linearization of the mapping (VI.40) are in the form

$$1+\varepsilon^2\tilde{\sigma}_i+o(\varepsilon^2), \qquad i=1,2,$$

where $\tilde{\sigma}_i$ are the eigenvalues of the matrix

$$\text{(VI.41)} \qquad 4\begin{pmatrix} -\mu_2 T\sigma_1+2\alpha_1\bar{\lambda}_0 & \alpha_1\bar{\lambda}_0 e^{2i\phi}+3\alpha_2\bar{\lambda}_0 e^{-2i\phi} \\ \bar{\alpha}_1\lambda_0 e^{-2i\phi}+3\bar{\alpha}_2\lambda_0 e^{2i\phi} & -\mu_2 T\bar{\sigma}_1+2\bar{\alpha}_1\lambda_0 \end{pmatrix}.$$

Reproducing now the arguments employed to discuss the eigenvalues of (V.36) we prove again the results of Theorem 5.

VI.5. Remarks about the Case $n=5$ ($\lambda_0^5=1$)

In the previous sections of this paper we have shown that subharmonic bifurcation of periodic solutions will occur at points of resonance when $n=1, 2, 3, 4$. For all other values of the critical Floquet multiplier λ_0 on the unit circle the periodic solution bifurcates into an invariant torus. SACKER (1964) was the first to prove the existence of an invariant torus for systems in $\mathbb{R}^n$ under the conditions stated above. Later RUELLE & TAKENS (1971) proved nearly the same result by a different method; in their work the resonant point $n=5$ was also excluded. The work of RUELLE & TAKENS has been extended to partial differential equations by IOOSS (1975) and by MARSDEN & MCCRACKEN (1976). We are now going to show how the method of RUELLE & TAKENS can be modified to include the resonant case $n=5$ among those which lead to an invariant torus.*

It is now well known that there is a change of coordinates in $\mathbb{C}$ which reduces ψ_μ to

$$\psi_\mu(z)=\lambda(\mu)\,z+\alpha_1(\mu)\,z^2\,\bar{z}+\alpha_3(\mu)\,\bar{z}^4+O(|z|^5). \tag{VI.42}$$

In polar coordinates: $z=r\,e^{i\phi}$, and $\psi_\mu(r\,e^{i\phi})=Re^{i\Phi}$; we have (VI.42) in the form

$$\begin{aligned} &R=|\lambda(\mu)|\,r+f_1(\mu)\,r^3+f_2(\mu,\phi)\,r^4+O(r^5),\\ &\Phi=\phi+\theta(\mu)+f_3(\mu)\,r^2+f_4(\mu,\phi)\,r^3+O(r^4),\\ &f_1(\mu)+i\,|\lambda(\mu)|\,f_3(\mu)=\alpha_1(\mu)\,e^{-i\theta(\mu)},\\ &f_2(\mu,\phi)+i\,|\lambda(\mu)|\,f_4(\mu,\phi)=\alpha_3(\mu)\,e^{-i\theta(\mu)}\,e^{-5i\phi},\\ &\lambda(\mu)=|\lambda(\mu)|\,e^{i\theta(\mu)},\qquad |\lambda(\mu)|=1-\mu\,\xi_1\,T+O(\mu^2),\\ &\theta(\mu)=-2\pi\,m/5-\mu\,\omega_1\,T+O(\mu^2),\qquad m=1,2,3 \text{ or } 4. \end{aligned} \tag{VI.43}$$

Assume now that $f_1(0)<0$ and define a new radius

$$r=\left(\frac{\mu\,\xi_1\,T}{f_1(\mu)}\right)^{\frac{1}{2}}(1+y\sqrt{\mu}),\qquad \text{for } \mu\geqq 0 \text{ only.} \tag{VI.44}$$

(If $f_1(0)>0$, we define the new radius with $-\mu$ instead of μ.) (VI.44) defines a map $(y,\phi)\mapsto(Y,\Phi)$ with

$$\begin{aligned} &Y=(1+2\mu\,\xi_1\,T)\,y+\mu f_5(\phi)+\mu^{\frac{3}{2}}H_\mu(y,\phi),\\ &\Phi=\phi+\theta_1(\mu)+\mu^{\frac{1}{2}}K_\mu(y,\phi) \end{aligned} \tag{VI.45}$$

and

$$\begin{aligned} &f_5(\phi)=(\xi_1\,T/f_1(0))^{\frac{3}{2}}f_2(0,\phi),\\ &\theta_1(\mu)=\theta(\mu)+\mu\,\xi_1\,Tf_3(0)/f_1(0). \end{aligned}$$

The functions H_μ and K_μ are regular in (y,ϕ) and, by assumption (H.3), $\xi_1<0$.

R.J. SACKER (1964) derived (VI.45) as equation (2.17), p. 28 of his thesis. He treated (VI.45) directly and proved the existence of an invariant circle for the map ψ_μ. We have already noted that this case, $\lambda_0^5=1$, was excluded in the paper of RUELLE & TAKENS (1971). We shall now exhibit a change of coordinates which extends the efficient method of proof of RUELLE & TAKENS to the resonant case with $n=5$. We must change coordinates to eliminate the term $\mu f_5(\phi)$ which has the period $2\pi/5$. Let

$$\tilde{y}=y+y_0(\phi)$$

where y_0 is $2\pi/5$ periodic. Then the new map $(\tilde{y},\phi)\mapsto(\tilde{Y},\Phi)$ will be of the form

$$\begin{aligned} &\tilde{Y}=(1+2\mu\,\xi_1\,T)\,\tilde{y}+\mu^{\frac{3}{2}}\tilde{H}_\mu(\tilde{y},\phi),\\ &\Phi=\phi+\theta_1(\mu)+\mu^{\frac{1}{2}}\tilde{K}_\mu(\tilde{y},\phi) \end{aligned} \tag{VI.46}$$

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* MARSDEN & MCCRACKEN (1976) remark in a footnote (p. 208) that "As D. Ruelle has pointed out, only $n=1,2,3,4$ is needed for the bifurcation theorems as can be seen from the proof in 6A." We have not seen how the proof in 6A would allow one to relax the condition (6.1) which clearly requires that $n=5$ be included in the exceptional set of excluded resonant points.

provided that y_0 is of class C^2 and satisfies

(VI.47) $$\theta_1 y_0'(\phi) - 2\xi_1 T y_0(\phi) + f_5(\phi) = 0, \qquad y_0(\phi + 2\pi/5) = y_0(\phi),$$

where θ_1 is defined by

$$\theta_1(\mu) = -2\pi m/5 + \mu\,\theta_1 + O(\mu^2).$$

The differential equation (VI.47) always has a solution, even when $\theta_1 = 0$ (exercise left to the little child of the reader). The method of RUELLE & TAKENS may now be applied to (VI.46) without excluding the resonant case $n=5$.

VII. Remarks about the Paper of IOOSS (1974)

The problem of bifurcation of nT-periodic solutions from a T-periodic one has been studied by IOOSS (1974b). In that study, however, it was assumed that some coefficients which actually vanish are not zero; for example, it was assumed that quadratic terms were non-zero for $n \geqq 4$. In this section we shall show how the method of IOOSS (1974b) yields the results proved in this paper. We start with a proof that quadratic terms do vanish when $n \neq 1$ or 3.

The Poincaré map (VI.1) may be written as

(VII.1) $$\Phi_\mu(u_0) = S_\mu(T)\,u_0 + B_\mu(u_0, u_0, T) + O(\|u_0\|^3_{D(A)}),$$

where

(VII.2) $$B_\mu(u_0, u_0, T) = -\int_0^T S_\mu(T, s)\,N_\mu^{(2)}[s; S_\mu(s)\,u_0, S_\mu(s)\,u_0]\,ds$$

and $N_\mu^{(2)}$ (the operator N in (III.6)) is the quadratic part in u of $N(t, \mu, u)$. The operators $S_\mu(t, s)$ are defined by the solution of the initial-value problem for $t > s$:

(VII.3) $$\frac{du}{dt} + L(t, \mu)\,u = 0, \qquad u(s) \text{ given in } D(A).$$

The solution of this problem may be expressed as

(VII.4) $$u(t) = S_\mu(t, s)\,u(s) \qquad \text{for } t \geqq s.$$

The regularity in (μ, t, s) of $S_\mu(t, s)$ has been established by IOOSS (1975); the monodromy operator $S_\mu(T)$ may be defined in terms of the operator

$$S_\mu(t) = S_\mu(t, 0) \qquad \text{for } t \geqq 0.$$

Now suppose (H.1), (H.2), (H.3) hold, and let us search for non-zero fixed points of Φ_μ^n in $D(A)$, for $\mu \in \mathscr{V}(0)$. We find that

(VII.5) $$\Phi_\mu^n(u_0) = S_\mu(nT)\,u_0 + B_\mu(u_0, u_0, nT) + O(\|u_0\|^3_{D(A)}).$$

When $\mu = 0$, 1 is an eigenvalue of $S_0(nT)$, simple for $n=1$ or 2, double for $n \geqq 3$ (see Lemma 3). Hence we look for bifurcation associated with the fixed-point problem

(VII.6) $$u_0 = \Phi_\mu^n(u_0).$$

The bifurcation equation holds in a one or two dimensional space $P_0 D(A)$. For instance, for $n \geqq 3$ it takes the form

$$\text{(VII.7)} \qquad \begin{aligned} -[n T\sigma_1 z\zeta(0) + n T\bar{\sigma}_1 \bar{z}\bar{\zeta}(0)]_\mu + P_0 B_0(Y, Y, n T) \\ + O(|\mu| \, \|Y\|^2 + \|Y\|^3) = 0, \end{aligned}$$

where, as in Section VI,

$$\text{(VII.8)} \qquad Y = P_0 u_0 = z\,\zeta(0) + \overline{z\zeta}(0).$$

(When $n=1$ or 2, the term $\overline{z\zeta}(0)$ does not appear in (VII.7) and (VII.8) and $\zeta(0)$ is a real vector in $D(A)$.) To analyze the quadratic terms in z^2, $|z|^2$, $\bar{z}^2$ in (VII.7) we must calculate quantities like

$$P_0 B_0[\zeta(0), \zeta(0), n T], \qquad P_0 B_0[\zeta(0), \bar{\zeta}(0), n T].$$

This calculation leads to expressions of the form

$$\text{(VII.9)} \qquad -\int_0^{nT} (S_0(n T, s) N_0^{(2)}[s; S_0(s)\,\zeta(0), S_0(s)\,\zeta(0)], \zeta^*(0))_H \, ds.$$

As in § VI,

$$\text{(VII.10)} \qquad S_0(s)\,\zeta(0) = e^{-2\pi i r s/T}\,\zeta(s),$$

where ζ is T-periodic and $r = m/n$ ($\zeta \in D^m(J)$).

It is necessary now to show how integrals like (VII.9) may be computed. In particular we need to show that

$$\text{(VII.11)} \qquad S_0^*(n T, s)\,\zeta^*(0) = e^{-2\pi i r s/T}\,\zeta^*(s), \qquad \zeta^* \in D^m(J^*).$$

For this purpose we consider the evolution problem

$$\text{(VII.12)} \qquad \begin{aligned} -\frac{dv}{dt} + L^*(t, \mu)\, v = 0, \quad \text{for } t < s, \\ v(s) \in D(A^*). \end{aligned}$$

The operator

$$\tilde{S}_\mu(t, s), \qquad t \leqq s$$

has the same regularity as $S_\mu(t, s)$ and is defined through (VII.12) in the same way that $S_\mu(t, s)$ is defined through (VII.3). In the case of Navier-Stokes equations, this family of operators acts from H into $D(A^*)$ for $t < s$ (see Iooss [1975]). Let us consider the scalar product in H

$$f(\tau) = (S_\mu(\tau, s)\, u_0, \tilde{S}_\mu(\tau, t)\, v_0)_H, \qquad s \leqq \tau \leqq t,$$

where $u_0 \in D(A)$ and $v_0 \in D(A^*)$. Then

$$\begin{aligned} f'(\tau) = -(L(\tau, \mu) S_\mu(\tau, s) u_0, \tilde{S}_\mu(\tau, t) v_0)_H \\ + (S_\mu(\tau, s) u_0, L^*(\tau, \mu) \tilde{S}_\mu(\tau, t)\, v_0)_H = 0 \end{aligned}$$

because $S_\mu(\tau,s)u_0 \in D(A)$ and $\tilde{S}_\mu(\tau,t)v_0 \in D(A^*)$. Hence $f(\tau)$ is independent of τ. Setting $\tau=t$ and $\tau=s$, we obtain the identity

(VII.13) $$(S_\mu(t,s)u_0, v_0)_H = (u_0, \tilde{S}_\mu(s,t)v_0)_H, \qquad t \geqq s.$$

(VII.13) is true $\forall u_0$ and v_0 in H because $D(A)$ and $D(A^*)$ are dense in H. This proves *Lemma* 7,

(VII.14) $$S_\mu^*(t,s) = \tilde{S}_\mu(s,t), \qquad t \geqq s.$$

Now, using (III.10), we find that

$$\tilde{S}_0(t,0)\,\zeta^*(0) = e^{-2\pi i r t/T}\zeta^*(t) \qquad \text{for } t \leqq 0.$$

Hence for $s \leqq nT$,

$$\tilde{S}_0(s-nT,0)\,\zeta^*(0) = e^{-2\pi i r(s-nT)/T}\zeta^*(s-nT) = e^{-2\pi i r s/T}\zeta^*(s).$$

But, by the T-periodicity of L^* in t, we have

$$\tilde{S}_0(s-nT,0) = \tilde{S}_0(s,nT),$$

and using (VII.14) we prove (VII.11).

It follows now from (VII.11) that the calculation of quantities like (VII.9) involves evaluation of integrals of functions of s on $[0,nT]$ of the form

$$e^{-2\pi i r s[k_1+k_2-k_3]/T}g(s),$$

where g is T-periodic and where $|k_1|+|k_2|+|k_3|=3$, k_1 and k_2 equals 1 or -1 (-1 if one ζ_0 is replaced by $\bar{\zeta}_0$) and $k_3=1$ or -1 (-1 if ζ_0^* is replaced by $\bar{\zeta}_0^*$). Non-zero coefficients have $k_1+k_2-k_3=nk$, $k \in \mathbb{Z}$. It is clear that for $n \neq 1$ or 3, all integrals like (VII.9) vanish and there are no quadratic terms in the bifurcation equation (VII.7).

All of the other coefficients in (VII.7) may be calculated in a similar way. In this way, we may prove all of theorems of IV, V and VI using the method of Iooss (1974d).

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