

*Offprint from "Archive for Rational Mechanics and Analysis",  
Volume 66, Number 2, 1977, P. 99—118  
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Printed in Germany*

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*Factorization Theorems, Stability  
and Repeated Bifurcation*

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## *Factorization Theorems, Stability and Repeated Bifurcation*

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In this paper I prove theorems about the stability of bifurcating solutions without restricting the study to small amplitudes. I do not even always require that the solutions which I call "bifurcating" form connected branches; they may be isolated branches which are not easily described, or not described at all, by conventional types of analysis of bifurcation. I begin in §I with a simple theory of bifurcation and stability of equilibrium solutions of evolution problems in  $R_1$ . This theory gives complete and rigorous results for stability and repeated branching which are in an extensive analogy to results which hold for the stability and repeated branching of steady solutions in Banach space. In the more general problem, I have in mind steady equilibrium solutions of nonlinear evolution equations possessing different patterns of spatial symmetry. These solutions are points in a Banach space and the families with different symmetries may be projected as plane curves (bifurcation curves). In  $R_1$  the projections and the solutions coincide and the theory simplifies enormously.

In §II, I consider the problem of bifurcating subharmonic solutions,  $nT$ -periodic solutions, of  $T$ -periodic nonautonomous systems. On each such solution branch there is a factorization theorem which relates the stability and subsequent bifurcation of that branch to its shape. I show that such factorization theorems hold even at eigenvalues with non-zero Jordan chains of generalized eigenvectors.

In §III, I show how factorization theorems may be used to characterize points of secondary and repeated  $nT$ -periodic bifurcation at a simple eigenvalue.

In §IV, I establish a factorization theorem for the stability of periodic solutions of autonomous systems. This theorem generalizes my earlier work (JOSEPH, 1976; JOSEPH & NIELD, 1976) on factorization theorems for the stability of the Hopf bifurcation. As in the non-autonomous problem treated in §3, the factorization theorem implies a necessary condition for repeated bifurcation at a simple eigenvalue; that is, at a simple zero Floquet exponent. This exponent (or the equivalent unit multiplier) always has a multiplicity greater than one and is typically a double eigenvalue with one proper eigenvector and one generalized eigenvector (a two-link Jordan chain).

The following persons have made informal contributions which helped to clarify and even to correct points of analysis in earlier drafts of this paper: M. CRANDALL, G. HABETLER, G. IOOSS, K. KIRCHGÄSSNER, S. ROSENBLAT and H. WEINBERGER. I was

stimulated to consider secondary and repeated bifurcation by an example of such bifurcation in  $R_2$  due to PIMBLEY (1969) cited in a preliminary draft of the paper by SIMON ROSENBLAT which appears in this number. The theory of stability and bifurcation in  $R_1$  given here and in ROSENBLAT's paper is joint work with Professor ROSENBLAT.

The work of BAUER, KELLER & REISS (1975), which takes up the problem of secondary bifurcation at eigenvalues of higher multiplicity, is complementary to this study of repeated bifurcation at a simple eigenvalue.

My research was supported by the U.S. Army Research Office and under NSF grant 19047.

### I. Stability and Repeated Bifurcation of Solutions of Nonlinear Evolution Equations in a Single Variable

To clarify the mathematical nature of this effort, I first construct a simple theory for evolution equations in  $R_1$  of the form

$$(I.1) \quad V_t + F(\mu, V) = 0$$

where  $F(\mu, 0) = 0$ ,  $F(0, V) \neq 0$  when  $V \neq 0$  and  $F$  together with its first two partial derivatives are continuous functions of  $\mu$  and  $V$  in  $R_1$ ; in particular,

$$(I.2) \quad F(\mu, V + W) = F(\mu, V) + F_V(\mu, V)W + \frac{1}{2}F_{VV}(\mu, V)W^2 + o(W^2)$$

for all  $V, W \in R_1$ . When  $F(\mu, V)$  is analytic we write  $o(W^2) = R(\mu, V, W)W^3$ .

Equilibrium solutions arising from autonomous evolution equations in  $R_1$  are necessarily steady. This means that the study of bifurcation in  $R_1$  is equivalent to finding branches of solutions of the equation

$$(I.3) \quad F(\mu, V) = 0.$$

Suppose that  $V = \varepsilon$  and  $\mu = \mu(\varepsilon)$  is a solution of (I.3) with  $\mu(0) = 0$ . Then

$$(I.4) \quad F(\mu, \varepsilon) = 0 = F_\mu(\mu, \varepsilon)d\mu + F_V(\mu, \varepsilon)d\varepsilon = 0.$$

I define a point of bifurcation to be a double point of (I.4)<sub>1</sub>; that is, a point through which there are two solutions of (I.4)<sub>1</sub> possessing distinct tangents. It is well-known and elementary (e.g., see COURANT, 1957, p. 210) that at such a point

$$(I.5) \quad F_\mu = F_V = 0$$

and

$$(I.6) \quad F_{V\mu}^2 - F_{VV}F_{\mu\mu} > 0$$

provided that at least one of the second derivatives is not null.

A disturbance  $W$  of  $V = \varepsilon$  with  $\mu = \mu(\varepsilon)$  satisfies the equation

$$\frac{dW}{dt} + F_V(\mu(\varepsilon), \varepsilon)W + O(W^2) = 0.$$

Linearizing for small disturbances  $W = e^{-\gamma t} W'$ , we find that  $\gamma(\varepsilon) = F_V(\mu(\varepsilon), \varepsilon)$ .

**Factorization Theorem 1 for the Stability of the solution  $V = \varepsilon, \mu = \mu(\varepsilon)$  of (I.1):**

$$(I.7) \quad \gamma(\varepsilon) = F_V(\mu(\varepsilon), \varepsilon) = -\mu_\varepsilon F_\mu(\mu(\varepsilon), \varepsilon) \equiv \mu_\varepsilon \hat{\gamma}(\varepsilon).$$

Moreover, when  $\varepsilon$  is small

$$(I.8) \quad \hat{\gamma}(\varepsilon) = -\varepsilon F_{V\mu}(0, 0) + O(\varepsilon^2).$$

**Proof.** The second equality in (I.7) follows from (I.4)<sub>2</sub>. The third equality is a definition of  $\hat{\gamma}(\varepsilon)$ , and (I.8) follows from expanding  $F(\mu(\varepsilon), \varepsilon)$  in powers of  $\varepsilon$  through terms of order one in  $\varepsilon$ . It is useful and usual to note that  $F_{V\mu}(0, 0) < 0$  expresses an assumption that  $V=0$  loses stability strictly as  $\mu$  increases past zero. The stability of  $V=0$  is governed in the linearized approximation by the equation  $V_t + F_V(\mu, 0)V = 0$ . Then, with  $V = e^{-\sigma t} V$ , we find that  $\sigma(\mu) = F_V(\mu, 0)$  and  $\sigma(0) = 0$ , so that  $\sigma_\mu(0) = F_{V\mu}(0, 0)$ . It then follows from (I.7) and (I.8) that if  $F_{V\mu}(0, 0) < 0$ , then locally, near  $\varepsilon=0$ , subcritical bifurcating solutions ( $\varepsilon \mu_\varepsilon < 0$ ) are unstable ( $\gamma(\varepsilon) < 0$ ) and supercritical solutions ( $\varepsilon \mu_\varepsilon > 0$ ) are stable ( $\gamma(\varepsilon) > 0$ ).

A point at which  $\mu_\varepsilon = 0$  is a stationary point of the bifurcation curve. A point at which  $\mu_\varepsilon$  changes sign is a *critical point* of the bifurcation curve. If  $\hat{\gamma}(\varepsilon) \neq 0$  at a critical point, then  $\gamma(\varepsilon)$  changes sign when  $\mu_\varepsilon$  does. If  $\hat{\gamma}(\varepsilon) \neq 0$  at a critical point, then  $F_\mu(\mu, \varepsilon) \neq 0$  and  $(\mu, \varepsilon)$  is not a point of bifurcation (see Fig. I.1).

Now we prove Theorem 2 which states that *points  $\varepsilon_0$  at which  $\hat{\gamma}(\varepsilon_0) = 0$  and  $\gamma_\varepsilon(\varepsilon_0) \neq 0$  are points of bifurcation*. At such points, (I.7) shows (I.5) holds and

$$(I.9) \quad \begin{aligned} \gamma_\varepsilon(\varepsilon_0) &= F_{VV}(\mu(\varepsilon_0), \varepsilon_0) + \mu_\varepsilon(\varepsilon_0) F_{V\mu}(\mu(\varepsilon_0), \varepsilon_0) \\ &= -\mu_\varepsilon(\varepsilon_0) F_{\mu V}(\mu(\varepsilon_0), \varepsilon_0) - \mu_\varepsilon^2 F_{\mu\mu}(\mu(\varepsilon_0), \varepsilon_0). \end{aligned}$$

(I.9) implies that at  $\varepsilon = \varepsilon_0$ ,

$$(I.10) \quad d\varepsilon^2 F_{VV} + 2d\varepsilon d\mu F_{V\mu} + d\mu^2 F_{\mu\mu} = 0.$$

Since  $\mu_\varepsilon(\varepsilon_0)$  is real, the discriminant of (I.10),

$$(I.11) \quad F_{V\mu}^2 - F_{VV} F_{\mu\mu} \geq 0,$$

is not negative. Suppose equality holds in (I.11). Then the second or third term in (I.9) must vanish, which is impossible since  $\gamma_\varepsilon(\varepsilon_0) \neq 0$ . It follows that the inequality in (I.11) is actually strict and  $(\mu(\varepsilon_0), \varepsilon_0)$  is a point of bifurcation. The ‘‘Hopf condition’’  $\gamma_\varepsilon(\varepsilon_0) \neq 0$  rules out ‘‘cusp’’ bifurcation (equality in (I.11)) and simultaneous crossing of more than two solutions (all second derivatives of  $F$  vanish at  $\varepsilon = \varepsilon_0$ ).

As one example of the foregoing, consider the equation

$$(I.12) \quad V_t + V(\mu V - 9)(\mu + 2V - V^2) = 0.$$

The bifurcating solutions are

(i)  $V = 0 \quad \forall \mu \in R_1,$

(ii)  $\mu = V^2 - 2V$

and

(iii)  $\mu = \frac{9}{V}.$

The curve (ii) has two bifurcation points and one critical point at  $(\mu, V)=(1, 1)$ . The stability of various branches of (I.12) are indicated in Fig. I.1. It is almost a miracle that examples of secondary bifurcation in  $R_1$  as simple as the one just given seem not to have been discussed in the literature on bifurcation. Of course, everyone knows that  $F(\mu, V)=0$  can have multiple solutions.

In the conventional definition, bifurcation is equivalent to the continuous branching of solutions which, in  $R_1$ , is the continuous branching of solutions of  $F(\mu, V)=0$ . I have added the further requirement, implicit in conventional studies, that the branches have distinct tangents at the point of bifurcation. The conventional understanding and the one used in this section is restrictive since it excludes isolated solutions of  $F(\mu, V)=0$  which are not ultimately connected through repeated branching to the solution  $V=0$ . The hyperbola  $\mu=9/V$  in the third quadrant of Fig. I.1 is just one type of isolated solution which can occur.

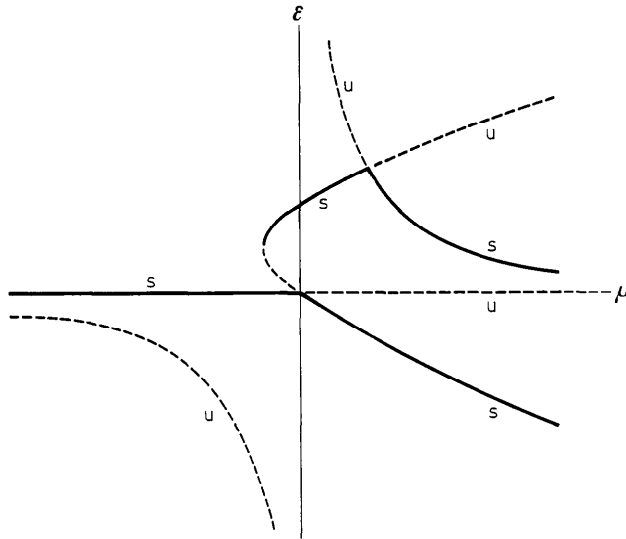


Fig. I.1. Stability and bifurcation of equilibrium solutions of  $V_t = V(9 - \mu V)(\mu + 2V - V^2)$ .

## II. Factorization Theorems for the Stability of Repeated Subharmonic Bifurcation of Forced $T$ -periodic solutions

We turn now to the problem of stability, bifurcation and repeated bifurcation of the nonlinear, nonautonomous, evolution problem

$$(II.1) \quad V_t + F(t, \mu, V) = 0.$$

Here  $F(t, \mu, V)$  is a nonlinear,  $T$ -periodic ( $F(t, \cdot, \cdot) = F(t + T, \cdot, \cdot)$ ) map from  $R_1 \times \mathbb{C} \times H$  into  $H$ , where  $H$  is a Hilbert space with natural scalar product  $(u, v)_H = (\overline{v}, u)_H$ , which carries real vectors  $V \in H$  into real vectors when  $\mu \in \mathbb{C}$  is real. It is assumed that  $F(t, \mu, V)$  possesses two continuous Fréchet derivatives in  $H$  at

every point  $V \in X$  and may be expressed as

$$(II.2) \quad F(t, \mu, V + W) = F(t, \mu, V) + F_V(t, \mu, V|W) + \frac{1}{2}F_{VV}(t, \mu, V|W, W) + o(\|W\|^2)$$

for  $t \in R_1$ ,  $V, W \in X = \text{domain } F_V \hookrightarrow H$  (compactly) and  $\mu$  in a neighborhood of real interval of  $\mathbf{C}$ . It is assumed that  $F(t, \mu, 0) = 0$  so that  $V = 0$  is a solution of (II.1).

We may identify (II.1) with a partial differential equation in which  $V$  is the difference between two solutions driven by prescribed  $T$ -periodic data. One of the two solutions is  $T$ -periodic and it accounts for the appearance of  $t$  in  $F(t, \mu, V)$ . The solution  $V = 0$  of (II.1) corresponds to a forced  $T$ -periodic solution of the original problem.

Now we shall assume that there are subharmonic solutions of (II.1) which satisfy the hypothesis

**H.1.**  $U(t, \varepsilon) = U(t + nT, \varepsilon)$ ,  $n \in \mathbf{N}^*$  (positive integers) and  $\mu = \mu(\varepsilon)$  are continuously differentiable functions of the amplitude  $\varepsilon$  on some open interval  $I(\varepsilon)$ . Moreover  $\mu_\varepsilon(\varepsilon) \neq 0$  except possibly at isolated points of  $I(\varepsilon)$ .

*Remark 1.* We do not require that  $I(\varepsilon)$  be a neighborhood of the origin so as to include isolated subharmonic solutions which are not bifurcating solutions in the usual sense.

To study the stability of  $U(t, \varepsilon)$ , we linearize (II.1) around  $U$ , using  $U + W$  in (II.2) and, following Floquet theory, we set  $W = e^{-\gamma t} \Gamma(t)$  where

$$(II.3) \quad -\gamma \Gamma + \mathbb{J}(\varepsilon) \Gamma = 0, \quad \Gamma(t) = \Gamma(t + nT)$$

and

$$(II.4) \quad \mathbb{J}(\varepsilon)(\cdot) = (\cdot)_t + F_V(t, \mu(\varepsilon), U(t, \varepsilon))(\cdot)$$

is a closed linear operator mapping  $X_{nT} = \{x: x \in X, x(t) = x(t + nT)\}$  into the Hilbert space  $H_{nT} = \{y: y \in H, y(t) = y(t + nT)\}$ . The scalar product on  $H_{nT}$  is

$$[u, v]_{nT} = \frac{1}{nT} \int_0^{nT} (u(t), v(t))_H dt, \quad \forall u, v \in H_{nT}.$$

Now I state the basic spectral assumptions:

**H.2.**  $\mathbb{J}(\varepsilon)$  is a Fredholm operator from a complex Banach space  $\text{dom}(\mathbb{J}) = X_{nT} \rightarrow H_{nT} \hookrightarrow X_{nT}$  (compactly with a compact resolvent from  $X_{nT} \rightarrow X_{nT}$ ). Since the inverse of  $\mathbb{J}$  is a compact operator, the spectrum of  $\mathbb{J}$  is entirely of eigenvalues of finite multiplicity and there is a unique adjoint operator  $\mathbb{J}^*$  from  $\text{dom}(\mathbb{J}^*) \rightarrow H_{nT}$ , defined by

$$(II.5) \quad [\mathbb{J}u, v]_{nT} = [u, \mathbb{J}^*v]_{nT}, \quad \forall u \in X, v \in X^*$$

whose spectrum consists of the same eigenvalues  $\gamma$  as  $\mathbb{J}$ . The eigenvectors  $\Gamma$  belonging to  $\gamma$  lie on the null space  $N_\gamma$  of  $-\gamma + \mathbb{J}$ . The number, say  $n_1 \geq 1$ , of the independent eigenvectors  $\Gamma$  is the dimension of the null space  $N_\gamma$ . The dimension of the null space  $N_\gamma^*$  of the adjoint operator  $-\bar{\gamma} + \mathbb{J}^*$  is also  $n_1$  and

$$(II.6) \quad -\bar{\gamma} \Gamma^* + \mathbb{J}^* \Gamma^* = 0, \quad \Gamma^*(t) = \Gamma^*(t + nT)$$

where  $\bar{\gamma}$  is the complex conjugate of  $\gamma$ .

**H.3.**  $\gamma(\varepsilon)$  is an algebraically simple eigenvalue of  $\mathbb{J}(\varepsilon)$  for all  $\varepsilon \in I(\varepsilon)$  except possibly on an exceptional set  $\{\varepsilon_j\}$  of isolated points across which  $\Gamma(t, \varepsilon)$  and  $\Gamma^*(t, \varepsilon)$  are continuous and at which  $\mu_\varepsilon(\varepsilon_j) \neq 0$ . H.2 and H.3 imply  $\gamma(\varepsilon)$  is continuous even at  $\varepsilon = \varepsilon_j$  and  $\gamma(\varepsilon)/\mu_\varepsilon(\varepsilon)$  is continuous at  $\varepsilon = \varepsilon_j$ . On  $I(\varepsilon) \setminus \{\varepsilon_j\}$ ,  $\gamma\{\varepsilon\}$  is simple and

$$(II.7) \quad [\Gamma(\varepsilon), \Gamma^*(\varepsilon)]_{nT} = f(\varepsilon) \neq 0.$$

The scalar product (II.7) can vanish at points in the exceptional set.

**Theorem 3.** Suppose H.1, H.2 and H.3 hold and assume that

$$(II.8) \quad [U_\varepsilon(\varepsilon), \Gamma^*(\varepsilon)]_{nT} \neq 0, \quad \varepsilon \in I(\varepsilon) \setminus \{\varepsilon_j\}$$

Then there is a unique continuous function  $\hat{\gamma}(\varepsilon)$  defined on all of  $I(\varepsilon)$  such that

$$(II.9) \quad \gamma(\varepsilon) = \mu_\varepsilon(\varepsilon) \hat{\gamma}(\varepsilon)$$

where, on  $I(\varepsilon) \setminus \{\varepsilon_j\}$ ,

$$(II.10) \quad \hat{\gamma}(\varepsilon) = -[F_\mu(t, \mu(\varepsilon), U(\varepsilon)), \Gamma^*(\varepsilon)]_{nT} / [U_\varepsilon, \Gamma^*]_{nT}.$$

Moreover,

$$(II.11) \quad \Gamma = b(\varepsilon) (U_\varepsilon(\varepsilon) + \mu_\varepsilon(\varepsilon) q(\varepsilon))$$

where  $b(\varepsilon)$  is a normalizing factor for  $\Gamma$  and

$$(II.12) \quad q(t, \varepsilon) = q(t + nT, \varepsilon)$$

is uniquely determined by

$$(II.13) \quad \hat{\gamma} U_\varepsilon + F_\mu(t, \mu(\varepsilon), U(\varepsilon)) + \{\gamma - \mathbb{J}(\varepsilon)\} q = 0$$

and

$$(II.14) \quad [q, \Gamma^*]_{nT} = 0.$$

**Proof.** Differentiating

$$(II.15) \quad U_t + F(t, \mu(\varepsilon), U(\varepsilon)) = 0$$

with respect to  $\varepsilon$ , we get

$$(II.16) \quad \mathbb{J} U_\varepsilon + \mu_\varepsilon F_\mu(t, \mu(\varepsilon), U(\varepsilon)) = 0.$$

Since  $\Gamma^*$  satisfies (II.6), we find, using (II.16), that

$$(II.17) \quad -\mu_\varepsilon [F_\mu, \Gamma^*]_{nT} = [\mathbb{J} U_\varepsilon, \Gamma^*]_{nT} = [U_\varepsilon, \mathbb{J}^* \Gamma^*]_{nT} = \gamma(\varepsilon) [U_\varepsilon, \Gamma^*]_{nT}.$$

Equation (II.17) holds at all points where  $\gamma(\varepsilon)$  is an algebraically simple

eigenvalue and also, by continuity, across points in the exceptional set where  $\gamma(\varepsilon)$  is not algebraically simple. Solving (II.17) for  $\gamma(\varepsilon)$ , we find (II.10). Now, combining (II.3) and (II.11), we get

$$-\mu_\varepsilon \hat{\gamma}(\varepsilon)(U_\varepsilon + \mu_\varepsilon q) + \mathbb{J}(U_\varepsilon + \mu_\varepsilon q) = 0.$$

Elimination of  $\mathbb{J}U_\varepsilon$  with (II.16) leads to

$$(II.18) \quad \mu_\varepsilon \{ \hat{\gamma} U_\varepsilon + F_\mu(t, \mu, U) + (\gamma - \mathbb{J}) q \} = 0.$$

The coefficient of  $\mu_\varepsilon$  in (II.18) vanishes when  $\mu_\varepsilon \neq 0$  and, by continuity, even when  $\mu_\varepsilon = 0$  at a point. This proves (II.13). Since  $-\gamma + \mathbb{J}$  is a Fredholm operator and (II.17) holds, (II.13) is uniquely solvable with  $[q, F^*]_{nT} = 0$  whenever  $\gamma(\varepsilon)$  is algebraically simple. This proves Theorem 3. In Theorem 5 we give the form of the factorization at a point  $\varepsilon_j$  in the exceptional set  $\{\varepsilon_j\}$ .

*Remark 2.* A complete local theory for analytic problems which contain Navier-Stokes problems as a special case has been given by IOOSS & JOSEPH (1977) under the assumptions that  $F$  is analytic,  $I(\varepsilon)$  is a neighborhood of the origin and that conditions analogous to H.2 and H.3 hold. They show that  $nT$ -periodic solutions bifurcate for values  $n=1, 2, 3, 4$  and no others. In all other cases, the  $T$ -periodic solution bifurcates into a torus.

The solutions on the torus are not well understood. In some cases these solutions may be regarded as close to quasi-periodic solutions with two frequencies:  $\omega_0 = 2\pi/T$  and a frequency  $\omega(\varepsilon)$  which varies with the amplitude (JOSEPH, 1973; IOOSS, 1975). The presence of a frequency  $\omega(\varepsilon)$  which varies continuously with  $\varepsilon$  is a topologically generic property of solutions on the torus, but the variation need not be differentiable. All this is in sharp contrast with the subharmonic bifurcating solutions which have fixed periods  $nT$ ,  $n=1, 2, 3, 4$  independent of  $\varepsilon$ .

*Remark 3.* In many problems  $\gamma(\varepsilon)$  is an algebraically simple eigenvalue of  $\mathbb{J}(\varepsilon)$  for nearly all values of  $\varepsilon \in I(\varepsilon)$ . The stability of the  $nT$ -periodic solution is controlled by the eigenvalue  $\gamma(\varepsilon)$  with the smallest  $\text{re } \gamma(\varepsilon)$ . At the origin,  $\gamma(0) = 0$  is a double semi-simple eigenvalue of  $\mathbb{J}(0)$ . The Floquet exponent  $-\sigma(\mu)$  corresponding to the Floquet representation  $Z = e^{-\sigma(\omega)t} \zeta(t)$ ,  $\zeta(t) = \zeta(t+T)$  of the solution of the linearized problem  $Z_t + F_V(t, \mu, 0)Z = 0$  for the stability of  $V=0$ , is a simple eigenvalue of the operator  $J(\mu) = d/dt + F_V(t, \mu, 0)$ , from  $X_T$  to  $H_T$ , at criticality.  $\mu=0$ ,  $\sigma(0) = 2\pi ir/T$ ,  $r \leq 0 < 1$ . If  $\zeta(t)$  satisfying  $\frac{-2\pi ir}{T} \zeta + J(0)\zeta = 0$  is an eigenfunction of  $J(0)$  with a simple eigenvalue  $2\pi ir/T$ , then  $\bar{\zeta}$  is also an eigenfunction of  $J(0)$  belonging to the eigenvalue  $-2\pi ir/T$ , and  $Z$  and  $\bar{Z}$  are independent solutions of the linearized problem with  $\mu=0$ . The Floquet multiplier  $\lambda(\mu) = e^{-\sigma(\omega)T}$  is an eigenvalue of the monodromy operator for the linearized problem. At criticality,  $\lambda(0) = e^{-2\pi ir}$  and  $\bar{\lambda}(0)$  are eigenvalues corresponding to solutions  $Z$  and  $\bar{Z}$ . These solutions are  $nT$ -periodic if and only if  $r = m/n$ ,  $m, n \in \mathbb{N}^*$ ,  $m < n$ . In two cases:  $m/n = 0/1, 1/2$ ,  $\lambda_0 = 1, -1$  is real valued and  $Z = \bar{Z}$ . These two cases and only these two lead to  $T$  and  $2T$  periodic solutions bifurcating at a simple eigenvalue zero of the operator  $\mathbb{J}(0)$ . For these two cases



( $n=1$  and  $n=2$ ), we find that as  $\varepsilon \rightarrow 0$ ,  $\mu(0) \rightarrow 0$

$$(\Gamma(\varepsilon), \Gamma^*(\varepsilon)) \rightarrow (Z(t), Z^*(t)) = (Z(t+nT), Z^*(t+nT)),$$

$$U(t, \varepsilon) \rightarrow \varepsilon Z + O(\varepsilon^2)$$

and

$$F_\mu(t, \mu, U) \rightarrow \varepsilon F_{V_\mu}(t, 0, 0|Z).$$

It is not hard to establish that

$$\sigma_\mu(0) = [F_{V_\mu}(t, 0, 0|Z), Z^*]_{nT}.$$

The results of IOOSS & JOSEPH (1977) which were summarized under Remark 3 and Theorem 3 imply

**Theorem 4** (IOOSS & JOSEPH, 1977). *The Floquet exponent  $\gamma(\varepsilon)$  for the stability of  $nT$ -periodic bifurcating solutions when  $n=1$  or  $n=2$  may be represented as  $\gamma(\varepsilon) = \mu_\varepsilon(\varepsilon) \hat{\gamma}(\varepsilon)$  where*

$$(II.19) \quad \lim_{\varepsilon \rightarrow 0} \hat{\gamma}(\varepsilon) = -\varepsilon [F_{V_\mu}(t, 0, 0|Z), Z^*]_{nT} = -\varepsilon \sigma_\mu(0).$$

*If  $V=0$  loses stability strictly as  $\mu$  increases through zero, then subcritical bifurcating solutions are unstable and supercritical bifurcating solutions are stable.*

**Proof.**  $[Z, Z^*]_{nT} = [U_\varepsilon, \Gamma^*]_{nT} = 1.$

*Remark 4.* When  $n=3$  or  $n=4$ , the analysis of the stability of the bifurcating subharmonic solutions requires analysis of perturbations of a semi-simple, double eigenvalue to separate the branches of  $\gamma(\varepsilon)$ . Without information equivalent to that given by such an analysis, it would not be possible to specify the linear combinations of  $(Z, \bar{Z})$  and  $(Z^*, \bar{Z}^*)$  which give the limiting  $\varepsilon \rightarrow 0$  values of  $\Gamma$  and  $\Gamma^*$ .

*Remark 5.* Theorem 3 implies that if  $\text{re } \hat{\gamma}(\varepsilon) \neq 0$  at a point where  $\mu_\varepsilon(\varepsilon)$  changes sign (critical point), then  $\text{re } \gamma(\varepsilon)$  changes sign across this point. This implies a change of stability at a critical point. As an application, we consider the  $3T$ -periodic bifurcating solutions studied by IOOSS & JOSEPH (1977). These solutions are unstable locally, for small  $\varepsilon$ , on both sides of criticality. Moreover,  $\gamma(\varepsilon)$  separates into two simple branches. Theorem 3 shows that the unstable subcritical solution probably regains stability when it turns around at a critical point (see Fig. I.1).

Now I will prove a factorization theorem which is valid at points in the exceptional set independent of the continuity assumption under H.3. The construction used in this theorem is more complicated than the one given in Theorem 3 and is less useful in applications. The construction emphasizes the difficulties which arise at eigenvalues which are not semi-simple. These difficulties, as we shall show in Theorem 8, cannot be avoided in problems of repeated bifurcation of  $T(\varepsilon)$ -periodic solutions from  $T(\varepsilon)$ -periodic solutions of autonomous problems; for such problems  $\gamma=0$  is never an algebraically simple eigenvalue of  $\mathbb{J}(\varepsilon)$ .

We start by listing some properties of the spectrum of the operator  $\mathbb{J}_\gamma \equiv -\gamma + \mathbb{J}$  (and  $\mathbb{J}_\gamma^* \equiv -\bar{\gamma} + \mathbb{J}^*$ ). Our assumption H.2 implies that this is a Fredholm operator and has a compact resolvent  $(\mathbb{J}_\gamma + \lambda)^{-1}$ ,  $\lambda \notin \Sigma \mathbb{J}_\gamma$ , mapping  $X_{n_T}$  into itself. Putting  $\lambda = 1$ , we write  $\mathbb{J}_\gamma u = \mathbb{J}_\gamma u + u - u = 0$ ; hence  $u = Tu$  where  $T = (\mathbb{J}_\gamma + 1)^{-1}$  is compact. We are therefore in the frame of the Riesz-Schauder theory. But it is more convenient, and is mathematically justified, to work directly with the closed operator  $\mathbb{J}_\gamma$ . (The facts of the Riesz-Schauder theory for  $\mathbb{J}_\gamma$  may be obtained by direct inversion of the Jordan chain equations for generalized eigenfunctions in the generalized null spaces (see RIESZ-NAGY, 1965, § 80 and SATTINGER, 1973, § 31).) We first define generalized null spaces

$$N_l = \{\psi : \mathbb{J}_\gamma^l \psi = 0\},$$

$$n_l = \dim N_l$$

for  $1 \leq l \leq v$ . We also have generalized null spaces for the adjoint  $\mathbb{J}_\gamma^*$  of  $\mathbb{J}_\gamma$ ,

$$N_l^* = \{\psi^* : \mathbb{J}_\gamma^{*l} \psi^* = 0\},$$

$$n_l^* = \dim N_l^*$$

for  $1 \leq l \leq v$ . The integer  $v \geq 1$  is the largest number for which the inclusion

$$N_1 \subset N_2 \subset \dots \subset N_v = N_{v+k} \quad \forall k \in \mathbb{N}$$

is strict. The multiplicity of the eigenvalue  $\gamma$  is defined by

$$n_v = \text{algebraic multiplicity of } \gamma,$$

$$n_1 = \text{geometric multiplicity of } \gamma$$

and, of course,  $n_v \geq n_1$ . The vectors  $\psi \in N_1$  are the proper eigenvectors of  $\gamma$ ; they satisfy (II.3). The vectors  $\psi \in N_l$ ,  $l > 1$ , are called generalized eigenvectors. There are no generalized eigenvectors when  $n_1 = n_v$ . In this case  $\gamma$  is a semi-simple eigenvalue of  $\mathbb{J}(\varepsilon)$ .

Generalized eigenvectors may be found as solutions of the Jordan chain equations. Suppose  $\psi_i^{(1)}$  are independent eigenfunctions of  $\mathbb{J}_\gamma$ . There are  $n_1 \geq 1$  of them

$$\mathbb{J}_\gamma \psi_i^{(1)} = 0, \quad i = 1, 2, \dots, n_1.$$

For each  $i$  we get Jordan chain equations

$$\mathbb{J}_\gamma \psi_i^{(2)} = (\mathbb{J}_\gamma + I) \psi_i^{(1)} = \psi_i^{(1)},$$

$$\mathbb{J}_\gamma \psi_i^{(3)} = (\mathbb{J}_\gamma + I) \psi_i^{(2)} = \psi_i^{(1)} + \psi_i^{(2)},$$

and so on up to the  $v_i \leq v$  the link chain

$$\mathbb{J}_\gamma \psi_i^{(v_i)} = (\mathbb{J}_\gamma + I) \psi_i^{(v_i-1)} = \psi_i^{(1)} + \psi_i^{(2)} + \dots + \psi_i^{(v_i-1)}$$

where, for some one or more of the values  $i = 1, \dots, n_1$ ,  $v_i = v$ . There are therefore  $m \geq v$  generalized eigenvectors in  $N_v$ . And there are also  $m \geq v$  generalized adjoint eigenvectors in  $N_v^*$  which may be generated as solutions of adjoint

Jordan chain equations

$$(II.21) \quad \mathbb{J}_{\gamma}^* \psi_j^{*(v)} = 0, \quad j = 1, 2, \dots, n_1,$$

$$\mathbb{J}_{\gamma}^* \psi_j^{*(v-1)} = (\mathbb{J}_{\gamma}^* + I) \psi_j^{*(v)} = \psi_j^{*(v)},$$

and

$$\mathbb{J}_{\gamma}^* \psi_j^{*(l-1)} = (\mathbb{J}_{\gamma}^* + I) \psi_j^{*(l)} = \psi_j^{*(l)} + \dots + \psi_j^{*(v)}.$$

For a fixed  $j$ ,  $\psi_j^{*(m)} = 0$  when  $m < v_j$ . It is easy to show by direct calculation using the Jordan chain equations that

$$[\psi_i^{(m)}, \psi_j^{*(m)}]_{nT} = 0 \quad \text{if } m \neq n$$

and that

$$[\psi_i^{(m)}, \psi_j^{*(m)}]_{nT} = [\psi_i^{(1)}, \psi_j^{*(1)}]_{nT}$$

for each and every integer  $m = 1, 2, \dots, v$ . Moreover we can always choose a canonical basis in  $N_1$  such that  $[\psi_i^{(1)}, \psi_j^{*(1)}]_{nT} = \delta_{ij}$ ; hence

$$(II.22) \quad [\psi_i^{(m)}, \psi_j^{*(m)}]_{nT} = \delta_{ij} \delta^{mn}.$$

Equation (II.22) shows that  $N_1 \perp N_1^*$  when  $v > 1$  (when  $\gamma$  is *not* a semi-simple eigenvalue of  $\mathbb{J}$ ).

The Fredholm alternative states that

$$(II.23) \quad (\mathbb{J} - \gamma)\theta = f \in H_{nT}$$

has a solution  $\theta \in \text{dom } \mathbb{J} = X_{nT}$ , unique to within the addition of eigenvectors on the null space  $N_1$ , if and only if  $f$  is orthogonal to the eigenvectors on  $N_1^*$ ; that is, if and only if

$$(II.24) \quad [f, \psi_i^*]_{nT} = 0, \quad i = 1, \dots, n_1.$$

**Theorem 5.** *Suppose H.1, H.2 and H.3 hold and let  $l$  be the largest positive integer,  $l = \max\{1, 2, \dots, v\}$ , such that*

$$(II.25) \quad [U_\varepsilon, \psi_i^{*(l)}]_{nT} \neq 0.$$

*Then there is a unique*

$$(II.26) \quad \hat{\gamma}(\varepsilon) = -[F_\mu, \psi_i^{*(l)}]_{nT} / [U_\varepsilon, \psi_i^{*(l)}]_{nT},$$

*independent of  $i = 1, 2, \dots, n_1$ , such that*

$$(II.27) \quad \gamma(\varepsilon) = \mu_\varepsilon(\varepsilon) \hat{\gamma}(\varepsilon).$$

*Moreover, there is a unique  $\Gamma \in N_1$  given by*

$$(II.28) \quad \Gamma(t, \varepsilon) = U_\varepsilon + \mu_\varepsilon q = \sum_{l=1}^{n_1} a_l \psi_l^{(1)}$$

*where*

$$(II.29) \quad a_l = [U_\varepsilon, \psi_l^{*(1)}]_{nT} + \mu_\varepsilon [q, \psi_l^{*(1)}]_{nT}$$

and  $q(t, \varepsilon) = q(t + nT)$  satisfies

$$(II.30) \quad (-\gamma + \mathbb{J})q = \hat{\gamma} U_\varepsilon + F_\mu(t, \mu, U).$$

Since (II.16) is solvable by assumption, the solvability requirement of the Fredholm alternative implies that

$$(II.31) \quad \gamma [U_\varepsilon, \psi_i^{(v)}]_{nT} + \mu_\varepsilon [F_\mu, \psi_i^{(v)}]_{nT} = 0$$

for each  $\psi_i^{*(v)} \in N_1^*$ . Since  $\gamma \in \mathbb{C}$  and  $\mu_\varepsilon \in \mathbb{R}$  are each assumed to a definite value, their ratio which is proportional to the ratio of the scalar products are independent of  $i$ . Or if the scalar products in (II.31) vanish, then using (II.21)<sub>2</sub> we find that

$$(II.32) \quad \gamma [U_\varepsilon, \psi_i^{*(v-1)}]_{nT} + \mu_\varepsilon [F_\mu, \psi_i^{*(v-1)}]_{nT} = 0.$$

Repeating the argument following (II.31), using the adjoint chain equations (II.21), we come to the first case where the scalar products in (II.32), with  $\psi_i^{*(v)}$  replacing  $\psi_i^{*(v-1)}$ , do not vanish. Then we find (II.26) and (II.27). Equation (II.30), which arises from the same factorization which leads to (II.13) in Theorem 3, is solvable because  $\mathbb{J}_\gamma$  is a Fredholm operator and (II.31) holds. The  $a_i$ , given uniquely by (II.29), follow from (II.22). This proves Theorem 5.

*Remark 6.* Theorems 3 and 5 make statements about one certain eigenvalue  $\gamma(\varepsilon)$  of  $\mathbb{J}(\varepsilon)$ . These statements are also statements about stability only if all the other Floquet exponents of  $\mathbb{J}(\varepsilon)$  have positive real parts  $\forall \varepsilon \in I(\varepsilon)$ . The theorems hold independent of possible changes in an ordering of the eigenvalues based on the sign of their real part; but in this case Theorems 3 and 5 are not physically interesting.

### III. Repeated Bifurcation of $nT$ -Periodic Solutions at a Simple Eigenvalue

We turn next to the problem of repeated bifurcation and seek a unique  $\tau = nT$ -periodic solution  $U(t, \varepsilon) + W(t)$  of the governing equation (II.1). The function  $W(t)$  satisfies  $W_t + F(t, \mu, U + W) = 0$  where  $F$  is given by (II.2) with  $\mu = \mu(\varepsilon)$  and  $V = U(t, \varepsilon) = U(t + \tau, \varepsilon)$ ; that is,

$$\mathbb{J}(\varepsilon)W + \frac{1}{2}F_{VV}(t, \mu(\varepsilon), U(t, \varepsilon); W, W) + R(t, \mu(\varepsilon), U(t, \varepsilon), W; W, W, W) = 0.$$

Since  $W$  is to be used to describe secondary (or repeated) bifurcation, we introduce an amplitude  $\delta$ ,  $\delta^2 = [W, W]$  and a function  $w(t, \delta)$  such that

$$W(t, \delta) = \delta w(t, \delta), \quad W(t, 0) = 0,$$

where  $w(t, \delta)$  satisfies

$$(III.1) \quad \mathbb{J}(\varepsilon)w + \frac{\delta}{2}F_{VV}(t, \mu(\varepsilon), U(t, \varepsilon); w, w) + \delta^2 R(t, \mu, U, \delta w; w, w, w) = 0,$$

$$(III.2) \quad w(t, \delta) = w(t + 2\pi, \delta)$$

and

$$(III.3) \quad [w, w] = 1.$$

We want to show that there is a unique  $w$  bifurcating from the  $\tau$ -periodic solution when  $\hat{\gamma}(\varepsilon_0) = 0$  and  $\gamma_\varepsilon(\varepsilon_0) \neq 0$ . If  $\gamma(\varepsilon_0) = 0$ , then

$$(III.4) \quad \mathbb{J}(\varepsilon_0) \Gamma_0 = 0,$$

and

$$(III.5) \quad \mathbb{J}^*(\varepsilon_0) \Gamma_0^* = 0$$

where  $\Gamma_0$  and  $\Gamma_0^*$  are  $\tau$ -periodic. In Theorem 6 we shall suppose that the operator  $F(t, \mu, U)$  is analytic in  $\mu$  and  $U$  in the usual sense and that the bifurcating solution is analytic in  $\varepsilon$  on  $I(\varepsilon)$ . In the rest of this section,  $[\cdot, \cdot]_{nT} \equiv [\cdot, \cdot]$ .

**Theorem 6.** *Let the hypotheses of Theorem 3 hold in the analytic case and assume that  $\gamma_\varepsilon(\varepsilon_0) \neq 0$  when*

$$(III.6) \quad \hat{\gamma}(\varepsilon_0) = - [F_\mu(\mu(\varepsilon_0), U(\varepsilon_0)), \Gamma_0^*] / [U_\varepsilon(\varepsilon_0), \Gamma_0^*] = 0$$

is an algebraically simple eigenvalue of  $\mathbb{J}(\varepsilon_0)$ . Then  $(\mu(\varepsilon_0), \varepsilon_0)$  is a point of secondary (or repeated) bifurcation. There are unique analytic functions of the amplitude  $\delta$ ,  $w(t, \delta) = w(t + \tau, \delta)$  and  $v(\delta)$  where  $\delta$  is the amplitude of  $W(t, \delta) = \delta w(t, \delta)$ ,  $\delta^2 = [W, W]_{nT}$  such that

$$(III.7) \quad w(t, 0) = \Gamma_0(t) = b(\varepsilon_0) (U_\varepsilon(t, \varepsilon_0) + \mu_\varepsilon(\varepsilon_0) q(t, \varepsilon_0))$$

and

$$(III.8) \quad \varepsilon = v(\delta), \quad \varepsilon_0 = v(0);$$

$w(t, \delta)$  and  $\varepsilon = v(\delta)$  satisfy (III.1), (III.2) and (III.3) when  $\delta$  is small. The new bifurcating solution is given parametrically by

$$(III.9) \quad \begin{aligned} V &= U(t, v(\delta)) + W(t, \delta), \\ \mu &= \mu(v(\delta)) = \bar{\mu}(\delta). \end{aligned}$$

**Proof.** We shall construct the functions (III.9) as a power series:

$$(III.10) \quad \begin{pmatrix} w(t, \delta) - w_0(t) \\ v(\delta) - \varepsilon_0 \end{pmatrix} = \sum_1 \delta^n \begin{pmatrix} w^{(n)}(t) \\ v^{(n)} \end{pmatrix}.$$

To find the coefficients in the series (III.10), we put  $\varepsilon = v(\varepsilon)$  into (III.1) and insert the series representation (III.10) into (III.1), (III.2) and (III.3). By identification, we find that

$$(III.11) \quad \mathbb{J}_0 w^{(0)} = 0, \quad [w^{(0)}, w^{(0)}] = 1;$$

$$(III.12) \quad \mathbb{J}_0 w^{(1)} + v^{(1)} \mathbb{J}_1 w^{(0)} + \frac{1}{2} F_{VV}(\mu(\varepsilon_0), U(t, \varepsilon_0); w^{(0)}, w^{(0)}) = 0, \quad [w^{(0)}, w^{(1)}] = 0;$$

$$(III.13) \quad \mathbb{J}_0 w^{(n)} + v^{(n)} \mathbb{J}_1 w^{(0)} + f^{(n)} = 0, \quad [w, w]^{(n)} = 0$$

where

$$w^{<n>}(t) = w^{<n>}(t + \tau), \quad n \geq 0,$$

$$\mathbb{J}_0 = \mathbb{J}(\varepsilon_0),$$

$$\mathbb{J}_n = \frac{1}{n!} \frac{d^n}{d\varepsilon^n} \mathbb{J}(\varepsilon)|_{\varepsilon = \varepsilon_0},$$

$f_n$  depends on  $w^{<l>}$  and  $v^{<v>}$  for  $v, l < n$ .

Since zero is a simple eigenvalue of  $\mathbb{J}_0$ , we find, comparing (III.11) and (III.4), that

$$w^{<0>}(t) = \Gamma_0(t) = b(\varepsilon_0) (U_\varepsilon(\varepsilon_0) + \mu_\varepsilon(\varepsilon_0) q(\varepsilon_0)).$$

Since  $\mathbb{J}_0$  is a Fredholm operator and zero is a simple eigenvalue of  $\mathbb{J}_0$ , (III.12) and (III.13) are solvable if

$$(III.14) \quad v^{<1>} [\mathbb{J}_1 \Gamma_0, \Gamma_0^*] + \frac{1}{2} [F_{VV}(\mu(\varepsilon_0), U(t, \varepsilon_0); \Gamma_0, \Gamma_0), \Gamma_0^*)] = 0$$

and

$$(III.15) \quad v^{<n>} [\mathbb{J}_1 \Gamma_0, \Gamma_0^*]_{nT} + [f^{<n>}, \Gamma_0^*] = 0.$$

To demonstrate that  $[\mathbb{J}_1 \Gamma_0, \Gamma_0^*] \neq 0$ , we differentiate the equation  $-\gamma \Gamma + \mathbb{J}(\varepsilon) \Gamma = 0$  at  $\varepsilon = \varepsilon_0$ :

$$(III.16) \quad \mathbb{J}_0 \Gamma_\varepsilon + \mathbb{J}_1 \Gamma_0 - \gamma_\varepsilon \Gamma_0 = 0.$$

Equation (III.16) is solvable if and only if

$$[\mathbb{J}_1 \Gamma_0, \Gamma_0^*] = \gamma_\varepsilon(\varepsilon_0) [\Gamma_0, \Gamma_0^*].$$

Hence, since  $\gamma_\varepsilon(\varepsilon_0) \neq 0$  by assumption,  $[\mathbb{J}_1 \Gamma_0, \Gamma_0^*] \neq 0$ , and we may solve (III.14) and (III.15) for  $v^{<n>}$ . Now  $\mathbb{J}_0$  is a Fredholm operator so that (III.12) and (III.13) are solvable in the form

$$w^{<n>} = A_n \Gamma_0 + \tilde{w}^{<n>}$$

where

$$\tilde{w}^{<n>} = -\mathbb{J}_0^{-1} \{v^{<n>} \mathbb{J}_1 \Gamma_0 + f_n\}$$

is on the complement of the null space of  $\mathbb{J}_0$ ; that is  $[\tilde{w}^{<n>}, \Gamma_0^*] = 0$ . Hence,

$$w^{<n>} = [w^{<n>}, \Gamma_0^*] \Gamma_0 + \tilde{w}^{<n>}$$

for  $n = 1, 2, \dots$

The convergence and uniqueness of the series (III.10) may be established by well-known methods using the implicit function theorem. These methods imitate exactly those used to establish the solvability of (III.12).

#### IV. Factorization Theorems for Periodic Bifurcating Solutions of Autonomous Problems

We now consider an autonomous evolution problem generated by steady forcing

$$(IV.1) \quad \frac{dV}{dt} + F(\mu, V) = 0$$

where  $F(\mu, V)$  is a nonlinear operator from  $X$  to  $H$  satisfying the same hypotheses as  $F(t, \mu, V)$ . Of course, since  $F(\mu, V)$  is independent of  $t$  when  $V$  is,  $F$  is a  $T$ -periodic operator for every  $T \in \mathbb{R}_1$ .

By a periodic bifurcating solution we understand a solution of Hopf's type; problem (IV.1) is satisfied by a  $2\pi$ -periodic function  $U(s, \varepsilon) = U(s + 2\pi, \varepsilon)$  of  $s = \omega(\varepsilon)t$  where  $\omega(\varepsilon)$  is a frequency depending on the amplitude  $\varepsilon$  of  $U$  and  $\mu = \mu(\varepsilon)$  is a bifurcation parameter whose graph is the bifurcation curve. The  $T(\varepsilon) = 2\pi/\omega(\varepsilon)$ -periodic bifurcating solution satisfied the equation

$$(IV.2) \quad \omega(\varepsilon) \dot{U} + F(\mu(\varepsilon), U(s, \varepsilon)) = 0$$

where  $\dot{U} = U_s$ .

The solution  $V=0$  of (IV.1) bifurcates into a Hopf solution when a pair of complex conjugate eigenvalues passes into the left-hand side of the complex  $\sigma$  plane with finite speed as  $\mu$  is increased past zero. Locally, near  $\varepsilon=0$ ,  $U(s, \varepsilon)$ ,  $\omega(\varepsilon)$  and  $\mu(\varepsilon)$  are analytic in  $\varepsilon$  when  $F(\mu, V)$  is an analytic operator from  $X$  to  $H$  when  $\mu$  is in a neighborhood of zero in  $\mathbb{C}$ . In fact, locally,  $\mu(\varepsilon)$ ,  $\omega(\varepsilon)$  are even analytic functions so that the Hopf bifurcation is always one-sided. To study the stability of the Hopf bifurcation, we linearize (IV.1) for small disturbances  $W(t)$  of  $U(s, \varepsilon)$ . Then using Floquet theory  $W = e^{-\gamma t} \Gamma(t)$ , we find the spectral problem

$$(IV.3) \quad (-\gamma + \mathcal{J}(\varepsilon)) \Gamma = 0, \quad \Gamma(s) = \Gamma(s + 2\pi)$$

where

$$(IV.4) \quad \mathcal{J}(\varepsilon) = \omega(\varepsilon) \frac{d}{ds} + F_V(\mu(\varepsilon), U(s, \varepsilon)|)$$

is an operator mapping  $X_{2\pi} = \text{dom}(\mathcal{J})$  into  $H_{2\pi}$ .

JOSEPH (1976) and JOSEPH & NIELD (1976) have given a factorization theorem, similar to that stated in Theorem 3, which applies, in the analytic case, to (IV.3) when  $\varepsilon$  is small. CRANDALL & RABINOWITZ (1971) have proved a similar theorem for the non-analytic case.

Now I will prove a global factorization theorem for nonautonomous problems which is the direct analogue of Theorem 3 for autonomous problems. The assumptions required for the global theorem are almost identical to those required under Theorem 3.

**H.4.**  $U(s, \varepsilon) = U(s + 2\pi, \varepsilon)$ ,  $\mu(\varepsilon)$  and  $\omega(\varepsilon)$  are continuously differentiable functions of  $\varepsilon$  on some open interval  $I(\varepsilon)$ .  $\mu_\varepsilon(\varepsilon) \neq 0$  on  $I(\varepsilon)$  except possibly at isolated points.

**H.5.**  $\mathcal{J}(\varepsilon)$  is a Fredholm operator from  $X_{2\pi} \rightarrow H_{2\pi}$  with a compact resolvent from  $X_{2\pi} \rightarrow X_{2\pi}$ .

**H.6.**  $\gamma(\varepsilon)$  is an algebraically simple eigenvalue of  $\mathcal{J}(\varepsilon)$  for all  $\varepsilon \in I(\varepsilon)$  except possibly on an exceptional set of isolated points across which  $\Gamma(s, \varepsilon)$  and  $\Gamma^*(s, \varepsilon)$  are continuous.

**Theorem 7.** Let H.4, H.5 and H.6 hold and assume that

$$(IV.5) \quad [U_\varepsilon(\varepsilon), \Gamma^*(\varepsilon)]_{2\pi} \neq 0, \quad \forall \varepsilon \in I(\varepsilon).$$

Then there is a unique continuous function  $\hat{\gamma}(\varepsilon)$  defined on all of  $I(\varepsilon)$  such that

$$(IV.6) \quad \gamma(\varepsilon) = \mu_\varepsilon \hat{\gamma}(\varepsilon)$$

where

$$(IV.7) \quad \hat{\gamma}(\varepsilon) = -[F_\mu(\mu, U), \Gamma^*]_{2\pi} / [U_\varepsilon, \Gamma^*]_{2\pi}.$$

Moreover

$$(IV.8) \quad \Gamma = b(\varepsilon) \left\{ -\frac{\omega_\varepsilon}{\gamma} \dot{U}(s, \varepsilon) + U_\varepsilon + \mu_\varepsilon q \right\}$$

where  $b(\varepsilon)$  is a normalizing factor for  $\Gamma$  and

$$q(s, \varepsilon) = q(s + 2\pi, \varepsilon)$$

is uniquely determined by

$$(IV.9) \quad \hat{\gamma} U_\varepsilon + F_\mu(\mu, U) + \{\gamma - \mathcal{J}(\varepsilon)\} q = 0$$

and

$$(IV.10) \quad [q, \Gamma^*]_{2\pi} = 0.$$

**Proof.** The proof of Theorem 7 follows easily along the lines of proof for Theorem 3 after account is taken of the fact that in the autonomous problem,

$$(IV.11) \quad \mathcal{J} \dot{U} = 0, \quad \forall \varepsilon \in I.$$

Equation (IV.11) shows that  $\dot{U}$  is always an eigenfunction on the null space of  $\mathcal{J}(\varepsilon)$ . The eigenfunction  $\dot{U}$  does not exist in nonautonomous problems and is responsible for the spectral complications to be specified in Theorem 8. To prove Theorem 7 we note that

$$(IV.12) \quad -\bar{\gamma} \Gamma^* + \mathcal{J}^*(\varepsilon) \Gamma^* = 0, \quad \Gamma^*(s) = \Gamma^*(s + 2\pi)$$

where  $[\mathcal{J}u, v]_{2\pi} = [u, \mathcal{J}^*v]_{2\pi} \forall u \in X_{2\pi}, v \in X_{2\pi}^*$ . Using (IV.11) and (IV.12), we find that

$$0 = [\mathcal{J} \dot{U}, \Gamma^*]_{2\pi} = [\dot{U}, \mathcal{J}^* \Gamma^*]_{2\pi} = \gamma [\dot{U}, \Gamma^*]_{2\pi}$$

so that  $[\dot{U}, \Gamma^*]_{2\pi} = 0$  when  $\gamma \neq 0$  and, by continuity, also when  $\gamma = 0$ . Noting next that

$$(IV.13) \quad \mathcal{J}_\varepsilon U = \omega_\varepsilon \dot{U} + \mu_\varepsilon F_\mu(\mu, U),$$



we find, using (IV.4), that

$$(IV.14) \quad \begin{aligned} \mu_\varepsilon [F_\mu(\mu, U), \Gamma^*]_{2\pi} &= -[\mathcal{J} U_\varepsilon, \Gamma^*]_{2\pi} = -[U_\varepsilon, \mathcal{J}^* \Gamma^*]_{2\pi} \\ &= -\gamma [U_\varepsilon, \Gamma^*]_{2\pi}. \end{aligned}$$

The assumption (IV.5) together with (IV.14) implies (IV.6). Equation (IV.9) follows from (IV.3) after eliminating  $\mathcal{J} U_\varepsilon$  with (IV.4), and (IV.9) can be solved subject to (VI.10) because  $-\gamma + \mathcal{J}$  is a Fredholm operator and (IV.7) holds. The formulae proved hold whenever  $\gamma$  is a simple eigenvalue of  $\mathcal{J}(\varepsilon)$  and also, by continuity, across points in the exceptional set where  $\gamma(\varepsilon)$  is not simple. This proves Theorem 7.

*Remark 7.* Theorem 6 does not require that  $I(\varepsilon)$  contain an interval around the point  $\varepsilon=0$ . I have already noted that the Hopf solution bifurcates at  $\varepsilon=0$ . This point of bifurcation is in the exceptional set;  $\gamma(0)=0$  is a semi-simple double eigenvalue of  $\mathcal{J}(0)$ . The double eigenvalue splits into two simple branches when  $\varepsilon$  is varied away from zero. On each of these branches, Theorem 7 applies. A factorization theorem which is very close to the one given in Theorem 7 but which is more appropriate to the situation that prevails near  $\varepsilon=0$  has been given by JOSEPH (1976) and by CRANDALL & RABINOWITZ (1977).

*Remark 8.* Theorem 7 shows that  $\gamma(\varepsilon)$  changes sign across every critical point (points across which  $\mu_\varepsilon$  changes sign) at which  $\hat{\gamma}(\varepsilon) \neq 0$ . At such points  $\Gamma \sim \dot{U}$  if  $\omega_\varepsilon \neq 0$ . If  $\omega_\varepsilon = 0$  at a critical point, then  $\Gamma \sim U_\varepsilon$  there.

The interested reader may wish to formulate and prove a factorization theorem for autonomous problems, analogous to Theorem 5 for nonautonomous problems, which is valid for eigenvalues of higher multiplicity and does not require assumption H.6. Theorem 8 asserts that the eigenvalue  $\gamma(\varepsilon)=0$  of  $\mathcal{J}(\varepsilon)$  is always of higher multiplicity.

**Theorem 8.** *The algebraic multiplicity of the eigenvalue  $\gamma(\varepsilon)=0$  of  $\mathcal{J}(\varepsilon)$  is at least two. Relative to such an eigenvalue, we have at least a two-link Jordan chain*

$$(IV.15) \quad \begin{aligned} \mathcal{J} \dot{U} &= 0, \\ \mathcal{J} \{ -(U_\varepsilon + \mu_\varepsilon q) / \omega_\varepsilon \} &= \dot{U} \end{aligned}$$

whenever  $\omega_\varepsilon \neq 0$  when  $\gamma(\varepsilon)=0$ . If  $\omega_\varepsilon(\varepsilon)=0$  when  $\gamma(\varepsilon)=0$ , then the geometric multiplicity of  $\gamma(\varepsilon)=0$  is at least two, and  $\dot{U}$  and  $U_\varepsilon + \mu_\varepsilon q$  are both eigenfunctions on the null space of  $\mathcal{J}(\varepsilon)$ . The proof of Theorem 8 follows by inspection of the equations

$$\mathcal{J} U_\varepsilon + \omega_\varepsilon \dot{U} + \mu_\varepsilon F_\mu(\mu, U) = 0$$

and

$$\mathcal{J} q = \hat{\gamma} U_\varepsilon + F_\mu(\mu, U)$$

which hold when  $\gamma = \mu_\varepsilon \hat{\gamma} = 0$ .

*Remark 9.* For the Hopf bifurcation  $\omega(\varepsilon)$  is an even function so that  $\omega_\varepsilon(0)=0$ . Then Theorem 8 implies that  $\gamma(0)=0$  is a semi-simple double eigenvalue of  $\mathcal{J}(0)$ .

I leave the study of repeated Hopf-bifurcation at a non-semi-simple eigenvalue to a future work and close with an example in  $R_2$ , barely more

complicated than the example in  $R_1$  studied in §I for which factorization theorems and repeated branching of periodic solutions may be computed explicitly.

Let  $F(\mu, V)$  and  $\omega(V)$  be independent analytic functions of  $\mu, V \in R_1$  such that  $\omega(0) = 1$ ,  $F(\mu, 0) = 0$ ,  $F(0, V) \neq 0$  if  $V \neq 0$ ,  $F_V(\mu, 0) \leq 0$  if  $\mu \leq 0$ . Consider the following problem in  $R_2$ :

$$(IV.16) \quad \begin{pmatrix} x_t \\ y_t \end{pmatrix} + F(\mu, x^2 + y^2) \begin{pmatrix} x \\ y \end{pmatrix} - \omega(x^2 + y^2) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0.$$

Every solution of (IV.16) has

$$(IV.17) \quad (x^2 + y^2)_t + F(\mu, x^2 + y^2) = 0$$

so that  $x^2 + y^2 = 0$  is stable when  $\mu < 0$  and is unstable when  $\mu > 0$ . A solution  $x^2 + y^2 = \varepsilon^2$  with constant radius bifurcates at the point  $(\mu, \varepsilon) = (0, 0)$ . This solution exists when  $\mu = \mu(\varepsilon^2)$  so long as

$$F(\mu, \varepsilon^2) = 0, \quad \mu \equiv \mu(\varepsilon^2)$$

and is given by

$$(IV.18) \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} X \\ Y \end{pmatrix} = \varepsilon \begin{pmatrix} \cos s \\ \sin s \end{pmatrix} = \mathbf{X}$$

where  $s = \omega(\varepsilon^2)t$ . This solution is stable whenever  $F_V(\mu(\varepsilon^2), \varepsilon^2) > 0$  and is unstable where  $F_V(\mu(\varepsilon^2), \varepsilon^2) < 0$ . It is of interest to formulate a Floquet problem for the stability of (IV.18).

We find that small disturbances  $\phi(t) = e^{-\gamma t} \Gamma(s)$  of (IV.18) are governed by

$$(IV.19) \quad -\gamma \Gamma + \mathcal{J}(\varepsilon) \Gamma = 0, \quad \Gamma(s) = \Gamma(s + 2\pi)$$

where  $\mathcal{J}(\varepsilon) = \omega(\varepsilon^2) \frac{d}{ds} + J(\varepsilon)$  and

$$J(\varepsilon) = 2\varepsilon^2 F_V(\mu, \varepsilon^2) \begin{pmatrix} \cos^2 s & \sin s \cos s \\ \sin s \cos s & \sin^2 s \end{pmatrix} \\ + 2\varepsilon^2 \omega'(\varepsilon^2) \begin{pmatrix} \sin s \cos s & \sin^2 s \\ -\cos^2 s & -\sin s \cos s \end{pmatrix} - \omega(\varepsilon^2) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

$$\omega'(\varepsilon^2) = \frac{d\omega(\varepsilon^2)}{d\varepsilon^2}.$$

It is easy to verify that  $\dot{\mathbf{X}} \equiv d\mathbf{X}/ds$  is a solution of (IV.19) with  $\gamma = 0$ . This solution and  $\mathbf{X}_\varepsilon \equiv d\mathbf{X}/d\varepsilon$  are independent.

The problem

$$(IV.20) \quad -\gamma \Gamma^* + \mathcal{J}^* \Gamma^* = 0, \quad \Gamma^*(s) = \Gamma^*(s + 2\pi)$$

where

$$\mathcal{J}^*(\varepsilon) = \omega(\varepsilon^2) \frac{d}{ds} + J^*(\varepsilon),$$

$$J^*(\varepsilon) = 2\varepsilon^2 F_V(\mu, \varepsilon^2) \begin{pmatrix} \cos^2 s & \sin s \cos s \\ \sin s \cos s & \sin^2 s \end{pmatrix} \\ + 2\varepsilon^2 \omega'(\varepsilon^2) \begin{pmatrix} \sin s \cos s & -\cos^2 s \\ \sin^2 s & -\sin s \cos s \end{pmatrix} - \omega(\varepsilon^2) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

is adjoint to (IV.19).

Since (IV.16) satisfies all the conditions for Hopf bifurcation, the factorization Theorem 7 applies. We may write the factorization as follows:

$$(IV.21) \quad \begin{pmatrix} \Gamma_1 \\ \Gamma_2 \end{pmatrix} = b \left\{ \frac{2\varepsilon \omega'}{\gamma} \begin{pmatrix} X \\ Y \end{pmatrix} + \begin{pmatrix} X_\varepsilon \\ Y_\varepsilon \end{pmatrix} + 2\varepsilon \mu_\varepsilon \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \right\}$$

and

$$(IV.22) \quad \gamma(\varepsilon^2) = \mu_\varepsilon(\varepsilon^2) \hat{\gamma}(\varepsilon).$$

Inserting (IV.21) and (IV.22) into (IV.19), we find that

$$(IV.23) \quad -(\hat{\gamma} \mathbf{X}_\varepsilon + F_\mu(\mu, \varepsilon^2) \mathbf{X}) + \varepsilon \{ -\gamma \mathbf{q} + \mathcal{J}(\varepsilon) \mathbf{q} \} = 0.$$

Equation (IV.23) may be simplified by putting  $\mathbf{X} = \varepsilon \mathbf{X}_\varepsilon$ .

We next note that when  $\varepsilon \neq 0$  is small,  $\gamma(\varepsilon^2)$  is a simple eigenvalue of (IV.19). (This follows from local analysis of Hopf bifurcation.) It follows from Fredholm theory that (IV.23) is uniquely solvable on the complement of the null space of the operator  $-\gamma + \mathcal{J}(\varepsilon)$  if and only if

$$(IV.24) \quad (\hat{\gamma} + \varepsilon F_\mu(\mu, \varepsilon^2)) \int_0^{2\pi} (\Gamma_1^* \cos s + \Gamma_2^* \sin s) ds = 0.$$

Moreover, it is readily verified that

$$\begin{pmatrix} \Gamma_1^* \\ \Gamma_2^* \end{pmatrix} = C_1 \begin{pmatrix} \cos s \\ \sin s \end{pmatrix}$$

if

$$(IV.25) \quad \gamma = 2\varepsilon^2 F_V(\mu, \varepsilon^2) = -\varepsilon \mu_\varepsilon F_\mu(\mu, \varepsilon^2) = \mu_\varepsilon(\varepsilon^2) \hat{\gamma}(\varepsilon).$$

Hence

$$\hat{\gamma} = -\varepsilon F_\mu(\mu(\varepsilon^2), \varepsilon^2),$$

and all solutions of (IV.23) with  $\gamma = 2\varepsilon^2 F_V(\mu, \varepsilon^2)$  are proportional to  $\Gamma$  and  $q_1 = q_2 = 0$ .

Returning now to (IV.21) with  $\gamma = -\mu_\varepsilon \varepsilon F_\mu(\mu, \varepsilon^2) = -2\mu'(\varepsilon^2) \varepsilon^2 F_\mu(\mu, \varepsilon^2)$ , we have

$$(IV.26) \quad \begin{aligned} \begin{pmatrix} \Gamma_1 \\ \Gamma_2 \end{pmatrix} &= \frac{-\omega'(\varepsilon^2)}{\varepsilon\sqrt{\omega'^2 + \mu'^2 F_\mu^2}} \begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} + \frac{\mu' F_\mu}{\sqrt{\omega'^2 + \mu'^2 F_\mu^2}} \begin{pmatrix} X_\varepsilon \\ Y_\varepsilon \end{pmatrix} \\ &= \frac{-\omega'(\varepsilon^2)}{\sqrt{\omega'^2 + \mu'^2 F_\mu^2}} \begin{pmatrix} -\sin s \\ \cos s \end{pmatrix} + \frac{\mu' F_\mu}{\sqrt{\omega'^2 + \mu'^2 F_\mu^2}} \begin{pmatrix} \cos s \\ \sin s \end{pmatrix} \end{aligned}$$

where  $\mu' F_\mu = \mu'(\varepsilon^2)F_\mu(\mu(\varepsilon^2), \varepsilon^2)$ .

It is of interest to consider the stability of (IV.18) from a different point view involving the monodromy matrix and its eigenvalues, the Floquet multipliers. A small disturbance  $\phi$  of  $\mathbf{X}$  satisfies

$$(IV.27) \quad \phi_t + J(\varepsilon)\phi = 0.$$

There are two and only two independent solutions of (IV.27),  $\phi^{(1)}$  and  $\phi^{(2)}$ . We choose  $\phi^{(1)}$  and  $\phi^{(2)}$  so that the fundamental solution matrix

$$\phi(t) = \begin{bmatrix} \phi_1^{(1)}(t) & \phi_1^{(2)}(t) \\ \phi_2^{(1)}(t) & \phi_2^{(2)}(t) \end{bmatrix}$$

satisfies  $\phi(0) = \mathbf{1}$ , where  $\mathbf{1}$  is the unit matrix. We find that

$$\phi^{(1)} = \frac{\omega'}{F_V} (1 - e^{-\gamma t}) \begin{bmatrix} -\sin s \\ \cos s \end{bmatrix} + \begin{bmatrix} \cos s \\ \sin s \end{bmatrix} e^{-\gamma t},$$

where  $\gamma$  satisfies (IV.25), and

$$\phi^{(2)} = \begin{bmatrix} -\sin s \\ \cos s \end{bmatrix} = \frac{1}{\varepsilon} \dot{\mathbf{X}}.$$

The Floquet multipliers  $\lambda$  are the eigenvalues of the monodromy matrix  $\phi\left(\frac{2\pi}{\omega}\right)$ ; that is, of the matrix

$$\begin{bmatrix} \phi^{(1)}\left(\frac{2\pi}{\omega}\right) & \phi^{(2)}\left(\frac{2\pi}{\omega}\right) \\ \phi_2^{(1)}\left(\frac{2\pi}{\omega}\right) & \phi_2^{(2)}\left(\frac{2\pi}{\omega}\right) \end{bmatrix} = \begin{bmatrix} e^{-2\pi\gamma/\omega} & 0 \\ \frac{\omega'}{F_V}(1 - e^{-2\pi\gamma/\omega}) & 1 \end{bmatrix}.$$

The eigenvalues of this matrix satisfy the equation  $(\lambda - 1)(\lambda - e^{-2\pi\gamma/\omega}) = 0$ . It follows that  $\lambda = 1$  and  $\lambda = e^{-2\pi\gamma/\omega}$  are algebraically simple eigenvalues of the monodromy matrix whenever  $e^{-2\pi\gamma/\omega} \neq 1$  and are algebraically double whenever  $\gamma = 2\varepsilon^2 F_V = 0$ . If  $\gamma = 0$  there are still two fundamental solutions of (IV.27):  $\phi^{(2)}$  and

$$\phi^{(1)} = 2\varepsilon^2 \omega'(\varepsilon^2) t \begin{bmatrix} -\sin s \\ \cos s \end{bmatrix} + \begin{bmatrix} \cos s \\ \sin s \end{bmatrix}.$$

Of these, only  $\phi^{(2)}$  is a proper  $2\pi$ -periodic eigenvector. Since  $\phi^{(2)} = \dot{\mathbf{X}}/\varepsilon$ ,  $\mathcal{J}\dot{\mathbf{X}} = 0$  when  $\gamma = 0$  and  $\mathcal{J}(-X_\varepsilon/\omega_\varepsilon) = \dot{\mathbf{X}}$ . Hence when  $\gamma = 0$ , we have a two-link Jordan chain in the frame of Theorem 8.

The example exhibits the following properties:

1. It undergoes Hopf bifurcation at  $\varepsilon = 0$ .
2. The factorization theorem holds for all values of  $\varepsilon$  for which  $\omega$  and  $F$  and its first derivatives are defined.
3.  $F(\mu, \varepsilon^2)$  and  $\omega(\varepsilon^2)$  are independent functions. In general,  $\mu'(\varepsilon^2)$  and  $\omega'(\varepsilon^2)$  do not vanish simultaneously.
4.  $\gamma = 0$  is always an algebraically double eigenvalue of  $\mathcal{J}(\varepsilon)$ . It is geometrically simple and algebraically double when  $\omega'(\varepsilon^2) \neq 0$  at points at which  $\gamma(\varepsilon) = 0$ . If  $\omega'(\varepsilon^2) = 0$  where  $\gamma(\varepsilon) = 0$ , then  $\gamma = 0$  is a geometrically and algebraically double eigenvalue.
5. For suitably chosen functions  $F(\mu, V)$ , we get secondary and repeated bifurcation of  $T(\varepsilon)$ -periodic solutions in  $t$  ( $2\pi$ -periodic solutions in  $s$ ) of constant radius  $\varepsilon$ . In fact, (IV.17) shows that the study of such bifurcation may be reduced to an equation in  $R_1$  whose bifurcation properties were characterized completely in §1.

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(Received February 9, 1977; revised July 26, 1977)